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All graphs of order $n \geq 11$ and diameter 2 with partition dimension $n-3$

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ABSTRACT

All graphs of order n with partition dimension 2, $n-2$, $n-1$, or n have been characterized. However, finding all graphs on n vertices with partition dimension other than these above numbers is still open. In this paper, we characterize all graphs of order $n \geq 11$ and diameter 2 with partition dimension $n-3$.

1. Introduction

Characterizing all graphs of order n with partition dimension k is a difficult problem. There are few results concerning this problem, in particular for k equal to $2, n$ or $n-1$ [1] and $n-2$ [2]. In this paper, we characterize all graphs with partition dimension $n-3$.

Let G be a connected graph. The *distance* of two vertices $u, v \in V(G)$, denoted by $d(u,v)$, is the length of shortest paths connecting u and v in G. For a subset of vertices $S \subset V(G)$, the distance between $u \in V(G)$ and S is defined by $d(u,S) = \min\{d(u,x) : x \in S\}$. The *eccentricity* of a vertex $u \in V(G)$, denoted by $\text{ecc}(u)$, is the maximum distance of vertex u to any other vertices of G, namely $\text{ecc}(u) = \max\{d(u,v) : v \in V(G)\}$. The *diameter* of G, denoted by $\text{diam}(G)$, is the maximum eccentricity of the vertices in G, or in short $\text{diam}(G) = \max\{\text{ecc}(u) : u \in V(G)\}$. Furthermore, $u \in V(G)$ is called a *peripheral* vertex if $\text{ecc}(u) = \text{diam}(G)$.

Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of $V(G)$. The *metric representation* of a vertex $u \in V(G)$ with respect to W is $r(u|W) = (d(u,w_1), d(u,w_2), \ldots, d(u,w_k))$. A set W is called a *resolving set* of G if the metric representations of any two vertices of G are distinct with respect to W. The cardinality of a minimum resolving set of graph G is called *metric dimension* of G and denoted by $\dim(G)$. Some results related to the metric dimension can be seen in [3, 4].

In [5] Chartrand et al. presented another kind of metric dimension concept, as follows. Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be a partition of a connected graph G. Define the *partition representation* of a vertex $u \in V(G)$ with respect to Π by $r(u|\Pi) = (d(u,S_1), d(u,S_2), \ldots, d(u,S_k))$, where $d(u,S_i) = \min\{d(u,x) : x \in S_i\}$ for $1 \leq i \leq k$. If any two vertices $u, v \in V(G)$ have distinct representations with respect to Π, namely $r(u|\Pi) \neq r(v|\Pi)$, then such a partition Π is called a *resolving partition* of G. The *partition dimension* of G, denoted by $pd(G)$, is the smallest cardinality of a resolving partition Π of G.

In general, for a connected graph G we have $pd(G) \leq \dim(G) + 1$. It is also natural to think that if two vertices $u, v \in V(G)$ have the same distance to all other vertices $V(G) \setminus \{u,v\}$, then these two vertices must be contained in distinct elements of any resolving partition Π of G. This result is shown as follows.

Remark 1 *([1]).* Let Π be a resolving partition of G and $u, v \in V(G)$. If $d(u,x) = d(v,x)$ for any $x \in V(G) \setminus \{u,v\}$, then u and v belong to distinct elements of Π.

In [1], Chartrand et al. characterized all connected graphs G of order n with partition dimension $2, n$ or $n-1$. They showed that for $n \geq 2$, the only graph with partition dimension 2 is a path and the only graph G with $pd(G) = n$ is the complete graph. Furthermore, they characterized all graphs of order $n \geq 3$ with partition dimension $n-1$, namely $K_{1,n-1}$, $K_n - e$ for any edge $e \in E(K_n)$, or $K_1 + (K_1 \cup K_{n-2})$. The characterization of connected graphs on $n \geq 9$ vertices with partition dimensions $n-2$ has been done by Tomescu [2]. He showed that there are only 23 graphs G of order $n \geq 9$ with $pd(G) = n-2$, namely $K_{2,n-2}$, $K_2 + \overline{K_{n-2}}$, $K_n - E(P_3)$, $K_n - E(K_3)$, $K_n - E(P_4)$, $K_1 + (K_1 \cup (K_{n-2} - e))$, $K_n - E(C_4)$, $K_{1,n-1} + e$, $K_n - E(2K_2)$, $K_{2,n-2} - e$, $K_n - E(K_{1,3} + e)$, $G_1, G_2, \ldots, G_{12}$, where e is any edge. The detail definitions of graphs $G_1, \ldots, G_{12}$ can be found in [2]. However, in this paper we prove that two of these above graphs, namely $K_{1,n-1} + e$ and $K_n - E(K_{1,3} + e)$, have partition dimension $n-3$ (not $n-2$). Furthermore, it is easy to verify that the graph F on $n \geq 9$ vertices obtained by connecting a vertex v to end vertex e of $K_{n-1} - e$ for any edge $e \in E(K_{n-1})$, has partition dimension $n-2$.

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In this paper, we study graphs of order n with partition dimension $n-3$. From [1], for all connected graphs G we have that $pd(G) \le n - \text{diam}(G) + 1$. Then, the graph G with partition dimension $n-3$ must have diameter 2, 3 or 4. In this paper, we characterize all connected graphs on $n \ge 11$ vertices with diameter 2 and partition dimension $n-3$. We will show that there are 114 non-isomorphic such graphs.

2. Main results

Before presenting the main results, we provide a useful property as follows.

Lemma 1. *For $n \ge 8$, let G be a graph on n vertices. If G does not contain the following three configurations:*

(1) *five vertices a, t_1, t_2, t_3 and t_4 forming $at_1, at_2 \in E(G)$ and $at_3, at_4 \notin E(G)$,*
(2) *six vertices a, b, t_1, t_2, t_3 and t_4 forming $at_1, bt_3 \in E(G)$ and $at_2, bt_4 \notin E(G)$, and*
(3) *four vertices t_1, t_2, t_3 and t_4 forming $t_1t_2 \in E(G)$ and $t_1t_4, t_2t_3, t_3t_4 \notin E(G)$,*

then G is isomorphic to either $\overline{K_n}$, K_n, $K_{1,n-1}$, $K_{n-1} \cup K_1$, $K_n - E(K_{1,n-2})$, or $K_n - e$ for any edge $e \in E(K_n)$.

Proof. Let K_r be a maximum clique of G for some integer $r \in [1, n]$. We consider four following cases.

Case 1. $r = 1$ or $r = n$. We can easily see that $G \cong \overline{K_n}$ or $G \cong K_n$, respectively.

Case 2. $r = 2$. Let $V(K_2) = \{x, y\}$ and $V(G) - V(K_2) = \{v_i : 1 \le i \le n-2\}$. If all vertices of K_2 are not adjacent to any vertex of $G - K_2$, then any two vertices of $G - K_2$ must be adjacent, since otherwise we have Configuration (3) in G. Hence $G - K_2$ induces K_{n-2}, but this contradicts that K_2 is the maximum clique in G. Now assume that there exists a vertex of K_2, namely a vertex x, such that x is adjacent to s vertices of $G - K_2$. If $1 \le s \le n-4$, then we have Configuration (1) in G, a contradiction. If $s = n-3$, namely $xv_i \in E(G)$ for all $1 \le i \le n-3$ and $xv_{n-2} \notin E(G)$, then $v_iv_j, yv_i \notin E(G)$ for all $1 \le i, j \le n-3$, since otherwise K_2 is not a maximum clique in G. Hence, we also obtain that $yv_{n-2}, v_iv_{n-2} \notin E(G)$ for any $1 \le i \le n-3$, if not we have Configuration (1) in G. However, $G \cong K_{1,n-2} \cup K_1$ and it contains Configuration (3) in G, a contradiction. Otherwise, assume that $s = n-2$. Note that $v_iv_j, yv_i \notin E(G)$ for any $1 \le i, j \le n-2$, since otherwise K_2 is not a maximum clique of G. Thus we obtain that $G \cong K_{1,n-1}$.

Case 3. $3 \le r \le n-2$. Let $V(K_r) = \{x_i : 1 \le i \le r\}$ and $V(G - K_r) = \{v_i : 1 \le i \le n-r\}$. Note that in this case, any vertex of K_r is not adjacent to at most one vertex of $G - K_r$ and any vertex of $G - K_r$ is not adjacent to at least one vertex of K_r, since otherwise we have Configuration (1) or K_{r+1} in G, respectively, a contradiction. Therefore, without loss of generality we can assume that $x_iv_i \notin E(G)$ for all $1 \le i \le \min\{r, n-r\}$ and $x_iv_j \in E(G)$ for all $i \ne j$, $1 \le i, j \le \min\{r, n-r\}$. However, it leads us to Configuration (2) in G, a contradiction.

Case 4. For $r = n-1$, let $V(G - K_{n-1}) = \{v\}$. Note that a vertex v is either adjacent to 0, 1 or $n-2$ vertices of K_{n-1}, since otherwise we have Configuration (1) or K_{n-1} is not a maximum clique of G, a contradiction. If v is not adjacent to any vertex of K_{n-1}, then $G \cong K_{n-1} \cup K_1$. If v is only adjacent to a single vertex of K_{n-1}, then $G \cong K_n - E(K_{1,n-2})$. Otherwise, v is adjacent to $n-2$ vertices of K_{n-1} and we obtain $G \cong K_n - e$. □

In the following result, we prove that there are exactly 114 non-isomorphic graphs G on $n \ge 11$ vertices and $\text{diam}(G) = 2$ such that $pd(G) = n-3$.

Theorem 1. *Let G be a connected graph of order $n \ge 11$ and $\text{diam}(G) = 2$. Then $pd(G) = n-3$ if and only if G is one of the following graphs:*

(i) *$\overline{K_{n-3}} + H$, where H is any graph on three vertices,*
(ii) *$K_1 + (K_{n-4} \cup H)$, where H is any graph on three vertices,*
(iii) *$K_1 + (K_{n-3} - e \cup H)$, where H is any graph on two vertices,*
(iv) *$K_1 + (K_{1,n-4} \cup H)$, where H is any graph on two vertices,*
(v) *$K_n - E(K_{1,n-4} \cup H)$, where H is any connected graph on three vertices,*
(vi) *$K_{n-5} + (K_2 \cup H)$, where H is any connected graph on three vertices,*
(vii) *$K_n - E(H)$, where H is any connected graph on four vertices other than C_4 and P_4,*
(vii) *$K_n - E(H)$, where H is either C_5, P_5, $K_{2,3}$, $K_2 \cup K_3$, $K_2 \cup P_3$, $3K_2$, $K_2 \cup C_4$, or $K_2 \cup P_4$,*
(ix) *$K_1 + (K_{2,n-4} \cup K_1)$,*
(x) *$K_n - E(K_{1,n-4})$,*
(xi) *$K_{1,n-1} + e$,*
(xii) *$K_n - E(K_{1,n-3} + e)$,*
(xiii) *Graphs $H_1, H_2, \ldots, H_{82}$.*

Proof. If G is one of the above graphs, then it is easy to verify that $pd(G) = n-3$. Now we are going to show the other direction. Let G be a connected graph of order $n \ge 11$ where $pd(G) = n-3$ and $\text{diam}(G) = 2$. Let x be a peripheral vertex of G with $\text{ecc}(x) = 2$. Denote $N_i(x)$ as the set of vertices of G with distance i to a vertex x, for $i = 1, 2$. Let $N_1(x) \supseteq \{u_1, u_2, u_3, u_4\}$ and $N_2(x) \supseteq \{v_1, v_2, v_3, v_4\}$. If $\min\{|N_1(x)|, |N_2(x)|\} \ge 4$, then $(x)(u_1, v_1)(u_2, v_2)(u_3, v_3)(u_4, v_4)\pi$ is a resolving $(n-4)$-partition, for a singleton partition π containing the vertices $V(G) \setminus \{x, u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$, a contradiction. Therefore, $\min\{|N_1(x)|, |N_2(x)|\} \le 3$. We consider the following subcases: (A) $|N_1(x)| = 3$, $|N_2(x)| = n-4$; (B) $|N_1(x)| = n-4$, $|N_2(x)| = 3$; (C) $|N_1(x)| = 2$, $|N_2(x)| = n-3$; (D) $|N_1(x)| = n-3$, $|N_2(x)| = 2$; (E) $|N_1(x)| = 1$, $|N_2(x)| = n-2$ and (F) $|N_1(x)| = n-2$, $|N_2(x)| = 1$.

(A) $|N_1(x)| = 3$ and $|N_2(x)| = n-4$. Let $N_1(x) = \{u_1, u_2, u_3\}$. If $N_2(x)$ contains 3 vertices a, b, c such that $ab \in E(G)$ and $ac \notin E(G)$, then we can define a resolving $(n-4)$-partition of G, namely $(x)(a)(b, c)(u_1, t_1)(u_2, t_2)(u_3, t_3)\pi$, for $t_1, t_2, t_3 \in N_2(x) \setminus \{a, b, c\}$ and a singleton partition π of the remaining vertices, a contradiction. Therefore, $N_2(x)$ induces (A1) $\overline{K_{n-4}}$ or (A2) K_{n-4}.

(A1) $N_2(x)$ induces $\overline{K_{n-4}}$. If there exists a vertex of $N_1(x)$, namely u_1, with $u_1a \notin E(G)$ and $u_1b \in E(G)$ for some $a, b \in N_2(x)$, then we have a resolving $(n-4)$-partition in G, namely $(x)(a, b)(u_1, t_1)(u_2, t_2)(u_3, t_3)\pi$, for $t_1, t_2, t_3 \in N_2(x)$ and a singleton partition π, a contradiction. Therefore, any vertex of $N_1(x)$ are adjacent to all vertices of $N_2(x)$ or some of them are not adjacent to any vertex of $N_2(x)$. If $u_1 \in N_1(x)$ is adjacent to all vertices of $N_2(x)$ and $u_2 \in N_1(x)$ is not adjacent to any vertex of $N_2(x)$, then we can also define a resolving $(n-4)$-partition of G, namely $(u_1, a_1)(u_2, a_2)(u_3, a_3)(x, a_4)\pi$, for $a_1, a_2, a_3, a_4 \in N_2(x)$ and a singleton partition π of the remaining vertices, a contradiction. Therefore, we can conclude that any vertex of $N_1(x)$ are adjacent to all vertices of $N_2(x)$. We obtain that $G \cong K_{3,n-3}$ if none of vertices of $N_1(x)$ are connected, or $G \cong (K_1 \cup K_2) + \overline{K_{n-3}}$ if $N_1(x)$ induces $K_1 \cup K_2$, or $G \cong P_3 + \overline{K_{n-3}}$ if $N_1(x)$ induces P_3, or $G \cong K_3 + \overline{K_{n-3}}$ if any two vertices of $N_1(x)$ are connected (Fig. 1).

(A2) $N_2(x)$ induces K_{n-4}. If there exist four distinct vertices $t_1, t_2, t_3, t_4 \in N_2(x)$ such that $u_1t_1, u_1t_2 \in E(G)$ but $u_1t_3, u_1t_4 \notin E(G)$, then we have a resolving $(n-4)$-partition of G, namely $(x)(u_1)(t_1, t_3)(t_2, t_4)(u_2, t_5)(u_3, t_6)\pi$, for some $t_5, t_6 \in N_2(x) \setminus \{t_1, t_2, t_3, t_4\}$ and a singleton partition π, a contradiction. Therefore, any vertex of $N_1(x)$ is either adjacent to at most one vertex of $N_2(x)$ or it is adjacent at least to $n-5$ vertices of $N_2(x)$. Note that for any $t \in N_2(x)$, there exists a vertex $u_i \in N_1(x)$ such that $u_it \in E(G)$, since otherwise $\text{diam}(G) = 3$.

Remark 2. Let $\{a, b, c\} \subset N_2(x)$. If we have one of the following five conditions in G:

1. u_1 is not adjacent to any vertex of $N_2(x)$, u_2 is only adjacent to vertex a in $N_2(x)$, and u_3 is adjacent to $n-5$ vertices of $N_2(x) \setminus \{a\}$, or

Table 1
Adjacency of three vertices $u_1, u_2, u_3 \in N_1(x)$ to the vertices of $N_2(x)$.

u_1	0	0	0	0	1	1	1	$n-5$	$n-5$	$n-4$
u_2	0	1	$n-5$	$n-4$	1	$n-5$	$n-4$	$n-5$	$n-4$	$n-4$
u_3	$n-4$	$n-4$	$n-4$	$n-4$	$n-4$	$n-4$	$n-4$	$n-4$	$n-4$	$n-4$

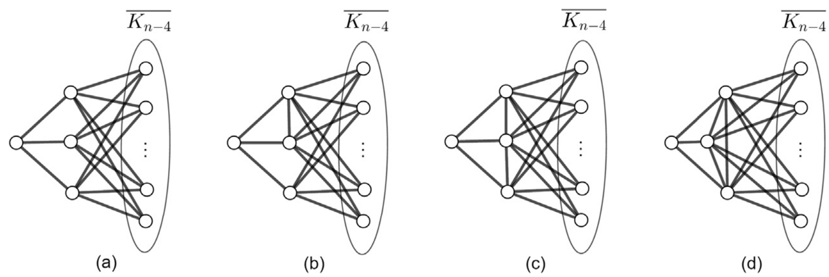

Fig. 1. Graph (a) $K_{3,n-3}$, (b) $(K_1 \cup K_2) + \overline{K_{n-3}}$, (c) $P_3 + \overline{K_{n-3}}$ and (d) $K_3 + \overline{K_{n-3}}$.

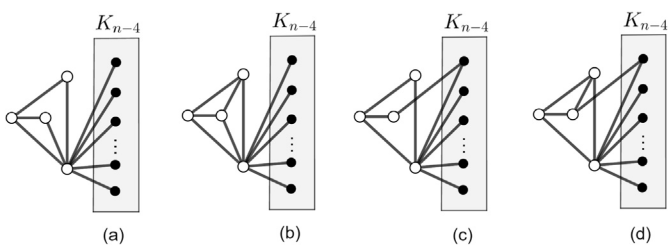

Fig. 2. Graph (a) $K_1 + (K_{n-4} \cup P_3)$, (b) $K_1 + (K_{n-4} \cup K_3)$, (c) H_{73} and (d) H_{74}.

2. u_1 is not adjacent to any vertex of $N_2(x)$, u_2 is adjacent to $n-5$ vertices of $N_2(x) \setminus \{a\}$, and u_3 is adjacent to $n-5$ vertices of $N_2(x) \setminus \{b\}$, or
3. u_1 is only adjacent to vertex a in $N_2(x)$, u_2 is only adjacent to vertex b in $N_2(x)$, and u_3 is adjacent to $n-5$ vertices of $N_2(x)$ other than a or b, or
4. u_1 and u_2 are only adjacent to vertex a in $N_2(x)$, and u_3 is adjacent to $n-5$ vertices of $N_2(x) \setminus \{a\}$, or
5. u_1 is only adjacent to vertex a in $N_2(x)$, u_2 is adjacent to $n-5$ vertices of $N_2(x) \setminus \{s\}$ where $s \in \{a, b\}$, and u_3 is adjacent to $n-5$ vertices of $N_2(x) \setminus \{c\}$, or
6. u_1 is adjacent to $n-5$ vertices of $N_2(x) \setminus \{a\}$, u_2 is adjacent to $n-5$ vertices of $N_2(x) \setminus \{s\}$, and u_3 is adjacent to $n-5$ vertices of $N_2(x) \setminus \{t\}$, where $s, t \in \{b, c\}$,

then there exists a resolving $(n-4)$-partition of G, namely $(a)(b)(c)(u_1, t_1)(u_2, t_2)(x, u_3, t_3)\pi$ for $t_1, t_2, t_3 \in N_2(x) \setminus \{a, b, c\}$ and a singleton partition π, a contradiction.

By the previous facts and Remark 2, the adjacency of three vertices $u_1, u_2, u_3 \in N_1(x)$ to the vertices of $N_2(x)$ are shown in the Table 1.

(A2.1) $u_1 \in N_1(x)$ is not adjacent to any vertex of $N_2(x)$. If $u_2 \in N_1(x)$ is also not adjacent to any vertex of $N_2(x)$, then u_3 is adjacent to all vertices of $N_2(x)$ and $u_1u_3, u_2u_3 \in E(G)$, since otherwise $\mathrm{diam}(G) = 3$. We obtain that $G \cong K_1 + (K_{n-4} \cup P_3)$ if $u_1u_2 \notin E(G)$ or $G \cong K_1 + (K_{n-4} \cup K_3)$ if $u_1u_2 \in E(G)$. If u_2 is only adjacent to a single vertex $t_1 \in N_2(x)$ and u_3 is adjacent to all vertices of $N_2(x)$, then $u_1u_3 \in E(G)$ and $u_2u_3 \notin E(G)$, since otherwise $\mathrm{diam}(G) = 3$ or $(u_2)(t_1, t_2)(u_1, t_3)(x, u_3, t_4)\pi$ is a resolving $(n-4)$-partition for $t_2, t_3, t_4 \in N_2(x) \setminus \{t_1\}$ and π is a singleton partition, respectively, a contradiction. We obtain that $G \cong H_{73}$ if $u_1u_2 \notin E(G)$ or $G \cong H_{74}$ if $u_1u_2 \in E(G)$ (Fig. 2).

If u_2 is only not adjacent to a single vertex $a \in N_2(x)$ and u_3 is adjacent to all vertices of $N_2(x)$, then $u_1u_3 \in E(G)$ and $u_2u_3 \in E(G)$, since otherwise $\mathrm{diam}(G) = 3$ or $(u_2)(x, u_3, t_1)(a, t_2)(u_1, t_3)\pi$ is a resolving $(n-4)$-partition, for $t_1, t_2, t_3 \in N_2(x) \setminus \{a\}$ and a singleton partition π, contradiction. We deduce that $G \cong H_{51}$ if $u_1u_2 \notin E(G)$ or $G \cong H_{60}$ if $u_1u_2 \in E(G)$. For the remaining cases, assume that both u_2 and u_3 are adjacent to all vertices of $N_2(x)$. Then u_1 is adjacent to at least one vertex of u_2 or u_3, since otherwise $\mathrm{diam}(G) = 3$. We obtain G as depicted in Fig. 3 (c)-(f).

(A2.2) $u_1 \in N_1(x)$ is only adjacent to vertex a in $N_2(x)$. Let u_2 be only adjacent to vertex b in $N_2(x)$ and u_3 be adjacent to all vertices of $N_2(x)$. If $a = b$ or $a \neq b$, then u_3 is not adjacent to both u_1 and u_2, since otherwise one of $(u_1)(a, t_1)(u_2, t_2)(x, u_3, t_3)\pi$, or $(u_2)(a, t_1)(u_1, t_2)(x, u_3, t_3)\pi$, or $(u_1)(u_2)(a, t_1)(b, t_2)(x, u_3, t_3)\pi$ is a resolving $(n-4)$-partition of G, for $t_1, t_2, t_3 \in N_2(x) \setminus \{a, b\}$, a contradiction. We deduce $G \cong H_{75}$ if $a = b$ and $u_1u_2 \notin E(G)$, or $G \cong H_{74}$ if $a = b$ and $u_1u_2 \in E(G)$, or $G \cong H_{76}$ if $a \neq b$ and $u_1u_2 \notin E(G)$, or $G \cong H_{77}$ if $a \neq b$ and $u_1u_2 \in E(G)$.

Now assume that u_2 is only not adjacent to a single vertex $b \in N_2(x)$ and u_3 is adjacent to all vertices of $N_2(x)$, so that $u_2u_3 \in E(G)$, since otherwise $(u_2)(x, u_3, t_1)(a, t_2)(u_1, t_3)\pi$ is a resolving $(n-4)$-partition, for $t_1, t_2, t_3 \in N_2(x) \setminus \{a, b\}$, a contradiction. If $a = b$, then u_1 is not adjacent to at least one of u_2 or u_3, since otherwise $(u_1)(a, t_1)(u_2, t_2)(x, u_3, t_3)\pi$ is a resolving $(n-4)$-partition, for $t_1, t_2, t_3 \in N_2(x) \setminus \{a\}$, a contradiction. We deduce G as depicted in Fig. 4 (e)-(g). If $a \neq b$, then $u_1u_3 \notin E(G)$, since otherwise $(u_1)(u_2)(a, t_1)(b, t_2)(x, u_3, t_3)$ is a resolving $(n-4)$-partition, for $t_1, t_2, t_3 \in N_2(x) \setminus \{a, b\}$, a contradiction. We obtain G as depicted in Fig. 4 (h)-(i).

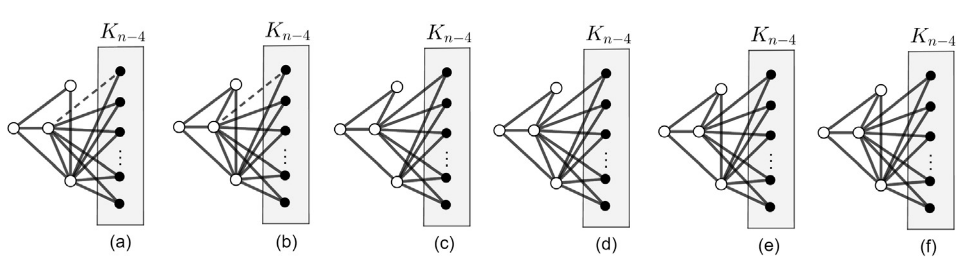

Fig. 3. Graph (a) H_{51}, (b) H_{60}, (c) H_{56}, (d) H_{47}, (e) H_{63} and (f) H_{48}.

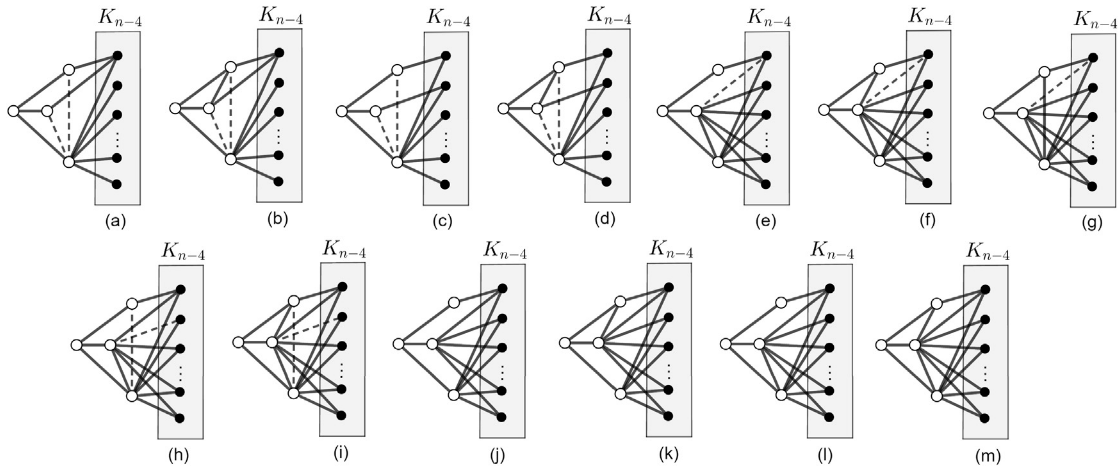

Fig. 4. Graph (a) H_{75}, (b) H_{74}, (c) H_{76}, (d) H_{77}, (e) H_{54}, (f) H_{61}, (g) H_{52}, (h) H_{58}, (i) H_{59}, (j) H_{55}, (k) H_{61}, (l) H_{46} and (m) H_{49}.

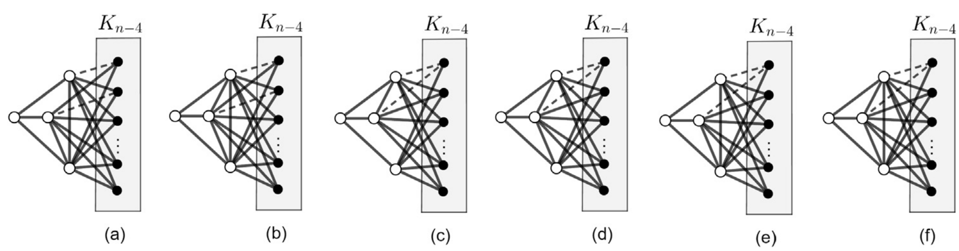

Fig. 5. Graph (a) H_{26}, (b) H_{17}, (c) H_{37}, (d) H_{27}, (e) H_{15} and (f) H_5.

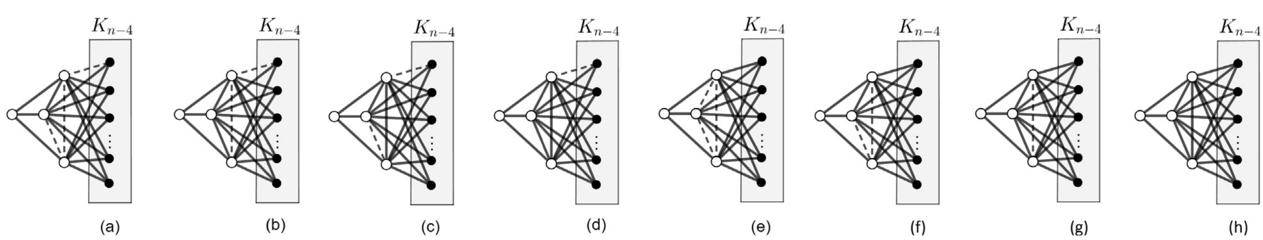

Fig. 6. Graph (a) H_{28}, (b) H_6, (c) H_{18}, (d) H_1, (e) $K_n - E(K_{1,n-4} \cup K_3)$, (f) $K_n - E(K_{1,n-4} \cup P_3)$, (g) H_2 and (h) $K_n - E(K_{1,n-4})$.

For the remaining cases, let both u_2 and u_3 be adjacent to all vertices of $N_2(x)$. Then, u_1 is not adjacent to at least one of u_2 or u_3, since otherwise $(u_1)(a,t_1)(u_2,t_2)(x,u_3,t_3)\pi$ is a resolving $(n-4)$-partition, for $t_1,t_2,t_3 \in N_2(x) \setminus \{a\}$, a contradiction. We obtain G as depicted in Fig. 4 (j)-(m).

(A2.3) u_1 is adjacent to $n-5$ vertices of $N_2(x) \setminus \{a\}$. Let u_2 be also adjacent to $n-5$ vertices of $N_2(x) \setminus \{b\}$ and u_3 be adjacent to all vertices of $N_2(x)$. If $a \neq b$ then u_3 is adjacent to both u_1 and u_2, since otherwise we have a resolving $(n-4)$-partition, namely $(u_1)(u_2)(a,t_1)(b,t_2)(x,u_3,t_3)\pi$, for $t_1,t_2,t_3 \in N_2(x) \setminus \{a,b\}$, a contradiction. We obtain G as depicted in Fig. 5 (a)-(b). If $a = b$, then $u_1,u_2 \in E(G)$ or (both u_1 and u_2 are adjacent to u_3), since otherwise $(u_1)(a,t_1)(u_2,t_2)(x,u_3,t_3)\pi$ or $(u_2)(a,t_1)(u_1,t_2)(x,u_3,t_3)\pi$) is a resolving $(n-4)$-partition, for $t_1,t_2,t_3 \in N_2(x) \setminus \{a\}$, a contradiction. We deduce G as depicted in Fig. 5 (c)-(f).

Now assume that both u_2 and u_3 are adjacent to all vertices of $N_2(x)$. Then, u_1 is adjacent to at least one of u_2 or u_3, since otherwise $(u_1)(a,t_1)(u_2,t_2)(x,u_3,t_3)\pi$ is a resolving $(n-4)$-partition, for $t_1,t_2,t_3 \in N_2(x) \setminus \{a\}$, a contradiction. We deduce $G \cong H_{28}$ if $u_2u_3 \notin E(G)$ and u_1 is only adjacent to one of u_2 or u_3, or $G \cong H_6$ if $u_2u_3 \in E(G)$ and u_1 is only adjacent to one of u_2 or u_3, or $G \cong H_{18}$ if $u_2u_3 \notin E(G)$ and u_1 is adjacent to both u_2 and u_3, or $G \cong H_1$ if $u_2u_3 \in E(G)$ and u_1 is adjacent to both u_2 and u_3, as depicted in Fig. 6 (a)-(d), respectively.

(A2.4) All vertices of $N_1(x)$ are adjacent to all vertices of $N_2(x)$. We deduce that $G \cong K_n - E(K_{1,n-4} \cup K_3)$ if $N_1(x)$ induces $\overline{K_3}$, or $G \cong K_n - E(K_{1,n-4} \cup P_3)$ if $u_1u_2 \in E(G)$ and $u_1u_3, u_2u_3 \notin E(G)$, or $G \cong H_2$ if $N_1(x)$ induces P_3, or $G \cong K_n - E(K_{1,n-4})$ if $N_1(x)$ induces K_3, as depicted in Fig. 6 (e)-(h).

(B) $|N_1(x)| = n-4$ and $|N_2(x)| = 3$. Let $N_2(x) = \{v_1,v_2,v_3\}$. If $N_1(x)$ contains three vertices a,b,c such that $ab \in E(G)$ and $ac \notin E(G)$, then $(x)(a)(b,c)(v_1,t_1)(v_2,t_2)(v_3,t_3)\pi$ is a resolving $(n-4)$-partition, for $t_1,t_2,t_3 \in N_1(x) \setminus \{a,b,c\}$ and a singleton partition π of the remaining vertices, a contradiction. Therefore, $N_1(x)$ induces (B1) $\overline{K_{n-4}}$ or (B2) K_{n-4}.

(B1) $N_1(x)$ induces $\overline{K_{n-4}}$. Note that for any vertex $v_i \in N_2(x)$, there exists $t \in N_1(x)$ such that $v_it \in E(G)$, and conversely for any $t \in N_1(x)$, there exists $v_i \in N_2(x)$ such that $tv_i \in E(G)$, since otherwise $\mathrm{diam}(G) = 3$. Without loss of generality, we can assume that $v_1a, v_2b, v_3c \in E(G)$ for some $a,b,c \in N_2(x)$. Then, $(a)(b)(c)(x,t_1)(v_1,t_2)(v_2,t_3)(v_3,t_4)\pi$ is a resolving $(n-4)$-partition for $t_1,t_2,t_3,t_4 \in N_2(x) \setminus \{a,b,c\}$ and a singleton partition π, a contradiction. Hence we can conclude that there exists no graphs G with $pd(G) = n-3$ where $N_1(x)$ induces $\overline{K_{n-4}}$.

(B2) $N_1(x)$ induces K_{n-4}. If there exist four distinct vertices $a,b,c,d \in N_1(x)$ such that $v_1a, v_1b \in E(G)$ but $v_1c, v_1d \notin E(G)$, then $(x)(v_1)(a,c)(b,d)(v_2,t_1)(v_3,t_2)\pi$ is a resolving $(n-4)$-partition, for $t_1,t_2 \in N_1(x) \setminus \{a,b,c,d\}$, a contradiction. Additionally, any vertex of $N_2(x)$ is adjacent to at least one vertex of $N_1(x)$, since otherwise $\mathrm{diam}(G) = 3$.

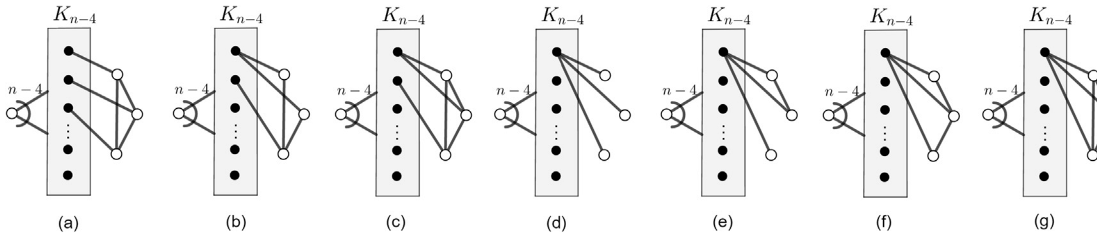

Fig. 7. Graph (a) H_{77}, (b) H_{75}, (c) H_{74}, (d) $K_1 + (K_{n-4} \cup \overline{K_3})$, (e) $K_1 + (K_{n-4} \cup (P_3 - e))$, (f) $K_1 + (K_{n-4} \cup P_3)$ and (g) $K_1 + (K_{n-4} \cup K_3)$.

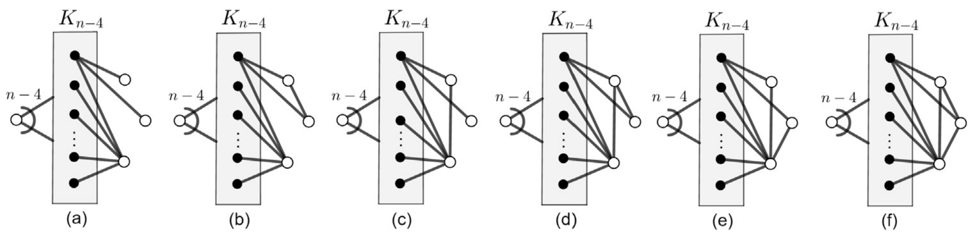

Fig. 8. Graph (a) $K_1 + (K_{n-3} - e \cup 2K_1)$, (b) $K_1 + (K_{n-3} - e \cup K_2)$, (c) H_{65}, (d) H_{51}, (e) H_{69} and (f) H_{60}.

Therefore, any vertex of $v_1, v_2, v_3 \in N_2(x)$ is either adjacent to 1, $n-5$ or $n-4$ vertices of $N_1(x)$. Now consider the following remarks.

Remark 3. Let $\{a,b,c\} \subset N_1(x)$, v_1 be only adjacent to vertex a in $N_1(x)$ and v_2 be only adjacent to vertex b in $N_1(x)$.

1. If $a = b$ and v_3 is adjacent to $n-5$ vertices of $N_1(x) \setminus \{a\}$, then $(v_3)(a,t_1)(t_2,v_2)(x,t_3,v_1)\pi$ is a resolving $(n-4)$-partition, for $t_1,t_2,t_3 \in N_1(x) \setminus \{a\}$ and a singleton partition π, a contradiction.
2. If $a = b$ and v_3 is adjacent to $n-5$ vertices of $N_1(x) \setminus \{c\}$ where $c \neq a$, then $(v_1)(v_3)(a,t_1)(c,t_2)(x,t_3,v_2)\pi$ is a resolving $(n-4)$-partition, for $t_1,t_2,t_3 \in N_1(x) \setminus \{a,c\}$ and a singleton partition π, a contradiction.
3. If $a \neq b$ and v_3 is adjacent to $n-5$ vertices of $N_1(x)$ other than a (or similarly other than b), then $(v_3)(a,t_1)(b,v_2)(x,t_2,v_1)\pi$ is a resolving $(n-4)$-partition, for $t_1,t_2 \in N_1(x) \setminus \{a,b\}$ and a singleton partition π, a contradiction.
4. If $a \neq b$ and v_3 is adjacent to $n-5$ vertices of $N_1(x) \setminus \{c\}$ where $c \neq a$ and $c \neq b$, then $(v_3)(x,a,v_1)(b,v_2)(c,t_1)\pi$ is a resolving $(n-4)$-partition, for $t_1 \in N_1(x) \setminus \{a,b,c\}$ and a singleton partition π, a contradiction.

Remark 4. Let $\{a,b,c\} \subset N_1(x)$, v_1 be only adjacent to vertex a in $N_1(x)$ and v_2 be adjacent to $n-5$ vertices of $N_1(x) \setminus \{b\}$.

1. If $a = b$ and v_3 is adjacent to $n-5$ vertices of $N_1(x) \setminus \{c\}$ with $c \neq a$, then $(v_2)(v_3)(a,t_1)(c,t_2)(x,t_3,v_1)\pi$ is a resolving $(n-4)$-partition, for $t_1,t_2,t_3 \in N_1(x) \setminus \{a,c\}$ and a singleton partition π, a contradiction.
2. If $a \neq b$ and v_3 is adjacent to $n-5$ vertices of $N_1(x) \setminus \{a\}$, then $(v_2)(v_3)(a,t_1)(b,t_2)(x,t_3,v_1)$ is a resolving $(n-4)$-partition for $t_1,t_2,t_3 \in N_1(x) \setminus \{a,b\}$ and a singleton partition π, a contradiction.
3. If ($a \neq b$, v_3 is adjacent to $n-5$ vertices of $N_1(x) \setminus \{b\}$ and $v_1v_3 \in E(G)$ (or similarly $v_1v_2 \in E(G)$)) or ($a \neq b$, v_3 is adjacent to all vertices of $N_1(x)$ and $v_1v_3 \in E(G)$), then $(v_1)(v_2)(a,t_1)(b,t_2)(x,v_3,t_3)\pi$ is a resolving $(n-4)$-partition, for $t_1,t_2,t_3 \in N_1(x) \setminus \{a,b\}$, a contradiction.
4. If v_3 is adjacent to $n-5$ vertices of $N_1(x) \setminus \{c\}$ with all a,b,c are distinct, then $(v_1)(v_2)(v_3)(a,t_1)(b,t_2)(x,c,t_3)$ is a resolving $(n-4)$-partition, for $t_1,t_2,t_3 \in N_1(x) \setminus \{a,b,c\}$ and a singleton partition π, a contradiction.
5. If $a \neq b$, v_3 is adjacent to all vertices of $N_1(x)$ and $v_1v_3 \in E(G)$, then $(v_1)(v_2)(a,t_1)(b,t_2)(x,t_3,v_3)\pi$ is a resolving $(n-4)$-partition, for $t_1,t_2,t_3 \in N_1(x) \setminus \{a,b\}$ and a singleton partition π, a contradiction.

Table 2
Adjacency of three vertices $v_1, v_2, v_3 \in N_2(x)$ to the vertices of $N_1(x)$.

v_1	1	1	1	1	1	$n-5$	$n-5$	$n-5$	$n-4$
v_2	1	1	$n-5$	$n-5$	$n-4$	$n-5$	$n-5$	$n-4$	$n-4$
v_3	1	$n-4$	$n-5$	$n-4$	$n-4$	$n-5$	$n-4$	$n-4$	$n-4$

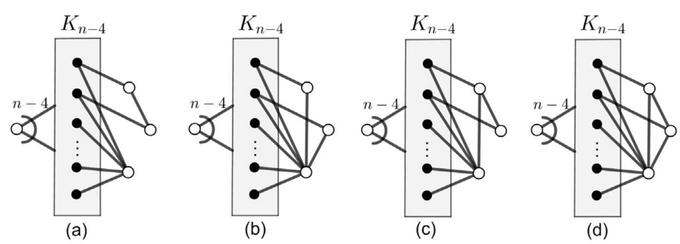

Fig. 9. Graph (a) H_{57}, (b) H_{68}, (c) H_{58} and (d) H_{59}.

Remark 5. Let $\{a,b,c\} \subset N_1(x)$, v_1 be adjacent to $n-5$ vertices of $N_1(x) \setminus \{a\}$ and v_2 be adjacent to $n-5$ vertices of $N_1(x) \setminus \{b\}$.

1. If $a = b$ and v_3 is adjacent to $n-5$ vertices of $N_1(x) \setminus \{c\}$ where $c \neq a$, then $(a)(v_3)(c,t_1)(v_1,t_2)(x,t_3,v_2)\pi$ is a resolving $(n-4)$-partition, for $t_1,t_2,t_3 \in N_1(x) \setminus \{a,c\}$ and a singleton partition π, a contradiction.
2. If $a \neq b$ and v_3 is adjacent to at least $n-5$ vertices of $N_1(x) \setminus \{a\}$ (or similarly v_3 is adjacent to $n-5$ vertices of $N_1(x) \setminus \{b\}$), then $(a)(v_2)(v_1,t_1)(b,t_2)(x,t_3,v_3)\pi$ is a resolving $(n-4)$-partition, for $t_1,t_2,t_3 \in N_1(x) \setminus \{a,b\}$, a contradiction.
3. If v_3 is adjacent to $n-5$ vertices of $N_1(x) \setminus \{c\}$ with all a,b,c are distinct, then $(a)(c)(v_2)(v_1,t_1)(b,t_2)(x,t_3,v_3)\pi$ is a resolving $(n-4)$-partition, for $t_1,t_2,t_3 \in N_1(x) \setminus \{a,b,c\}$, a contradiction.

Remark 6. Let $\{a,b\} \subset N_1(x)$, v_1 be adjacent to $n-5$ vertices of $N_1(x) \setminus \{a\}$, v_2 be adjacent to $n-5$ vertices of and $N_1(x) \setminus \{b\}$, and v_3 is adjacent to all vertices of $N_1(x)$.

1. If $a = b$, $v_1v_2 \notin E(G)$ and v_3 is not adjacent to one of v_1 or v_2, then $(v_1)(a,t_1)(v_2,t_2)(v_3,t_3)(x,t_4)\pi$ or $(v_2)(a,t_1)(v_1,t_2)(v_3,t_3)(x,t_4)\pi$ is a resolving $(n-4)$-partition, for $t_1,t_2,t_3,t_4 \in N_1(x) \setminus \{a\}$, a contradiction.
2. If $a \neq b$ and v_3 is neither adjacent to v_1 nor v_2, then $(v_1)(v_2)(a,t_1)(b,t_2)(x,t_3,v_3)\pi$ or $(a)(v_2)(x,t_1,v_1)(b,t_2)(t_3,v_3)\pi$ is a resolving $(n-4)$-partition, for $t_1,t_2,t_3 \in N_1(x) \setminus \{a,b\}$, a contradiction.

Therefore, without loss of generality, the adjacency of any vertex of $v_1, v_2, v_3 \in N_2(x)$ to $N_1(x)$ is given in Table 2.

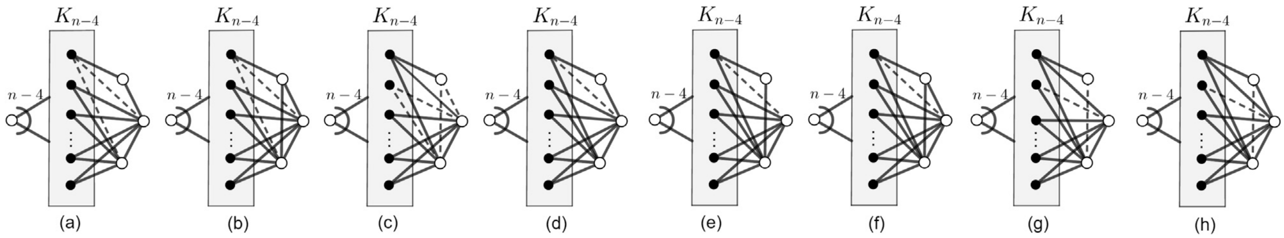

Fig. 10. Graph (a) H_{38}, (b) H_{37}, (c) H_{39}, (d) H_{30}, (e) H_{29}, (f) H_{27}, (g) H_{25} and (h) H_{31}.

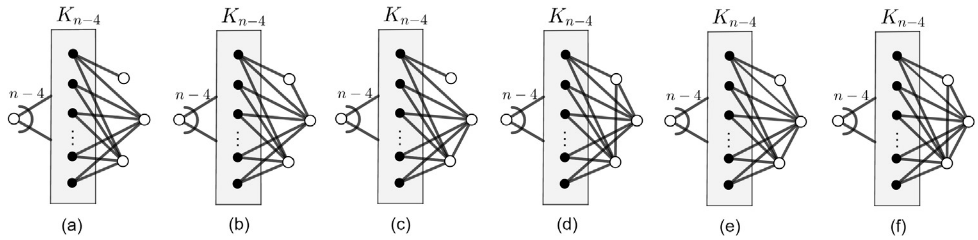

Fig. 11. Graph (a) H_{13}, (b) H_{14}, (c) H_7, (d) H_{15}, (e) H_8 and (f) H_5.

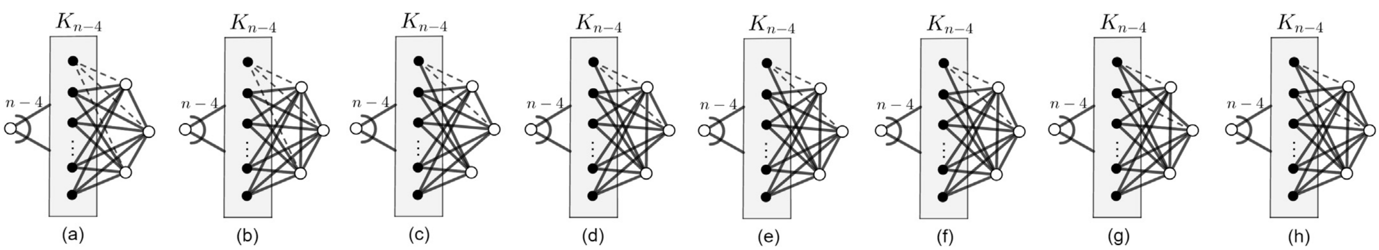

Fig. 12. Graph (a) $K_{n-5}+(K_2\cup P_3)$, (b) $K_{n-5}+(K_2\cup K_3)$, (c) H_{40}, (d) H_{32}, (e) H_{16}, (f) H_9, (g) H_{33} and (h) H_{19}.

(B2.1) $v_1 \in N_2(x)$ is only adjacent to vertex $a \in N_1(x)$. Let each v_2 and v_3 be only adjacent to vertex $b \in N_1(x)$ and $c \in N_1(x)$, respectively. If all vertices a, b, c are distinct, then $v_1v_2, v_1v_3, v_2v_3 \in E(G)$ since otherwise $\mathrm{diam}(G)=3$. We deduce $G \cong H_{77}$. If only two of a, b, c are equal, namely $a=b$, then $v_1v_3, v_2v_3 \in E(G)$ since otherwise $\mathrm{diam}(G)=3$. We deduce $G \cong H_{75}$ if $v_1v_2 \notin E(G)$ or $G \cong H_{74}$ if $v_1v_2 \in E(G)$. If all a, b, c are equal, then we deduce G as depicted in Fig. 7 (d)-(g).

Now assume that v_2 is adjacent to a single vertex $b \in N_1(x)$ and v_3 is adjacent to all vertices of $N_1(x)$. If $a=b$, then we deduce G as in Fig. 8 (a)-(f). If $a \neq b$, then $v_1v_2 \in E(G)$ or $v_1v_3, v_2v_3 \in E(G)$, since otherwise $\mathrm{diam}(G)=3$. We deduce G as depicted in Fig. 9 (a)-(d).

Now suppose that v_2 is adjacent to $n-5$ vertices of $N_1(x)$ and v_3 is adjacent to at least $n-5$ vertices of $N_1(x)$. In this case, $v_2v_3 \in E(G)$ since otherwise $(v_2)(a,t_1)(v_1,t_2)(v_3,t_3)(x,t_4)\pi$ is a resolving $(n-4)$-partition for $t_1, t_2, t_3 \in N_1(x) \setminus \{a\}$ and a singleton partition π, a contradiction. If v_2 is also adjacent to $n-5$ vertices of $N_1(x) \setminus \{a\}$ and v_3 is adjacent to $n-5$ vertices of $N_1(x) \setminus \{c\}$, then $a=c$ by considering Remark 4 (1) and v_1 is adjacent to at least one of v_2 or v_3, since otherwise $\mathrm{diam}(G)=3$. We deduce $G \cong H_{38}$ if v_1 is only adjacent to one of v_2 or v_3, or $G \cong H_{37}$ if v_1 is adjacent to both v_2 and v_3. If v_2 is adjacent to $n-5$ vertices of $N_1(x) \setminus \{b\}$ with $a \neq b$, and v_3 is adjacent to $n-5$ vertices of $N_1(x) \setminus \{b\}$, then $v_1v_2, v_1v_3 \notin E(G)$, by considering Remark 4 (3). We deduce $G \cong H_{39}$. Otherwise, v_2 is adjacent to $n-5$ vertices of $N_1(x) \setminus \{b\}$ and v_3 is adjacent to all vertices of $N_1(x)$. If $a=b$, then v_1 is adjacent to at least one of v_2 or v_3, since otherwise $\mathrm{diam}(G)=3$. We deduce G as depicted in Fig. 10 (d)-(f). If $a \neq b$, then $v_1v_3 \notin E(G)$ by considering Remark 4 (5) and hence $G \cong H_{25}$ for $v_1v_2 \notin E(G)$ or $G \cong H_{31}$ for $v_1v_2 \in E(G)$.

For the remaining case, let both v_2 and v_3 be adjacent to all vertices of $N_1(x)$. We deduce G as depicted in Fig. 11 (a)-(f).

(B2.2) v_1 is adjacent to $n-5$ vertices of $N_1(x) \setminus \{a\}$. If each v_2 and v_3 are also adjacent to $n-5$ vertices of $N_1(x) \setminus \{b\}$ and $N_1(x) \setminus \{c\}$, respectively, then all a, b, c are equal, by considering Remark 5. In this case, then $N_2(x)$ contains P_3, since otherwise $(v_1)(a,t_1)(v_2,t_2)(v_3,t_3)(x,t_4)\pi$, or $(v_2)(a,t_1)(v_1,t_2)(v_3,t_3)(x,t_4)\pi$, or $(v_3)(a,t_1)(v_1,t_2)(v_2,t_3)(x,t_4)\pi$ is a resolving $(n-4)$-partition, a contradiction. We obtain $G \cong K_{n-5}+(K_2\cup P_3)$ if $N_2(x)$ induces P_3, or $G \cong K_{n-5}+(K_2\cup K_3)$ if $N_2(x)$ induces K_3. Now assume that v_2 is adjacent to $n-5$ vertices of $N_1(x) \setminus \{b\}$ and v_3 is adjacent to all vertices of $N_2(x)$. If $a=b$, then $v_1v_2 \in E(G)$ or v_3 is adjacent to both v_1 and v_2, by considering Remark 6(1). We deduce G as depicted in Fig. 12 (c)-(f). Otherwise, $a \neq b$ and so that v_3 is adjacent to both v_1 and v_2 by considering Remark 6(2) and we deduce G as depicted in Fig. 12 (g)-(h).

For the remaining case, let both v_2 and v_3 be adjacent to all vertices of $N_1(x)$. Then, v_1 is adjacent to at least one of v_2 or v_3, since otherwise $(v_1)(a,t_1)(v_2,t_2)(v_3,t_3)(x,t_4)\pi$ is a resolving $(n-4)$-partition for $t_1, t_2, t_3, t_4 \in N_1(x) \setminus \{a\}$, a contradiction. We deduce G as depicted in Fig. 13 (a)-(d).

(B2.3) All vertices of $N_2(x)$ are adjacent to all vertices of $N_1(x)$. We deduce that $G \cong K_n - E(K_4)$ if $N_2(x)$ induces $\overline{K_3}$, or $G \cong K_n - E(K_4 - e)$ if $v_1v_2, v_1v_3 \notin E(G)$ and $v_2v_3 \in E(G)$, or $G \cong K_n - E(K_{1,3}+e)$ if $v_1v_2, v_1v_3 \in E(G)$ and $v_2v_3 \notin E(G)$ or $G \cong K_n - E(K_{1,3})$ if $N_2(x)$ induces K_3, as depicted in Fig. 13 (e)-(h).

(C) $|N_1(x)|=2$ and $|N_2(x)|=n-3$. Let $N_1(x)=\{u_1,u_2\}$. If $N_2(x)$ contains five vertices z, a, b, c, d such that $za, zb \in E(G)$ and $zc, zd \notin E(G)$, then $(x)(z)(a,c)(b,d)(u_1,t_1)(u_2,t_2)\pi$ is a resolving $(n-4)$-partition, for some $t_1, t_2 \in N_2(x) \setminus \{z,a,b,c,d\}$ and a singleton partition π, a contradiction. Therefore, any vertex of $N_2(x)$ is either adjacent to at most one vertex of $N_2(x)$ or it is adjacent to at least $n-5$ vertices of $N_2(x)$. On the other hand, if there exist $a, b, a_1, a_2, b_1, b_2 \in N_2(x)$ such that $aa_1, bb_1 \in E(G)$ and $aa_2, bb_2 \notin E(G)$, then $(x)(a)(b)(a_1,a_2)(b_1,b_2)(u_1,t_1)(u_2,t_2)\pi$ is a resolving $(n-4)$-partition, for some $t_1, t_2 \in N_2(x) \setminus \{a,b,a_1,a_2,b_1,b_2\}$ and a singleton partition π, a contradiction. Furthermore, if there exist $a_1, a_2, b_1, b_2 \in N_2(x)$ such that $a_1b_1 \in E(G)$ and $a_1b_2, a_2b_1, a_2b_2 \notin E(G)$, then $(x)(a_1,a_2)(b_1,b_2)(u_1,t_1)(u_2,t_2)\pi$ is a resolving $(n-4)$-partition, for

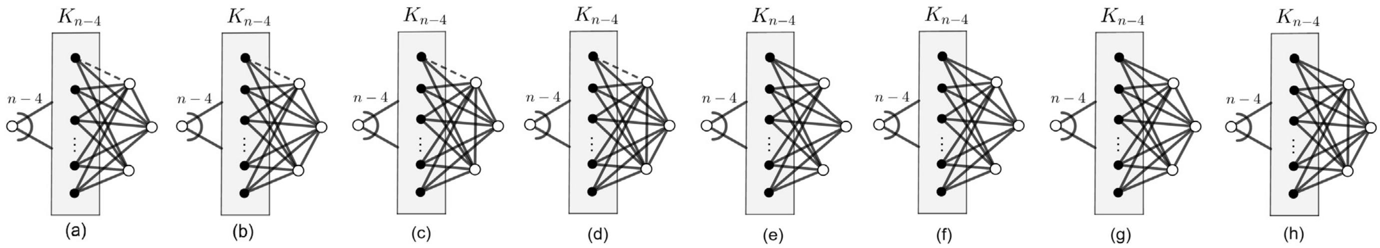

Fig. 13. Graph (a) H_{34}, (b) H_{10}, (c) H_{20}, (d) H_3, (e) $K_n - E(K_4)$, (f) $K_n - E(K_4 - e)$, (g) $K_n - E(K_{1,3} + e)$ and (h) $K_n - E(K_{1,3})$.

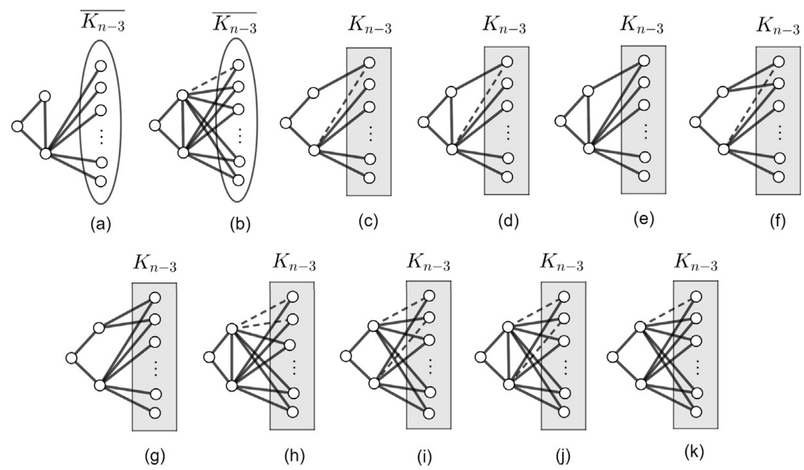

Fig. 14. Graph (a) $K_{1,n-1} + e$, (b) H_{50}, (c) H_{64} (d) H_{56}, (e) H_{47}, (f) H_{54}, (g) H_{46}, (h) H_{11}, (i) H_{35}, (j) H_{21} and (k) H_{12}.

some $t_1, t_2 \in N_2(x) \setminus \{a_1, a_2, b_1, b_2\}$ and a singleton partition π, a contradiction. Therefore, by considering Lemma 1, $N_2(x)$ induces one of the graphs (C1) $\overline{K_{n-3}}$, (C2) K_{n-3}, (C3) $K_{1,n-4}$, (C4) $K_{n-4} \cup K_1$, (C5) $K_{n-3} - E(K_{1,n-5})$, or (C6) $\overline{K_{n-3} - e}$.

(C1) $N_2(x)$ induces $\overline{K_{n-3}}$. If there exists a vertex of $N_1(x)$, namely u_1, and $a_1, a_2, b_1, b_2 \in N_2(x)$ such that $u_1a_1, u_1a_2 \notin E(G)$ and $u_1b_1, u_1b_2 \in E(G)$, then $(x)(a_1, b_1)(a_2, b_2)(u_1, t_1)(u_2, t_2)\pi$ is a resolving $(n-4)$-partition, for $t_1, t_2 \in N_2(x) \setminus \{a_1, a_2, b_1, b_2\}$ and a singleton partition π, a contradiction. Therefore any vertex of $N_1(x)$ is either adjacent to at most one vertex of $N_2(x)$ or it is adjacent to at least $n-4$ vertices of $N_2(x)$.

If $u_1 \in N_1(x)$ is not adjacent to any vertex of $N_2(x)$, then u_2 is adjacent to all vertices of $N_2(x)$ since otherwise $\text{diam}(G) = 3$, and $u_1u_2 \in E(G)$ since otherwise $\text{diam}(G) = 3$. We obtain $G \cong K_{1,n-1} + e$. If u_1 is only adjacent to a single vertex $a \in N_2(x)$, then u_2 is adjacent to all vertices of $N_2(x)$ and $u_1u_2 \in E(G)$, since otherwise $\text{diam}(G) = 3$. However, $(x, t_1)(u_1, t_2)(u_2, a, t_3)\pi$ is a resolving $(n-4)$-partition, for $t_1, t_2, t_3 \in N_2(x) \setminus \{a\}$ and a singleton partition π, a contradiction. If each u_1 and u_2 are adjacent to $n-4$ vertices of $N_2(x) \setminus \{a\}$ and $N_2(x) \setminus \{b\}$, respectively, with $a \neq b$, then $u_1u_2 \in E(G)$ since otherwise $\text{diam}(G) = 3$. However, $(x)(u_1, t_1)(u_2, t_2)(a, t_3)\pi$ is a resolving $(n-4)$-partition, for $t_1, t_2, t_3 \in N_2(x) \setminus \{a, b\}$ and a singleton partition π, a contradiction. If u_1 is only not adjacent to a vertex $a \in N_2(x)$ and u_2 is adjacent to all vertices of $N_2(x)$, then $u_1u_2 \in E(G)$ and we obtain $G \cong H_{50}$. Now we consider that both $u_1, u_2 \in N_1(x)$ are adjacent to all vertices of $N_2(x)$. We deduce $G \cong K_{2,n-2}$ if $u_1u_2 \notin E(G)$ or $G \cong K_2 + \overline{K_{n-2}}$ if $u_1u_2 \in E(G)$. However for these two graphs, $pd(G) = n-2$ by [2].

(C2) $N_2(x)$ induces K_{n-3}. If there exists a vertex of $N_1(x)$, namely u_1, and $a_1, a_2, a_3, b_1, b_2, b_3 \in N_2(x)$ such that $u_1a_i \in E(G)$ and $u_1b_i \notin E(G)$ for all $1 \leq i \leq 3$, then $(x)(u_1)(a_1, b_1)(a_2, b_2)(a_3, b_3)(u_2, t)\pi$ is a resolving $(n-4)$-partition, for $t \in N_2(x) \setminus \{a_1, a_2, a_3, b_1, b_2, b_3\}$ and a singleton partition π, a contradiction. Therefore, any vertex of $N_1(x)$ is either adjacent to at most two vertices of $N_2(x)$ or it is adjacent to at least $n-5$ vertices of $N_2(x)$.

If u_1 is not adjacent to any vertex of $N_2(x)$, then u_2 is adjacent to all vertices of $N_2(x)$ since otherwise $\text{diam}(G) = 3$, and $u_1u_2 \in E(G)$ since otherwise $\text{diam}(G) = 3$. We obtain $G \cong G_8$, but $pd(G_8) = n-2$ by [2]. Now assume that u_1 is adjacent to a single vertex $a \in N_2(x)$. Then u_2 is adjacent to at least $n-4$ vertices of $N_2(x)$. If u_2 is adjacent to $n-4$ vertices of $N_2(x) \setminus \{a\}$, then we obtain $G \cong H_{64}$ if $u_1u_2 \notin E(G)$ or $G \cong H_{56}$ if $u_1u_2 \in E(G)$, as depicted in Fig. 14 (c)-(d). Otherwise, suppose that u_2 is adjacent to all vertices of $N_2(x)$. We obtain $G \cong G_7$ if $u_1u_2 \notin E(G)$ or $G \cong H_{47}$ if $u_1u_2 \in E(G)$. However by [2], $pd(G_7) = n-2$.

Let u_1 be only adjacent to two vertices $a, b \in N_2(x)$. Then u_2 is adjacent to at least $n-5$ vertices of $N_2(x)$, since otherwise $\text{diam}(G) = 3$. If u_2 is only adjacent to $n-5$ vertices of $N_2(x) \setminus \{a, b\}$, then $(u_2)(x, t_1)(u_1, t_2)(a, t_3)(b, t_4)\pi$ is a resolving $(n-4)$-partition, for $t_1, t_2, t_3, t_4 \in N_2(x) \setminus \{a, b\}$ and a singleton partition π, a contradiction. If u_2 is adjacent to either $n-4$ vertices of $N_2(x) \setminus \{a\}$ or it is adjacent to all vertices of $N_2(x)$, then $u_1u_2 \notin E(G)$ since otherwise $(u_1)(x, t_1)(u_2, t_2)(a, t_3)(b, t_4)\pi$ is also a resolving $(n-4)$-partition, a contradiction. Hence we obtain $G \cong H_{54}$ if u_2 is adjacent to $n-5$ vertices of $N_2(x) \setminus \{a\}$ or $G \cong H_{46}$ if u_2 is adjacent to all vertices of $N_2(x)$.

Now assume that u_1 is adjacent to $n-5$ vertices of $N_2(x) \setminus \{a, b\}$ for some $a, b \in N_2(x)$. If u_2 is not adjacent to all vertices of $N_2(x)$, then there exists $c \in N_2(x)$ different from a and b such that $u_2c \notin E(G)$. However, $(u_1)(u_2)(x, t_1)(a, t_2)(b, t_3)(c, t_4)\pi$ is a resolving $(n-4)$-partition, for $t_1, t_2, t_3, t_4 \in N_2(x) \setminus \{a, b, c\}$, a contradiction. Therefore, u_2 is adjacent to all vertices of $N_2(x)$. Furthermore, if $u_1u_2 \notin E(G)$, then $(u_1)(x, u_2, t_1)(a, t_2)(b, t_3)\pi$ is also a resolving $(n-4)$-partition, for $t_1, t_2, t_3 \in N_2(x) \setminus \{a, b\}$, a contradiction. Therefore, $u_1u_2 \in E(G)$ and we obtain $G \cong H_{11}$.

Let u_1 be only not adjacent to a vertex $a \in N_2(x)$. If u_2 is also only not adjacent to a single vertex $b \in N_2(x)$ where $a \neq b$, then we obtain $G \cong H_{35}$ if $u_1u_2 \notin E(G)$, or $G \cong H_{21}$ if $u_1u_2 \in E(G)$ as depicted in Fig. 14 (i)-(j). Otherwise, assume that u_2 is adjacent to all vertices of $N_2(x)$. Then $u_1u_2 \notin E(G)$ since otherwise $G \cong G_6$ and $pd(G_6) = n-2$ by [2]. We deduce $G \cong H_{12}$. If both u_1 and u_2 are adjacent to all vertices of $N_2(x)$, then we obtain $G \cong G_1$ if $u_1u_2 \notin E(G)$ or $G \cong G_2$ if $u_1u_2 \in E(G)$. However, for these two graphs G we have $pd(G) = n-2$ by [2].

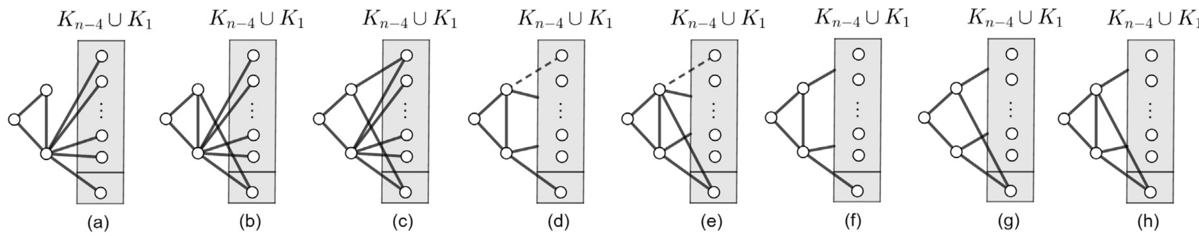

Fig. 15. Graph (a) $K_1 + (K_{n-4} \cup (P_3 - e))$, (b) $K_1 + (K_{n-4} \cup P_3)$, (c) H_{75}, (d) H_{65}, (e) H_{69}, (f) H_{43}, (g) H_{72} and (h) H_{44}.

(C3) $N_2(x)$ induces $K_{1,n-4}$. Let $V(N_2(x)) = \{t, t_i : 1 \le i \le n-4\}$ and $E(G) = \{tt_i : 1 \le i \le n-4\}$. However, $(x, t_1)(u_1, t_2)(u_2, t_3)(t, t_4)\pi$ is a resolving $(n-4)$-partition, a contradiction. Therefore, there is no graph G with $pd(G) = n-3$ satisfying (C3).

(C4) $N_2(x)$ induces $K_{n-4} \cup K_1$. Let $V(N_2(x)) = \{t, t_i : 1 \le i \le n-4\}$ and $E(G) = \{t_i t_j : 1 \le i < j \le n-4\}$. If there exist $t_1, t_2, t_3, t_4 \in N_2(x)$ and $u_1 \in N_1(x)$ such that $u_1 t_1, u_1 t_2 \in E(G)$ but $u_1 t_3, u_1 t_4 \notin E(G)$, then $(x)(u_1)(t_1, t_3)(t_2, t_4)(u_2, t_5)(t, t_6)\pi$ is a resolving $(n-4)$-partition, a contradiction. Therefore any vertex of $N_1(x)$ is adjacent to at most one vertex of $N_2(x) \setminus \{t\}$ or it is adjacent to at least $n-5$ vertices of $N_2(x) \setminus \{t\}$.

If u_1 is not adjacent to any vertex of $N_2(x) \setminus \{t\}$, then u_2 is adjacent to all vertices of $N_2(x) \setminus \{t\}$ and $u_1 u_2, u_2 t \in E(G)$, since otherwise $\mathrm{diam}(G) = 3$. We deduce $G \cong K_1 + (K_{n-4} \cup (P_3 - e))$ if $tu_1 \notin E(G)$ or $G \cong K_1 + (K_{n-4} \cup P_3)$ if $tu_1 \in E(G)$. If u_1 is only adjacent to a single vertex $t_1 \in N_2(x) \setminus \{t\}$ and u_2 is adjacent to $n-5$ vertices of $N_2(x) \setminus \{t_1, t\}$, then $u_1 t, u_2 t \in E(G)$ since otherwise diam(G) = 3. However, $(t_1)(x, t_2)(u_1, t_3)(u_2, t_4)(t, t_5)\pi$ is a resolving $(n-4)$-partition, a contradiction. Therefore, if u_1 is only adjacent to a single vertex $t_1 \in N_2(x) \setminus \{t\}$, then u_2 is adjacent to all vertices $N_2(x) \setminus \{t\}$, $u_2 t \in E(G)$ and ($u_1 u_2 \in E(G)$ or $u_1 t \in E(G)$). However, if $u_1 u_2 \in E(G)$, then $(u_1)(t_1, t_2)(x, t_3)(u_2, t_4)(t, t_5)\pi$ is a resolving $(n-4)$-partition, a contradiction. Therefore we deduce $G \cong H_{75}$.

Now assume that u_1 is not adjacent to a single vertex $t_1 \in N_2(x)$ and it is adjacent to other vertices $t_i \in N_2(x)$ for all $2 \le i \le n-4$. If u_2 is also only not adjacent to other single vertex $t_2 \in N_2(x)$, then $(u_1)(u_2)(t_1, t_3)(t_2, t_4)(x, t_5)(t, t_6)\pi$ is a resolving $(n-4)$-partition, a contradiction. Therefore, u_2 is adjacent to all vertices of $N_2(x) \setminus \{t\}$, $u_2 t \in E(G)$ and ($u_1 u_2 \in E(G)$ or $u_1 t \in E(G)$). If $u_1 u_2 \notin E(G)$, then $(u_1)(t_1, t_2)(x, t_3)(u_2, t_4)(t, t_5)$ is a resolving $(n-4)$-partition, a contradiction. Hence we obtain $G \cong H_{65}$ if $u_1 u_2 \in E(G)$ and $u_1 t \notin E(G)$, or $G \cong H_{69}$ if $u_1 u_2, u_1 t \in E(G)$. Otherwise, let both u_1 and u_2 be adjacent to all vertices of $N_1(x) \setminus \{t\}$. Then, $u_i t, u_1 u_2 \in E(G)$ for some $1 \le i \le 2$, or $tu_1, tu_2 \in E(G)$. Hence we deduce $G \cong H_{43}$ if $u_1 t, u_1 u_2 \in E(G)$ and $u_2 t \notin E(G)$, or $G \cong H_{72}$ if $u_1 t, u_2 t \in E(G)$ and $u_1 u_2 \notin E(G)$, or $G \cong H_{44}$ if $u_1 u_2, u_1 t, u_2 t \in E(G)$ (Fig. 15 (f)-(h)).

(C5) $N_2(x)$ induces $K_{n-3} - E(K_{1,n-5})$. Let $V(N_2(x)) = \{v, w, w_i : 1 \le i \le n-5\}$ and $E(N_2(x)) = \{vw, vw_i, w_i w_j : 1 \le i, j \le n-5\}$. If there exist $u_1 \in N_1(x)$ and $w_1, w_2, w_3, w_4 \in N_2(x)$ such that $u_1 w_1, u_1 w_2 \in E(G)$ but $u_1 w_3, u_1 w_4 \notin E(G)$, then $(x)(u_1)(w)(w_1, w_3)(w_2, w_4)(u_2, w_5)(v, w_6)\pi$ is a resolving $(n-4)$-partition, a contradiction. Therefore, any vertex of $N_1(x)$ is either adjacent to at most one vertex of $w_i \in N_2(x)$ or it is adjacent at least $n-6$ vertices of $w_i \in N_2(x)$, for $1 \le i \le n-5$.

(C5.1) u_1 is not adjacent to any vertex $w_i \in N_2(x)$ and so that u_2 is adjacent to all vertices $w_i \in N_2(x)$ for $1 \le i \le n-5$. If u_2 is not adjacent to any other vertices $v, w \in N_2(x)$, then $u_1 v, u_1 w, u_1 u_2 \in E(G)$ since otherwise $\mathrm{diam}(G) = 3$. However, $(v)(x, w_1)(u_1, w_2)(u_2, w_3)(w, w_4)\pi$ is a resolving $(n-4)$-partition, a contradiction. If u_2 is also adjacent to a single vertex $w \in N_2(x)$, then $(w)(v, w_1)(x, w_2)(u_1, w_3)(u_2, w_4)\pi$ is a resolving $(n-4)$-partition, a contradiction. Otherwise, u_2 is also adjacent to a single vertex $v \in N_2(x)$ and $u_2 w \notin E(G)$, so that $u_1 w \in E(G)$. If $vu_1, u_1 u_2 \in E(G)$, then $(u_1)(w, w_1)(u_2, w_2)(v, w_3)(x, w_4)\pi$ is a resolving $(n-4)$-partition, a contradiction. This implies that u_1 is adjacent to at most one of the vertex v or u_2. If u_1 is not adjacent to any v or u_2, then $\mathrm{diam}(G) = 3$, a contradiction. Otherwise, u_1 is only adjacent to one of the vertex u_2 or v, so that we deduce $G \cong H_{73}$ for these two conditions.

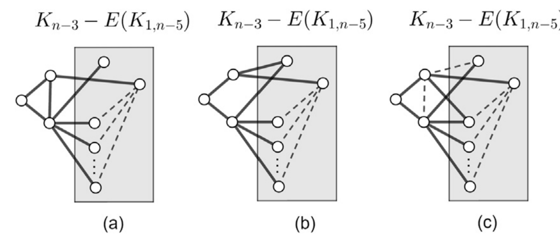

Fig. 16. Graph (a) H_{73}, (b) H_{73} and (c) H_{76}.

(C5.2) If (u_1 is only adjacent to a single vertex $w_1 \in N_2(x)$ and u_2 is adjacent to $n-6$ vertices $w_i \in N_2(x)$ for all $2 \le i \le n-5$) or (u_1 and u_2 are not adjacent to distinct vertices w_1 and w_2, respectively, and they are adjacent to other $n-6$ vertices of $w_i \in N_2(x)$), then $(w)(u_2)(w_1, w_2)(u_1, w_3)(v, w_4)(x, w_5)\pi$ or $(w)(u_1)(u_2)(w_1, w_3)(w_2, w_4)(v, w_5)(x, w_6)\pi$ is a resolving $(n-4)$-partition, a contradiction.

(C5.3) u_1 is adjacent to a single vertex $w_1 \in N_2(x)$ and u_2 is adjacent to all vertices $w_i \in N_2(x)$ for all $1 \le i \le n-5$. If u_2 is not adjacent to any other vertices $v, w \in N_2(x)$ or it is adjacent to a single vertex $w \in N_2(x)$, then $(u_1)(u_2)(w_1, w_2)(v, w_3)(w, w_4)(x, w_5)\pi$ or $(u_1)(w)(w_1, w_2)(v, w_3)(u_2, w_4)(x, w_5)\pi$ is a resolving $(n-4)$-partition, respectively, a contradiction. Otherwise u_2 is adjacent to a vertex $v \in N_2(x)$ but it is not adjacent to a vertex $w \in N_2(x)$ so that $u_1 w \in E(G)$. For this case, if $u_1 u_2 \in E(G)$ or $u_1 v \in E(G)$, then we obtain a resolving $(n-4)$-partition, namely $(u_1)(w)(w_1, w_2)(v, w_3)(u_2, w_4)(x, w_5)\pi$ or $(u_1)(w_1, w_2)(v, w_3)(w, u_2)(x, w_4)$, respectively. Hence, $u_1 u_2, u_1 v \notin E(G)$ and we deduce $G \cong H_{76}$ as depicted in Fig. 16 (c).

(C5.4) u_1 is only not adjacent to a single vertex $w_1 \in N_2(x)$ and it is adjacent to all remaining vertices $w_i \in N_2(x)$ for all $i \ne 1$, and u_2 is adjacent to all vertices $w_i \in N_2(x)$ for all $1 \le i \le n-5$. If u_2 is adjacent to w or it is not adjacent to u_1, then $(u_1)(w)(w_1, w_2)(v, w_3)(u_2, w_4)(x, w_5)\pi$ is a resolving $(n-4)$-partition, a contradiction. Therefore, $u_2 w \notin E(G)$ and $u_1 u_2, u_1 w \in E(G)$. Hence we only need to consider the adjacency of a vertex v to the vertices $u_1, u_2 \in N_1(x)$. Note that v is adjacent to at least one of $u_1, u_2 \in N_1(x)$, since otherwise $\mathrm{diam}(G) = 3$. If v is not adjacent to one of u_1 or u_2, then $(u_1)(w_1, w_2)(u_2, w)(v, w_3)(x, w_4)\pi$ or $(u_1)(u_2)(w_1, w_2)(w, w_3)(v, w_4)(x, w_5)\pi$ is a resolving $(n-4)$-partition, a contradiction. Therefore, v is adjacent to both $u_1, u_2 \in N_1(x)$ and we deduce $G \cong H_{67}$ as depicted in Fig. 17 (a).

Now assume that both u_1 and u_2 are adjacent to all vertices $w_i \in N_2(x)$ for $1 \le i \le n-5$. If w is adjacent to both u_1 and u_2, then $(w)(u_1, w_1)(u_2, w_2)(v, w_3)(x, w_4)\pi$ is a resolving $(n-4)$-partition, a contradiction. Furthermore, if both v and w are not adjacent to a single vertex $u_1 \in N_1(x)$ (or similarly to a single vertex $u_2 \in N_1(x)$) and $u_1 u_2 \notin E(G)$, then $(u_1)(u_2, w_1)(v, w_2)(w, w_3)(x, w_4)\pi$ (or $(u_2)(u_1, w_1)(v, w_2)(w, w_3)(x, w_4)\pi$) is also a resolving $(n-4)$-partition, a contradiction. Therefore without loss of generality, we can assume that w is adjacent to $u_1 \in N_1(x)$ and it is not adjacent to $u_2 \in N_1(x)$. If $vu_1 \in E(G)$ and $vu_2 \in E(G)$, then $u_1 u_2 \in E(G)$ and we obtain $G \cong H_{66}$ as depicted in Fig. 17 (b). If $vu_1 \notin E(G)$ and $vu_2 \in E(G)$, then $u_1 u_2 \in E(G)$ since otherwise $(u_1)(u_2, w_1)(w, w_2)(v, w_3)(x, w_4)\pi$ is a resolving $(n-4)$-partition, a contradiction. We deduce $G \cong H_{70}$ as depicted in Fig. 17 (c). Otherwise $vu_1, vu_2 \in E(G)$ and we obtain $G \cong H_{70}$ if $u_1 u_2 \notin E(G)$ or $G \cong H_{45}$ if $u_1 u_2 \in E(G)$.

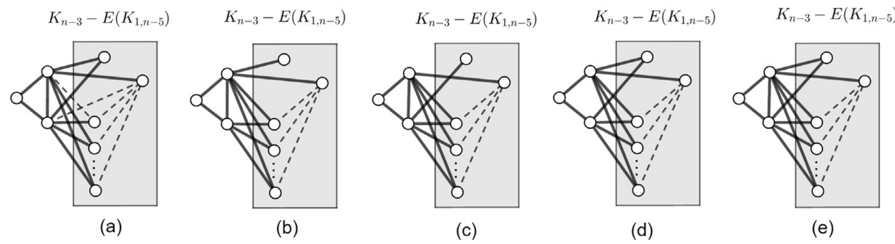

Fig. 17. Graph (a) H_{67}, (b) H_{66}, (c) H_{70}, (d) H_{70} and (e) H_{45}.

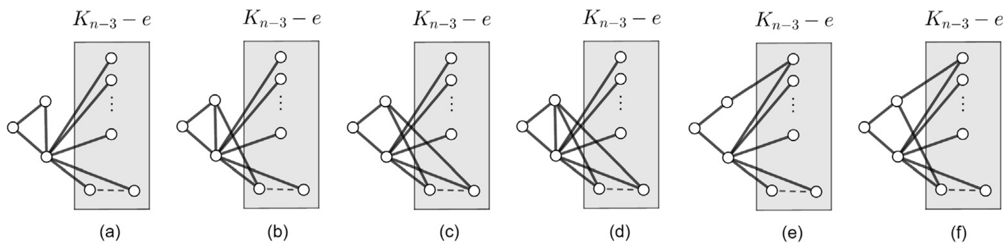

Fig. 18. Graph (a) $K_1 + (K_{n-3} - e \cup K_2)$, (b) H_{51}, (c) H_{55}, (d) H_{53}, (e) H_{57} and (f) H_{58}.

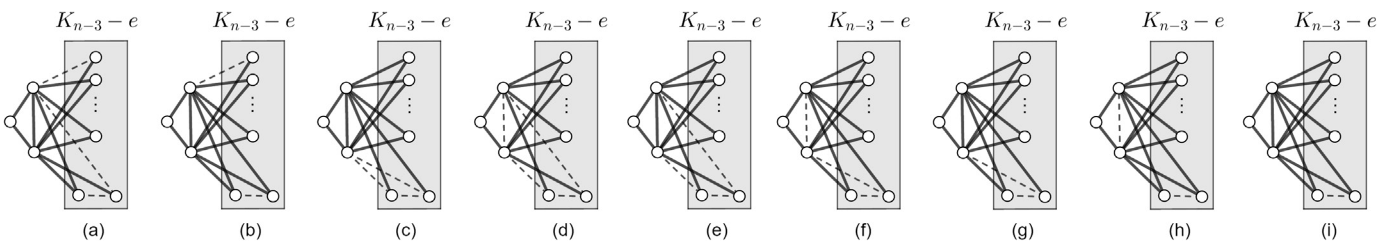

Fig. 19. Graph (a) H_{31}, (b) H_{22}, (c) H_{14}, (d) H_{38}, (e) H_{29}, (f) H_{30}, (g) H_8, (h) H_{23}, and (i) $K_n - E(K_{1,n-3} + e)$.

(C6) $N_2(x)$ induces $K_{n-3} - e$. Let $e = ab$ and other vertices of $N_2(x)$ by t_i for $1 \le i \le n-5$. If there exists $u_1 \in N_1(x)$ such that $u_1t_1, u_1t_2 \in E(G)$ and $u_1t_3, u_1t_4 \notin E(G)$, then $(x)(u_1)(a)(t_1,t_3)(t_2,t_4)(u_2,t_5)(b,t_6)\pi$ is a resolving $(n-4)$-partition, a contradiction. Therefore, any vertex of $N_1(x)$ is either adjacent to at most one vertex of $N_2(x) \setminus \{a,b\}$ or it is adjacent to at least $(n-6)$ vertices of $N_2(x) \setminus \{a,b\}$. Furthermore, if one vertex of $N_1(x)$, namely u_1, is not adjacent to at least one vertex $t_1 \in N_2(x) \setminus \{a,b\}$ and one other vertex $u_2 \in N_1(x)$ is not adjacent to a vertex $a \in N_2(x)$ (similarly to a vertex $b \in N_2(x)$), then $(a)(t_1)(u_1,t_2)(b,t_3)(u_2,t_4)(x,t_5)\pi$ (or $(b)(t_1)(u_1,t_2)(a,t_3)(u_2,t_4)(x,t_5)\pi$) is a resolving $(n-4)$-partition, a contradiction.

(C6.1) u_1 is not adjacent to any vertex of $N_2(x) \setminus \{a,b\}$. Then u_2 is adjacent to all vertices of $N_2(x)$. If u_1 is not adjacent to two remaining vertices $a, b \in N_2(x)$, then $u_1u_2 \in E(G)$ and we obtain $G \cong K_1 + (K_{n-3} - e \cup K_2)$. If u_1 is either adjacent to a single vertex $a \in N_2(x)$ or $b \in N_2(x)$, then $u_1u_2 \in E(G)$ and we obtain $G \cong H_{51}$. Otherwise, u_1 is adjacent to both $a, b \in N_2(x)$ and we deduce $G \cong H_{55}$ if $u_1u_2 \notin E(G)$ or $G \cong H_{53}$ if $u_1u_2 \in E(G)$ (Fig. 18 (a)-(d)).

(C6.2) u_1 is adjacent to a single vertex $t_1 \in N_2(x) \setminus \{a,b\}$. If u_2 is adjacent to $n-4$ vertices $N_1(x)$ other than t_1, then $(a)(t_1)(u_1,t_2)(u_2,t_3)(x,t_4)(b,t_5)\pi$ is a resolving $(n-4)$-partition, a contradiction. Therefore, u_2 is adjacent to all vertices of $N_2(x)$. If u_1 is not adjacent to other two vertices $a, b \in N_2(x)$ or it is only adjacent to $a \in N_2(x)$ (or similarly to $b \in N_2(x)$), then $u_1u_2 \notin E(G)$, since otherwise we have a resolving $(n-4)$-partition, namely $(a)(u_1)(t_1,t_2)(u_2,t_3)(x,t_4)(b,t_5)\pi$ (or $(b)(u_1)(t_1,t_2)(u_2,t_3)(x,t_4)(a,t_5)\pi$). Hence we deduce $G \cong H_{57}$ if $u_1a, u_1b \notin E(G)$ or $G \cong H_{58}$ if u_1 is either adjacent to a or b. Otherwise, u_1 is adjacent to both a and b, but $(u_1)(u_2)(t_1,t_2)(a,t_3)(b,t_4)(x,t_5)\pi$ is a resolving $(n-4)$-partition, a contradiction.

(C6.3) u_1 is only not adjacent to a single vertex $t_1 \in N_2(x) \setminus \{a,b\}$. If u_2 is adjacent to $n-4$ vertices of $N_2(x) \setminus \{t_2\}$, then $(u_1)(u_2)(a)(t_1,t_3)(t_2,t_4)(x,t_5)(b,t_6)\pi$ is a resolving $(n-4)$-partition, a contradiction. Therefore, u_2 is adjacent to all vertices $N_2(x)$. In this case, $u_1u_2 \in E(G)$, since otherwise we also have a resolving $(n-4)$-partition, namely $(u_1)(a)(t_1,t_2)(u_2,t_3)(b,t_4)(x,t_5)\pi$. Furthermore, if u_1 is not adjacent to both $a, b \in N_2(x)$, then $(u_1)(u_2)(t_1,t_2)(a,t_3)(b,t_4)(x,t_5)\pi$ is a resolving $(n-4)$-partition, a contradiction. Thus we obtain $G \cong H_{31}$ if u_1 is only adjacent to one of vertices a or b, or $G \cong H_{22}$ if u_1 is adjacent to both vertices a and b (Fig. 19 (a)-(b)).

(C6.4) u_1 and u_2 are adjacent to $n-5$ vertices of $N_2(x) \setminus \{a,b\}$. Therefore we only need to consider adjacency of vertices $N_1(x) \cup \{a,b\}$. If both a and b are only adjacent to a single vertex of $N_1(x)$, namely u_1, then $u_1u_2 \in E(G)$ since otherwise $(u_2)(u_1,t_1)(a,t_2)(b,t_3)(x,t_4)\pi$ is a resolving $(n-4)$-partition, a contradiction. Thus we obtain $G \cong H_{14}$. If a and b are adjacent to different vertices of $N_1(x)$, namely $au_1, bu_2 \in E(G)$ and $au_2, bu_1 \notin E(G)$, then we obtain $G \cong H_{38}$ or $G \cong H_{29}$ for $u_1u_2 \notin E(G)$ or $u_1u_2 \in E(G)$, respectively. Now assume that one vertex of a or b is adjacent to all vertices $N_1(x)$ and one other vertex is only adjacent to a single vertex of $N_1(x)$, namely $au_1, au_2, bu_1 \in E(G)$ and $bu_2 \notin E(G)$. Then we deduce $G \cong H_{30}$ or $G \cong H_8$ for $u_1u_2 \notin E(G)$ or $u_1u_2 \in E(G)$, respectively. Otherwise, both a and b are adjacent to all vertices of $N_1(x)$, and thus $G \cong H_{23}$ or $G \cong K_n - E(K_{1,n-3} + e)$ for $u_1u_2 \notin E(G)$ or $u_1u_2 \in E(G)$, respectively.

(D) $|N_1(x)| = n-3$ and $|N_2(x)| = 2$. Let $N_2(x) = \{v_1, v_2\}$. By a similar reason to Subcase (C), if $N_1(x)$ contains vertices z_1, z_2, a, b, c, d such that

(i) $z_1a, z_1b \in E(G)$ and $z_1c, z_1d \notin E(G)$, or
(ii) $z_1a, z_2b \in E(G)$ and $z_1c, z_2d \notin E(G)$, or
(iii) $ab \in E(G)$ and $ad, bc, cd \notin E(G)$,

then $(x)(z_1)(z_2)(a,c)(b,d)(v_1,t_1)(v_2,t_2)\pi$ is a resolving $(n-4)$ partition, for $t_1, t_2 \in N_1(x) \setminus \{z_1, z_2, a, b, c, d\}$, a contradiction. Therefore, by considering Lemma 1, $N_1(x)$ induces one of the graphs (D1) $\overline{K_{n-3}}$, (D2) K_{n-3}, (D3) $K_{1,n-4}$, (D4) $K_{n-4} \cup K_1$, (D5) $K_{n-3} - E(K_{1,n-5})$, or (D6) $K_{n-3} - e$.

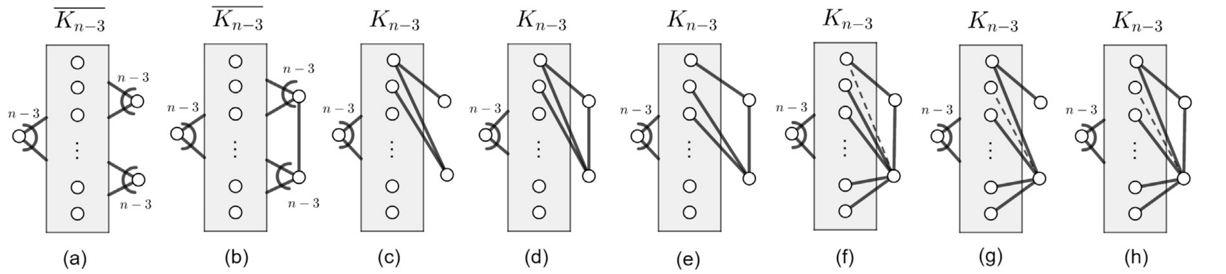

Fig. 20. Graph (a) $K_{3,n-3}$, (b) $(K_1 \cup K_2) + \overline{K_{n-3}}$, (c) H_{43}, (d) H_{47}, (e) H_{46}, (f) H_{12}, (g) H_7 and (h) H_{11}.

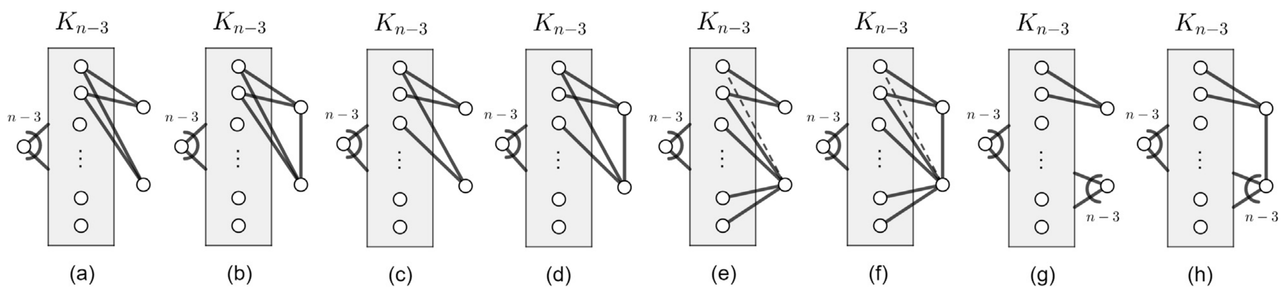

Fig. 21. Graph (a) H_{44}, (b) H_{48}, (c) H_{45}, (d) H_{49}, (e) H_8, (f) H_6, (g) H_4 and (h) H_1.

(D1) $N_1(x)$ induces $\overline{K_{n-3}}$. Note that for any vertex $v_i \in N_2(x), 1 \le i \le 2$, there exists at least one vertex $t \in N_1(x)$ such that $v_i t \in E(G)$, and conversely for any $t \in N_1(x)$ there exist $v_i \in N_2(x)$ such that $tv_i \in E(G)$, since otherwise diam$(G) = 3$. If there exists a vertex of $N_2(x)$, namely v_1, and $a, b, c, d \in N_1(x)$ such that $v_1 a, v_1 b \in E(G)$ and $v_1 c, v_1 d \notin E(G)$, then $(x)(v_1, t_1)(a, c)(b, d)(v_2, t_2)\pi$ is a resolving $(n-4)$-partition, for $t_1, t_2 \in N_1(x) \setminus \{a, b, c, d\}$, a contradiction. Therefore, any vertex of $N_2(x)$ is adjacent to 1, $n-4$ or $n-3$ vertices of $N_1(x)$. Now consider the following 4 conditions.

1. If v_1 is adjacent to a single vertex $t_1 \in N_1(x)$ and v_2 is adjacent to $n-4$ vertices of $N_1(x) \setminus \{t_1\}$, or
2. if v_1 is adjacent to a single vertex $t_1 \in N_1(x)$ and v_2 is adjacent to all vertices of $N_1(x)$, or
3. if each v_1 and v_2 are only not adjacent to a single vertex $t_1 \in N_1(x)$ and $t_2 \in N_1(x)$, respectively, or
4. if v_1 is only not adjacent to a single vertex $t_1 \in N_1(x)$ and v_2 is adjacent to all vertices of $N_1(x)$,

then $(x, y)(v_1, t)(t_1, t_3)(v_2, t_2)\pi$ is a resolving $(n-4)$-partition, for $y, t, t_3 \in N_1(x) \setminus \{t_1, t_2\}$, a contradiction. Thus, we can conclude that any vertex of $N_2(x)$ is adjacent to all vertices $N_1(x)$. We deduce $G \cong K_{3,n-3}$ if $v_1 v_2 \notin E(G)$ or $G \cong (K_1 \cup K_2) + \overline{K_{n-3}}$ if $v_1 v_2 \in E(G)$, as depicted in Fig. 20 (a)-(b).

(D2) $N_1(x)$ induces K_{n-3}. If there exists a vertex $v_1 \in N_2(x)$ and $a_1, a_2, a_3, b_1, b_2, b_3 \in N_1(x)$ such that $v_1 a_i \in E(G)$ and $v_1 b_i \notin E(G)$ for all $1 \le i \le 3$, then $(x)(v_1)(a_1, b_1)(a_2, b_2)(a_3, b_3)(v_2, t)\pi$ is a resolving $(n-4)$-partition, for $t \in N_2(x) \setminus \{a_1, a_2, a_3, b_1, b_2, b_3\}$ and a singleton partition π, a contradiction. Therefore, any vertex of $N_2(x)$ is either adjacent to at most two vertices of $N_2(x)$ or it is adjacent to at least $n-5$ vertices of $N_1(x)$.

(D2.1) v_1 is only adjacent to a single vertex $t \in N_1(x)$. If v_2 is also only adjacent to a vertex $t \in N_1(x)$, then $G \cong G_9$ or $G \cong G_8$ for $v_1 v_2 \notin E(G)$ or for $v_1 v_2 \in E(G)$, respectively. But, $pd(G_9) = pd(G_8) = n-2$ by [2]. If v_2 is only adjacent to a single vertex $s \in N_1(x)$ where $s \ne t$, then $v_1 v_2 \in E(G)$, since otherwise diam$(G) = 3$. However, we obtain $G \cong G_7$ and $pd(G_7) = n-2$ by [2]. If v_2 is adjacent to two vertices $s, t \in N_1(x)$, then $G \cong H_{43}$ or $G \cong H_{47}$ for $v_1 v_2 \notin E(G)$ or for $v_1 v_2 \in E(G)$, respectively. If v_2 is adjacent to two vertices $s, r \in N_1(x)$ distinct from t, then $v_1 v_2 \in E(G)$ and thus $G \cong H_{46}$. If v_2 is adjacent to $(n-5)$ vertices of $N_1(x) \setminus \{s, t\}$ (or $N_1(x) \setminus \{r, s\}$ where $r, s \ne t$), then $(v_2)(t, t_1)(s, t_2)(v_2, t_3)(x, t_4)\pi$ (or $(v_1)(v_2)(t, t_1)(r, t_2)(s, t_3)(x, t_4)\pi$) is a resolving $(n-4)$-partition, for $t_1, t_2, t_3, t_4 \in N_1(x) \setminus \{t, r, s, \}$, a contradiction. If v_2 is adjacent to $(n-4)$ vertices of $N_1(x) \setminus \{t\}$, then $v_1 v_2 \in E(G)$ since otherwise diam$(G) = 3$ and thus $G \cong H_{12}$. If v_2 is adjacent to $(n-4)$ vertices of $N_1(x) \setminus \{s\}$ for $s \ne t$, then $G \cong H_7$ or $G \cong H_{11}$ for $v_1 v_2 \notin E(G)$ or for $v_1 v_2 \in E(G)$, respectively. Otherwise, v_2 is adjacent to all vertices of $N_1(x)$ and we obtain $G \cong K_1 + (K_1 \cup K_{n-2} - e)$ or $G \cong G_6$ for $v_1 v_2 \notin E(G)$ or for $v_1 v_2 \in E(G)$, respectively. But $pd(K_1 + (K_1 \cup K_{n-2} - e)) = pd(G_6) = n-2$ by [2], a contradiction.

(D2.2) v_1 is only adjacent to two vertices $s, t \in N_1(x)$. If v_2 is also only adjacent to two vertices $s, t \in N_1(x)$, then $G \cong H_{44}$ or $G \cong H_{48}$ for $v_1 v_2 \notin E(G)$ or for $v_1 v_2 \in E(G)$, respectively. If v_2 is only adjacent to two vertices $r, s \in N_1(x)$ where $r \ne t$, then $G \cong H_{45}$ or $G \cong H_{49}$ for $v_1 v_2 \notin E(G)$ or for $v_1 v_2 \in E(G)$, respectively. If v_2 is only adjacent to two vertices $p, q \in N_1(x)$ distinct from two vertices $s, t \in N_1(x)$, then we have a resolving $(n-4)$ partition, namely $(v_1)(v_2)(s, s_1)(t, t_1)(p, p_1)(q, q_1)\pi$ for $s_1, t_1, p_1, q_1 \in N_1(x) \setminus \{s, t, p, q\}$, a contradiction. If v_2 is adjacent to $n-5$ vertices of $N_1(x) \setminus \{p, q\}$ where $p \ne s$ and q may equal to t (or $N_1(x) \setminus \{s, t\}$), then $(v_1)(v_2)(s, s_1)(t, t_1)(p, p_1)(x, q_1)$ (or $(v_2)(s, s_1)(t, t_1)(v_1, t_2)(x, t_3)\pi$) is a resolving $(n-4)$-partition, for $p_1, q_1, s_1, t_1 \in N_1(x) \setminus \{s, t, p, q\}$ (or $s_1, t_1, t_2, t_3 \in N_1(x) \setminus \{s, t\}$), a contradiction. If v_2 is adjacent to $n-4$ vertices of $N_1(x) \setminus \{p\}$ where $p \ne s, t$, then we have a resolving $(n-4)$ partition, namely $(v_1)(v_2)(s, s_1)(t, t_1)(p, p_1)(x, x_1)\pi$ for $s_1, t_1, p_1, x_1 \in N_1(x) \setminus \{s, t, p\}$, a contradiction. If v_2 is adjacent to $n-4$ vertices of $N_1(x) \setminus \{s\}$, then we obtain $G \cong H_8$ or $G \cong H_6$ for $v_1 v_2 \notin E(G)$ or $v_1 v_2 \in E(G)$, respectively. Otherwise, v_2 is adjacent to all vertices of $N_1(x)$ and we deduce $G \cong H_4$ or $G \cong H_1$ for $v_1 v_2 \notin E(G)$ or $v_1 v_2 \in E(G)$, respectively (Fig. 21).

(D2.3) v_1 is adjacent to $(n-5)$ vertices of $N_1(x) \setminus \{t_1, t_2\}$. Suppose that v_2 is not adjacent to at least one vertex $t_3 \in N_1(x)$ different from t_1 and t_2. However, $(v_1)(v_2)(t_1, t_4)(t_2, t_5)(t_3, t_6)(x, t_7)\pi$ is a resolving $(n-4)$-partition, for $t_4, t_5, t_6 \in N_1(x) \setminus \{t_1, t_2, t_3\}$, a contradiction. Therefore, if v_2 is not adjacent to some vertices of $N_1(x)$, then they are elements of $\{t_1, t_2\}$. Now, for the following conditions: (v_2 is also adjacent to $n-5$ vertices of $N_1(x) \setminus \{t_1, t_2\}$), or ($v_2$ is adjacent to $n-4$ vertices $N_1(x) \setminus \{t_1\}$), or (v_2 is adjacent to all vertices of $N_1(x)$), then $v_1 v_2 \in E(G)$ since otherwise $(v_1)(t_1, t_3)(t_2, t_4)(v_2, t_5)(x, t_6)\pi$ is a resolving $(n-4)$-partition, for $t_3, t_4, t_5, t_6 \in N_1(x) \setminus \{t_1, t_2\}$ a contradiction. Hence we deduce $G \cong K_n - E(K_{2,3})$, or $G \cong H_9$, or $G \cong H_3$ for the previous three conditions, respectively (Fig. 22 (a)–(c)).

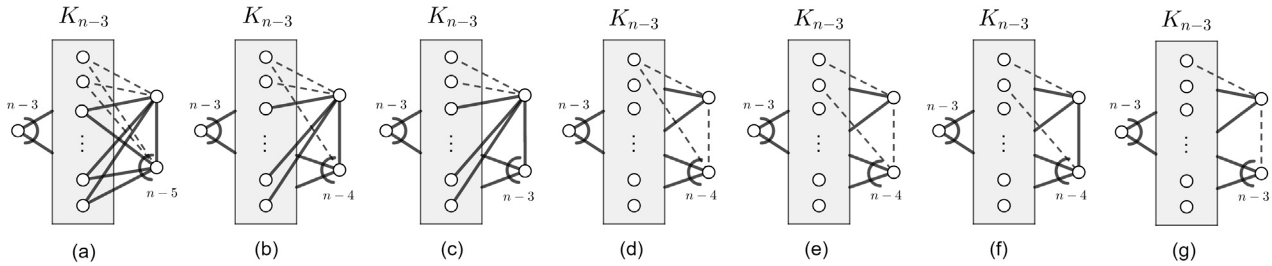

Fig. 22. Graph (a) $K_n - E(K_{2,3})$, (b) H_9, (c) H_3, (d) $K_n - E(K_4 - e)$, (e) H_{82}, (f) $K_n - E(P_5)$ and (g) $K_n - E(K_{1,3} + e)$.

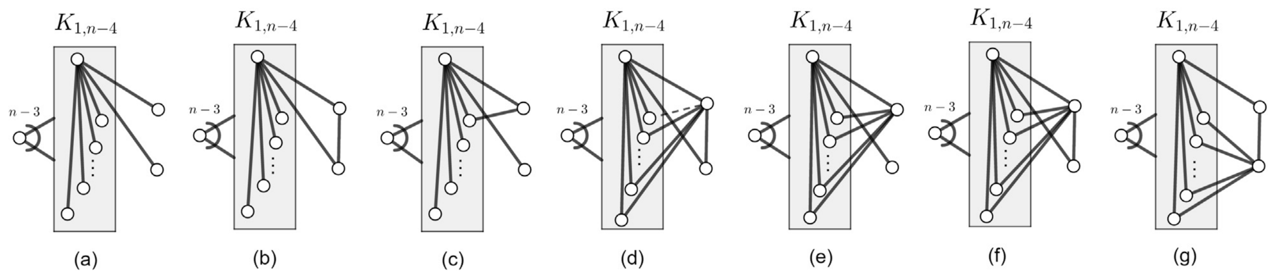

Fig. 23. Graph (a) $K_1 + (K_{1,n-4} \cup 2K_1)$, (b) $K_1 + (K_{1,n-4} \cup K_2)$, (c) H_{78}, (d) H_{79}, (e) $K_1 + (K_{2,n-4} \cup K_1)$, (f) H_{80} and (g) H_{81}.

(D2.4) v_1 is adjacent to $(n-4)$ vertices of $N_1(x) \setminus \{t_1\}$. If v_2 is also adjacent to $(n-4)$ of $N_1(x) \setminus \{t_1\}$, then $v_1v_2 \notin E(G)$, since otherwise $G \cong K_n - E(C_4)$ and $pd(K_n - E(C_4)) = n-2$ by [2]. Thus we deduce $G \cong K_n - E(K_4 - e)$. If v_2 is adjacent to $n-4$ vertices of $N_1(x) \setminus \{t_2\}$ for $t_1 \neq t_2$, then we obtain $G \cong H_{82}$ or $G \cong K_n - E(P_5)$ for $v_1v_2 \notin E(G)$ or $v_1v_2 \in E(G)$, respectively. If v_2 is adjacent to all vertices of $N_1(x)$, then $v_1v_2 \notin E(G)$, since otherwise $G \cong K_n - E(P_4)$ and $pd(K_n - E(P_4)) = n-2$ by [2]. We deduce $G \cong K_n - E(K_{1,3} + e)$. Now for the remaining condition, assume that both v_1 and v_2 are adjacent to all vertices of $N_1(x)$. However, we obtain that $G \cong K_n - E(K_3)$ if $v_1v_2 \notin E(G)$ or $G \cong K_n - E(P_3)$ if $v_1v_2 \in E(G)$ and $pd(K_n - E(K_3)) = pd(K_n - E(P_3)) = n-2$ by [2], a contradiction.

(D3) $N_1(x)$ induces $K_{1,n-4}$. Let $V(N_1(x)) = \{t, t_i : 1 \leq i \leq n-4\}$ and $E(N_1(x)) = \{tt_i : 1 \leq i \leq n-4\}$. Note that if a vertex of $N_2(x)$ is not adjacent to a vertex $t \in N_1(x)$, then it is adjacent to all vertices $t_i \in N_1(x)$ for $1 \leq i \leq n-4$, since otherwise $\mathrm{diam}(G) = 3$. Furthermore, if each v_1 and v_2 are adjacent to at least one vertex $t_i \in N_1(x)$ and $t_j \in N_1(x)$, respectively, for $1 \leq i \leq j \leq n-4$, then $(t_i)(t_j)(t,t_1)(v_1,t_2)(x,t_4)\pi$ is a resolving $(n-4)$-partition, for $i, j \neq 1,2,3,4$, a contradiction. Therefore, there exists at most one vertex of $N_2(x)$ which is adjacent the vertices $t_i \in N_1(x)$ for some $1 \leq i \leq n-4$. Furthermore, if there exists a vertex of $N_2(x)$, namely v_1, and $t_1, t_2, t_3, t_4 \in N_2(x)$ such that $v_1t_1, v_2t_2 \in E(G)$ and $v_1t_3, v_1t_4 \notin E(G)$, then $(v_1)(t_1,t_3)(t_2,t_4)(t,t_5)(x,t_6)\pi$ is a resolving $(n-4)$-partition, a contradiction. This implies that any vertex of $N_2(x)$ is adjacent to at most one vertex of $t_i \in N_1(x)$ or it is adjacent to at least $n-5$ vertices of $t_i \in N_1(x)$.

Let both v_1 and v_2 are adjacent to a vertex $t \in N_1(x)$. If v_1 and v_2 are not adjacent to any other vertex $t_i \in N_1(x)$, we deduce $G \cong K_1 + (K_{1,n-4} \cup 2K_1)$ or $G \cong K_1 + (K_{1,n-4} \cup K_2)$ for $v_1v_2 \notin E(G)$ or $v_1v_2 \in E(G)$, respectively. If one vertex of $N_1(x)$, namely v_1, is also adjacent to a single vertex $t_1 \in N_1(x)$ or it is only not adjacent to a single vertex $t_1 \in N_1(x)$, then $v_1v_2 \notin E(G)$ or $v_1v_2 \in E(G)$, respectively. Since otherwise we have a resolving $(n-4)$-partition, namely $(v_1)(t_1,t_2)(t,t_3)(v_2,t_4)(x,t_5)\pi$. Hence for this case, v_2 is not adjacent to any other vertex $N_1(x)$ and we obtain $G \cong H_{78}$ or $G \cong H_{79}$. If a vertex of $N_1(x)$, namely v_1, is adjacent to all vertices $t_i \in N_1(x)$ for $1 \leq i \leq n-4$, then v_2 is not adjacent to any vertex $t_i \in N_1(x)$ and we obtain $G \cong K_1 + (K_{2,n-4} \cup K_1)$ for $v_1v_2 \notin E(G)$ or $G \cong H_{80}$ for $v_1v_2 \in E(G)$. Otherwise, assume that v_1 is adjacent to a vertex $t \in N_1(x)$ and v_2 is not adjacent to $t \in N_1(x)$, so that v_2 is adjacent to all vertices $t_i \in N_1(x)$ for $1 \leq i \leq n-4$. This implies that v_1 is not adjacent to any other vertex $t_i \in N_1(x)$ and $v_1v_2 \in E(G)$, since otherwise $\mathrm{diam}(G) = 3$. We deduce $G \cong H_{81}$ (Fig. 23).

(D4) $N_1(x)$ induces $K_{n-4} \cup K_1$. Let $V(N_1(x)) = \{t, t_i : 1 \leq i \leq n-4\}$ and $E(N_1(x)) = \{t_it_j : 1 \leq i < j \leq n-4\}$. If there exists a vertex of $N_2(x)$, namely v_1, such that $v_1t_1, v_1t_2 \in E(G)$ and $v_1, t_3, v_1t_4 \notin E(G)$, then $(x)(v_1)(t_1,t_3)(t_2,t_4)(t,t_5)(v_2,t_6)\pi$ is a resolving $(n-4)$-partition, a contradiction. Therefore, any vertex of $N_2(x)$ is adjacent to 1, $n-5$ or $n-4$ vertices of $N_1(x) \setminus \{t\}$.

Let v_1 be only adjacent to a single vertex $t_1 \in N_1(x) \setminus \{t\}$. If v_2 is also only adjacent to a single $t_1 \in N_1(x)$, then $tv_1, tv_2 \in E(G)$ or $tv_i, v_1v_2 \in E(G)$ for some $i = 1,2$, since otherwise $\mathrm{diam}(G) = 3$. We deduce $G \cong H_{75}$ if $tv_1, tv_2 \in E(G)$ and $v_1v_2 \notin E(G)$, or $G \cong H_{73}$ if t is only adjacent to one vertex of v_1 or v_2 and $v_1v_2 \in E(G)$, or $G \cong H_{74}$ if $tv_1, tv_2, v_1v_2 \in E(G)$. Similarly, if v_2 is only adjacent to a single vertex $t_2 \in N_1(x) \setminus \{t\}$, then $tv_1, tv_2 \in E(G)$ or $tv_i, v_1v_2 \in E(G)$ for some $i = 1,2$. We deduce $G \cong H_{76}$ if ($tv_1, tv_2 \in E(G)$ and $v_1v_2 \notin E(G)$) or (t is only adjacent to one vertex of v_1 or v_2 and $v_1v_2 \in E(G)$), or $G \cong H_{77}$ if $tv_1, tv_2, v_1v_2 \in E(G)$. If v_2 is adjacent to $(n-5)$ vertices of $N_1(x) \setminus \{t, t_1\}$ or $N_1(x) \setminus \{t, t_2\}$, then $(v_2)(t_1,t_2)(v_2,t_3)(t,t_4)(x,t_5)\pi$ or $(v_1)(v_2)(t_1,t_3)(t_2,t_4)(t,t_5)(x,t_6)\pi$ is a resolving $(n-4)$-partition, respectively, a contradiction. Otherwise, v_2 is adjacent to $(n-4)$ vertices $N_1(x) \setminus \{t\}$. In this case t is adjacent to all vertices of N_2, or it is adjacent to a single vertex of N_2 and $v_1v_2 \in E(G)$, since otherwise $\mathrm{diam}(G) = 3$. We deduce $G \cong H_{55}$ if $tv_1, tv_2 \in E(G)$ and $v_1v_2 \notin E(G)$, or $G \cong H_{54}$ if $tv_1, v_1v_2 \in E(G)$ and $tv_2 \notin E(G)$, or $G \cong H_{70}$ if $tv_2, v_1v_2 \in E(G)$ and $tv_1 \notin E(G)$, or $G \cong H_{61}$ if $tv_1, tv_2, v_1v_2 \in E(G)$ (Fig. 24).

Now assume that v_1 is adjacent to $(n-5)$ vertices of $N_1(x) \setminus \{t, t_1\}$. If v_2 is adjacent to $(n-5)$ vertices of $N_1(x) \setminus \{t, t_2\}$, then $(v_1)(v_2)(t_1,t_3)(t_2,t_4)(t,t_5)(x,t_6)\pi$ is a resolving $(n-4)$-partition, a contradiction. If v_2 is adjacent to $(n-5)$ vertices of $N_1(x) \setminus \{t, t_1\}$ or it is adjacent to all $(n-4)$ vertices of $N_1(x) \setminus \{t\}$, then $v_1v_2 \in E(G)$, since otherwise we have a resolving $(n-4)$-partition, namely $(v_1)(t_1,t_2)(t,t_3)(v_2,t_4)(x,t_5)\pi$, a contradiction. Hence, (for v_2 is adjacent to $(n-5)$ vertices of $N_1(x) \setminus \{t, t_1\}$, we deduce $G \cong H_{38}$ if t is only adjacent to one of v_1 or v_2, or $G \cong H_{37}$ if $tv_1, tv_2 \in E(G)$) and (for v_2 is adjacent to $(n-4)$ vertices of $N_1(x) \setminus \{t\}$, we deduce $G \cong H_{35}$ if $tv_1 \in E(G)$ and $tv_2 \notin E(G)$, or $G \cong H_{30}$ if $tv_2 \in E(G)$ and $tv_1 \notin E(G)$, or $G \cong H_{28}$ if $tv_1, tv_2 \in E(G)$). Otherwise, assume that both v_1 and v_2 are adjacent to $(n-4)$ vertices of $N_1(x) \setminus \{t\}$. Then $tv_1, tv_2 \in E(G)$ or (t is adjacent to one of $v_1, v_2 \in N_2(x)$ and $v_1v_2 \in E(G)$), since otherwise $\mathrm{diam}(G) = 3$. We deduce $G \cong K_n - E(K_{1,n-4} \cup K_3)$ if $tv_1, tv_2 \in E(G)$

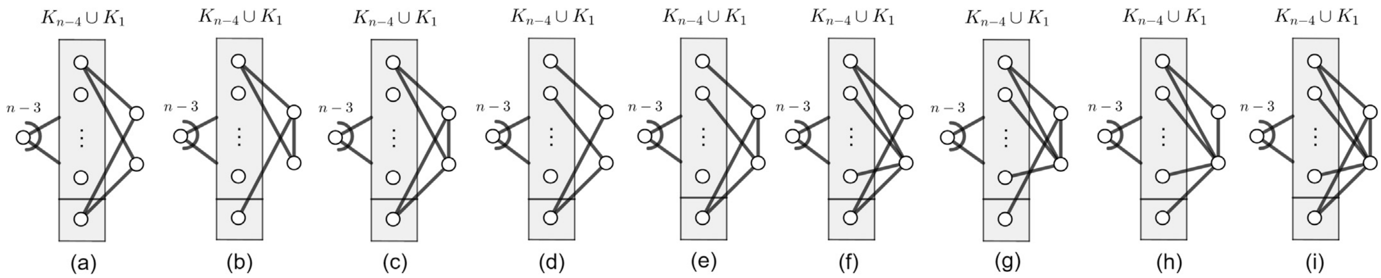

Fig. 24. Graph (a) H_{75}, (b) H_{73}, (c) H_{74}, (d) H_{76}, (e) H_{77}, (f) H_{55}, (g) H_{54}, (h) H_{70} and (i) H_{61}.

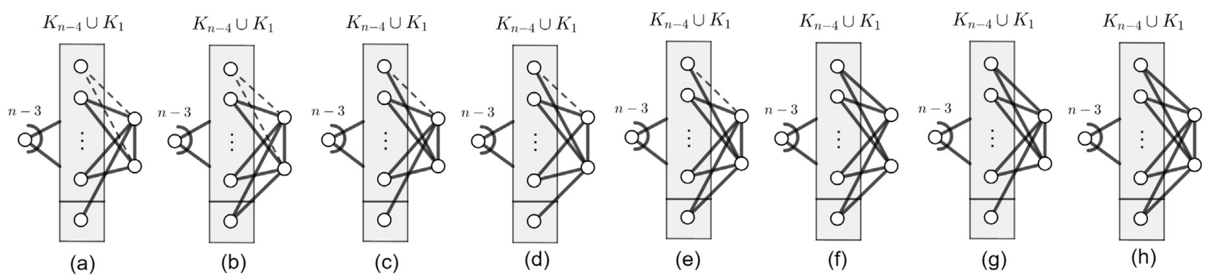

Fig. 25. Graph (a) H_{38}, (b) H_{37}, (c) H_{35}, (d) H_{30}, (e) H_{28}, (f) $K_n - E(K_{1,n-4} \cup K_3)$, (g) H_{12} and (h) $K_n - E(K_{1,n-4} \cup P_3)$.

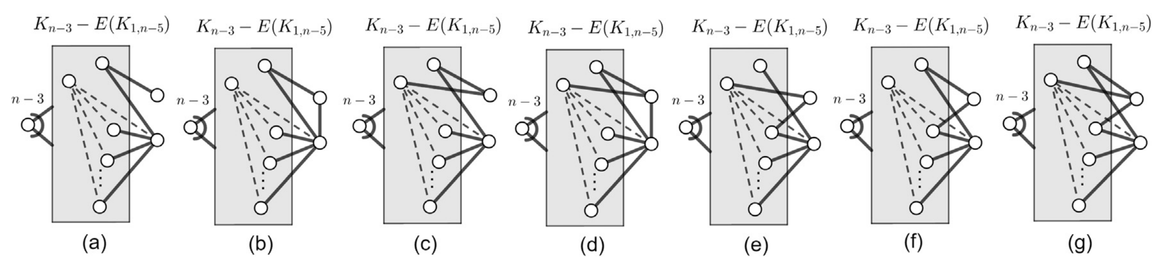

Fig. 26. Graph (a) H_{65}, (b) H_{66}, (c) H_{51}, (d) H_{52}, (e) H_{58}, (f) H_{71} and (g) H_{62}.

and $v_1v_2 \notin E(G)$, or $G \cong H_{12}$ if $tv_1, v_1v_2 \in E(G)$ and $tv_2 \notin E(G)$, or $G \cong K_n - E(K_{1,n-4} \cup P_3)$ if $tv_1, tv_2, v_1v_2 \in E(G)$ (Fig. 25).

(D5) $N_1(x)$ induces $K_{n-3} - E(K_{1,n-5})$. Let $V(N_1(x)) = \{v, w, w_i : 1 \le i \le n-5\}$ and $E(N_1(x)) = \{vw, vw_i, w_iw_j : 1 \le i, j \le n-5\}$. If there exist a vertex of $N_2(x)$, namely v_1, and $w_1, w_2, w_3, w_4 \in N_1(x)$ such that $v_1w_1, v_1w_2 \in E(G)$ but $v_1w_3, v_1w_4 \notin E(G)$, then $(x)(v_1)(w)(w_1, w_3)(w_2, w_4)(v_2, w_5)(v, w_6)\pi$ is a resolving $(n-4)$-partition, a contradiction. Therefore, any vertex of $N_2(x)$ is either adjacent to at most one vertex of $w_i \in N_1(x)$ or it is adjacent to at least $n-6$ vertices of $w_i \in N_1(x)$, for $1 \le i \le n-5$.

(D5.1) v_1 is not adjacent to any vertex $w_i \in N_1(x), 1 \le i \le n-5$, so that v_1 is adjacent to a vertex $v \in N_1(x)$, since otherwise $\text{diam}(G) = 3$. If v_2 is not adjacent to at least one vertex $w_1 \in N_1(x)$, then we have a resolving $(n-4)$-partition, namely $(w)(w_1)(v_1, w_2)(v_2, w_3)(v, w_4)(x, w_5)\pi$, a contradiction. Therefore, v_2 is adjacent to all vertices $w_i \in N_1(x)$ for all $1 \le i \le n-5$. Furthermore, we have that $v_2w \notin E(G)$ and $v_2v \in E(G)$, since otherwise $(w)(v_1, w_1)(v_2, w_2)(v, w_3)(x, w_4)\pi$ or $(v_2)(v_1, w_1)(w, w_2)(v, w_3)(x, w_4)\pi$ is a resolving $(n-4)$-partition, a contradiction. We deduce $G \cong H_{65}$ if $wv_1, v_1v_2 \notin E(G)$, or $G \cong H_{66}$ if $wv_1 \notin E(G)$ and $v_1v_2 \in E(G)$, or $G \cong H_{51}$ if $wv_1 \in E(G)$ and $v_1v_2 \notin E(G)$, or $G \cong H_{52}$ if $wv_1, v_1v_2 \in E(G)$, as depicted in Fig. 26 (a)-(d), respectively.

(D5.2) v_1 is only adjacent to a single vertex $w_1 \in N_1(x)$ and $v_1w_i \notin E(G)$ for all other remaining $i \ne 1$. If (v_2 is also only adjacent to a single vertex $w_1 \in N_1(x)$ and $v_2w_j \notin E(G)$ for all $j \ne 1$) or (v_2 is only adjacent to a single vertex $w_2 \in N_1(x)$ and $v_2w_j \notin E(G)$ for all $j \ne 2$), then we have a resolving $(n-4)$-partition, namely $(w)(v_1, w_1)(v_2, w_2)(v, w_3)(x, w_4)\pi$, a contradiction. If v_2 is only not adjacent to a single vertex $w_1 \in N_1(x)$ and $v_2w_j \in E(G)$ for all $j \ne 1$ (or v_2 is only not adjacent to a single vertex $w_2 \in N_1(x)$ and $v_2w_j \in E(G)$ for all $j \ne 2$), then $(w)(w_1)(v_1, w_2)(v_2, w_3)(v, w_4)(x, w_5)\pi$ (or $(w)(w_2)(v_1, w_1)(v_2, w_3)(v, w_4)(x, w_5)\pi$) is a resolving $(n-4)$-partition, a contradiction. Therefore, v_2 is adjacent to all vertices $w_i \in N_1(x)$ for all $1 \le i \le n-5$. In this case, $v_2w, v_1v_2 \notin E(G)$ and $v_2v \in E(G)$, since otherwise $(w)(v_1, w_1)(v_2, w_2)(v, w_3)(x, w_4)\pi$, or $(v_2)(v_1, w_1)(w, w_2)(v, w_3)(x, w_4)\pi$, or $(v_1)(w)(w_1, w_2)(v_2, w_3)(v, w_4)(x, w_5)\pi$ is a resolving $(n-4)$-partition, a contradiction. This implies that v_1 is adjacent to at least one of the vertex w or v, since otherwise $\text{diam}(G) = 3$. We deduce $G \cong H_{58}$ if $v_1w \in E(G)$ and $v_1v \notin E(G)$, or $G \cong H_{71}$ if $v_1v \in E(G)$ and $v_1w \notin E(G)$, or $G \cong H_{62}$ if $v_1v, v_1w \in E(G)$.

(D5.3) v_1 is only not adjacent to a single vertex $w_1 \in N_1(x)$ and $v_1w_i \in E(G)$ for all other $i \ne 1$. If (v_2 is only not adjacent to a single vertex $w_1 \in N_1(x)$ and $v_2w_j \in E(G)$ for all $j \ne 1$) or (v_2 is only not adjacent to a single vertex $w_2 \in N_1(x)$ and $v_2w_j \in E(G)$ for all $j \ne 2$), then we have a resolving $(n-4)$-partition, namely $(w)(w_1)(v_1, w_2)(v_2, w_3)(v, w_4)(x, w_5)\pi$ or $(w)(w_1)(w_2)(v_1, w_3)(v_2, w_4)(v, w_5)(x, w_6)\pi$, respectively, a contradiction. Therefore, v_2 is adjacent to all vertices $w_i \in N_1(x)$ for all $1 \le i \le n-5$. In this case, $v_2w \notin E(G)$ and $v_1v, v_2v, v_1v_2 \in E(G)$, since otherwise $(w)(w_1)(v_1, w_2)(v_2, w_3)(v, w_4)(x, w_5)\pi$, or $(v_1)(w_1, w_2)(w, v_2)(v, w_3)(x, w_4)\pi$, or $(v_1)(v_2)(w_1, w_2)(v, w_3)(w, w_4)(x, w_5)\pi$, or $(w)(v_1)(w_1, w_2)(v_2, w_3)(v, w_4)(x, w_5)\pi$ is a resolving $(n-4)$-partition, a contradiction. We deduce $G \cong H_{31}$ if $v_1w \notin E(G)$ or $G \cong H_{26}$ if $v_1w \in E(G)$, as depicted in Fig. 27 (a)-(b), respectively. For the remaining case, assume that both v_1 and v_2 are adjacent to all vertices $w_i \in N_1(x)$ for all $1 \le i \le n-5$. Then, other vertex $w \in N_1(x)$ is adjacent to at most one vertex of $v_1, v_2 \in N_2(x)$, since otherwise $(w)(v_1, w_1)(v_2, w_2)(v, w_3)(x, w_4)\pi$ is a resolving $(n-4)$-partition, a contradiction. If w is not adjacent to any vertex $v_1, v_2 \in N_2(x)$, then v is adjacent to both v_1 and v_2, since otherwise $\text{diam}(G) = 3$. In this

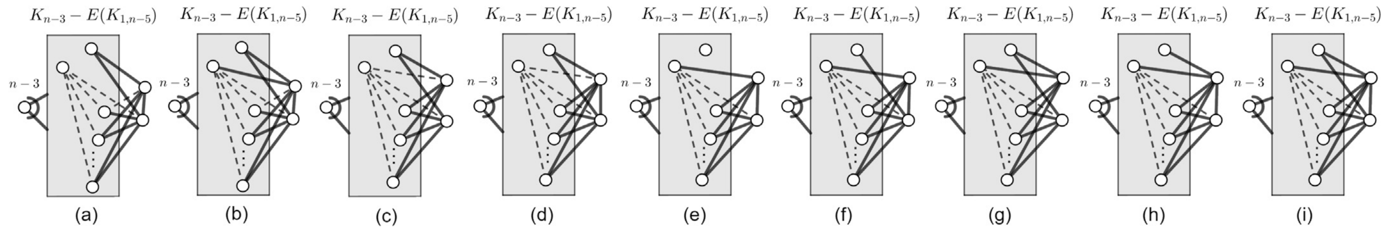

Fig. 27. Graph (a) H_{31}, (b) H_{26}, (c) H_{14}, (d) H_{11}, (e) H_{37}, (f) H_{28}, (g) H_{15}, (h) H_{27} and (i) H_6.

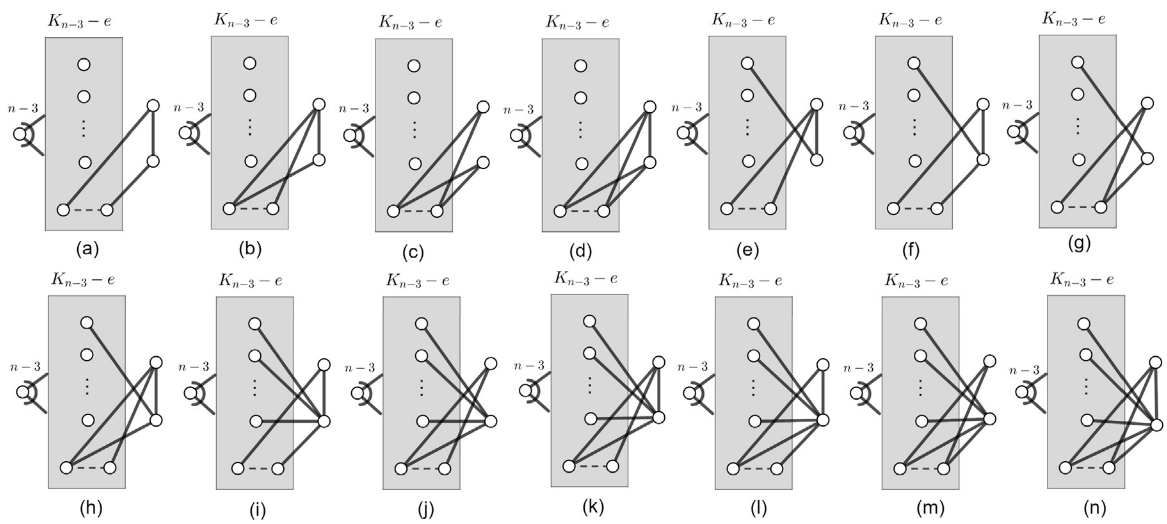

Fig. 28. Graph (a) H_{64}, (b) H_{56}, (c) H_{72}, (d) H_{63}, (e) H_{55}, (f) H_{54}, (g) H_{70}, (h) H_{61}, (i) H_{35}, (j) H_{30}, (k) H_{28}, (l) H_{21}, (m) H_{23} and (n) H_{18}.

case we deduce $G \cong H_{14}$ if $v_1v_2 \notin E(G)$ or $G \cong H_{11}$ if $v_1v_2 \in E(G)$, as depicted in Fig. 27 (c)-(d), respectively. Now assume that w is only adjacent to one vertex of $N_2(x)$, namely $v_1w \in E(G)$ and $v_2w \notin E(G)$. Then, $v_1v_2 \in E(G)$ or $vv_2 \in E(G)$, since otherwise $\mathrm{diam}(G) = 3$ and v_1 is adjacent to at least one of the vertex v or v_2, since otherwise $(v_1)(v_2, w_1)(v, w_2)(w, w_3)(x, w_4)\pi$ is a resolving $(n-4)$-partition, a contradiction. We deduce $G \cong H_{37}$ if $v_1v_2 \in E(G)$ and $v_1v, v_2v \notin E(G)$, or $G \cong H_{28}$ if $v_1v_2, vv_2 \in E(G)$ and $vv_1 \notin E(G)$, or $G \cong H_{15}$ if $vv_1, vv_2 \in E(G)$ and $v_1v_2 \notin E(G)$, or $G \cong H_{27}$ if $v_1v_2, vv_1 \in E(G)$ and $vv_2 \notin E(G)$, or $G \cong H_6$ if $v_1v_2, vv_1, vv_2 \in E(G)$, as depicted in Fig. 27 (e)-(i), respectively.

(D6) $N_1(x)$ induces $K_{n-3} - e$. Let $e = ab$ and other vertices of $N_1(x)$ by t_i where $1 \le i \le n-5$. If there exists a vertex of $N_2(x)$, namely v_1, such that $v_1t_1, v_1t_2 \in E(G)$ and $v_1t_3, v_1t_4 \notin E(G)$, then $(x)(v_1)(a)(t_1, t_3)(t_2, t_4)(b, t_5)(v_2, t_6)\pi$ is a resolving $(n-4)$-partition, a contradiction. Therefore any vertex of $N_2(x)$ is adjacent to at most one vertex of $N_1(x) \setminus \{a, b\}$ or it is adjacent to at least $n-6$ vertices of $N_1(x) \setminus \{a, b\}$.

(D6.1) v_1 is not adjacent to any vertex of $N_1(x) \setminus \{a, b\}$. If v_2 is also not adjacent to any vertex of $N_1(x) \setminus \{a, b\}$, then ($v_1$ and v_2 are adjacent to different vertex of a and b, and $v_1v_2 \in E(G)$), or (one of the vertex of $N_1(x)$ is adjacent to both $a, b \in N_1(x)$, one other vertex of $N_1(x)$ is at least adjacent to one vertex $a, b \in N_1(x)$ and $v_1v_2 \in E(G)$), or (all vertices of $N_1(x)$ are adjacent to both $a, b \in N_1(x)$), since otherwise $\mathrm{diam}(G) = 3$. We deduce G as depicted in Fig. 28 (a)-(d). If v_2 is adjacent to a single vertex $t_1 \in N_1(x) \setminus \{a, b\}$ and $v_2t_i \notin E(G)$ for all remaining $i \ne 1$, then at least one end vertex of e is not adjacent to v_2 since otherwise $(v_2)(t_1, t_2)(a, t_3)(b, t_4)(x, v_1)\pi$ is a resolving $(n-4)$-partition, a contradiction. Hence we obtain G as depicted in Fig. 28 (e)-(h). Now suppose that v_2 is only not adjacent to a single vertex $t_1 \in N_1(x)$ and $v_2t_i \in E(G)$ for all $i \ne 1$. However, we obtain that $(v_1)(v_2)(t_1, t_2)(a, t_3)(b, t_4)(x, t_5)\pi$ or $(v_2)(t_1, t_2)(a, t_3)(b, t_4)(x, v_1)\pi$ is a resolving $(n-4)$-partition, a contradiction. Otherwise, assume that v_2 is adjacent to all vertices $t_i \in N_1(x)$ for all $1 \le i \le n-5$. Then v_2 is adjacent to at least one end vertex of e or $v_1v_2 \notin E(G)$, since otherwise $(v_2)(a, t_1)(b, v_1, t_2)(x, t_3)\pi$ is a resolving $(n-4)$-partition, a contradiction. We deduce G as depicted in Fig. 28 (i)-(n).

(D6.2) v_1 is adjacent to a single vertex $t_1 \in N_1(x)$ and $v_1t_i \notin E(G)$ for all $i \ne 1$. If v_2 is also adjacent to a single vertex $t_1 \in N_1(x)$ and $v_2t_i \notin E(G)$ for all $i \ne 1$, then v_1 (or similarly v_2) is not adjacent to at least one end vertex of e, since otherwise $(v_1)(x, t_1)(v_2, t_2)(a, t_3)(b, t_4)\pi$ (or $(v_2)(x, t_1)(v_1, t_2)(a, t_3)(b, t_4)\pi$) is a resolving $(n-4)$-partition, a contradiction. We deduce $G \cong K_1 + (K_{n-3} - e \cup 2K_1)$ if both v_1 and v_2 are not adjacent to any end vertex of e and $v_1v_2 \notin E(G)$, or $G \cong K_1 + (K_{n-3} - e \cup K_2)$ if both v_1 and v_2 are not adjacent to any end vertex of e and $v_1v_2 \in E(G)$, or $G \cong H_{65}$ if one of v_1 or v_2 is only adjacent to one end vertex of e and $v_1v_2 \notin E(G)$, or $G \cong H_{51}$ if one of v_1 or v_2 is only adjacent to one end vertex of e and $v_1v_2 \in E(G)$, or $G \cong H_{69}$ if one of end vertex e is adjacent to both v_1 and v_2, and $v_1v_2 \notin E(G)$, or $G \cong H_{60}$ if one of end vertex e is adjacent to both v_1 and v_2, and $v_1v_2 \in E(G)$, or $G \cong H_{66}$ if each v_1 and v_2 are adjacent to different end vertex of e and $v_1v_2 \notin E(G)$, or $G \cong H_{52}$ if each v_1 and v_2 are adjacent to different end vertex e and $v_1v_2 \in E(G)$. If v_2 is only adjacent to a single vertex $t_2 \in N_1(x)$ and $v_2t_i \notin E(G)$ for all $i \ne 2$, then both v_1 and v_2 are adjacent at most to a single vertex a or b, since otherwise $(v_1)(v_2)(t_1, t_3)(t_2, t_4)(a, t_5)(b, t_6)\pi$ is a resolving $(n-4)$-partition, a contradiction. Furthermore, if both v_1 and v_2 are not adjacent to a vertex a (or similarly to a vertex b), then $v_1v_2 \in E(G)$ since otherwise $\mathrm{diam}(G) = 3$. We deduce $G \cong H_{57}$ if both v_1 and v_2 are not adjacent to any end vertex of e and $v_1v_2 \in E(G)$, or $G \cong H_{58}$ if only one of v_1 or v_2 is adjacent to an end vertex of e and $v_1v_2 \in E(G)$, or $G \cong H_{68}$ if both v_1 and v_2 are adjacent to a single end vertex of e and $v_1v_2 \notin E(G)$, or $G \cong H_{59}$ if both v_1 and v_2 are adjacent to one end vertex of e and $v_1v_2 \in E(G)$ (Fig. 29).

Now suppose that (v_2 is only not adjacent to a vertex $t_1 \in N_1(x)$ and $v_2t_i \in E(G)$ for all other $i \ne 1$) or (v_2 is only not adjacent to a vertex $t_2 \in N_1(x)$ and $v_2t_i \in E(G)$ for all other $i \ne 2$). However, $(a)(v_2)(t_1, t_2)(v_1, t_3)(b, t_4)(x, t_5)\pi$ is a resolving $(n-4)$-partition, a contradiction. Otherwise, assume that v_2 is adjacent to all vertices $t_i \in N_1(x)$

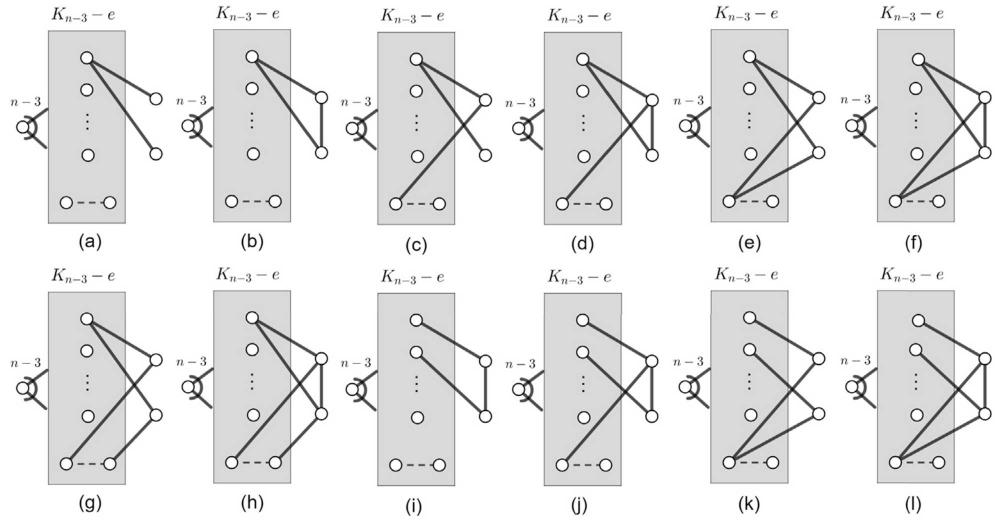

Fig. 29. Graph (a) $K_1+(K_{n-3}-e\cup 2K_1)$, (b) $K_1+(K_{n-3}-e\cup K_2)$, (c) H_{65}, (d) H_{51}, (e) H_{69}, (f) H_{60}, (g) H_{66}, (h) H_{52}, (i) H_{57}, (j) H_{58}, (k) H_{68} and (l) H_{59}.

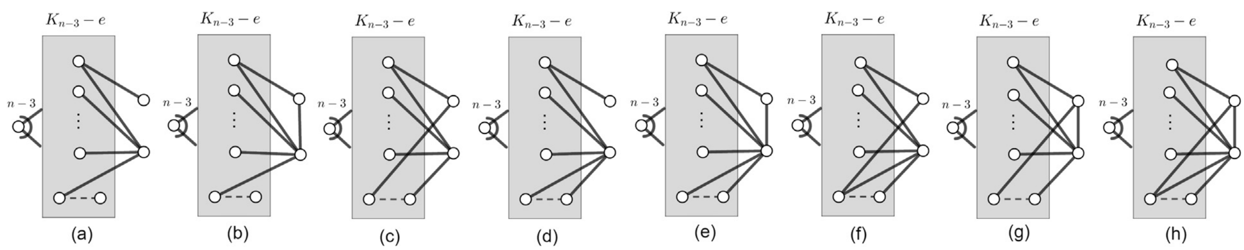

Fig. 30. Graph (a) H_{25}, (b) H_{31}, (c) H_{31}, (d) H_{24}, (e) H_{22}, (f) H_{22}, (g) H_{26} and (h) H_{17}.

for all $1\le i\le n-5$. In this case, v_1 is adjacent to at most one end vertex of e and v_2 is adjacent to at least one end vertex of e, and also (if an end vertex of e is adjacent to both v_1 and v_2, then other end vertex of e is also adjacent to v_2), since otherwise $(v_1)(v_2)(t_1,t_2)(a,t_3)(b,t_4)(x,t_5)\pi$ is a resolving $(n-4)$-partition, a contradiction. We deduce $G\cong H_{25}$ if v_1 is not adjacent to any end vertex of e, v_2 is only adjacent to one end vertex e and $v_1v_2\notin E(G)$, or $G\cong H_{31}$ if (v_1 is not adjacent to any end vertex of e, v_2 is only adjacent to one end vertex e and $v_1v_2\in E(G)$) or (v_1 and v_2 are adjacent to different end vertex of e and $v_1v_2\notin E(G)$), or $G\cong H_{24}$ if v_1 is not adjacent to any end vertex of e, v_2 is adjacent to end vertices of e and $v_1v_2\notin E(G)$, or $G\cong H_{22}$ if (v_1 is not adjacent to any end vertex of e, v_2 is adjacent to end vertices of e and $v_1v_2\in E(G)$) or (v_1 is adjacent to one end vertex of e, v_2 is adjacent to end vertices of e, and $v_1v_2\notin E(G)$), or $G\cong H_{26}$ if each v_1 and v_2 are adjacent to distinct end vertex of e and $v_1v_2\in E(G)$, or $G\cong H_{17}$ if v_1 is adjacent to one end vertex of e, v_2 is adjacent to end vertices of e, and $v_1v_2\in E(G)$ (Fig. 30).

(D6.3) v_1 is only not adjacent to a single vertex $t_1\in N_1(x)$ and $v_1t_i\in E(G)$ for all remaining $i\neq 1$. If v_2 is also only not adjacent to a single vertex $t_1\in N_1(x)$ and $v_2t_i\in E(G)$ for all remaining $i\neq 1$, then $v_1v_2\in E(G)$ and end vertices of e are adjacent to both v_1 and v_2, since otherwise we have a resolving $(n-4)$-partition, namely $(v_1)(a)(t_1,t_2)(v_2,t_3)(b,t_4)(x,t_5)\pi$ or $(v_1)(v_2)(t_1,t_2)(a,t_3)(b,t_4)(x,t_5)$, a contradiction. Hence we deduce $G\cong K_n-E(C_4\cup K_2)$. If v_2 is also only not adjacent to a single vertex $t_2\in N_1(x)$ and $v_2t_i\in E(G)$ for all remaining $i\neq 2$, then $(v_1)(v_2)(a)(t_1,t_3)(t_2,t_4)(b,t_5)(x,t_6)\pi$ is a resolving $(n-4)$-partition, a contradiction. Otherwise, assume that v_2 is adjacent to all vertices $t_i\in N_1(x)\setminus\{a,b\}$ for all $1\le i\le n-5$. In this case, $v_1v_2\in E(G)$, v_1 is adjacent to at least one of a or b, and v_2 is adjacent to both a and b, since otherwise $(v_1)(a)(t_1,t_2)(v_2,t_3)(b,t_4)(x,t_5)\pi$ or $(v_1)(t_1,t_2)(a,t_3)(b,t_4)(x,t_5)\pi$ is a resolving $(n-4)$-partition, a contradiction. We deduce $G\cong H_{19}$ if v_1 is only adjacent to one end vertex of e, or $G\cong K_n-E(K_2\cup P_4)$ if v_2 is adjacent to end vertices of e. Now let both v_1 and v_2 be adjacent to all vertices $t_i\in N_1(x)$ for all $1\le i\le n-5$. Then $v_1v_2\in E(G)$ or any vertex of $N_2(x)$ is adjacent to at least one end vertex of e, since otherwise $(v_1)(v_2,t_1)(a,t_2)(b,t_3)(x,t_4)\pi$ or $(v_2)(v_1,t_1)(a,t_2)(b,t_3)(x,t_4)\pi$ is a resolving $(n-4)$-partition, a contradiction. If $v_1v_2\notin E(G)$, then we deduce G as depicted in Fig. 31 (d)-(g). Otherwise, we deduce G as depicted in Fig. 31 (h)-(n).

(E) $|N_1(x)|=1$ and $|N_2(x)|=n-2$. Let $N_1(x)=\{u\}$ and so that u is adjacent to all vertices of $N_2(x)$, since otherwise $\mathrm{diam}(G)\ge 3$. If $N_2(x)$ induces K_{n-2} or $\overline{K_{n-2}}$, then $G\cong K_1+(K_1\cup K_{n-2})$ or $G\cong K_{1,n-1}$, respectively. However for these two different graphs G, $pd(G)=n-1$ by [1], a contradiction. Otherwise, there exists a vertex $z\in N_2(x)$ such that $2\le|N_1(z)|\le n-3$. By a similar reason with the previous case with z as a peripheral vertex, then $\min\{|N_1(z)|,|N_2(z)|\}\le 3$, since otherwise there exists a resolving $(n-4)$-partition. Therefore, we obtain that $|N_1(z)|,|N_2(z)|\in\{2,3,n-3,n-4\}$ and we are again in one of the Case (A), (B), (C) or (D).

(F) $|N_1(x)|=n-2$ and $|N_2(x)|=1$. Let $N_2(x)=\{v\}$. Then, x is adjacent to all vertices of $N_1(x)$ and v is adjacent to at least one vertex of $N_1(x)$. If $1\le|N_1(v)|\le n-3$, then $|N_1(v)|\in\{1,2,3,n-3,n-4\}$ and we are again in one of the Cases (A), (B), (C), (D) or (E) with v as a peripheral vertex. Now we assume that $|N_1(v)|=n-2$ or in other words v is adjacent to all vertices of $N_1(x)$. Consider the vertices in $N_1(x)$. If any two different vertices in $N_1(x)$ are adjacent, then $G\cong K_n-e$ but $pd(K_n-e)=n-1$ [1]. If there exists a vertex $z\in N_1(x)$ such that z is not adjacent to at least two vertices $N_1(x)$, then $|N_1(z)|\le n-3$ and we back in one of Cases (A), (B), (C) or (D) with z as a peripheral vertex. Otherwise we assume that $N_1(x)$ form a matching M. If $M=1$, then $G\cong K_n-E(2K_2)$ but $pd(K_n-E(2K_2))=n-2$ by [2]. If $M=2$, then $G\cong K_n-E(3K_2)$. If $M\ge 3$, then there exist $a,a_1,a_2,b,b_1,b_2,c,c_1,c_2\in N_1(x)$ such that $aa_1,bb_1,cc_1\in E(G)$ but $aa_2,bb_2,cc_2\notin E(G)$. However, $(x)(a)(b)(c)(a_1,a_2)(b_1,b_2)(c_1,c_2)(v,t)\pi$ is a resolving $(n-4)$-partition, for a vertex $t\in N_1(x)\setminus\{a,a_1,a_2,b,b_1,b_2,c,c_1,c_2\}$, a contradiction. □

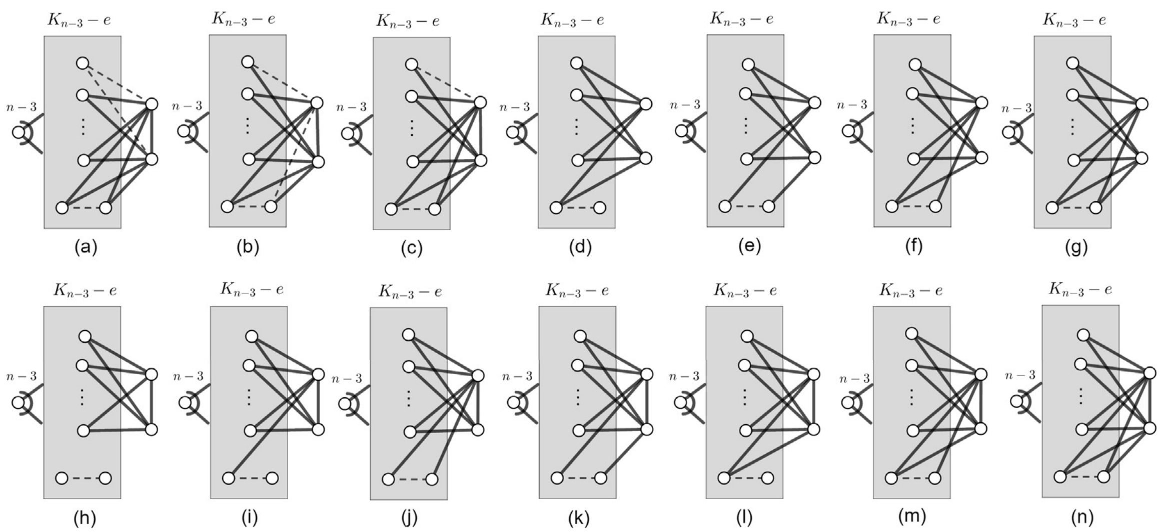

Fig. 31. Graph (a) $K_n - E(K_2 \cup C_4)$, (b) H_{19}, (c) $K_n - E(K_2 \cup P_4)$, (d) H_{36}, (e) H_{32}, (f) H_{20}, (g) $K_n - E(K_2 \cup K_3)$, (h) H_{41}, (i) H_{32}, (j) H_{20}, (k) $K_n - E(C_5)$, (l) H_{42}, (m) $K_n - E(P_5)$ and (n) $K_n - E(K_2 \cup P_3)$.

Declarations

Author contribution statement

Edy Tri Baskoro: Conceived and designed the experiments; Analyzed and interpreted the data; Wrote the paper.

Debi Oktia Haryeni: Conceived and designed the experiments; Performed the experiments; Wrote the paper.

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Competing interest statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

Appendix A

Graphs of order n obtained from $K_{n-1} - e$ by adding one new vertex adjacent to:

H_1: three vertices, exactly two vertices with maximum degree;
H_2: three vertices, two of them are the end points of e;
H_3: $n-4$ vertices, exactly one vertex is the end point of e;
H_4: two vertices with maximum degree;

Graphs of order n obtained from $K_{n-1} - E(P_3)$ by adding one new vertex adjacent to:

H_5: three vertices, two are the end points of P_3 and one with maximum degree;
H_6: three vertices with different degrees;
H_7: one vertex with maximum degree;
H_8: two vertices, one with maximum degree and one is the end point of P_3;
H_9: $n-4$ vertices, one with minimum degree and $n-5$ vertices with maximum degree;
H_{10}: $n-4$ vertices, one is the end point of P_3 and $n-5$ vertices with maximum degree;
H_{11}: two vertices, one with minimum degree and one with maximum degree;
H_{12}: two vertices of P_3 with different degrees;

Graphs of order n obtained from $K_{n-1} - E(K_3)$ by adding one new vertex adjacent to:

H_{13}: one vertex with maximum degree;
H_{14}: two vertices with different degrees;
H_{15}: three vertices, exactly two with minimum degree;
H_{16}: $n-4$ vertices, exactly one with minimum degree;

Graphs of order n obtained from $K_{n-1} - E(2K_2)$ by adding one new vertex adjacent to:

H_{17}: three vertices, two are the end points of different edges of $E(2K_2)$ and one with maximum degree;
H_{18}: three end points of $E(2K_2)$;
H_{19}: $n-4$ vertices, two are the end points of different edges of $E(2K_2)$ and $n-6$ vertices with maximum degree;
H_{20}: $n-4$ vertices, exactly one with minimum degree;
H_{21}: two end points of different edges of $E(2K_2)$;
H_{22}: two vertices with different degrees;
H_{23}: two non-adjacent vertices;
H_{24}: one vertex with maximum degree;

Graphs of order n obtained from $K_{n-1} - E(P_4)$ by adding one new vertex adjacent to:

H_{25}: one vertex with maximum degree;
H_{26}: three vertices, one with maximum degree and two are internal vertices of P_4;
H_{27}: three vertices of P_4, two are the end points of P_4;
H_{28}: three vertices of P_4, two are the internal vertices of P_4;
H_{29}: two end points of P_4;
H_{30}: two vertices of P_4 with different degrees;
H_{31}: two vertices, one with minimum degree and one with maximum degree;
H_{32}: $n-4$ vertices, one with minimum degree and $n-5$ with maximum degree;
H_{33}: $n-4$ vertices, two are the end points of P_4 and $n-6$ with maximum degree;
H_{34}: $n-4$ vertices, one is the end point of P_4 and $n-5$ with maximum degree;
H_{35}: two internal vertices of P_4;

H_{36}: $n-4$ vertices, one is the end point of P_4 and $n-5$ with maximum degree;

Graphs of order n obtained from $K_{n-1} - E(C_4)$ by adding one new vertex adjacent to:

H_{37}: three vertices with minimum degree;
H_{38}: two vertices with minimum degrees;
H_{39}: one vertex with maximum degree;
H_{40}: $n-4$ vertices, one with minimum degree and $n-5$ with maximum degree;

Graphs of order n obtained from $K_{n-1} - E(K_{1,3} + e)$ by adding one new vertex adjacent to:

H_{41}: $n-4$ vertices, one with minimum degree and $n-5$ with maximum degree;

Graphs of order n obtained from $K_{n-1} - E(K_{1,3})$ by adding one new vertex adjacent to:

H_{42}: $n-3$ vertices, $n-5$ with maximum degree, and two with different degrees;

Graphs of order n obtained from K_{n-2} by connecting two new vertices x and y with:

H_{43}: exactly two vertices a and b in K_{n-2} such that $(a,x),(a,y),(b,x)$ are new edges;
H_{44}: exactly two vertices a and b in K_{n-2} such that $(a,x),(a,y),(b,x)$, (b,y) are new edges;
H_{45}: exactly three vertices a,b and c in K_{n-2} such that $(a,x),(a,y),(b,x)$, (c,y) are new edges;
H_{46}: exactly three vertices a,b and c in K_{n-2} such that $(a,x),(b,x),(c,y)$, (x,y) are new edges;
H_{47}: H_{43} by adding new edge (x,y);
H_{48}: H_{44} by adding new edge (x,y);
H_{49}: H_{45} by adding new edge (x,y);

Graphs of order n obtained from $\overline{K_{n-2}}$:

H_{50}: $(K_2 + \overline{K_{n-2}}) - e$, where e is an edge connecting a vertex of K_2 and $\overline{K_{n-2}}$;

Graphs of order n obtained from $K_{n-2} - e$ by connecting a path $P_2 := (x,y)$ with:

H_{51}: three new edges $(a,x),(c,x),(c,y)$, where a is one of the end-points of e and c is a vertex of K_{n-2} with maximum degree;
H_{52}: four new edges $(a,x),(b,y),(c,x),(c,y)$, where a and b are the end-points of e and c is a vertex of K_{n-2} with maximum degree;
H_{53}: four new edges $(a,x),(b,x),(c,x),(c,y)$, where a and b are the end-points of e and c is a vertex of K_{n-2} with maximum degree;
H_{54}: three new edges $(a,x),(b,y),(c,y)$, where a and b are the end-points of e and c is a vertex of K_{n-2} with maximum degree;
H_{55}: three new edges $(a,x),(b,x),(c,y)$, where a and b are the end-points of e and c is a vertex of K_{n-2} with maximum degree;
H_{56}: three new edges $(a,x),(b,x),(b,y)$, where a and b are the end-points of e;
H_{57}: two new edges $(c,x),(d,x)$, where c and d are vertices with maximum degree;
H_{58}: three new edges $(a,x),(c,x),(d,y)$, where a is one of the end-points of e and c and d are vertices of K_{n-2} with maximum degree;
H_{59}: four new edges $(a,x),(a,y),(c,x),(d,y)$, where a is one of the end-points of e and c and d are vertices of K_{n-2} with maximum degree;
H_{60}: four new edges $(a,x),(a,y),(c,x),(c,y)$, where a is one of the end-points of e and c is a vertex of K_{n-2} with maximum degree;
H_{61}: four new edges $(a,x),(b,x),(a,y),(c,y)$, where a and b are the end-points of e and c is a vertex of K_{n-2} with maximum degree;
H_{62}: four new edges $(a,x),(c,x),(c,y),(d,y)$, where a is one of the end-points of e and c and d are vertices of K_{n-2} with maximum degree;
H_{63}: four new edges $(a,x),(a,y),(b,x),(b,y)$, where a and b are the end-points of e;
H_{64}: two new edges $(a,x),(b,y)$, where a and b are the end-points of e;
H_{65}: H_{51} by removal of (x,y);
H_{66}: H_{52} by removal of (x,y);
H_{67}: H_{58} by removal of (x,y);
H_{68}: H_{59} by removal of (x,y);
H_{69}: H_{60} by removal of (x,y);
H_{70}: H_{61} by removal of (x,y);
H_{71}: H_{62} by removal of (x,y);
H_{72}: H_{63} by removal of (x,y);

Graphs of order n obtained from K_{n-3} by connecting a path $P_3 = (x,y,z)$ with:

H_{73}: three new edges $(a,x),(a,y),(b,z)$, where a and b are any two distinct vertices of K_{n-3};
H_{74}: H_{73} by adding a new edge (x,z);
H_{75}: three new edges $(a,x),(a,z),(b,y)$, where a and b are any two distinct vertices of K_{n-3};
H_{76}: three new edges $(a,x),(b,y),(c,z)$, where a,b and c are any three distinct vertices of K_{n-3};
H_{77}: H_{76} by adding a new edge (x,z);

Graphs of order n obtained from $K_{1,n-2}$ by adding a new vertex adjacent to:

H_{78}: $n-3$ vertices of $K_{1,n-2} + e$, including one vertex with maximum degree and exactly one of the end points of e;

Graphs of order n obtained from $K_{1,n-4}$:

H_{79}: the graph $(\overline{K_2} + K_{1,n-4}) - e$ where e is an edge connecting a vertex $\overline{K_2}$ and a pendant vertex of $K_{1,n-4}$, added by one new vertex adjacent to two vertices of $(\overline{K_2} + K_{1,n-4}) - e$, namely the center of $K_{1,n-4}$ and one of end points e of $\overline{K_2}$;
H_{80}: $\overline{K_2} + K_{1,n-4}$ added by one new vertex adjacent to the center of $K_{1,n-4}$ and one vertex of $\overline{K_2}$;
H_{81}: $(\overline{K_2} + K_{1,n-4}) - e$ where e is an edge connecting a vertex $\overline{K_2}$ and a center of $K_{1,n-4}$, and added by one new vertex adjacent to two end points of e;

Graphs of order n obtained from K_n:

H_{82}: $K_n - E(P_5 + e)$, where e is an edge connecting two vertices P_5 of degree 2;

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