

RESEARCH ARTICLE

Analysis of transmission dynamics of COVID-19 via closed-form solutions of a susceptible-infectious-quarantined-diseased model with a quarantine-adjusted incidence function

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We analyze the disease control and prevention strategies in a susceptible-infectious-quarantined-diseased (SIQD) model with a quarantine-adjusted incidence function. We have established the closed-form solutions for all the variables of SIQD model with a quarantine-adjusted incidence function provided $\beta \neq \gamma + \alpha$ by utilizing the classical techniques of solving ordinary differential equations (ODEs). The epidemic peak and time required to attain this peak are provided in closed form. We have provided closed-form expressions for force of infection and rate at which susceptible becomes infected. The management of epidemic perceptive using control and prevention strategies is explained as well. The epidemic starts when $\rho_0 > 1$, the peak of epidemic appears when number of infected attains peak value when $\rho_0 = 1$, and the disease dies out $\rho_0 < 1$. We have provided the comparison of estimated and actual epidemic peak of COVID-19 in Pakistan. The forecast of epidemic peak for the United states, Brazil, India, and the Syrian Arab Republic is given as well.

KEYWORDS

dynamical systems in biology, exact solutions

MSC CLASSIFICATION

37N25; 83C15

1 | INTRODUCTION

The novel coronavirus (COVID-19) emerged in the early December 2019 in the city of Wuhan located in Hubei Province of China, and within a month, it spread outside China. On January 31, 2020, the World Health Organization (WHO) declared COVID-19 as the global emergency. The containment strategies of COVID-19 adopted by several countries include hand hygiene, social distancing, isolation, quarantine, and lockdown. It is crucial for the mathematicians, statisticians, and scientists to develop appropriate models to uncover the essential aspects of COVID-19 spread and its transmission dynamics. The information about the total number of confirmed, infected, recovered, and mortality cases is essential for the public health decision makers in pandemic preparedness.

Many mathematical models have been developed. The mathematical models in epidemiology are governed by a system of first-order ODEs. The classical susceptible-infected-recovered (SIR) model developed by Kermack-McKendrick¹ is widely used by researchers in several studies, and many extensions of model are pro-

posed. The susceptible-infectious-recovered-deceased (SIRD) model is utilized for forecasting of COVID-19 by Mansoor et al² for Pakistan, Chatterjee et al³ for India, and Al-Raeei⁴ for the United States, Russia, China, and the Syrian Arab Republic. Several models are utilized to estimate and predict the COVID-19 which include SIR model,⁵ susceptible-exposed-infectious-removed (SEIR),^{6,7} susceptible-asymptomatic-infected-removed (SAIR),⁸ and extensions of these models.

The dynamic properties of COVID-19 models are analyzed by utilizing different methods. Yadav et al⁹ utilized machine learning methods to analyze COVID-19 epidemic. Ali et al¹⁰ applied optimal control theory to help public health decision makers to implement different control programs for the containment of this deadly virus. A small number of papers have utilized classical techniques to establish the closed-form solutions of epidemiology models.^{11,12} The stability analysis of the dynamical systems, numerical methods, and data drive approach are also utilized by several researchers to investigate transmission dynamics of COVID 19.

Hethcote et al¹³ studied the effects of quarantine in six endemic models for infectious diseases. The susceptible-infectious-quarantined-susceptible (SIQS) and susceptible-infectious-quarantined-removed (SIQR) models were analyzed by taking simple mass action incidence, the standard incidence, and the quarantine-adjusted incidence. In Chinviriyasit S and Chinviriyasit W,¹⁴ global stability of an SIQ epidemic model with simple mass action incidence, constant immigration, and natural death is investigated. The SIQ model¹⁴ is extended to susceptible-infectious-quarantined-diseased (SIQD) model by adding a death compartment, and a quarantine-adjusted incidence function is considered. To the best of our knowledge, SIQD model with a quarantine-adjusted incidence function has not been considered in literature to analyze the transmission dynamics of COVID-19. Our newly developed SIQD model with a quarantine-adjusted incidence function is analyzed by utilizing the closed-form solutions. It is worthy to mention here that we focus on the infectious group rather than the exposed or asymptomatic to understand the epidemic peak, force of infection (FOI), and rate at which susceptible becomes infected by utilizing the closed-form solutions.

The detailed outline of paper is as follows. The SIQD model with quarantine-adjusted incidence function is formulated in Section 2. The basic reproduction number is derived by utilizing the next generation operator approach by van den Driessche and Watmough.¹⁵ The classical techniques of solving ODEs are utilized to establish closed-form solutions for all the variables of SIQD model with a quarantine-adjusted incidence function. In Section 3, we have provided the closed-form expressions for epidemic peak and maximum number of infected individuals. The closed-form expressions for the FOI and rate at which susceptibles become infected are given in closed form. In Section 4, we have provided different control and prevention policies to slow down pace of epidemic. In Section 5, the model is utilized to study the COVID-19 epidemic in Pakistan. Moreover, we have forecasted epidemic peak for USA, Brazil, India, and the Syrian Arab Republic. The concluding remarks are presented in Section 6.

2 | THE SIQD MODEL WITH QUARANTINE-ADJUSTED INCIDENCE FUNCTION

We consider a model of COVID-19 transmission in a constant population of size N which consists of four compartments susceptible $S(t)$, infected $I(t)$, quarantined $Q(t)$, deceased $\Delta(t)$, and total population $N = S(t) + Q(t) + I(t) + \Delta(t)$. In literature, several types of incidence functions are considered to study the dynamics of infectious diseases; see, for example, Bailey,¹⁶ Jacquez et al,¹⁷ and Watson.¹⁸ Hethcote et al¹³ studied the SIQS and SIQR models by taking simple mass action incidence, the standard incidence, and the quarantine-adjusted incidence. We follow the idea of Hethcote et al¹³ and consider a quarantine-adjusted incidence function. The actively mixing population which is the pool of people with whom a susceptible can meet for our SIQD model is $N - Q - D = S + I$. The expression for the quarantine-adjusted incidence function is $(\beta SI / S + I)$, and it is obtained by replacing the denominator N in the standard incidence $(\beta SI / N)$ by actively mixed population $N - Q - D = S + I$. The FOI is $\beta I / S + I$. The model can be expressed as a system of following four ODEs:

$$\dot{S} = -\frac{\beta SI}{S + I}, \quad (1)$$

$$\dot{I} = \frac{\beta SI}{S + I} - \gamma I - \alpha I, \quad (2)$$

$$\dot{Q} = \gamma I - \mu Q, \quad (3)$$

$$\dot{\Delta} = \alpha I + \mu Q, \quad (4)$$

where dot above a variable denotes time differentiation and initial conditions are $S(0) = S_0$, $I(0) = I_0$, $Q(0) = Q_0$, and $\Delta(0) = N - S(0) - I(0) - Q(0)$. The parameters are defined as follows: $\beta > 0$ is contact rate of susceptible individuals with infected individuals, $\gamma > 0$ is isolation rate of infected individuals, $\alpha > 0$ is the virus-induced mortality rate of infected class, and $\mu > 0$ is the virus-induced mortality rate of quarantined class.

2.1 | The basic reproduction number for the SIQD model with quarantine-adjusted incidence function

Next, we compute the basic reproduction number ρ_0 for this model by utilizing the next generation operator approach by van den Driessche and Watmough.¹⁵ It is an essential dimensionless quantity which provides the threshold in the analysis of a disease to predict outbreak and to evaluate its control strategies. Let $e_0 = (S^*, 0, 0, 0)$ represent the disease free equilibrium (DFE). The associated F and V matrices of SIQD model with quarantine-adjusted incidence functions (1)–(4) for the next generation operator approach are given as

$$F = \left[\begin{array}{c} \frac{\beta S}{S+I} - \frac{\beta SI}{(S+I)^2} \end{array} \right] \tag{5}$$

and

$$V = [\gamma + \alpha]. \tag{6}$$

The the basic reproduction number ρ_0 is the only eigenvalue of FV^{-1} at the DFE $e_0 = (S^*, 0, 0, 0)$ and is given by

$$\rho_0 = \frac{\beta}{\gamma + \alpha}. \tag{7}$$

It is important to mention here that the epidemic starts if $\dot{I}|_{t=0} > 0$. Setting $t = t_0 = 0$ in Equation (2), we have

$$\dot{I}|_{t=0} = I_0 \left(\frac{\beta S_0}{S_0 + I_0} - \gamma - \alpha \right) > 0 \tag{8}$$

provided $\beta S_0/S_0 + I_0 - \gamma - \alpha > 0$. This condition in terms of the basic reproduction number is $\rho_0 > 1 + I_0/S_0$. The epidemic starts when $\rho_0 > 1$ at initial time t_0 , the peak of epidemic appears when $I(t)$ attains peak value I_p at time t_p when $\rho_0 = 1$, and the disease dies out $\rho_0 < 1$ at time t_d .

2.2 | Closed-form solutions for SIQD model with quarantine-adjusted incidence function

In this section, we establish the closed-form solutions of system of Equations (1)–(4). It is important to mention here that Equations (1) and (2) form a coupled system of nonlinear ODEs in terms of variables S and I . Equations (3) and (4) are linear ODEs for variables Q and Δ , respectively. We can solve Equations (3) and(4) if variable I is known. First, we solve Equations (1) and (2) for variables S and I . Then we use value of I in ODEs (3) and (4) to find $S(t)$ and $\Delta(t)$.

Introduce a new variable

$$Z = \frac{S}{I} \tag{9}$$

then

$$\frac{\dot{Z}}{Z} = \frac{\dot{S}}{S} - \frac{\dot{I}}{I}. \tag{10}$$

With the aid of Equations (1) and (2), Equation (10) takes following form:

$$\dot{Z} + (\beta - \gamma - \alpha)Z = 0. \tag{11}$$

Equations (1) and (2) can be expressed as a system of linear ODEs in terms of variable $Z(t)$

$$\frac{\dot{S}}{S} = -\frac{\beta}{1+Z}, \tag{12}$$

$$\frac{\dot{I}}{I} = \frac{\beta Z}{Z+1} - \gamma I - \alpha I, \tag{13}$$

where variable Z satisfies a linear ODE given by (11). It is straightforward to derive $Z(t)$ from the linear ODE (11) and is given by

$$Z(t) = Z_0 e^{-(\beta-\gamma-\alpha)t}, \quad Z(0) = Z_0, \tag{14}$$

where $Z_0 = S(0)/I(0)$ and $\beta \neq \gamma + \alpha$. We substitute the expression for the variable $Z(t)$ from (14) in ODE (12), and resulting the ODE can be expressed in variable separable as follows:

$$\int \frac{1}{S} dS = - \int \frac{\beta}{1 + S_0 e^{-(\beta-\gamma-\alpha)t}} dt. \tag{15}$$

After some simplifications and using initial condition $S(0) = S_0$, Equation (15) yields

$$S(t) = S_0 \left(\frac{S_0 + I_0}{S_0} \right)^{\frac{\beta}{\beta-\gamma-\alpha}} \left(1 + \frac{I_0}{S_0} e^{(\beta-\gamma-\alpha)t} \right)^{-\frac{\beta}{\beta-\gamma-\alpha}}. \tag{16}$$

Substituting expression for $Z(t)$ from (14) and $S(t)$ from (16) in Equation (9), the closed-form solution for $I(t)$ is given by

$$I(t) = I_0 \left(\frac{S_0 + I_0}{S_0} \right)^{\frac{\beta}{\beta-\gamma-\alpha}} \left(1 + \frac{I_0}{S_0} e^{(\beta-\gamma-\alpha)t} \right)^{-\frac{\beta}{\beta-\gamma-\alpha}} e^{(\beta-\gamma-\alpha)t}. \tag{17}$$

It is worthy to mention here that, alternatively, we can establish closed-form solution for the variable $I(t)$ given in (17) by solving ODE (13) subject to initial condition $I(0) = I_0$. We substitute the expression for the variable $I(t)$ from (17) in ODE (3), and we arrive at a linear ODE which finally yields

$$Q(t) = Q_0 e^{-\mu t} + \gamma e^{-\mu t} \int_0^t I_0 \left(\frac{S_0 + I_0}{S_0} \right)^{\frac{\beta}{\beta-\gamma-\alpha}} \left(1 + \frac{I_0}{S_0} e^{(\beta-\gamma-\alpha)t} \right)^{-\frac{\beta}{\beta-\gamma-\alpha}} e^{(\beta-\gamma-\alpha+\mu)t} dt, \tag{18}$$

and finally, Δ is given by

$$\Delta(t) = N - S(t) - I(t) - Q(t). \tag{19}$$

Alternatively, same expression for $\Delta(t)$ can be derived by solving ODE (4). The closed-form solutions for all variables of model can be summarized as follows:

$$\begin{aligned} S(t) &= S_0 \left(\frac{S_0 + I_0}{S_0} \right)^{\frac{\beta}{\beta-\gamma-\alpha}} \left(1 + \frac{I_0}{S_0} e^{(\beta-\gamma-\alpha)t} \right)^{-\frac{\beta}{\beta-\gamma-\alpha}}, \\ I(t) &= I_0 \left(\frac{S_0 + I_0}{S_0} \right)^{\frac{\beta}{\beta-\gamma-\alpha}} \left(1 + \frac{I_0}{S_0} e^{(\beta-\gamma-\alpha)t} \right)^{-\frac{\beta}{\beta-\gamma-\alpha}} e^{(\beta-\gamma-\alpha)t}, \\ Q(t) &= Q_0 e^{-\mu t} + \gamma e^{-\mu t} \int_0^t I_0 \left(\frac{S_0 + I_0}{S_0} \right)^{\frac{\beta}{\beta-\gamma-\alpha}} \left(1 + \frac{I_0}{S_0} e^{(\beta-\gamma-\alpha)t} \right)^{-\frac{\beta}{\beta-\gamma-\alpha}} e^{(\beta-\gamma-\alpha+\mu)t} dt, \\ \Delta(t) &= N - S(t) - I(t) - Q(t), \end{aligned} \tag{20}$$

and $\beta \neq \gamma + \alpha$.*

2.3 | The equilibrium point for the model

The model attains its equilibrium point when $\rho_0 < 1$, and thus, $\beta < \gamma + \alpha$. We take limit $t \rightarrow \infty$ for closed-form solutions of all variables in Equation (20) and arrive at the following equilibrium point for the model provided $\beta < \gamma + \alpha$:

$$\begin{aligned} S^* &= S_0 \left(\frac{S_0 + I_0}{S_0} \right)^{\frac{\beta}{\beta-\gamma-\alpha}}, \quad I^* = 0, \quad Q^* = 0, \\ \Delta^* + S^* &= N. \end{aligned} \tag{21}$$

The endemic will end as $I(t) \rightarrow 0$ with $S(t)$ approaching some positive value if $t \rightarrow \infty$.

*The closed-form solutions for all variables of model for the special case when $\mu = 0$ are given in Appendix A1.

3 | THE CLOSED-FORM EXPRESSIONS OF SOME IMPORTANT INDICATORS OF EPIDEMIC

In this section, we have provided the closed-form expressions for the epidemic peak and maximum number of infected individuals. The closed-form expressions for the FOI and rate at which susceptibles become infected are also given in analytical form.

3.1 | The epidemic peak and maximum number of infected individuals

It is worthy to mention here that we can find analytical expression for time t where peak of infected curve occurs and it is termed as the epidemic peak. Differentiate $I(t)$ given in (20) with respect to t , we get

$$\frac{dI}{dt} = \frac{S_0(\beta - \gamma - \alpha) - I_0(\gamma + \alpha)e^{(\beta - \gamma - \alpha)t}}{I_0e^{(\beta - \gamma - \alpha)t} + S_0} \times \left[I_0 \left(\frac{S_0 + I_0}{S_0} \right)^{\frac{\beta}{\beta - \gamma - \alpha}} \left(1 + \frac{I_0}{S_0} e^{(\beta - \gamma - \alpha)t} \right)^{-\frac{\beta}{\beta - \gamma - \alpha}} e^{(\beta - \gamma - \alpha)t} \right]. \quad (22)$$

Setting $dI/dt = 0$, we get

$$t_p = \frac{1}{\beta - \gamma - \alpha} \ln \left(\frac{S_0(\beta - \gamma - \alpha)}{I_0(\gamma + \alpha)} \right), \quad (23)$$

and $t_p > 0$ as $\beta/\gamma + \alpha > 1$ and $S_0 > I_0$. We check second-order derivative at this peak value of t , and it is given by

$$\frac{d^2I}{dt^2} = -\frac{S_0}{\beta}(\beta - \gamma - \alpha)^3 \left(\frac{S_0 + I_0}{S_0} \frac{\gamma + \alpha}{\beta} \right)^{\frac{\beta}{\beta - \gamma - \alpha}}, \quad (24)$$

which is negative as $\beta/\gamma + \alpha > 1$. The maximum number of infected cases I_p is reported for this value of t and is given by

$$I_p = \frac{S_0}{\gamma + \alpha}(\beta - \gamma - \alpha) \left(\frac{S_0 + I_0}{S_0} \frac{\gamma + \alpha}{\beta} \right)^{\frac{\beta}{\beta - \gamma - \alpha}}. \quad (25)$$

Equations (23) and (25) can be expressed in terms of the basic reproduction number as follows:

$$t_p = \ln \left(\frac{S_0}{I_0}(\rho_0 - 1) \right)^{\frac{\rho_0}{\beta(\rho_0 - 1)}}, \quad (26)$$

and

$$I_p = S_0(\rho_0 - 1) \left(\frac{S_0 + I_0}{\rho_0 S_0} \right)^{\frac{\rho_0}{\rho_0 - 1}}. \quad (27)$$

3.2 | The FOI and rate at which susceptibles become infected

The FOI $\Omega = \beta I/S + I$ is the rate at which the susceptibles are infected. We utilize the closed-form solutions for $S(t)$ and $I(t)$ from (20) to find following closed-form expressions for the FOI:

$$\Omega = \frac{\beta}{1 + \frac{S_0}{I_0} e^{-(\beta - \gamma - \alpha)t}}. \quad (28)$$

The expression for FOI given in (28) can also be expressed in terms of the basic reproduction number ρ_0 and is given by

$$\Omega = \frac{\beta}{1 + \frac{S_0}{I_0} e^{-\frac{\beta(\rho_0 - 1)}{\rho_0} t}}. \quad (29)$$

TABLE 1 The range of parameters for the SIQD model with a quarantine-adjusted incidence function taken from clinical studies and modeling

Parameter	Range for parameters (day ⁻¹)	References
β	0.4–1.5	Previous studies ^{23–25}
γ	0.2–0.8	Previous studies ^{10,25,28}
α	0.001–0.1	Previous studies ^{25–27}
μ	0.001	Assumed

The rate at which susceptibles become infected is $\beta SI/S + I$, which is $\Omega \times S(t)$, and is given as

$$\Phi = \frac{\beta}{1 + \frac{S_0}{I_0} e^{-(\beta-\gamma-\alpha)t}} \times S_0 \left(\frac{S_0 + I_0}{S_0} \right)^{\frac{\beta}{\beta-\gamma-\alpha}} \left(1 + \frac{I_0}{S_0} e^{(\beta-\gamma-\alpha)t} \right)^{-\frac{\beta}{\beta-\gamma-\alpha}} \tag{30}$$

This can be further simplified as follows:

$$\Phi = \beta I_0 S_0 (S_0 + I_0)^{\frac{\beta}{\beta-\gamma-\alpha}} (S_0 + I_0 e^{(\beta-\gamma-\alpha)t})^{-\frac{\beta}{\beta-\gamma-\alpha}-1} e^{(\beta-\gamma-\alpha)t} \tag{31}$$

The rate at which susceptibles[†] become infected given (31) can be expressed as

$$\Phi = \beta I_0 S_0 (S_0 + I_0)^{\frac{\rho_0}{\rho_0-1}} \left(S_0 + I_0 e^{\frac{\beta(\rho_0-1)}{\rho_0}t} \right)^{-\frac{\rho_0}{\rho_0-1}-1} e^{\frac{\beta(\rho_0-1)}{\rho_0}t} \tag{32}$$

4 | THE SIQD WITH A QUARANTINE-ADJUSTED INCIDENCE FUNCTION FRAMEWORK TO MANAGE EPIDEMIC

In this section, we use the specific values of parameters and closed-form solutions of all variables provided in (20) to explore the results derived in Sections 2 and 3. The important indicators of understanding the transmission dynamic of disease which are peak time of epidemic (23), epidemic peak (25), FOI (28), and rate at which susceptibles become infected (31) are studied with the aid of simulations. We also provide an effective framework to manage the epidemic.

It is important to provide the range of parameters of SIQD model with a quarantine-adjusted incidence function in the context of the existing literature. The literature on the estimates of the basic reproduction number for the COVID-19 suggests that it varies from 1.4 to 5.7. According to WHO,¹⁹ it can take values between 1.4 and 2.5. Zhao et al²⁰ provided the preliminary estimates of the basic reproduction number of novel coronavirus between 3.6 and 4.0 and between 2.24 and 3.58. In China, ρ_0 was 5.6015 before sealing Wuhan city, and it was reduced to 3.4094 as of February 25, 2020.²¹ Musa et al²² estimated ρ_0 as 2.37 in Africa.

The estimated range of the parameters β , γ , and α for the SIQD model with a quarantine-adjusted incidence function is presented in Table 1. The reader is referred to see previous studies^{10,23–28} and references therein for further details.[‡]

The following comparative static analysis of the basic reproduction number,

$$\frac{\partial \rho_0}{\partial \beta} = \frac{1}{\alpha + \gamma} > 0, \quad \frac{\partial \rho_0}{\partial \gamma} = -\frac{\beta}{(\alpha + \gamma)^2} < 0, \tag{33}$$

reveals that it has a direct relationship with infection rate β and inverse relationship with rate of isolation γ . The epidemic control strategies suggest to lower the value of the reproduction number and reduce it to less than one, that is, $\rho_0 < 1$. This can be achieved in different ways. When government enforces strict control measures in the form of lockdown (social distancing), then infection rate β declines. The height of the epidemic peak I_p reduces, the time span t_p to attain this height increases, and ρ_0 decreases. This slows down the transmission dynamics of the disease. And when community

[†]It is worthy to mention here that the closed-form solutions of all variables (20), peak time of epidemic (23), epidemic peak (25), force of infection (28), and rate at which susceptibles become infected (31) are all important indicators of understanding the transmission dynamic of disease. These are valid to apply for any country's real data and forecast epidemic peak. These will provide insight for flattening curve and appropriate strategies to get rid of this pandemic.

[‡]Carcione et al⁶ in a study of SEIRD model provided high values of $\beta = 10^3$ as the uncertainties are related to parameter β , and it varies with time.

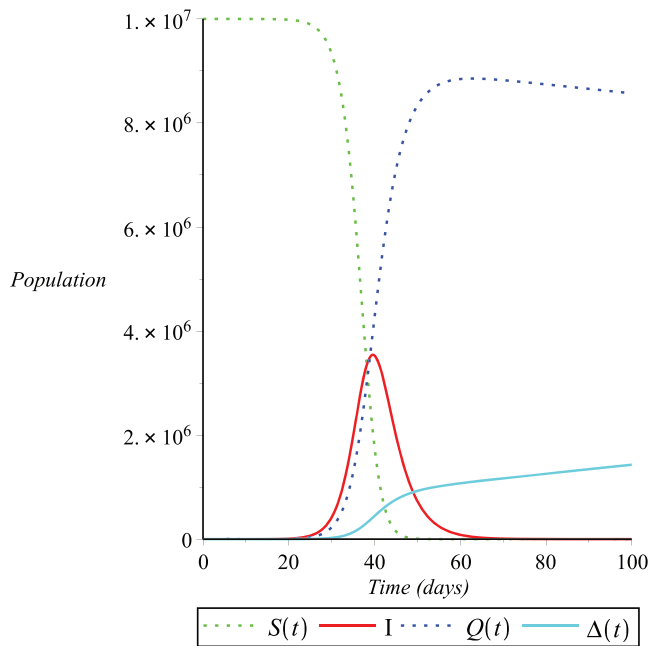


FIGURE 1 The graphs of $S(t)$, $I(t)$, $Q(t)$, and $\Delta(t)$ for SIQD model with a quarantine-adjusted incidence function [Colour figure can be viewed at wileyonlinelibrary.com]

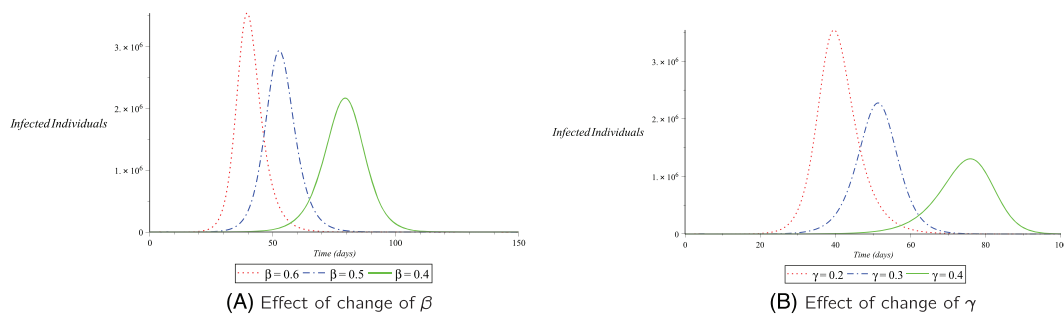


FIGURE 2 Effect of change of parameters on epidemic peak I_p and time span t_p [Colour figure can be viewed at wileyonlinelibrary.com]

follows proper prevention measures (quarantine, self isolation, mask, disinfection of surfaces, and hands hygiene) then γ increases, and thus, it lowers the basic reproduction number. We can reduce the basic reproduction number either by reducing β or by increasing γ . One can also use appropriate combination of reducing β and increasing γ to lower the value of ρ_0 .

The comparison of control strategies by government (lockdown) and prevention strategies (quarantine) is presented using simulations.

4.1 | Simulations to study the effect of change of parameters on the epidemic peak and transmission of disease

We consider the following parameters as an example to explore different control strategies mentioned as above: N is 10 million, $\alpha = 0.02$, $\beta = 0.6$, $\gamma = 0.2$, $\mu = 0.001$, and initial conditions are $S(0) = N - 5$, $I(0) = 5$, $Q(0) = 0$, $\Delta(0) = 0$. The graphical analysis of all variables $S(t)$, $I(t)$, $Q(t)$, and $\Delta(t)$ using closed-form solutions (20) is given in Figure 1.[§]

Figure 2A represents the effect of change of β while keeping other parameters as fixed. The effect of change of γ while keeping other parameters as fixed is given in Figure 2B. A closer look at Figure 2A,B shows that when β reduces (lockdown) or γ increases (quarantine and prevention), the epidemic peak reduces, and time span to attain this peak increases. This is an effective strategy to control spread of disease.

[§]We have used Maple 18 to produce these graphs.

FIGURE 3 The effect of change of β and γ in a fixed proportion on epidemic peak I_p and time span t_p [Colour figure can be viewed at wileyonlinelibrary.com]

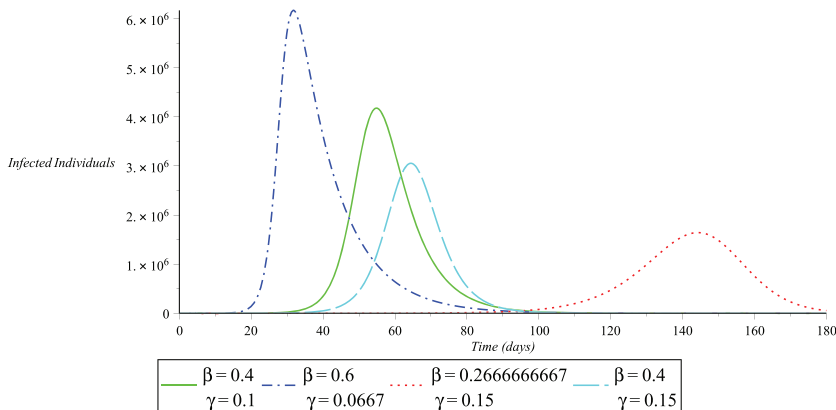
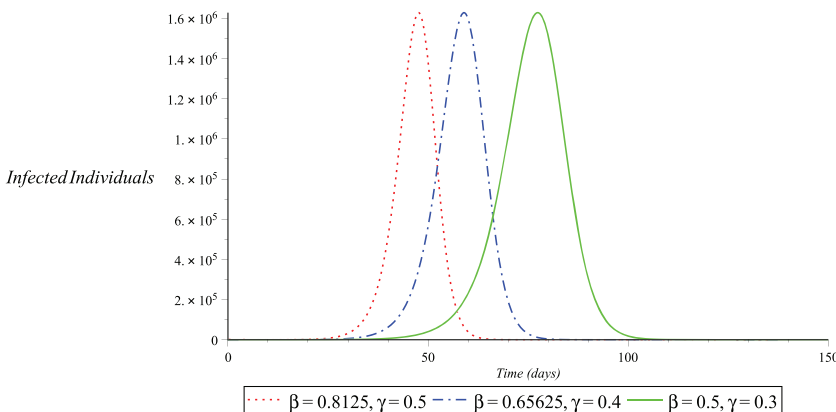


FIGURE 4 Effect of change of β and γ for fixed value of $\rho_0 = 1.5625$ [Colour figure can be viewed at wileyonlinelibrary.com]



One can also use appropriate combination of reducing β and increasing γ to lower the value of ρ_0 . We have set initially $\beta = 0.4$ and $\gamma = 0.1$, and this is shown in Figure 3 in green solid line. The value of $\rho_0 = 3.33$; the epidemic peak is on Day 55 with 4 178 373 number of infected individuals. Then we changed parameters by a fixed proportion $\phi = 1.5$ according to following three scenarios:

- (i) Increase β to $\phi\beta$ and reduce γ to γ/ϕ . This is shown in Figure 3 in blue dashed line. The value of $\rho_0 = 6.92$; the epidemic peak is on Day 32 with 6 171 341 number of infected individuals.
- (ii) Reduce β to β/ϕ and increase γ to $\phi\gamma$. This is shown in Figure 3 in red dotted line. The value of $\rho_0 = 1.6$; the epidemic peak is on Day 144 with 1 642 341 number of infected individuals. Scenario 2 is the best scenario to reduce epidemic peak and increase time span to attain the epidemic peak.
- (iii) Increase γ to $\phi\gamma$ with no change in β . This is shown in Figure 3 in cyan long dash line. The value of $\rho_0 = 2.35$; the epidemic peak is on Day 65 with 3 054 914 number of infected individuals.

Scenario 2 or 3 is the best scenario to reduce epidemic peak and increase time span to attain the epidemic peak. We can conclude that increasing γ (quarantine and prevention) strategy is an effective way to slow down transmission dynamics than the strategy of reducing β (lockdown). The prolonged lockdown results in social and economic loss. One should use appropriate combination of reducing β (lockdown) and increasing γ (quarantine and prevention) to lower the value of ρ_0 .

Another important strategy is to set the value of basic reproduction number at some reasonable fixed level close to the threshold value where transmission dynamics is slow. And then change parameters β and γ to reduce epidemic peak and increase the time span to attain this epidemic peak. This scenario is presented in Figure 4. This is a good strategy if the health care facilities are adequate to treat only I_* infected for time period t_* and is an effective way to develop herd immunity. To prevent second epidemic wave, it is necessary to slow down transmission dynamics instead of trying to quickly flatten the curve by strict lockdown measures. One should use appropriate combination of reducing β and increasing γ to maintain ρ_0 which will eventually help to attain the DFE.

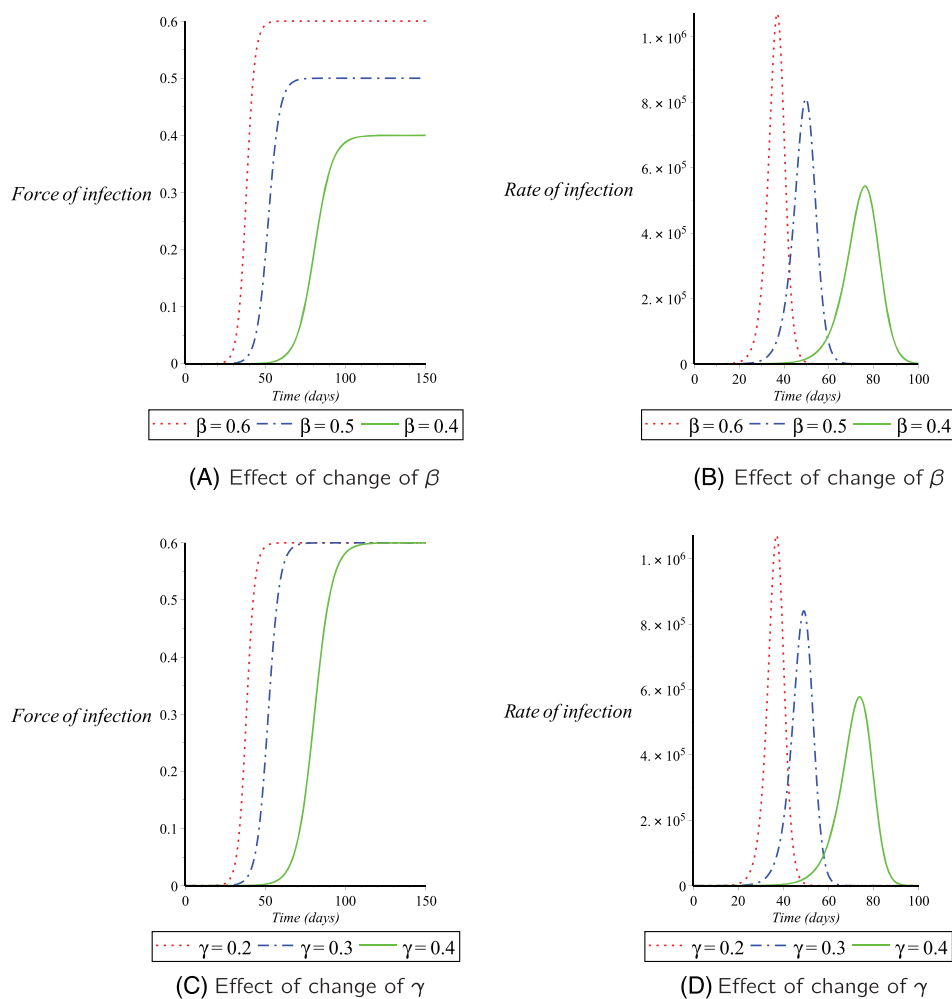


FIGURE 5 Effect of change of parameters on force of infection and rate of infection [Colour figure can be viewed at wileyonlinelibrary.com]

4.2 | Effect of change of parameters on the FOI and rate of infection

Next, we analyze the effect of change of parameters on the FOI and rate of infection. Figure 5A,B represents the effect of change of β while keeping other parameters as fixed. The FOI and rate of infection slow down as β reduces. This means that lockdown works efficiently for a short period in reducing force and rate of infection but at the cost of social and economics loss. But this is not an effective strategy in long run.

The effect of change of γ while keeping other parameters as fixed on the FOI and rate of infection is given in Figure 5C,D. A closer look at Figure 5C shows that when γ increases (quarantine and prevention), the time span to arrive at a certain point increases, but intensity of force remains same. The rate at which susceptible moves to infected class is reduced with an increase in γ .

The control and prevention policies include lockdown by government, quarantine, self isolation, social distancing, mask, disinfection of surfaces, and hands hygiene and are helpful in containment of corona virus. The effective control and prevention policies help in slow down pace of epidemic. The slow transmission dynamics will be helpful for a country to improve health care facilities by increasing capacity of beds, intensive care units, ventilators, and hiring more health care staff to deal with large number of patients.

Lockdown policies implemented by government in most of countries have negative impacts in terms of social and economic activities. The saving lives strategy has resulted in social and economic losses: decline in earnings, depreciation in gross domestic product (GDP), high unemployment rates, stagnant revenue, increased costs, and loss of social capital. We conclude that quarantine and prevention strategy measures are more efficient to slow down transmission dynamics than strict lockdown. The prolonged lockdown results in social and economic loss. One should use appropriate combinations of lockdown and prevention measures to flatten the curve of epidemic. The partial lockdown with effective quarantine and prevention measures is best way to handle the epidemic.

5 | THE SIMULATIONS OF SIQD WITH A QUARANTINE-ADJUSTED INCIDENCE FUNCTION BY UTILIZING LEAST SQUARE METHOD

In this section, we illustrate the simulation of the SIQD model with a quarantine-adjusted incidence function by utilizing the closed-form solutions of all variables provided in (20) for the new coronavirus disease. We had concluded in Section 4 that reducing infection rate β (lockdown) and increasing isolation rate γ (quarantine and prevention) in a fixed proportion are best ways to handle the epidemic. This can be done by reducing β to β/ϕ and increasing γ to $\phi\gamma$ for some fixed proportion ϕ . First, we analyze the estimated and actual epidemic peak for Pakistan. Then we apply the model for the United States, Brazil, India, and the Syrian Arab Republic. First, we fit the collected data of the new coronavirus disease to find the infection rate β , isolation rate γ , virus-induced mortality rate of infected individuals α , and scaling factor ϕ for the model. We assume that the virus-induced mortality rate of quarantined class μ is $10^{-7} \text{ days}^{-1}$ for the pandemic. We apply the least square method for the fitting of the collected cases with the the aid of closed-form solutions of all variables provided in (20). It is worthy to mention here that we have used the least square method with a general function form instead of a linear function. In Figure 6, we illustrate the flowchart of the fitting.

5.1 | The SIQD with a quarantine-adjusted incidence function to analyze the estimated and actual epidemic peak for Pakistan

We study the epidemic peak of Pakistan which has already arrived on July 1, 2020. We use the collected cases of the new coronavirus disease up to date August 2, 2020, to fit the parameters of the new coronavirus disease.^{29,30} The start date of pandemic in Pakistan is February 26, 2020, and we take this as t_0 . The initial values of the susceptible, infected, quarantine, and death cases are $S_0 = N - I_0 - Q_0 - \Delta_0$, $I_0 = 2$, $Q_0 = D_0 = 0$. The values of parameters estimated by utilizing the closed-form solution and least square method on actual data of Pakistan are given in Table 2.

The basic reproduction number for Pakistan for set of parameters given in Table 2 is $\rho_0 = 1.0013$. A closer look at Figure 7 reveals that the infection cases increase up to the Day 126, and total number of infected cases are 108 193. Now, we compare these estimated values with actual data of coronavirus cases in Pakistan. The actual peak occurred on Day 127, and number of infected at this peak value are 108 642. We conclude that our model provided close estimates to the actual data.

5.2 | The SIQD with a quarantine-adjusted incidence function to forecast epidemic peak for USA, Brazil, India, and the Syrian Arab Republic

First, we fit the collected cases of the pandemic for the United States where we use the collected cases of the new coronavirus disease up to date August 2, 2020, to fit the parameters of the new coronavirus disease.^{31,32} The first case of the pandemic in the United States appeared on January 13, 2020, and from this day, we use the initial values where the ini-

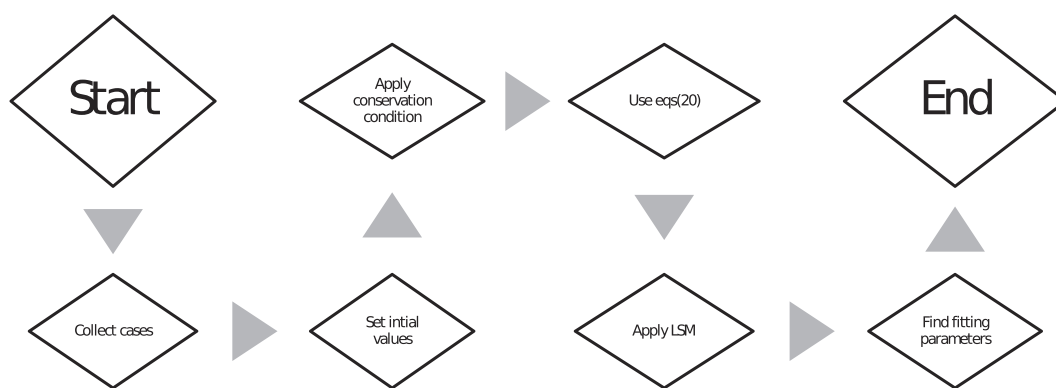


FIGURE 6 Flowchart of the fitting

TABLE 2 The parameter values for the SIQD model with quarantine-adjusted incidence function to estimate COVID-19 transmission dynamics for Pakistan

β (day^{-1})	γ (day^{-1})	α (day^{-1})	μ (day^{-1})	ϕ	N
71.0248	0.70909	0.0213	10^{-7}	100	220 892 331

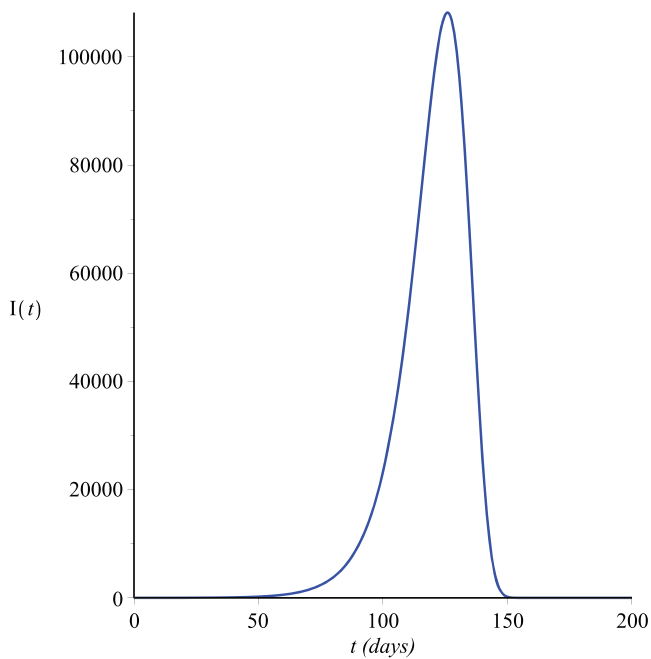


FIGURE 7 The graph of active infected cases $I(t)$ for Pakistan [Colour figure can be viewed at wileyonlinelibrary.com]

	USA	Brazil	India	The Syrian Arab Republic
β (day^{-1})	2.129	6.9	37.2711	17.6764
γ (day^{-1})	0.20469	0.68090	0.3719029	0.176009
α (day^{-1})	0.0203	0.0230	0.0189	0.02
μ (day^{-1})	10^{-7}	10^{-7}	10^{-7}	10^{-7}
ϕ	10	10	100	100
N	33 100 2647	212 559 409	1 380 004 385	17 500 657
t_0	Jan 13, 2020	Feb 26, 2020	Jan 30, 2020	March 22, 2020
I_0	1	1	1	1
Q_0	0	0	0	0
Δ_0	0	0	0	0

TABLE 3 The parameter values for the SIQD model with a quarantine-adjusted incidence function to forecast COVID-19 transmission dynamics for the United States, Brazil, India, and the Syrian Arab Republic

tial value of the infection cases is $I_0 = 1$ and the initial values of the quarantine and death $Q_0 = D_0 = 0$. The parameters of model for the United States are given in Table 3. A similar procedure is carried out for the collected cases of the pandemic for Brazil,³³ India,³⁴ and the Syrian Arab republic,^{35,36} and collected cases of the new coronavirus disease up to date August 2, 2020 are considered to fit the parameters of the new coronavirus disease. The initial conditions, parameter values, and scaling factor for the SIQD model with a quarantine-adjusted incidence function to forecast COVID-19 transmission dynamics for the United States, Brazil, India, and the Syrian Arab Republic are summarized in Table 3. The graphs for the active cases of infected individuals for United States, Brazil, India, and the Syrian Arab republic are presented in Figure 8. The peak time t_p (days), number of infected at peak I_p , and basic reproduction number are provided in Table 4.

In this section, we discussed the forecasting of the new coronavirus disease based on the closed-form solutions (20) of the SIQD model with a quarantine-adjusted incidence function for five countries, namely, Pakistan, the United States, Brazil, India, and the Syrian Arab Republic. We used the collected data of the infectious cases, the total cases, and the mortality cases up to date August 2 in the five countries to calculate the infection rate β , isolation rate γ , virus-induced mortality rate of infected individuals α , and scaling factor ϕ for the model. We assume that the virus-induced mortality rate of quarantined class μ is $10^{-7} days^{-1}$ for the pandemic. We used the least square method with the general function for this estimating. Also, we computed the peak of the infection function and the predict date of this peak for the five countries. We have compared the estimated and actual epidemic peak t_p and the number of infected individuals I_p at the peak for Pakistan. We conclude that our model provided close estimates to the actual data. Moreover, we have computed the basic reproduction number based on the fitting parameter for this model which lies in the range [1.0013–1.0299]. The smallest value of the basic reproduction number is for Pakistan, and the highest value is for the United States. It is important to mention here that highest number of the new coronavirus appeared in the United States up to date of writing this work.

FIGURE 8 The graphs of $I(t)$ for SIQD model with a quarantine-adjusted incidence function for USA, Brazil, India, and the syrian Arab republic [Colour figure can be viewed at wileyonlinelibrary.com]

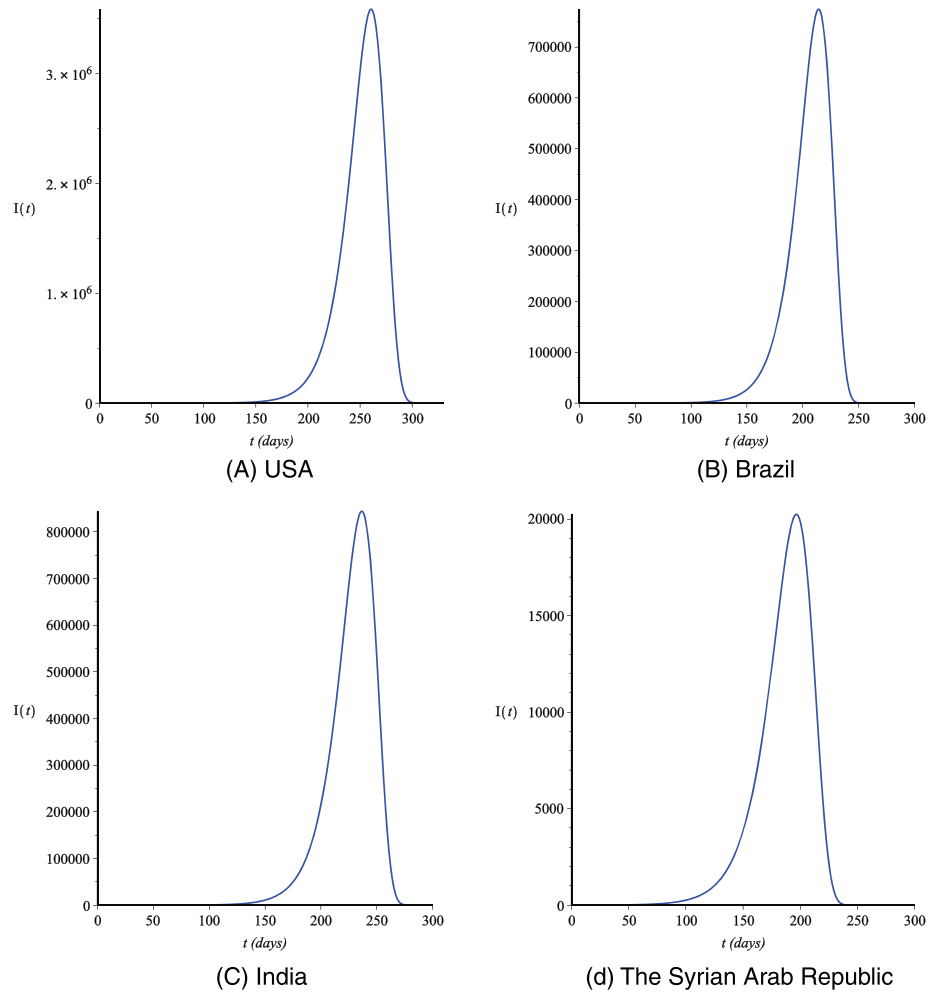


TABLE 4 The values of t_p , I_p , and ρ_0 for the SIQD model with a quarantine-adjusted incidence function to forecast COVID-19 transmission dynamics for the United States, Brazil, India, and the Syrian Arab Republic

Country	t_p (days)	I_p	ρ_0
USA	261	3 586 865	1.0299
Brazil	214	774 449	1.0099
India	237	843 987	1.0017
Syrian Arab Republic	197	20 246	1.0031

6 | CONCLUDING REMARKS

We have developed an SIQD model with a quarantine-adjusted incidence function. The closed-form solutions for all variables of model are established by utilizing the classical techniques of solving ODEs, and these hold provided $\beta \neq \gamma + \alpha$. We have provided the closed-form expressions for FOI, rate at which susceptible becomes infected, the epidemic peak and time required to attain this peak. The management of epidemic perceptive using control and prevention strategies is explained as well. The epidemic starts when $\rho_0 > 1$, the peak of epidemic appears when number of infected attains peak value when $\rho_0 = 1$, and the disease dies out $\rho_0 < 1$. The epidemic control strategies suggest to lower the value of the reproduction number and reduce it to less than, that is, $\rho_0 < 1$. This can be achieved in different ways. The epidemic peak reduces, and time span to attain this peak increases by reducing infection rate β or increasing isolation rate of infected individuals γ . Another effective strategy to reduce epidemic peak and increase the time span to attain this epidemic peak is to set the value of basic reproduction number at some fixed level close to the threshold value and then change parameters β and γ in a fixed proportion. The effect of change of parameter on the FOI and rate of infection is also presented in detail. We have provided the forecasting of the new coronavirus disease based on the closed-form solutions of SIQD model with a quarantine-adjusted incidence function and the scaling factor for five countries, namely, Pakistan, the United States, Brazil, India, and the Syrian Arab Republic.

The effective control and prevention policies help in slow down pace of epidemic. The slow transmission dynamics will be helpful for a country to improve health care facilities by increasing capacity of beds, intensive care units, ventilators, and hiring more health care staff to deal with large number of patients. One can also use appropriate combination of reducing infection rate (lockdown) and increasing quarantine rate to lower the value of the reproduction number to its threshold value. This will help in epidemic end. This model can be extended by adding exposed or asymptomatic compartment and will be considered in a future work.

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APPENDIX A

The closed-form solutions for $\mu = 0$ case for all the variables of model can be summarized as follows:

$$\begin{aligned}
 S(t) &= S_0 \left(\frac{S_0 + I_0}{S_0} \right)^{\frac{\beta}{\beta-\gamma-\alpha}} \left(1 + \frac{I_0}{S_0} e^{(\beta-\gamma-\alpha)t} \right)^{-\frac{\beta}{\beta-\gamma-\alpha}}, \\
 I(t) &= I_0 \left(\frac{S_0 + I_0}{S_0} \right)^{\frac{\beta}{\beta-\gamma-\alpha}} \left(1 + \frac{I_0}{S_0} e^{(\beta-\gamma-\alpha)t} \right)^{-\frac{\beta}{\beta-\gamma-\alpha}} e^{(\beta-\gamma-\alpha)t}, \\
 Q(t) &= Q_0 + \frac{\gamma(S_0 + I_0)}{\gamma + \alpha} - \frac{\gamma(S_0 + I_0)^{\frac{\beta}{\beta-\gamma-\alpha}} (S_0 + I_0 e^{(\beta-\gamma-\alpha)t})^{-\frac{\gamma+\alpha}{\beta-\gamma-\alpha}}}{(\gamma + \alpha)}, \\
 \Delta(t) &= N - S(t) - I(t) - Q(t),
 \end{aligned}
 \tag{A1}$$

and $\beta \neq \gamma + \alpha$. The closed-form solution given in (A2) can be rewritten in terms of the basic reproduction number as follows:

$$\begin{aligned}
 S(t) &= S_0 \left(\frac{S_0 + I_0}{S_0} \right)^{\frac{\rho_0}{\rho_0-1}} \left(1 + \frac{I_0}{S_0} e^{\frac{\beta(\rho_0-1)}{\rho_0} t} \right)^{-\frac{\rho_0}{\rho_0-1}}, \\
 I(t) &= S_0 \left(\frac{S_0 + I_0}{S_0} \right)^{\frac{\rho_0}{\rho_0-1}} \left(1 + \frac{I_0}{S_0} e^{\frac{\beta(\rho_0-1)}{\rho_0} t} \right)^{-\frac{\rho_0}{\rho_0-1}} e^{\frac{\beta(\rho_0-1)}{\rho_0} t}, \\
 Q(t) &= Q_0 + \frac{\gamma \rho_0 (S_0 + I_0)}{\beta} - \frac{\gamma \rho_0}{\beta} (S_0 + I_0)^{\frac{\rho_0}{\rho_0-1}} \left(S_0 + I_0 e^{\frac{\beta(\rho_0-1)}{\rho_0} t} \right)^{-\frac{1}{\rho_0-1}}, \\
 \Delta(t) &= N - S(t) - I(t) - Q(t),
 \end{aligned}
 \tag{A2}$$

and $\beta \neq \gamma + \alpha$.

For this case, we will arrive at the following equilibrium point for the model provided $\beta < \gamma + \alpha$:

$$\begin{aligned} S^* &= S_0 \left(\frac{S_0 + I_0}{S_0} \right)^{\frac{\beta}{\beta - \gamma - \alpha}}, \quad I^* = 0, \\ Q^* &= Q_0 + \frac{\gamma (S_0 + I_0)}{\gamma + \alpha} - \frac{\gamma (S_0 + I_0)^{\frac{\beta}{\beta - \gamma - \alpha}} (S_0)^{-\frac{\gamma + \alpha}{\beta - \gamma - \alpha}}}{(\gamma + \alpha)}, \\ \Delta^* &= N - S^* - I^* - Q^*. \end{aligned} \tag{A3}$$

The endemic will end as $I \rightarrow 0$ with S approaching some positive value if $t \rightarrow \infty$.