

Impedance Data Validation by Kramers-Kronig Relations – Derivation and Implications

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Explicitly based on causality, linearity (superposition) and stability (time invariance) and implicit on continuity (consistency), finiteness (convergence) and uniqueness (single valuedness) in the time domain, Kramers-Kronig (KK) integral transform (KKT) relations for impedances are derived as pure mathematical constructs in the complex frequency domain using the two-sided (bilateral) Laplace integral transform (LT) reduced to the Fourier domain for sufficiently rapid exponential decaying, bounded impedances.

Novel anti KK relations are also derived to distinguish LTI (linear, time invariant) systems from non-linear, unstable and acausal systems. All relations can be used to test KK transformability on the LTI principles of linearity, stability and causality of measured and model data by Fourier transform (FT) in impedance spectroscopy (IS).

Also, integral transform relations are provided to estimate (conjugate) impedances at zero and infinite frequency particularly useful to normalise data and compare data. Also, important implications for IS are presented and suggestions for consistent data analysis are made which generally apply likewise to complex valued quantities in many fields of engineering and natural sciences.

Impedance spectroscopy (IS) is a powerful tool nowadays readily available for the *in-situ* characterization of materials, interfaces and devices (systems) whether of conductive or dielectric nature in many branches particularly of physics, chemistry, biology and engineering.^[1–5]

Note, the term *impedance* is due to Hendrik Wade Bode who combined the words *impedance* and *admittance*.^[6] We mean by impedance apart from impedance and admittance commonly denoted by respectively Z and Y also complex capacitance and inductance denoted by respectively C and L whether in the real respectively imaginary time domain, $t \in \mathbb{R}$ and jt , complex frequency domain (Laplace space), $s = \sigma + j\omega$, $\sigma, \omega \in \mathbb{R}$, $(\pm j)^2 = -1$ or the real respectively imaginary frequency domain (Fourier space), ω and $j\omega$. More generally, impedance could also include dielectric quantities

such as complex permittivity (dielectric constant), ϵ , susceptibility, χ and modulus, M as well as transfer functions based on stimuli (excitation) other than electrical.^[1–3]

But prior to a meaningful analysis of conventional impedance measurements and likewise of similar complex valued quantities (absorbance, transmittance, reflectance, complex compliance, relaxation modulus) in many other fields of natural sciences and engineering,^[7–17] the acquired data have to conform to the very principles of linear, time invariant (LTI) systems: linearity, stability, boundedness, continuity (consistency), uniqueness and causality.^[1–5] This may either be checked experimentally or numerically, for example, using fast Fourier transformation (FFT)^[18–21] on Kramers-Kronig (KK) relations^[22–24] in the positive frequency domain ($\omega \geq 0$) or the Hilbert integral transform (HT)^[7,25–28] in the whole domain ($|\omega| \geq 0$), the subject herein.

Likewise, models explaining or supplementing measured data should always be checked for self-consistency. Estimating impedances (or transfer functions) at zero and infinite frequencies virtually inaccessible by measurements to allow for normalisation to compare data obtained at different frequency ranges or when data are rejected, for example, as too noisy or upon numerical KK (HT) validation may prove useful in subsequent analysis. Populating the bandwidth more densely or extend it numerically beyond the measured data may be needed in some cases before numerical models are applied.

Impedance is determined by applying uncorrelated time varying arbitrary (AC) perturbations of small, non-zero amplitude (input), $x(t,t') = x(t-t')$ onto a steady state (DC). The measured response (output), $y(t) \sim x(t+t_0)$ relates to the input shifted by $0 < t_0 \in \mathbb{R}$. Applying the bilateral Laplace integral transforms (LT, forward, \mathcal{L}),^[29,30]

$$\begin{aligned} \mathcal{L}\{I(t)\}(s) &\equiv I(s) = \int_{-\infty}^{+\infty} I(t)e^{-st} dt, \\ \mathcal{L}\{I^*(t)\}(s^*) &\equiv I^*(s^*) = \int_{-\infty}^{+\infty} I^*(t)e^{-s^*t} dt \end{aligned} \quad (1)$$

having corresponding inverse (backward, \mathcal{L}^{-1}) transforms,

$$\begin{aligned} \mathcal{L}^{-1}\{I(s)\}(t) &\equiv I(t) = \int_{\sigma-j\infty}^{\sigma+j\infty} I(s)e^{st} \frac{ds}{2\pi j}, \\ \mathcal{L}^{-1}\{I^*(s^*)\}(t) &\equiv I^*(t) = \int_{\sigma-j\infty}^{\sigma+j\infty} I^*(s^*)e^{st} \frac{ds}{2\pi j} \end{aligned} \quad (2)$$

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with their region of convergence (ROC), $\sigma_1 < \Re\{s\} < \sigma_0$, to the bounded input-bounded output (BIBO),

$$y(t) = \int_{-\infty}^t x(t')I(t')dt', \forall t' \leq t \in \mathbb{R}, x, y \in \mathbb{R} \quad (3)$$

provides from past, $t \in]-\infty, 0^-]$ to present, t for the complex valued immittance,

$$I(s) = \frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{x(t)\}}(s) \equiv \frac{Y}{X}(s), |X(s)| \neq 0, X, Y \in \mathbb{C}, \quad (4)$$

where $0 < \Re\{s\}$ is not smaller than the real part of **all** isolated immittance singularities (poles), $\Re\{p_i\} \leq \Re\{s\}, \forall i \in \mathbb{N}$.

Note, the existence of the improper integral (1) requires $I(t)$ to be absolutely convergent, $|I(t)e^{\sigma t}| \rightarrow 0$ as $t \rightarrow \infty$ that is, $I(t)$ needs to decay (exponentially) faster than $e^{\sigma t}$ grows which means, $I(t)$ must be of exponential order,^[7]

$$\lim_{t \rightarrow \infty} e^{-\sigma_0 t} |I(t)| < \infty, \quad \Re\{s\} < \sigma_0 \in \mathbb{R},$$

$$\lim_{t \rightarrow -\infty} e^{\sigma_1 t} |I(t)| < \infty, \quad \Re\{s\} > \sigma_1 \in \mathbb{R},$$

finite (bounded) and continuous whether uniformly (nowhere singular in the entire domain) or piecewise (sectionally). The latter means that $I(t)$ is nowhere singular in the neighbourhood of t_0 formed by $t \pm t_0$ (Hölder continuity),^[7,8]

$$\frac{|I(t) - I(t_0)|}{|t - t_0|^m} < \infty, \quad \forall t \neq t_0 \in \mathbb{R}, \quad 0 < m \leq 1$$

to probe a non-essential singularity (discontinuity), $|I(t \rightarrow t_0)| \rightarrow \infty$ near t_0 for its eventual exclusion from the domain of $I(t)$ and hence from the range of the integrand of (1); otherwise, this improper integral only exists in a principal value sense as is the case for an essential (non-removable) singularity.^[31]

Similarly, the existence of the Bromwich integral (2) requires $I(s)$ to be of exponential order, finite and continuous that is at least Hölder continuous meaning $I(s)$ is nowhere singular in the neighbourhood of s_0 formed by $s \pm s_0$,

$$\frac{|I(s) - I(s_0)|}{|s - s_0|^n} < \infty, \quad \forall s \neq s_0 \in \mathbb{C}, \quad 0 < n \leq 1$$

to probe a non-essential singularity, $|I(s \rightarrow s_0)| \rightarrow \infty$ near s_0 for its exclusion from the domain of $I(s)$ and hence from the range of the integrand of (2).

Given **all** these conditions met, the convergence,

$$\lim_{|s| \rightarrow \infty} I(s) = 0 \quad (5)$$

follows from the initial value theorem (IVT) of the bilateral LT,^[32-34]

$$\lim_{t \rightarrow 0} \{I(t) - I(-t)\} = \lim_{|s| \rightarrow \infty} sI(s) \quad (6)$$

applied to the LT pairs (1) & (2).

Note, Hölder continuity is implicitly used in subtracted HT (KK) relations^[16,17,35] to isolate poles or to enhance the convergence of the transformed quantities.

Further, the LT pairs (1) & (2), given reality, $I(t) = I^*(t), \forall t \in \mathbb{R}$, imply

$$I(s) = I^*(s^*), \forall s \in \mathbb{C} \quad (7)$$

for the conjugate immittance, $I^*(s^*) \in \mathbb{C}$ of conjugate frequency, $s^* = \sigma - j\omega$ with magnitude (modulus), $|s| \equiv \sqrt{ss^*}$.

However, immittances whether of dynamical systems with $I(t=0) = I(t=0^+) - I(t=0^-) \neq 0$ or systems at rest with $I(t=0) = 0$ which conform to LTI theory need both to obey **all** of the following fundamental principles:^[1-5]

1. Linearity (superposition) that is

$$\sum_k a_k y(t) = \sum_k a_k \int_{-\infty}^{+\infty} x(t-t')I(t')dt', \forall t' \leq t \in \mathbb{R} \quad (8)$$

to yield (4) in the s domain by the linearity property (theorem).^[29,30]

2. Stability (time invariance by dilation, translation & rotation) that is

$$y(\tau) = a e^{s t_0} \int_{-a\infty}^{+a\infty} x(\tau - \tau')I(\tau')d\tau', \quad (9)$$

$$\forall \tau' \leq \tau = a(t - t_0), \quad t_0 \leq t \in \mathbb{R}$$

with t multiplied (dilated) by $|a| > 1$ (expansion) or $|a| < 1$ (contraction), shifted (translated) by $t_0 \in \mathbb{R}$ and/or rotated by $\text{Arg } a = \pm \frac{\pi}{2}$ to also yield (4) in the s domain by the similarity and shifting theorems.^[29,30]

3. Primitive causality where an effect (output) cannot precede **its** cause (input), that is

$$y(t - t') = 0, \quad \forall t \leq t' \in \mathbb{R} \quad (10)$$

to yield vanishing (4) in the s domain. *Naively, in (flat) spacetime, relativistic causality requires $t - t_c < t'$ for (10) to hold in a homogeneous medium of thickness λ traversed in time $t_c = \lambda/c$ by an information bearing signal at speed c ; then, the relations derived herein need multiplication by $e^{s t_c}$ in Laplace space and by $e^{-j\omega t_c}$ in Fourier space.*

As already noted, finite, continuous and convergent immittance are prerequisite for (8) & (9) to hold that is, their integrands are not divergent. Also, (10) means the lower integration limits in (3), (8) and (9) can be set to $t=0^-$; then, initial data are consequences solely of the past, $t \leq 0^-$.

Experimentally, linearity is usually preserved when using small stimuli of sufficient amplitude to measure a response with reasonably low levels of noise. It is readily verified by taking a

multiple (fraction) of the stimulus to produce a response of same multiple (fraction) in amplitude. The output of a stable system is bounded for bounded input or decays sufficiently rapid for an impulse input with the system returning to its original state when the input is removed. This can be verified by applying repetitively the same input to measure each time, within experimental error, the same output or by sweeping frequencies back and forth to only notice an insignificant hysteresis in the response. Immittances of real systems which exhibit losses by storage (capacitive and/or inductive) and dissipation (resistance), are generally bounded at the discrete frequencies selected in the measurement and thus, singularities are unfortunately in most cases not directly detectable experimentally. Also, finiteness (5) is inaccessible by experiment given the finite range of frequencies in any IS measurement.

When a perturbation frequency is at or close where the measured system resonates, the system response is amplified to eventually saturate the input of the measurement device reading it with the result of the onset of an unbounded immittance. Thus, applying a decaying time varying stimulus, $e^{-\sigma_0 t} x(t)$, $0 < \sigma_0 \in \mathbb{R}$ and the response of the resonating system is damped to result in a bounded immittance when σ_0 is chosen sufficiently large that the input of the measurement device is no longer saturated by the response read.

Most important, we use below in (11), (12) and (13) which respectively represent time domain linearity, stability and causality, the fact that (7)–(10) remain invariant under time reversal (reflection), $t \rightarrow -t$ (time rotation by $-\pi$) and in imaginary time, $\pm jt$ (time rotation by $\pm \frac{\pi}{2}$).

Generally, (8), (9) & (10) are appropriate for time domain use along with (3) & (4) but they are inappropriate to derive HT relations in the s domain; instead,

$$j[I(t) - I(t=0)] = h(t)j[I(t) - I(t=0)], \forall t \in \mathbb{R}, \quad (11)$$

$$I(jt) - I(jt=j0) = h(jt)[I(jt) - I(jt=j0)], \forall t \in \mathbb{R}, \quad (12)$$

and

$$I(t) - I(t=0) = h(t)[I(t) - I(t=0)], \forall t \in \mathbb{R}, \quad (13)$$

are respectively used with the right continuous (Heaviside) unit step function, $h(t) = 0.5(1 + \text{sgn}(t))$ ^[36] and signum function, $\text{sgn}(t \neq 0) = \frac{t}{|t|}$ and $\text{sgn}(t=0) = 0$.^[36]

Advantageously, (11)–(13) are independent of actual initial data, $I(t=0)$ thus past immittances, $I(t < 0)$. The special case, $I(t=0^+) \mp I(t=0^-) = 0$ demands for inertial systems an impulse input, $x(t=0) = \delta(t=0)$ where $\delta(t) = 0$, $\forall t \neq 0$ is the Dirac delta distribution with translation property,

$$I(t) = \mathcal{P} \int_{-\infty}^{+\infty} \delta(t-t')I(t')dt', \forall t \in \mathbb{R} \quad (14)$$

where \mathcal{P} signifies the principal value taken at $|t-t'|=0$ and any pole of $I(t)$.

Applying forward LT to the relations (11)–(13) and using its linearity, scaling and convolution properties (theorems) yields

$$I(s) = \mathcal{H}_{\mathbb{C}}\{I(s)\}(w), \forall s \in \overline{\mathbb{C}^+} \quad (15)$$

in the closed upper half complex s plane (UHP), $\overline{\mathbb{C}^+} := \{s \in \mathbb{C} : 0 \leq \Im m\{s\}\}$ considering vanishing $I(s=\sigma+j\infty)$ due to finiteness (5) with

$$\mathcal{H}_{\mathbb{C}}^{\pm}\{I(s)\}(w) = \pm \mathcal{P} \int_{\gamma-j\infty}^{\gamma+j\infty} \frac{I(w)dw}{s-w} \frac{1}{2\pi j}, \frac{\omega}{|\omega|} = \frac{\nu}{|\nu|} \neq 0,$$

the HT (forward, $\mathcal{H}_{\mathbb{C}}^+$ and backward, $\mathcal{H}_{\mathbb{C}}^-$) with $w = \gamma + j\nu$, $\gamma, \nu \in \mathbb{R}$.

Further, obedience to (11)–(13) may also be verified proving,

$$j[I(t) - I(t=0)] \neq h(t)[I(t) - I(t=0)], \forall t \in \mathbb{R}, \quad (16)$$

$$I(jt) - I(jt=j0) \neq h(jt)[I(jt) - I(jt=j0)], \forall t \in \mathbb{R}, \quad (17)$$

and

$$I(-t) - I(t=0) \neq h(t)[I(t) - I(t=0)], \forall t \in \mathbb{R}, \quad (18)$$

the anti relations corresponding to (11)–(13) which by applying the forward LT read in the UHP respectively

$$jI(s) \neq \mathcal{H}_{\mathbb{C}}\{I(s)\}(w), \forall s \in \overline{\mathbb{C}^+}, \quad (19)$$

$$-jI(-js) \neq \mathcal{H}_{\mathbb{C}}\{I(s)\}(w), \forall s \in \overline{\mathbb{C}^+} \quad (20)$$

and

$$-I(-s) \neq \mathcal{H}_{\mathbb{C}}\{I(s)\}(w), \forall s \in \overline{\mathbb{C}^+} \quad (21)$$

for a non-linear, stable & causal system, a linear, unstable & causal system and a linear, stable & acausal system. Relation (15) and the anti relations (19)–(21) may readily be derived for conjugate immittance using parity (7).

Remark, all these (anti) relations are useful to verify the consistency of IS data modelled in the complex frequency domain but are inconvenient for IS data measured at real frequencies constituting the ω domain.

Then, under the assumption that the immittance in the ω domain is at least Hölder continuous and also bounded, $|I(t)| < \infty$ to comply with the pendant of convergence (5) at $|\sigma| \rightarrow 0$,

$$\lim_{|\omega| \rightarrow \infty} I(\omega) = 0 \quad (22)$$

when these limits exist and the ROC encloses the imaginary axis, $\sigma_1 < 0 < \sigma_0$, we derive HT relations and anti relations for the immittance in the ω domain using the Sochocki-Plemelj formulae.^[8,37-40] At $-js, js^* \rightarrow \omega$, they relate when existent the non-tangential boundary values of $I(s)$ and $\mathcal{H}_{\mathbb{C}}\{I(s)\}(\nu)$ from the UHP interior, $\overline{\mathbb{C}^+}$ (from above the clockwise by $\frac{\pi}{2}$ rotated complex s plane) and its exterior, $\mathbb{C} \setminus \overline{\mathbb{C}^+}$ (from below the clockwise by $\frac{\pi}{2}$ rotated complex s plane) to their ω domain pendants, $I(\omega)$ and $\mathcal{H}_{\mathbb{R}}\{I(\omega)\}(\nu)$ by

$$\lim_{\sigma \rightarrow 0^+} \{I(-js) - I(js^*)\} = I(\omega), \quad (23)$$

$$\lim_{\sigma, \gamma \rightarrow 0^+} \{\mathcal{H}_{\mathbb{C}}\{I(-js)\}(-j\nu) + \mathcal{H}_{\mathbb{C}}\{I(js^*)\}(j\nu^*)\} = -j\mathcal{H}_{\mathbb{R}}\{I(\omega)\}(\nu).$$

Applied to the left hand side (LHS) and the right hand side (RHS) of (15) taking $s, \omega \rightarrow -js, -j\nu$ & $s^*, \omega^* \rightarrow js^*, j\nu^*$ and $\sigma, \gamma \rightarrow 0^+$ in unison, it yields

$$(I \pm I^*)(\omega) = -j\mathcal{H}_{\mathbb{R}}\{(I \mp I^*)(\omega)\}(\nu), \quad (24)$$

$$= -j\mathcal{K}_{\mathbb{R}^+}\{(I \mp I^*)(\omega)\}(\nu)$$

when $I(s)$ is analytic (infinitely complex differentiable) on $\overline{\mathbb{C}^+}$ excluding poles and thus, for (Hölder) continuous and finite (convergent) $I(s)$.

$$\mathcal{H}_{\mathbb{R}}^{\pm}\{(I \pm I^*)(\omega)\}(\nu) = \pm \mathcal{P} \int_{-\infty}^{+\infty} \frac{(I \mp I^*)(\nu) d\nu}{\omega - \nu} \frac{d\nu}{\pi}, \quad \frac{\omega}{|\omega|} = \frac{\nu}{|\nu|} \quad (25)$$

is the standard form HT on the real line, \mathbb{R} and

$$\mathcal{K}_{\mathbb{R}^+}^{\pm}\{(I \pm I^*)(\omega)\}(\nu) = \pm 2 \mathcal{P} \int_0^{\infty} \frac{\nu^{(1 \pm 1)/2} (I \mp I^*)(\nu) d\nu}{\omega - (1 \mp 1)/2 (\omega^2 - \nu^2)} \frac{d\nu}{\pi}, \quad \frac{\omega}{|\omega|} = \frac{\nu}{|\nu|} \quad (26)$$

is the KK integral transform (KKT)^[22-24] on the half line, $\mathbb{R}^+ := \{x \in \mathbb{R} : x \geq 0\}$ readily derived from (25) using parity

$$I(\omega) = I^*(-\omega), \quad \forall \omega \quad (27)$$

which is obtained when applying the jump condition, the first of the two formulae (23) and its conjugate to respectively the LHS and RHS of (7).

Moreover, using the well known complex identities,^[31]

$$\Re\{I(\omega)\} \pm j\Im\{I(\omega)\} = \frac{(I + I^*) \pm (I - I^*)}{2}(\omega) \quad (28)$$

and (24) simplifies to the linear dispersion relations,^[1-5]

$$\Re\{I(\omega)\} = \mathcal{K}_{\mathbb{R}^+}^+\{\Im\{I(\omega)\}\}(\nu) = \mathcal{H}_{\mathbb{R}}^+\{\Im\{I(\omega)\}\}(\nu), \quad (29)$$

$$\Im\{I(\omega)\} = \mathcal{K}_{\mathbb{R}^+}^-\{\Re\{I(\omega)\}\}(\nu) = \mathcal{H}_{\mathbb{R}}^-\{\Re\{I(\omega)\}\}(\nu).$$

Here, $I(\omega) = |I(\omega)| e^{j\arg(I(\omega))}$ with modulus, $|I(\omega)| = \sqrt{\Re^2 I(\omega) + \Im^2 I(\omega)}$ and phase (argument),

$$\theta(\omega) = \arg I(\omega) = \tan^{-1} \left(\frac{\Im\{I(\omega)\}}{\Re\{I(\omega)\}} \right), \quad -\pi \leq \arg I(\omega) < \pi, \quad \arg I(\omega) = \arg |I(\omega)| + 2\pi k, \quad k \in \mathbb{Z},$$

is continuous, irreducible and rational,

$$I(\omega) = K \left(\prod_{i \in \mathbb{N}} \omega - z_i \right) \left(\prod_{j \in \mathbb{N}} \omega - p_j \right)^{-1}, \quad z_i, p_j \in \mathbb{C}, \quad K \in \mathbb{R}$$

with gain, $K \neq 0$. Unless zero pole cancellation occurs, $z_i = p_j$ (counting multiplicities), finite $\ln |I(\omega)| = \ln |K| + j \arg I(\omega)$ requires all zeros, z_i not to lie in the open UHP, $z_i \notin \mathbb{C}^+ := \{s \in \mathbb{C} : 0 < \Im\{s\}\}$ as such zero imply non-minimum phase (NMP).^[41] Also, bounded $\ln |I(\omega)|$ requires all poles to lie in the closed left half of the complex s plane (LHP), $p_j \in \overline{\mathbb{C}^+} := \{s \in \mathbb{C} : \Re\{s\} \leq 0\}$ as any open right half plane (RHP) pole, $p_j \in \mathbb{C}_- := \{s \in \mathbb{C} : 0 < \Re\{s\}\}$ or multiple poles on the imaginary axis, $p_j \in \mathbb{C}_0 := \{s \in \mathbb{C} : \Re\{s\} = 0\}$ yield $|I(\omega \rightarrow \infty)| \rightarrow \infty$ by the residue theorem.^[31,42]

Since constants vanish under HT (KK) transformation, the number of all zeros, N_z and poles, N_p with the winding number, $k = |N_z - N_p|$ by the Principle Argument theorem,^[31,42] must be known to uniquely establish phase from modulus and vice versa using (29) with $\ln |I(\omega)|$ and $\theta(\omega)$ in place of respectively $\Re\{I(\omega)\}$ and $\Im\{I(\omega)\}$.

Note, $|k|$ is at maximum unity for minimum phase (MP) systems, $(2k-1)\pi \leq 2\theta_{MP}(\omega) < (k+1)\pi$. For NMP systems either maximum phase, $(2k-1)\pi \leq \theta_{NMP}(\omega) < (k+1)\pi$ (e.g. transmission line, ω varying lump elements) or all-pass, $|I_{NMP}(\omega)| = 1, \forall \omega$, all UHP zeros must be known a priori to uniquely establish the said two quantities using in addition $I_{MP}(\omega) = B(\omega) I_{NMP}(\omega)$ ^[35,41] with finite Blaschke product,^[43,44] $B(\omega) = \lambda \prod (\omega - z_i)(1 - z_i^* \omega)^{-1}, |z_i| \neq 0, \forall z_i \in \mathbb{C}^+$ where $|\lambda| = 1$ and $|B(\omega)| = 1, \forall \omega$.

In the HT relations (24), we take the static (low frequency) limits, $\omega \rightarrow 0^{\pm}$ for the immittance, $j\omega I(\omega)$ and the instantaneous (high frequency) limits, $|\omega| \rightarrow \pm \infty$ for the immittance, $I(\omega)$ to find using parity (27) and the complex conjugation property (theorem) of HT^[7] along with $\omega \delta(\omega) = 0, \forall \omega$ and (14),

$$-\lim_{\omega \rightarrow 0} j\omega \{I(\omega) \pm I(-\omega)\} = \int_{-\infty}^{\infty} [I(\nu) \mp I(-\nu)] \frac{d\nu}{\pi}, \quad (30)$$

$$\lim_{\omega \rightarrow \infty} \{I(\omega) \mp I(-\omega)\} = -\lim_{\omega \rightarrow 0} j\omega \{I(\omega) \pm I(-\omega)\}$$

where respectively symmetry, $I(\omega = 0^{\pm}) - I(\omega = 0^{\mp}) = 0$ and anti-symmetry, $I(\omega = 0^{\pm}) + I(\omega = 0^{\mp}) = 0$ are implied by a vanishing difference and sum.

Using (23) separately on the LHS and RHS of the inequalities (19)–(21) while taking $\sigma, \gamma \rightarrow 0^+$ in unison and use (26) & (28), we find the KK anti relations,

$$\Im\{I^*(\omega)\} \neq \mathcal{K}_{\mathbb{R}^+}^+\{\Im\{I(\omega)\}\}(\nu), \quad (31)$$

$$\Re\{I^*(\omega)\} \neq \mathcal{K}_{\mathbb{R}^+}^-\{\Re\{I(\omega)\}\}(\nu),$$

$$\Im\{I^*(j\omega)\} \neq \mathcal{K}_{\mathbb{R}^+}^-\{\Im\{I(\omega)\}\}(\nu), \quad (32)$$

$$\Re\{I^*(j\omega)\} \neq \mathcal{K}_{\mathbb{R}^+}^+\{\Re\{I(\omega)\}\}(\nu)$$

and

$$\begin{aligned} \Re\{I(\omega)\} &\neq \mathcal{K}_{\Re+}^{-}\{\Im\{I(\omega)\}\}(\nu), \\ \Im\{I(\omega)\} &\neq \mathcal{K}_{\Re+}^{+}\{\Re\{I(\omega)\}\}(\nu). \end{aligned} \quad (33)$$

Then, the implications drawn from these anti relations when compared to the KK relations (29) for linear, stable and causal systems to become non-linear, unstable or acausal are as follows: Non-linearity in IS data is exhibited by the real part of the immittance to become imaginary and the imaginary part to become real and negated. Instability in such data is exhibited by the real part to become imaginary and negated and the imaginary part to become real. Anti-causality in IS data is exhibited by the real and imaginary parts to become negated.

Obviously, it implies no change to the magnitude of the immittance but its phase and loss (dissipation) factor (LF). First, the phase changes by $-\pi$ (counterclockwise) when a causal system becomes acausal while the LF does not change. Second, the loss factor changes from $\tan\theta(\omega)$ for a linear and stable system to $-\cot\theta(\omega)$ when the system becomes nonlinear or unstable.

In a companion communication,^[45] we use the convolution property (theorem) of Fourier transform (FT)^[30] to numerically HT (KKT) validate within computed fast FT (FFT) errors exemplary exact and experimental IS data using these implications on magnitude, phase and LF.

Another implication is that the RHS of (24) & (29) are recovered from their respective LHS (and *vice versa*) using for single valued (unique) $I(\omega)$ the inverse HT (KK) transform^[7] showing the intimacy and complementarity of the immittance and its conjugate in contrast to (31)–(33) and demonstrating the interdependence of the real and imaginary parts *vis-à-vis* their conjugates.

Although the HT (KKT) relations are pure mathematical constructs, they cumulatively reflect the underlying LTI principles and the transform properties (theorems).^[7,30,36]

They also reveal that the information of the entire positive spectrum, $|\omega| \in [0, \infty[$ is already contained in a single frequency measurement and that data at additional frequencies may principally be predicted numerically, for example, to compensate within numerical error for non-uniform spread in measurement bandwidth or modelled data but also when data are rejected failing KK (HT) validation in violation of one or another LTI principle^[1–5] or to more densely populate the frequency spectrum and extending it beyond the measured range. In turn, measured spectra are finite by nature and thus inevitably approximate the true immittance.

In summary, we derived from first principles of LTI theory based on the (imaginary) time domain equalities (11)–(13) and (analytic) integral transform properties (theorems)^[7,30,36] linear HT (24) and KK relations (29)^[22–24] in Fourier space as pure mathematical constructs. For an inertial LTI system, they are applicable to any complex valued quantity measured in not too noisy environment and when represented by rational nominator and denominator functions. These relations display the interdependence of the real and imaginary immittance parts and their conjugates of finite, linear, stable and causal systems whether dynamical or at rest to allow for IS data validation by

FFT^[18,20,21] supplemented where necessary by the constrains (30).

Suggestively, the latter may also viably regularise parameter identification and estimation for LTI systems including determining the distribution of their relaxation (retardation) times and energies.^[1–5]

Also, we present anti-relations (31)–(33) able to reveal non-linear, unstable and acausal frequency data. All these relations apply likewise under rotation of (complex) frequency by $\frac{\pi}{2} \pm 2\pi k, k \in \mathbb{Z}$ (dilation by $\exp(j[\frac{\pi}{2} \pm 2\pi k])$) or $\pm\omega, \pm\nu(\pm s, \pm\omega) \rightarrow \pm j\omega, \pm j\nu(\pm js, \pm j\nu)$.

Besides IS of finite energy systems excited by uncorrelated inputs, the derived relations applicable to rational immittances real valued in the (imaginary) time domain with all poles in the RHP, may find use to check for self-consistent data and to constrain models in other applications too:^[7–17] *perhaps not anywhere near the Planck scales or not close to the boundary of (flat) spacetime?*

Further, caution is to be taken in using these relations to validate magnitude and phase rather than real and imaginary parts especially on IS data of active systems with energy inputs and outputs to ensure uniqueness.^[35,41] Similar caution is advised for immittance compositions involving non-integer powers.

Remark, in non-electrical IS, the time domain may be replaced by the domain of wavelength to obtain dispersion relations in terms of wave number. Similar arguments hold for space variables, momentum and energy while relativistically, locality considerations are necessary too besides the Lorentz transformations.

Finally, we would not without further study be ready to assume that the self-coherent derivation of the HT (KK) relations made herein may straightforwardly be extended to, for example, higher dimensions and hypercomplex domains, (non-integer) immittance differentiation & integration and large signal, non-linear IS.

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Conflict of Interest

The author declares no conflict of interest.

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