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ORIGINAL PAPER

On the space of Laplace transformable distributions

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Abstract
We show that the space $\mathcal{S}'(\Gamma)$ of Laplace transformable distributions, where $\Gamma \subseteq \mathbb{R}^d$ is a non-empty convex open set, is an ultrabornological (PLS)-space. Moreover, we determine an explicit topological predual of $\mathcal{S}'(\Gamma)$.

Keywords Laplace transform · Distributions · Ultrabornological (PLS)-spaces · Short-time Fourier transform

Mathematics Subject Classification Primary 46F05 · 46A13 · Secondary 81S30

1 Introduction

Schwartz introduced the space $\mathcal{S}'(\Gamma)$ of Laplace transformable distributions as

$$\mathcal{S}'(\Gamma) = \{f \in \mathcal{D}'(\mathbb{R}^d) \mid e^{-\xi\cdot x} f(x) \in \mathcal{S}'(\mathbb{R}^d_x)\ \forall \xi \in \Gamma\},$$

where $\Gamma \subseteq \mathbb{R}^d$ is a non-empty convex set [1, p. 303]. This space is endowed with the projective limit topology with respect to the mappings $\mathcal{S}'(\Gamma) \to \mathcal{S}'(\mathbb{R}^d)$, $f \mapsto e^{-\xi\cdot x} f(x)$ for $\xi \in \Gamma$. The second author together with Kunzinger and Ortner [2] recently presented two new proofs of Schwartz's exchange theorem for the Laplace transform of vector-valued distributions [3, Prop. 4.3, p. 186]. Their methods required them to show that $\mathcal{S}'(\Gamma)$ is complete, nuclear and dual-nuclear [2, Lemma 5]. Following a suggestion of Ortner, in this article, we further study the locally convex structure of the space $\mathcal{S}'(\Gamma)$.

In order to be able to apply functional analytic tools such as De Wilde's open mapping and closed graph theorems [4, Theorem 24.30 and Theorem 24.31] or the theory of the derived projective limit functor [5], it is important to determine when a space is ultrabornological. This is usually straightforward if the space is given by a suitable inductive limit; in fact,

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ultrabornological spaces are exactly the inductive limits of Banach spaces [4, Proposition 24.14]. The situation for projective limits, however, is more complicated. Particularly, this applies to the class of (PLS)-spaces (i.e., countable projective limits of (DFS)-spaces). The problem of ultrabornologicity has been extensively studied in this class, both from an abstract point of view as for concrete function and distribution spaces; see the survey article [6] of Domański and the references therein.

In the last part of his doctoral thesis [7, Chap. II, Thm. 16, p. 131], Grothendieck showed that the convolutor space $\mathcal{O}'_C$ is ultrabornological. He proved that $\mathcal{O}'_C$ is isomorphic to a complemented subspace of the sequence space $s\widehat{\otimes}s'$ and verified directly that the latter space is ultrabornological. Much later, a different proof was given by Larcher and Wengenroth using homological methods [8]. The first author and Vindas [9] extended this result to a considerably wider setting by studying the locally convex structure of a general class of weighted convolutor spaces. More precisely, they characterized when such spaces are ultrabornological and determined explicit topological preduals for them. One of their main tools is a topological description of these convolutor spaces in terms of the short-time Fourier transform (STFT).

In this work, we will identify $\mathcal{S}'(\Gamma)$ with a particular instance of the convolutor spaces considered in [9]. To this end, we make a detailed study of the mapping properties of the STFT on $\mathcal{S}'(\Gamma)$. Once this identification has been established, we use Theorem 1.1 from [9] (see also Theorem 4.2 below) to show that $\mathcal{S}'(\Gamma)$ is an ultrabornological (PLS)-space and that it admits a weighted (LF)-space of smooth functions on $\mathbb{R}^d$ as a topological predual.

2 Weighted spaces of continuous functions

For formulating the mapping properties of the STFT we recall the following notions from [9,10].

Each non-negative function v on $\mathbb{R}^d$ defines a weighted seminorm on $C(\mathbb{R}^d)$ by

$$\|f\|_v := \sup_{x\in\mathbb{R}^d} |f(x)|\, v(x).$$

We endow the space

$$Cv(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) \mid \|f\|_v < \infty\}$$

with this seminorm; it is a Banach space if v is positive and continuous. A pointwise decreasing sequence $\mathcal{V} = (v_N)_{N\in\mathbb{N}}$ of positive continuous functions on $\mathbb{R}^d$ is called a *decreasing weight system*. With this, we define the (LB)-space

$$\mathcal{V}C(\mathbb{R}^d) := \varinjlim_{N\in\mathbb{N}} Cv_N(\mathbb{R}^d).$$

We consider the following condition on a decreasing weight system $\mathcal{V}$, see [10, p. 114]:

$$\forall N \in \mathbb{N}\, \exists M > N : \lim_{|x|\to\infty} \frac{v_M(x)}{v_N(x)} = 0. \tag{V}$$

The *maximal Nachbin family associated with* $\mathcal{V}$ is defined to be the family $\overline{V} = \overline{V}(\mathcal{V})$ consisting of all non-negative upper semicontinuous functions v on $\mathbb{R}^d$ such that

$$\forall N \in \mathbb{N} : \sup_{x\in\mathbb{R}^d} \frac{v(x)}{v_N(x)} < \infty.$$

The *projective hull of* $\mathcal{V}C(\mathbb{R}^d)$ is defined as

$$C\overline{V}(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) \mid \|f\|_v < \infty \; \forall v \in \overline{V}\}.$$

and endowed with the locally convex topology generated by the system of seminorms $\{\|\cdot\|_v \mid v \in \overline{V}\}$. The spaces $\mathcal{V}C(\mathbb{R}^d)$ and $C\overline{V}(\mathbb{R}^d)$ always coincide as sets and, if $\mathcal{V}$ satisfies condition (V), also as locally convex spaces [10, Thm. 1.3 (d), p. 118].

A pointwise increasing sequence $\mathcal{W} = (w_N)_{N\in\mathbb{N}}$ of positive continuous functions on $\mathbb{R}^d$ is called an *increasing weight system*. Given such a system, we define the Fréchet space

$$\mathcal{W}C(\mathbb{R}^d) := \varprojlim_{N\in\mathbb{N}} Cw_N(\mathbb{R}^d).$$

We consider the following conditions on an increasing weight system $\mathcal{W}$:

$$\forall N \in \mathbb{N}\, \exists M > N : \lim_{|x|\to\infty} \frac{w_N(x)}{w_M(x)} = 0, \tag{2.1}$$

$$\forall N \in \mathbb{N}\, \exists M > N : \frac{w_N}{w_M} \in L^1(\mathbb{R}^d), \tag{2.2}$$

$$\forall N \in \mathbb{N}\, \exists M_1, M_2 \geq N\, \exists C > 0\, \forall x, y \in \mathbb{R}^d : w_N(x+y) \leq C w_{M_1}(x) w_{M_2}(y). \tag{2.3}$$

In the next lemma, we obtain a concrete representation of the ε-tensor product of weighted spaces of continuous functions.

Lemma 2.1 *Let* $\mathcal{W} = (w_N)_{N\in\mathbb{N}}$ *be an increasing weight system and* $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ *a decreasing weight system satisfying* (V)*. Then, we have the identification*

$$\mathcal{W}C(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon \mathcal{V}C(\mathbb{R}^d_\xi) = \{f \in C(\mathbb{R}^{2d}_{x,\xi}) \mid \forall N \in \mathbb{N}\, \exists n \in \mathbb{N} : \|f\|_{w_N\otimes v_n} < \infty\},$$

where we set $\|f\|_{w\otimes v} := \sup_{(x,\xi)\in\mathbb{R}^{2d}} |f(x,\xi)|\, w(x)v(\xi)$ *for non-negative functions* w, v *on* $\mathbb{R}^d$*. Moreover,* $f \in C(\mathbb{R}^{2d}_{x,\xi})$ *belongs to* $\mathcal{W}C(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon \mathcal{V}C(\mathbb{R}^d_\xi)$ *if and only if* $\|f\|_{w_N\otimes v} < \infty$ *for all* $N \in \mathbb{N}$ *and* $v \in \overline{V}$*. Consequently, the topology of* $\mathcal{W}C(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon \mathcal{V}C(\mathbb{R}^d_\xi)$ *is generated by the system of seminorms* $\{\|\cdot\|_{w_N\otimes v} \mid N \in \mathbb{N}, v \in \overline{V}\}$.

Proof This follows from the fact that the ε-tensor product commutes with projective limits and [10, Thm. 3.1 (c), p. 137]. □

3 The short-time Fourier transform on $\mathcal{D}'(\mathbb{R}^d)$

The translation and modulation operators are denoted by $T_x f(t) = f(t-x)$ and $M_\xi f(t) = e^{2\pi i\xi\cdot t} f(t)$ for $x, \xi \in \mathbb{R}^d$. The *short-time Fourier transform (STFT)* of a function $f \in L^2(\mathbb{R}^d)$ with respect to a window function $\psi \in L^2(\mathbb{R}^d)$ is defined as

$$V_\psi f(x,\xi) := (f, M_\xi T_x \psi)_{L^2} = \int_{\mathbb{R}^d} f(t)\overline{\psi(t-x)} e^{-2\pi i\xi\cdot t}\, \mathrm{d}t, \qquad (x,\xi) \in \mathbb{R}^{2d},$$

where $(\cdot,\cdot)_{L^2}$ denotes the inner product on $L^2(\mathbb{R}^d)$. We have that $\|V_\psi f\|_{L^2(\mathbb{R}^{2d})} = \|\psi\|_{L^2}\|f\|_{L^2}$. In particular, the mapping $V_\psi : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ is continuous. The adjoint of V_ψ is given by the weak integral

$$V_\psi^* F = \int\int_{\mathbb{R}^{2d}} F(x,\xi) M_\xi T_x \psi \,\mathrm{d}x\,\mathrm{d}\xi, \qquad F \in L^2(\mathbb{R}^{2d}).$$

If $\psi \neq 0$ and $\gamma \in L^2(\mathbb{R}^d)$ is a synthesis window for ψ, that is, $(\gamma, \psi)_{L^2} \neq 0$, then

$$\frac{1}{(\gamma, \psi)_{L^2}} V_\gamma^* \circ V_\psi = \mathrm{id}_{L^2(\mathbb{R}^d)} .$$

We refer to [11] for further properties of the STFT.

Next, we explain how the STFT can be extended to the space of distributions; see [9, Sect. 2] for details and proofs. We set $\mathcal{V}_{\mathrm{pol}} = ((1 + |\cdot|)^{-N})_{N\in\mathbb{N}}$. Fix a window function $\psi \in \mathcal{D}(\mathbb{R}^d)$. For $f \in \mathcal{D}'(\mathbb{R}^d)$ we define

$$V_\psi f(x, \xi) := \langle f, \overline{M_\xi T_x \psi}\rangle, \qquad (x, \xi) \in \mathbb{R}^{2d}.$$

Clearly, $V_\psi f$ is a continuous function on $\mathbb{R}^{2d}$. In fact,

$$V_\psi : \mathcal{D}'(\mathbb{R}^d) \to C(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon \mathcal{V}_{\mathrm{pol}} C(\mathbb{R}^d_\xi)$$

is a well-defined continuous mapping [9, Lemma 2.2]. We *define* the adjoint STFT of an element $F \in C(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon \mathcal{V}_{\mathrm{pol}} C(\mathbb{R}^d_\xi)$ as the distribution

$$\langle V_\psi^* F, \varphi\rangle := \int\int_{\mathbb{R}^{2d}} F(x, \xi) V_{\overline{\psi}}\varphi(x, -\xi)\, \mathrm{d}x\, \mathrm{d}\xi, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Then,

$$V_\psi^* : C(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon \mathcal{V}_{\mathrm{pol}} C(\mathbb{R}^d_\xi) \to \mathcal{D}'(\mathbb{R}^d)$$

is a well-defined continuous mapping by [9, Prop. 2.2]. Finally, if $\psi \neq 0$ and $\gamma \in \mathcal{D}(\mathbb{R}^d)$ is a synthesis window for ψ, then the following reconstruction formula holds [9, Prop. 2.4]:

$$\frac{1}{(\gamma, \psi)_{L^2}} V_\gamma^* \circ V_\psi = \mathrm{id}_{\mathcal{D}'(\mathbb{R}^d)} . \tag{3.1}$$

4 Duals of inductive limits of weighted spaces of smooth functions

Let v be a non-negative function on $\mathbb{R}^d$ and $n \in \mathbb{N}$. We define $\mathcal{B}^n_v(\mathbb{R}^d)$ as the seminormed space consisting of all $\varphi \in C^n(\mathbb{R}^d)$ such that

$$\|\varphi\|_{v,n} := \max_{|\alpha|\le n} \sup_{x\in\mathbb{R}^d} \left|\partial^\alpha \varphi(x)\right| v(x) < \infty.$$

As before, $\mathcal{B}^n_v(\mathbb{R}^d)$ is a Banach space if v is positive and continuous. Let $\mathcal{W} = (w_N)_{N\in\mathbb{N}}$ be an increasing weight system. We define the (LF)-space

$$\mathcal{B}_{\mathcal{W}^\circ}(\mathbb{R}^d) := \varinjlim_{N\in\mathbb{N}} \varprojlim_{n\in\mathbb{N}} \mathcal{B}^n_{1/w_N}(\mathbb{R}^d).$$

We endow the dual space $\mathcal{B}'_{\mathcal{W}}(\mathbb{R}^d) := (\mathcal{B}_{\mathcal{W}^\circ}(\mathbb{R}^d))'$ with the strong topology. If $\mathcal{W}$ satisfies (2.1), then $\mathcal{D}(\mathbb{R})$ is densely and continuously included in $\mathcal{B}_{\mathcal{W}^\circ}(\mathbb{R}^d)$ and therefore $\mathcal{B}'_{\mathcal{W}}(\mathbb{R}^d)$ is a vector subspace of $\mathcal{D}'(\mathbb{R}^d)$.

On the other hand, we define the convolutor space

$$\mathcal{O}'_{C,\mathcal{W}}(\mathbb{R}^d) := \{f \in \mathcal{D}'(\mathbb{R}^d) \mid f * \varphi \in \mathcal{W}C(\mathbb{R}^d)\ \forall \varphi \in \mathcal{D}(\mathbb{R}^d)\}.$$

For $f \in \mathcal{O}'_{C,\mathcal{W}}(\mathbb{R}^d)$ fixed, the mapping

$$\mathcal{D}(\mathbb{R}^d) \to \mathcal{W}C(\mathbb{R}^d), \quad \varphi \mapsto f * \varphi$$

is continuous, as follows from the closed graph theorem. We endow $\mathcal{O}'_{C,\mathcal{W}}(\mathbb{R}^d)$ with the topology induced via the embedding

$$\mathcal{O}'_{C,\mathcal{W}}(\mathbb{R}^d) \to L_\beta(\mathcal{D}(\mathbb{R}^d), \mathcal{W}C(\mathbb{R}^d)), \quad f \mapsto [\varphi \mapsto f * \varphi],$$

where β denotes the topology of uniform convergence on bounded sets.

In [9] the structural and topological properties of the spaces $\mathcal{B}'_{\mathcal{W}}(\mathbb{R}^d)$ and $\mathcal{O}'_{C,\mathcal{W}}(\mathbb{R}^d)$ are discussed. We now present the main results of this paper and refer to [9] for more details and proofs.[1]

Proposition 4.1 [9, Prop. 4.2] *Let $\mathcal{W}$ be an increasing weight system satisfying (2.1), (2.2) and (2.3) and let $\psi \in \mathcal{D}(\mathbb{R}^d)$. Then, the mappings*

$$V_\psi : \mathcal{O}'_{C,\mathcal{W}}(\mathbb{R}^d) \to \mathcal{W}C(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon \mathcal{V}_{\mathrm{pol}}C(\mathbb{R}^d_\xi)$$

and

$$V^*_\psi : \mathcal{W}C(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon \mathcal{V}_{\mathrm{pol}}C(\mathbb{R}^d_\xi) \to \mathcal{O}'_{C,\mathcal{W}}(\mathbb{R}^d)$$

are well-defined and continuous.

Theorem 4.2 [9, Thm. 3.4, Thm. 4.6 and Thm. 4.15] *Let $\mathcal{W} = (w_N)_{N\in\mathbb{N}}$ be an increasing weight system satisfying (2.1), (2.2) and (2.3). Then, $\mathcal{B}'_{\mathcal{W}}(\mathbb{R}^d) = \mathcal{O}'_{C,\mathcal{W}}(\mathbb{R}^d)$ as sets and the inclusion mapping $\mathcal{B}'_{\mathcal{W}}(\mathbb{R}^d) \to \mathcal{O}'_{C,\mathcal{W}}(\mathbb{R}^d)$ is continuous. Moreover, the following statements are equivalent:*

(i) *$\mathcal{B}'_{\mathcal{W}}(\mathbb{R}^d) = \mathcal{O}'_{C,\mathcal{W}}(\mathbb{R}^d)$ as locally convex spaces.*
(ii) *$\mathcal{O}'_{C,\mathcal{W}}(\mathbb{R}^d)$ is an ultrabornological (PLS)-space.*
(iii) *The (LF)-space $\mathcal{B}_{\mathcal{W}^\circ}(\mathbb{R}^d)$ is complete.*
(iv) *$\mathcal{W}$ satisfies*

$$\forall N \in \mathbb{N}\, \exists M \geq N\, \forall P \geq M\, \exists \theta \in (0,1)\, \exists C > 0\, \forall x \in \mathbb{R}^d : \\ w_N(x)^{1-\theta} w_P(x)^\theta \leq C w_M(x). \tag{4.1}$$

Remark 4.3 Condition (4.1) is closely connected with D. Vogt's condition (Ω) that plays an essential role in the structure and splitting theory for Fréchet spaces.

5 The space $\mathcal{S}'(\Gamma)$

Our next goal is to characterize $\mathcal{S}'(\Gamma)$ in terms of the STFT.

Let $\emptyset \neq \Gamma \subseteq \mathbb{R}^d$ be open and convex. We denote by $\mathrm{CCS}(\Gamma)$ the family of all non-empty compact convex subsets of Γ and by $\mathfrak{B}(\mathcal{S}(\mathbb{R}^d))$ the family of all bounded subsets of $\mathcal{S}(\mathbb{R}^d)$. The topology of $\mathcal{S}'(\Gamma)$ can easily be described by a system of concrete seminorms which essentially is due to Schwartz [1, p. 301]; for this, note that the system of convex hulls of finite sets is cofinal in $\mathrm{CCS}(\Gamma)$:

[1] To be precise, the spaces considered in [9], denoted there by $(\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'$ and $\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}})$, differ from $\mathcal{B}'_{\mathcal{W}}(\mathbb{R}^d)$ and $\mathcal{O}'_{C,\mathcal{W}}(\mathbb{R}^d)$ defined above. However, if $\mathcal{W}$ satisfies (2.1), (2.2) and (2.3), then $\mathcal{B}'_{\mathcal{W}}(\mathbb{R}^d) = (\dot{\mathcal{B}}_{\mathcal{W}^\circ}(\mathbb{R}^d))'$ and $\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}}) = \mathcal{O}'_{C,\mathcal{W}}(\mathbb{R}^d)$; the first equality is clear, while the second one follows from [9, Prop. 6.2]. Moreover, under these conditions, all statements and proofs from [9] remain valid if one replaces $L^1_{\mathcal{W}}(\mathbb{R}^d)$ by $\mathcal{W}C(\mathbb{R}^d)$.

Lemma 5.1 [1, p. 301] *Let $\emptyset \neq \Gamma \subseteq \mathbb{R}^d$ be open and convex. For all $K \in \mathrm{CCS}(\Gamma)$ and $B \in \mathfrak{B}(\mathcal{S}(\mathbb{R}^d))$ we have that*

$$p_{K,B}(f) := \sup_{\eta \in K} \sup_{\varphi \in B} \left| \langle e^{-\eta \cdot x} f(x), \varphi(x) \rangle \right| < \infty, \qquad f \in \mathcal{S}'(\Gamma).$$

Moreover, the topology of $\mathcal{S}'(\Gamma)$ is generated by the system of seminorms $\{p_{K,B} \mid K \in \mathrm{CCS}(\Gamma), B \in \mathfrak{B}(\mathcal{S}(\mathbb{R}^d))\}$.

We need to introduce some additional terminology. Given a non-empty compact convex subset K of $\mathbb{R}^d$, we define its *supporting function* as

$$h_K(x) = \max_{\eta \in K} x \cdot \eta, \qquad x \in \mathbb{R}^d.$$

It is clear from the definition that h_K is subadditive and positive homogeneous of degree one. In particular, h_K is convex. Supporting functions have the following elementary properties.

Lemma 5.2 [12, Cor. 1.8.2 and Prop. 1.8.3] *Let K_1 and K_2 be non-empty compact convex subsets of $\mathbb{R}^d$.*

(a) *$K_1 \subseteq K_2$ if and only if $h_{K_1}(x) \leq h_{K_2}(x)$ for all $x \in \mathbb{R}^d$.*
(b) *$h_{K_1+K_2}(x) = h_{K_1}(x) + h_{K_2}(x)$ for all $x \in \mathbb{R}^d$.*

Example 5.3 For $r > 0$ we have $h_{\overline{B}(0,r)}(x) = r\,|x|$ for all $x \in \mathbb{R}^d$, where $\overline{B}(0, r)$ denotes the closed ball in $\mathbb{R}^d$ centered at the origin with radius r. Next, let K be a non-empty compact convex subset of $\mathbb{R}^d$ and $\varepsilon > 0$. We set $K_\varepsilon = K + \overline{B}(0, \varepsilon)$. Lemma 5.2 and the above yield that $h_{K_\varepsilon}(x) = h_K(x) + \varepsilon\,|x|$ for all $x \in \mathbb{R}^d$.

Let $\emptyset \neq \Gamma \subseteq \mathbb{R}^d$ be open and convex and let $(K_N)_{N\in\mathbb{N}} \subset \mathrm{CCS}(\Gamma)$ be such that $K_N \subseteq K_{N+1}$ for all $N \in \mathbb{N}$ and $\Gamma = \bigcup_N K_N$. Lemma 5.2 yields that $\mathcal{W} = (e^{h_{-K_N}})_{N\in\mathbb{N}}$ is an increasing weight system. We set $C_\Gamma(\mathbb{R}^d) := \mathcal{W}C(\mathbb{R}^d)$. Clearly, the definition of $C_\Gamma(\mathbb{R}^d)$ is independent of the chosen sequence $(K_N)_{N\in\mathbb{N}}$. The next result is the key observation of this article.

Proposition 5.4 *Let $\emptyset \neq \Gamma \subseteq \mathbb{R}^d$ be open and convex and let $\psi \in \mathcal{D}(\mathbb{R}^d)$. Then, the mappings*

$$V_\psi : \mathcal{S}'(\Gamma) \to C_\Gamma(\mathbb{R}^d_x) \widehat{\otimes}_\varepsilon \mathcal{V}_{\mathrm{pol}} C(\mathbb{R}^d_\xi)$$

and

$$V^*_\psi : C_\Gamma(\mathbb{R}^d_x) \widehat{\otimes}_\varepsilon \mathcal{V}_{\mathrm{pol}} C(\mathbb{R}^d_\xi) \to \mathcal{S}'(\Gamma)$$

are well-defined and continuous.

We need some preparation for the proof of Proposition 5.4. Firstly, Lemma 2.1 implies that the topology of $C_\Gamma(\mathbb{R}^d_x) \widehat{\otimes}_\varepsilon \mathcal{V}_{\mathrm{pol}} C(\mathbb{R}^d_\xi)$ is generated by the system of seminorms

$$\|f\|_{K,v} := \sup_{(x,\xi)\in\mathbb{R}^{2d}} |f(x,\xi)|\, e^{h_{-K}(x)} v(\xi) < \infty, \qquad K \in CCS(\Gamma), v \in \overline{V}(\mathcal{V}_{\mathrm{pol}}).$$

For $k, n \in \mathbb{N}$ we write

$$\|\varphi\|_{\mathcal{S}^n_k} := \max_{|\alpha|\leq n} \sup_{x\in\mathbb{R}^d} \left|\partial^\alpha \varphi(x)\right| (1 + |x|)^k, \qquad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

The topology of $\mathcal{S}(\mathbb{R}^d)$ is generated by the system of seminorms $\{\| \cdot \|_{\mathcal{S}^n_k} \mid k, n \in \mathbb{N}\}$. We now give two technical lemmas.

Lemma 5.5 *Let $\psi \in \mathcal{D}(\mathbb{R}^d)$, $K \subset \mathbb{R}^d$ be compact, $v \in \overline{V}(\mathcal{V}_{\text{pol}})$ and $\varepsilon > 0$. Then,*

$$\{e^{\eta\cdot(t-x)}\overline{M_\xi T_x \psi}(t)e^{-\varepsilon|x|}v(\xi) \mid (x,\xi) \in \mathbb{R}^{2d}, \eta \in K\} \in \mathfrak{B}(\mathcal{S}(\mathbb{R}^d_t)).$$

Proof Choose $r > 0$ such that $\operatorname{supp}\psi \subseteq \overline{B}(0,r)$ and $R \geq 1$ such that $K \subseteq \overline{B}(0,R)$. For all $k, n \in \mathbb{N}$ we have that

$$\begin{aligned}
&\sup_{(x,\xi)\in\mathbb{R}^{2d}} \sup_{\eta\in K} e^{-\varepsilon|x|}v(\xi)\|e^{\eta\cdot(t-x)}\overline{M_\xi T_x\psi}(t)\|_{\mathcal{S}^n_{k,t}} \leq \sup_{(x,\xi)\in\mathbb{R}^{2d}} \sup_{\eta\in K} e^{-\varepsilon|x|}v(\xi)\cdot \\
&\max_{|\alpha|\leq n} \sup_{x\in\mathbb{R}^d} \sum_{\beta\leq\alpha}\sum_{\gamma\leq\beta}\binom{\alpha}{\beta}\binom{\beta}{\gamma}|\eta|^{|\alpha|-|\beta|}e^{\eta\cdot(t-x)}(2\pi|\xi|)^{|\gamma|}\left|\partial^{\beta-\gamma}\overline{\psi}(t-x)\right|(1+|t|)^k \\
&\leq e^{Rr}(8\pi R)^n \max_{|\alpha|\leq n}\|\partial^\alpha\overline{\psi}\|_{L^\infty}(1+r)^k \sup_{x\in\mathbb{R}^d} e^{-\varepsilon|x|}(1+|x|)^k \sup_{\xi\in\mathbb{R}^d} v(\xi)(1+|\xi|)^n \\
&< \infty.
\end{aligned}$$

□

Lemma 5.6 *Let $\psi \in \mathcal{D}(\mathbb{R}^d)$ and $\eta \in \mathbb{R}^d$. Then, for all $k, n \in \mathbb{N}$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$,*

$$\left|V_{\overline{\psi},t}(e^{-\eta\cdot t}\varphi(t))(x,-\xi)\right| \leq \frac{C_{\eta,k,n,\psi}e^{-\eta\cdot x}\|\varphi\|_{\mathcal{S}^n_k}}{(1+|x|)^k(1+|\xi|)^n}, \qquad (x,\xi)\in\mathbb{R}^{2d},$$

where

$$C_{\eta,k,n,\psi} = 4^n(1+\sqrt{d})^n\max\{1,|\eta|^n\}\max_{|\alpha|\leq n}\|\partial^\alpha\psi\|_{L^\infty}\int_{\operatorname{supp}\psi} e^{-\eta\cdot t}(1+|t|)^k\mathrm{d}t.$$

In particular, $\sup_{\eta\in K} C_{\eta,k,n,\psi} < \infty$ *for all $K \subset \mathbb{R}^d$ compact.*

Proof We have that

$$\begin{aligned}
&\left|V_{\overline{\psi},t}(e^{-\eta\cdot t}\varphi(t))(x,-\xi)\right|(1+|x|)^k(1+|\xi|)^n \\
&\leq (1+\sqrt{d})^n\max_{|\alpha|\leq n}\left|\xi^\alpha V_{\overline{\psi},t}(e^{-\eta\cdot t}\varphi(t))(x,-\xi)\right|(1+|x|)^k \\
&\leq (1+\sqrt{d})^n(1+|x|)^k\max_{|\alpha|\leq n}\sum_{\beta\leq\alpha}\sum_{\gamma\leq\beta}\binom{\alpha}{\beta}\binom{\beta}{\gamma}\cdot \\
&\int_{\mathbb{R}^d}|\eta|^{|\gamma|}e^{-\eta\cdot t}\left|\partial^{\beta-\gamma}\varphi(t)\right|\left|\partial^{\alpha-\beta}\psi(t-x)\right|\mathrm{d}t \\
&\leq (1+\sqrt{d})^n(1+|x|)^k\max_{|\alpha|\leq n}\sum_{\beta\leq\alpha}\sum_{\gamma\leq\beta}\binom{\alpha}{\beta}\binom{\beta}{\gamma}\cdot \\
&\int_{\operatorname{supp}\psi}|\eta|^{|\gamma|}e^{-\eta\cdot(t+x)}\left|\partial^{\beta-\gamma}\varphi(t+x)\right|\left|\partial^{\alpha-\beta}\psi(t)\right|\mathrm{d}t \\
&\leq C_{\eta,k,n,\psi}e^{-\eta\cdot x}\|\varphi\|_{\mathcal{S}^n_k}.
\end{aligned}$$

□

Proof of Proposition 5.4 (i) $V_\psi : \mathcal{S}'(\Gamma) \to C_\Gamma(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon\mathcal{V}_{\text{pol}}C(\mathbb{R}^d_\xi)$ *is well-defined and continuous*: Let $K \in \operatorname{CCS}(\Gamma)$ and $v \in \overline{V}(\mathcal{V}_{\text{pol}})$ be arbitrary. Choose $\varepsilon > 0$ so small that $K_\varepsilon \in \operatorname{CCS}(\Gamma)$ and pick, for $x \in \mathbb{R}^d$ fixed, $\eta_x \in K$ such that $h_{-K}(x) \leq (-\eta_x\cdot x)+1$. Example 5.3 implies that, for all $f \in \mathcal{S}'(\Gamma)$ and $(x,\xi)\in\mathbb{R}^{2d}$,

$$\begin{aligned}\left|V_\psi f(x,\xi)\right| e^{h_{-K}(x)}v(\xi) &= \left|\langle e^{-(\eta_x-\varepsilon\frac{x}{|x|})\cdot t}f(t), e^{(\eta_x-\varepsilon\frac{x}{|x|})\cdot t}\overline{M_\xi T_x\psi}(t)\rangle\right| e^{h_{-K}(x)}v(\xi)\\ &\le e\left|\langle e^{-(\eta_x-\varepsilon\frac{x}{|x|})\cdot t}f(t), e^{(\eta_x-\varepsilon\frac{x}{|x|})\cdot (t-x)}\overline{M_\xi T_x\psi}(t)\rangle\right| e^{-\varepsilon|x|}v(\xi)\\ &\le e p_{K_\varepsilon,B}(f),\end{aligned}$$

where

$$B=\{e^{\tau\cdot(t-x)}\overline{M_\xi T_x\psi}(t)e^{-\varepsilon|x|}v(\xi)\mid (x,\xi)\in\mathbb{R}^{2d},\tau\in K_\varepsilon\}\in\mathfrak{B}(\mathcal{S}(\mathbb{R}^d_t))$$

by Lemma 5.5.

(ii) $V_\psi^*\colon C_\Gamma(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon\mathcal{V}_{\text{pol}}C(\mathbb{R}^d_\xi)\to\mathcal{S}'(\Gamma)$ *is well-defined and continuous*: We start by showing that $V_\psi^*F\in\mathcal{S}'(\Gamma)$ for all $F\in C_\Gamma(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon\mathcal{V}_{\text{pol}}C(\mathbb{R}^d_\xi)$. Lemma 5.6 implies that, for all $\eta\in\Gamma$,

$$\langle f_\eta,\varphi\rangle=\int\int_{\mathbb{R}^{2d}}F(x,\xi)V_{\overline{\psi},t}(e^{-\eta\cdot t}\varphi(t))(x,-\xi)\,\mathrm{d}x\,\mathrm{d}\xi,\qquad \varphi\in\mathcal{S}(\mathbb{R}^d),$$

is a well-defined continous linear functional on $\mathcal{S}(\mathbb{R}^d)$. Since $e^{-\eta\cdot t}V_\psi^*F(t)=f_\eta(t)|_{\mathcal{D}(\mathbb{R}^d)}$, we obtain that $e^{-\eta\cdot t}V_\psi^*F(t)\in\mathcal{S}'(\mathbb{R}^d)$ and that

$$\langle e^{-\eta\cdot t}V_\psi^*F(t),\varphi(t)\rangle=\int\int_{\mathbb{R}^{2d}}F(x,\xi)V_{\overline{\psi},t}(e^{-\eta\cdot t}\varphi(t))(x,-\xi)\,\mathrm{d}x\,\mathrm{d}\xi,\qquad \varphi\in\mathcal{S}(\mathbb{R}^d).$$

Next, we show that V_ψ^* is continuous. Let $K\in\text{CCS}(\Gamma)$ and $B\in\mathfrak{B}(\mathcal{S}(\mathbb{R}^d))$ be arbitrary. Choose $\varepsilon>0$ so small that $K_\varepsilon\in\text{CCS}(\Gamma)$. Lemma 5.6 implies that there is $v\in\overline{V}(\mathcal{V}_{\text{pol}})$ such that

$$\left|V_{\overline{\psi}}(e^{-\eta\cdot t}\varphi(t))(x,-\xi)\right|\le e^{h_{-K}(x)}v(\xi),\qquad (x,\xi)\in\mathbb{R}^{2d},$$

for all $\eta\in K$ and $\varphi\in B$. Set $w(\xi)=v(\xi)(1+|\xi|)^{d+1}\in\overline{V}(\mathcal{V}_{\text{pol}})$. Example 5.3 implies that, for all $F\in C_\Gamma(\mathbb{R}^d_x)\widehat{\otimes}_\varepsilon\mathcal{V}_{\text{pol}}C(\mathbb{R}^d_\xi)$,

$$\begin{aligned}p_{K,B}(V_\psi^*F)&\le\sup_{\eta\in K}\sup_{\varphi\in B}\int\int_{\mathbb{R}^{2d}}|F(x,\xi)|\left|V_{\overline{\psi},t}(e^{-\eta\cdot t}\varphi(t))(x,-\xi)\right|\,\mathrm{d}x\,\mathrm{d}\xi\\ &\le\int\int_{\mathbb{R}^{2d}}|F(x,\xi)|\,e^{h_{-K}(x)}v(\xi)\,\mathrm{d}x\,\mathrm{d}\xi\le C\|F\|_{K_\varepsilon,w},\end{aligned}$$

where

$$C=\int_{\mathbb{R}^d}e^{-\varepsilon|x|}dx\int_{\mathbb{R}^d}\frac{1}{(1+|\xi|)^{d+1}}\mathrm{d}\xi.$$

□

We now combine Theorem 4.1 with the results from Sect. 4 to study the space $\mathcal{S}'(\Gamma)$. Let $\emptyset\ne\Gamma\subseteq\mathbb{R}^d$ be open and convex and let $(K_N)_{N\in\mathbb{N}}\subset\text{CCS}(\Gamma)$ be such that $K_N\subseteq K_{N+1}$ for all $N\in\mathbb{N}$ and $\Gamma=\bigcup_N K_N$. For $\mathcal{W}=(e^{h_{-K_N}})_{N\in\mathbb{N}}$ we set $\mathcal{B}'_\Gamma(\mathbb{R}^d):=\mathcal{B}'_{\mathcal{W}}(\mathbb{R}^d)$ and $\mathcal{O}'_{C,\Gamma}(\mathbb{R}^d)=\mathcal{O}'_{C,\mathcal{W}}(\mathbb{R}^d)$. Clearly, these definitions are independent of the chosen sequence $(K_N)_{N\in\mathbb{N}}$. We are ready to state and prove our main theorem.

Theorem 5.7 *Let $\emptyset\ne\Gamma\subseteq\mathbb{R}^d$ be open and convex. Then, $\mathcal{S}'(\Gamma)=\mathcal{B}'_\Gamma(\mathbb{R}^d)=\mathcal{O}'_{C,\Gamma}(\mathbb{R}^d)$ as locally convex spaces and $\mathcal{S}'(\Gamma)$ is an ultrabornological (PLS)-space.*

Proof Let $(K_N)_{N\in\mathbb{N}} \subset \mathrm{CCS}(\Gamma)$ be such that $K_N \subseteq K_{N+1}$ for all $N \in \mathbb{N}$ and $\Gamma = \bigcup_N K_N$. Set $\mathcal{W} = (e^{h_{-K_N}})_{N\in\mathbb{N}}$. Lemma 5.2 and Example 5.3 imply that $\mathcal{W}$ satisfies (2.1), (2.2) and (2.3). Hence, in view of the reconstruction formula (3.1), the topological identity $\mathcal{S}'(\Gamma) = \mathcal{O}'_{C,\Gamma}(\mathbb{R}^d)$ follows from Proposition 4.1 and Proposition 5.4. Since $\mathcal{W}$ also satisfies (4.1) (again by Lemma 5.2 and Example 5.3), the other statements are a direct consequence of Theorem 4.2. □

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