



## Research article

# A practical proposal to obtain solutions of certain variational problems avoiding Euler formalism

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## ABSTRACT

The aim of this article is to show the way to get both, exact and analytical approximate solutions for certain variational problems with moving boundaries but without resorting to Euler formalism at all, for which we propose two methods: the Moving Boundary Conditions Without Employing Transversality Conditions (MWTC) and the Moving Boundary Condition Employing Transversality Conditions (METC). It is worthwhile to mention that the first of them avoids the concept of transversality condition, which is basic for this kind of problems, from the point of view of the known Euler formalism. While it is true that the second method will utilize the above mentioned conditions, it will do through a systematic elementary procedure, easy to apply and recall; in addition, it will be seen that the Generalized Bernoulli Method (GBM) will turn out to be a fundamental tool in order to achieve these objectives.

## 1. Introduction

In this brief introduction the required aspects of the variational calculus for this work are presented, we will begin by exposing the case of problems with fixed boundaries and later on we will expose what concerns to variational problems with moving boundaries [1, 2, 3, 4].

## 1.1. Introduction to variational calculus

Calculus of variations is a field of mathematical analysis which is concerned with finding maxima or minima of functionals [2, 5]. Functional is a term which refers to a mapping from a vector space into a field such as the real numbers. When the vector space is a space of functions then functionals are frequently expressed in terms of definite integrals, denominated functional integrals. In this case, a functional is characterized for making to correspond a function with a number, while a common function like  $y = f(x)$ , makes a number correspond with another one. Calculus of variations is not only relevant from the-

oretical point of view, but many laws of physics are expressed in terms of a variational principle. In this case, a certain functional has to reach its maximum or minimum value in the physical considered process. To these variational principles belong the law of conservation of energy (see our case study 2, as example), the law of conservation of momentum, the Fermat principle in optics, the Castiglianos principle in the theory of elasticity. In modern physics, Einstein used the calculus of variations in his works on general relativity and in quantum mechanics Schrödinger used it to discover his famous wave equation, among many other examples. Despite to the fact that some variational problems were solved through especial methods (like for instance, Bernoulli's solution to the Brachistochrone problem, which is of particular interest for this work [6, 7] (see (4))), it was Euler who presented the variational calculus as a coherent branch of the analysis by discovering the basic differential equation for an extremization curve. It is clear that the amplitude and relevance of the variational calculus justify the research on the subject; especially the one that contributes to facilitate both the variational problems formulation as well as the solution methods of these.

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This work proposes two ways to approach certain variational problems with moving boundaries without resorting to Euler equations; nevertheless there exist other methods to solve variational problems without Euler's formulation, for instance [8], employed He's brackets and Ritz method; and the case of variable two-end problems is considered in [9]. On the other hand, the question of the inverse problem of the variational principle which aims to replace the Lagrange multiplier method, is considered in [10, 11]. Other similar variational problems of interest are presented in [12, 13, 14, 15, 16].

Many of the results of this work are based on the Generalized Bernoulli method (GBM) [17] that proposed a technique which generalize the Johann Bernoulli's solution of the brachistochrone problem. GBM was employed to find the different equation for other, variational problems by using just elementary calculus methods. As a matter of fact, [17] showed that GBM is equivalent to Euler's equation for the case for the case where one of the variables does not appear explicitly in the functional and given that it is applied without knowing the Euler's theory is a good method for practical purposes.

The rest of the paper is organized as follows. In Subsection 1.2, a brief review of the basic idea of variational problems with fixed boundary conditions is provided. Subsection 1.3 provides the most relevant results of variational problems with variable endpoint conditions. Additionally, Section 2 presents the basic idea of Generalized Bernoulli Method (GBM). Besides, the original contributions of this work are presented in Section 3. While Section 4, exposes the methodology employed in this article. Section 5 applies the ideas and methodologies explained in the above sections, by solving in detail five cases study. Section 6 provides a detailed discussion of the issues addressed in this work. Finally, the conclusions of the relevant issues studied in this article, as well as the proposal of future works are given in Section 7.

## 1.2. Fixed boundary conditions

The problem of fixed boundary conditions consists in extremizing integrals of the form.

$$S[y] = \int_{x_1}^{x_2} f(x, y, y') dx, \quad (1)$$

where the coordinates of the endpoints remain fixed.

The goal is finding a function  $y(x)$ , which satisfies the boundary conditions; and maximizes or minimizes (1) (although this article mainly deals with moving boundary conditions problems, the case study 2 deals partially with functionals as (1) (see (51)).

A simple example of application would be regarding the problem of finding the curve of shortest length joining two given points. For this case (1) adopts the form

$$S[y] = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx, \quad (2)$$

that is

$$f(x, y, y') = \sqrt{1 + y'^2}. \quad (3)$$

Although the intuitive answer for this problem is a straight line between the proposed points, there exist many cases by which the answer is not so obvious. For example, the Brachistochrone problem which is considered as the antecedent to the calculus of variations. This problem involves finding the vertical curve, without friction, that joins two fixed points through which a particle slides in the shortest time [1, 2, 5, 17, 18].

It is possible to show that the integral which has to be minimized for this case is

$$S[y] = \int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}} dx, \quad (4)$$

where  $g$  is the acceleration of gravity.

Thus, the function  $f$  for this problem adopts the form

$$f(x, y, y') = \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}. \quad (5)$$

The systematic way to find a function which extremizes a functional like (1) is through the Euler equation of the calculus of variations [1, 2, 5].

$$\frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) - \left( \frac{\partial f}{\partial y} \right) = 0. \quad (6)$$

However, there exist some relevant particular cases of this equation.

If the variable  $y$  does not appear explicitly in function  $f$ , (6) adopts the form [1, 2, 5]

$$\frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) = 0, \quad (7)$$

or

$$\frac{\partial f}{\partial y'} = c, \quad (8)$$

where  $c$  is a constant of integration.

If  $f$  does not depend explicitly on  $x$ , then (6) is expressed as

$$y' \frac{\partial f}{\partial y'} - f = k, \quad (9)$$

for some integration constant  $k$ .

A relevant fact is that these alternative forms of Euler equation are expressed in terms of conserved quantities.

## 1.3. Variable endpoint conditions

In the previous subsection, we mentioned problems in which both boundary conditions are specified; nevertheless, there are variational problems where is required to determine one or more boundary conditions as part of the sought solution. Next, we summarize the basic elements about the conditions of transversality which will provide the way to determine the unknown boundary points in the Euler formalism.

To understand the idea of these problems, suppose that it is proposed a variational problem where one of the boundaries remains fixed and the other is at some point of a known curve. Then, the solution of the Euler equation that joins different end points with the fixed one will yield in different extreme values for the functional that defines the problem. To a large extent, the problem would be determining the right terminal point.

Assuming that it is required to extremize (1) with the condition that endpoints are constrained on two curves, then the most general transversality conditions that can be employed in order to determine the unknown end points  $(a, y_a)$  and  $(b, y_b)$  are given by [2, 5]

$$\left[ \left( f - y' \frac{\partial f}{\partial y'} \right) \delta_v x + \left( \frac{\partial f}{\partial y'} \right) \delta_v y \right]_{x=a} = 0, \quad (10)$$

$$\left[ \left( f - y' \frac{\partial f}{\partial y'} \right) \delta_v x + \left( \frac{\partial f}{\partial y'} \right) \delta_v y \right]_{x=b} = 0, \quad (11)$$

where  $\delta_v x$  and  $\delta_v y$  denote the variation in  $x$  and  $y$  along the boundary curve [2, 5].

Of course, in the case where one of the boundaries is fixed and the other unspecified, then we will only use any of the previous conditions. For sake of simplicity, in most of our case studies, we will assume that for example the left boundary condition is specified and the right boundary condition remains undetermined; therefore, some particular cases of (11) will be considered below (for the case where the left boundary is unspecified and the right is fixed, the corresponding transversality condition would be obtained identically).

If the boundary point  $(b, y_b)$  can be shifted on the horizontal straight  $y = y_b$ , then  $\delta_v y = 0$  and the transversality condition (11) would adopt the simpler form

$$\left[ \left( f - y' \frac{\partial f}{\partial y'} \right) \right]_{x=b} = 0, \quad (12)$$

whereas if moving boundary is on a vertical line  $x = x_b$ , then  $\delta_b x = 0$  and transversality condition (11) would become in

$$\left[ \left( \frac{\partial f}{\partial y'} \right) \right]_{x=b} = 0. \quad (13)$$

## 2. Basic idea of GBM method

Next, we present the basic ideas behind the GBM method, [17] in order to use it directly to write the Euler equation using a systematic procedure based on elementary calculus without resorting to the known Euler formalism. Later on, we will explain the novelties of this work in relation to GBM and other aspects.

[17] proposed a generalization of the Bernoulli's solution of the brachistochrone problem (GBM method) and resulted that such methodology allows finding directly the Euler-Lagrange equation without resorting to know Euler's formalism of calculus of variations for the case where one of the variables do not appear explicitly in the functional (cyclic variable). For example, the method would be useful for the case of problems such as (2) and (4). In the first case, both variables  $x$  and  $y$  are absent, while (5) does not depend explicitly on  $x$ . Therefore, the differential equations provided by GBM method correspond to the cases indicated above with (8) and (9).

Although [17] explains in detail how GBM works, next we provide a brief summary of the steps that should be followed.

Assuming for example integrals of the form

$$\int_{x_1}^{x_2} f(y, y') dx. \quad (14)$$

GBM procedure is expressed as follows.

Step 1. Express the integrand of (14) in terms of increments (substituting  $x$  differential as a differential).

Step 2. Differentiate the expression resulting from the previous step with respect to  $\delta x$ , rewriting in terms of  $\delta y/\delta x$  ratio and equating to a constant.

Step 3. Consider the limit  $\delta x \rightarrow 0$  to the expression obtained in the previous step in order to obtain the Euler equation of the problem. As a matter of fact, these steps will be shown frequently in the case studies proposed in this paper in order to show in detail the GBM methodology.

At this time it is convenient to mention that [17] showed the performance of GBM through several case studies. Another novelty of this work is that it presents a general demonstration which, as will be seen below, does not require solving particular cases.

## 3. Original contributions of this work

In order to achieve the objectives of this work it was necessary to propose some novelties, which will be explained below.

### 3.1. Formal justification of why GBM works to find the Euler equation in some relevant cases

Next we will deduce (9) by using GBM. With that purpose we start directly of (14).

In accordance with GBM, considering the integrand from (14) in terms of increments

$$f \left( y, \frac{\delta y}{\delta x} \right) \delta x, \quad (15)$$

where we have introduced the notation for increments adopted in [17].

Since the last expression is not an explicit function of  $x$ , we differentiate (15) with respect to  $\delta x$  and finally we will equate the result to a constant, thus

$$\frac{df \left( y, \frac{\delta y}{\delta x} \right) \delta x}{d(\delta x)} = f \left( y, \frac{\delta y}{\delta x} \right) + \delta x \left( -\frac{\delta y}{(\delta x)^2} \right) f_{y'} \left( y, \frac{\delta y}{\delta x} \right), \quad (16)$$

we note that in the previous step we use the chain rule.

Simplifying (16) we get

$$\frac{df \left( y, \frac{\delta y}{\delta x} \right) \delta x}{d(\delta x)} = f \left( y, \frac{\delta y}{\delta x} \right) - \left( \frac{\delta y}{\delta x} \right) f_{y'} \left( y, \frac{\delta y}{\delta x} \right). \quad (17)$$

Following GBM method, we take the limit as  $\delta x \rightarrow 0$  in such a way that the right hand side of (17) adopts the form.

$$f \left( y, \frac{\delta y}{\delta x} \right) - \left( \frac{\delta y}{\delta x} \right) f_{y'} \left( y, \frac{\delta y}{\delta x} \right) \rightarrow f(y, y') - y' f_{y'}(y, y'). \quad (18)$$

Finally, equating to a constant  $c'$ , the expression after the arrow adopts the form

$$y' \frac{\partial f}{\partial y'} - f = c, \quad (19)$$

where  $c = -c'$  and we adopted a simplified functional notation.

Of course (19) is the Euler equation (9) for the case where  $f$  does not depend explicitly on  $x$ .

Next, we will find (8) by using GBM.

We start directly from

$$\int_{x_1}^{x_2} f(x, y') dx. \quad (20)$$

Following GBM algorithm, the integrand of (20) in terms of increments is given by

$$f \left( x, \frac{\delta y}{\delta x} \right) \delta x. \quad (21)$$

Since the last expression is not an explicit function of  $y$ , we will differentiate (21) with respect to  $\delta y$  and finally we will equate the result to a constant, in such a way that

$$\frac{df \left( x, \frac{\delta y}{\delta x} \right) \delta x}{d(\delta y)} = \delta x \left( \frac{1}{(\delta x)} \right) f_{y'} \left( x, \frac{\delta y}{\delta x} \right), \quad (22)$$

or

$$\frac{df \left( x, \frac{\delta y}{\delta x} \right) \delta x}{d(\delta y)} = f_{y'} \left( x, \frac{\delta y}{\delta x} \right). \quad (23)$$

Following GBM method, we take the limit as  $\delta x \rightarrow 0$  so that the right hand side of (23) adopts the form.

$$f_{y'} \left( x, \frac{\delta y}{\delta x} \right) \rightarrow f_{y'}(x, y'). \quad (24)$$

Therefore, in accordance with GBM, we equate to a constant  $c'$ , the expression after the arrow in such a way that

$$\frac{\partial f}{\partial y'} = c, \quad (25)$$

where  $c = -c'$ .

(25) is the Euler equation (8) for the case where  $f$  does not depend explicitly on  $y$ .

### 3.2. Proposal for the solution of variational problems with moving boundary conditions without employing transversality conditions

**MWTC method.** As far as we know, the only way to solve problems with variable end points is by using the Euler formalism to determinate both, the Euler differential equation and the transversality conditions. This paper proposes two methodologies as an alternative way of solving this type of problems but without resorting to Euler's known formalism, as already explained. We will see that the Moving Boundary Conditions Without Employing Transversality Conditions (MWTC) method

proposes to replace transversality conditions for a simple but effective idea; substituting the solution of the obtained Euler equation (either exact or approximate) in the functional (see Section 4). An advantage of this procedure is that it is direct and systematic. Besides, when GBM is used also to find the Euler equation, the procedure offers the possibility of finding a solution to the problem without resorting to Euler's formalism at all (for instance, the first method does not even employ the concept of transversality anywhere in the process). It turns out that this method is not always applicable because the integral resulting from the aforementioned substitution process is not always soluble. However, we will see that when the method works, it is possible to resort to elementary differential calculus methods to obtain the solution of the proposed problem. In a sequence, also is possible to obtain, by using elementary methods (unlike the methodologies that employ the transversality conditions), the maximum or minimum character of the obtained solution.

### 3.3. Relation of GBM method with transversality conditions

**METC method.** Unlike MWTC, we will see that the Moving Boundary Condition Employing Transversality Conditions (METC) just like Euler formalism also determines the conditions of transversality, but in a different way. It turns out that GBM besides being useful to find Euler's equation of the problem also it will result appropriate to determine the transversality conditions. It is noteworthy to mention that the procedure followed is essentially the same employed for GBM in order to find the Euler equation from the functional integral. As a matter of fact, this practical extension of GBM method is another of the novel contributions of this work. In order to understand this new GBM application we compare the conditions of transversality expressed from the point of view of the Euler formalism (12) and (13) with the corresponding results (19) and (25) obtained for GBM in order to obtain Euler equations. From the expressions between brackets and the left hand side of equations (19) and (25) we note that it is possible to extend the application of GBM in order to find the conditions of transversality. However, it is necessary to mention at this point a formal difference regarding this remark that does not affect the GBM applications in practice. As it was mentioned already, the GBM application to find the Euler equation requires that one of the variables  $x$  or  $y$  do not explicitly appear in the functional (1) while to transversality conditions (10)–(13) such requirements are not imposed. The above means that it is possible to apply GBM to find such transversality conditions even when the integrand of (1) depends on all the variables  $f(x, y, y')$ . From a formal point of view, it means we could repeat the GBM procedure presented in (14)–(25) even with  $f = f(x, y, y')$ . From the above, it is important to remark, that although the application of GBM to find the Euler equation is restricted to the cases already mentioned (14) and (20), its application to find the conditions of transversality is completely general. Thus, it is possible to write the conditions of transversality (10) and (11) in terms of GBM algorithm as follows

$$\left[ \frac{df\left(x, y, \frac{\delta y}{\delta x}\right)\delta x}{d(\delta x)}\delta_v x + \frac{df\left(x, y, \frac{\delta y}{\delta x}\right)\delta x}{d(\delta y)}\delta_v y \right]_{x=a} = 0, \quad (26)$$

$$\left[ \frac{df\left(x, y, \frac{\delta y}{\delta x}\right)\delta x}{d(\delta x)}\delta_v x + \frac{df\left(x, y, \frac{\delta y}{\delta x}\right)\delta x}{d(\delta y)}\delta_v y \right]_{x=b} = 0, \quad (27)$$

where  $\delta_v x$  and  $\delta_v y$  denote again the variation in  $x$  and  $y$  along the boundary curve.

Following variational theory, for the case where  $\delta_v x$  and  $\delta_v y$  are dependent, then, assuming that for instance, the right boundary point  $(b, y_b)$  can move through the curve  $y_b = f(b)$  then,  $\delta_v y_b = f'(b)\delta_v x_b$ .

Substituting this last result into (27) and considering that  $\delta_v x_b$  varies arbitrarily, the transversality condition is easily obtained (see case studies 3, 4, 5).

It is important to remark that in practice the mentioned expressions are easy to recall; The result from deriving with respect to  $\delta x$  is multiplied by  $\delta_v x$  and the result obtained from deriving with respect to  $\delta y$  is multiplied by  $\delta_v y$ .

For the case where one of the boundaries is fixed and the other unspecified, we would use only one of the previous conditions. Such as it was mentioned before, we will assume that for example, the left boundary condition is specified and the right boundary condition remains undetermined, therefore we will require consider particular cases of (11). For the case where the left boundary is unspecified and the right is fixed, the corresponding transversality conditions would be obtained identically.

If the boundary point  $(b, y_b)$  can be shifted on the horizontal straight line  $y = y_b$ , then  $\delta_v y = 0$  and the transversality condition (27) adopts the form

$$\left[ \frac{df\left(x, y, \frac{\delta y}{\delta x}\right)\delta x}{d(\delta x)} \right]_{x=b} = 0, \quad (28)$$

whereas if the moving boundary is on a vertical line  $x = x_b$ , then  $\delta_v x = 0$  and transversality condition (27) becomes in

$$\left[ \frac{df\left(x, y, \frac{\delta y}{\delta x}\right)\delta x}{d(\delta y)} \right]_{x=b} = 0. \quad (29)$$

Note the symmetry of the previous equations.

In practice, it is easy to remember (28) and (29) cases. On one hand, if boundary point is on a horizontal straight line, then in order to obtain the transversality condition, differentiate with respect to  $\delta x$ . On the other hand, if boundary point is on a vertical straight line, then to obtain the transversality condition differentiate with respect to  $\delta y$ .

## 4. Methodology

As mentioned earlier, this article proposes two methods or procedures in order to expose and solve some variational problems with moving boundaries, without resorting to Euler formalism, by using a direct methodology based on elementary operations.

Both methods are distinguished in the way of determining the unknown end point, in fact, despite the practical advantages of the first method which does not consider the notion of transversality condition at all; its implementation is more direct but its application is not always possible (see subsection 3.2), and for these cases method 2 could be used, which employs the known transversality conditions, but starting from GBM to find them easily. Once that Euler equation and transversality condition are known by using GBM, the rest of the problem is performed in the same way as known Euler procedure (see Section 1.1).

**MWTC method.** This article takes advantage of the practical nature of GBM method previously explained, in order to obtain the Euler-Lagrange equation in an elementary and systematic way. Subsequently, instead of calculating transversality conditions, the proposal of this work consists in substituting the solution of the obtained equation (either exact or approximate) into the functional integral. From the point of view of this proposal, it can be considered that the former process is done in two stages. The first extremization occurs when Euler-Lagrange equation is obtained and solved. It results that, given two points, its solution passing through them corresponds to the curve that extremizes functional  $S$  for these points. Now suppose, to exemplify, that the right end point can move along a known curve in such a way that the coordinates of that boundary are not determined until this moment. Then, it is required to re-extremize  $S$ , that is, to determine which is the curve that extremizes  $S$  of all the curves that start from the same point and end on the aforementioned curve. Since the solution to the Euler equa-



tion will depend at least on one of the coordinates of the unknown end; then, by replacing it in  $S$  and after integrating, it will be possible to express functional  $S$  in terms of some unknown parameters of the problem to be determined. In order to express it in terms of a single parameter (for the case of one variable end point), we will resort to the fact that the extremal and boundary curves must intersect at the point required to find.

In this way, it is possible to resort to elementary differential calculus methods to identify the value of the aforementioned parameters, among them, the value of the unknown coordinate. Although we proposed for sake of simplicity to solve problems mostly with only one variable end point, the final problem will exemplify the methodology for a problem of two moving end points (in this case is possible to express  $S$  in terms of just two parameters). A remarkable fact is that in cases where this method can be applied it is possible to obtain, with relative ease (unlike the methodology that uses the transversality conditions), the maximum or minimum character of the obtained solution.

With the end to ease the understanding of MWTC method, we resume the above in the following steps:

1. – Employ GBM in order to obtain Euler equation.
2. – Substitute the solution of Euler equation (exact or approximate) into the functional integral  $S$  that defines the variational problem.
3. – Perform the integration indicated in the previous step to express  $S$  as a common function of some unknown parameters.
4. – Employ the condition that the extremal and boundary curves are intersected at the points required to be found by the problem, in order to get mathematical expressions among the mentioned parameters.
5. – Use the equations deduced in the above point in order to express  $S$  in terms of one (or two) parameters.
6. – Obtain the critical points of  $S$ , to identify the values of the parameters.
7. – Resort to elementary differential calculus to determinate if the aforementioned parameters maximize or minimize the value of  $S$ .
8. – Finally we employ this information to identify the extremal(s) of the problem.

**METC method.** This method starts again by obtaining the Euler-Lagrange equation through the GBM method. Unlike method 1, and as it is done by using Euler's known formalism, method 2 also determines the conditions of transversality, but it turns out that, as mentioned above, GBM is appropriate not only to find Euler's equation of the problem but also to determine, employing essentially the same procedure explained in [17], the transversality conditions. As a matter of fact, this is a practical extension of the utility of GBM, and as it was already mentioned is one of the novel contributions of this work. When comparing the conditions of transversality expressed from the point of view of the Euler formalism (10), (11) with the corresponding (26) and (27) of this work, it is clear that in the proposed form, they are more symmetrical and simpler to apply and recall, it is sufficient differentiating with respect to the corresponding increment, depending on the case study and after the increments quotient form is recovered, such as it was already explained. Once the differential equation and the conditions of transversality of the problem are established, what remains is to proceed in the usual way (see Section 1.1) as it is done using Euler's formalism in order to obtain the solution [2, 5].

Next, we resume METC method as follows:

1. – Employ GBM in order to obtain Euler equation.
2. – We employ again GBM method, in order to determinate the transversality conditions of the problem.
3. – The equation of Euler, the conditions of transversality and other possible conditions that could arise if one of the boundaries is fixed, determine a problem of boundary conditions to solve.
4. – The solution of this problem provides an exact or approximate solution to the propose problem (see case studies 3–5).

## 5. Applications

Next, we will apply the ideas and methodologies explained in the above sections with as much detail as possible, according to the idea of this work, but without resorting at all to the known formalism of Euler (Subsections 1.1, 1.2, 1.3).

**Example 1.** Finding the extremum for functional [5].

$$S[y] = \int_0^{x_0} (1 + y'^2) dx, \quad (30)$$

subject to boundary condition

$$y(0) = 0, \quad (31)$$

and where the right end point can move along the curve

$$y(x) = 1/x. \quad (32)$$

We will employ the first methodology explained in Section 4.

**WTC method.** In accordance with GBM [17]:

Step 1. We express the integrand in terms of increments.

$$\left(1 + \left(\frac{\delta y}{\delta x}\right)^2\right) \delta x \rightarrow \delta x + \frac{(\delta y)^2}{\delta x}. \quad (33)$$

Step 2. Since the integrand does not depend explicitly neither on  $x$  nor on  $y$ , we could differentiate (33) irrespectively respect to  $\delta x$  or  $\delta y$ .

Differentiating respect to  $\delta y$ , rewriting in terms of  $\delta y/\delta x$ , and equating to a constant  $c$ , we obtain

$$\frac{\delta y}{\delta x} = c. \quad (34)$$

Step 3. Considering  $\delta x \rightarrow 0$ , (34) adopts the form

$$y'(x) = c, \quad (35)$$

this is the differential equation for extrema.

After separating variables and integrating, the solution of elementary differential equation (35) is expressed as

$$y = cx + b, \quad (36)$$

where  $c$ , is an integration constant; thus the extrema are straight lines.

Taking up the boundary condition (31), then  $b = 0$ , and (36) adopts the form

$$y = cx, \quad (37)$$

therefore, the derivative of (37) results

$$y' = c. \quad (38)$$

In accordance with the method 1, we substitute (38) into (30) to get

$$S(c, x_0) = \int_0^{x_0} (1 + c^2) dx, \quad (39)$$

where we have already considered that functional  $S$  is from here on a function of parameter  $c$  and the abscissa of the right end point  $x_0$ .

After evaluating (39), it is obtained

$$S(c, x_0) = (1 + c^2) \int_0^{x_0} dx = (1 + c^2) x_0. \quad (40)$$

Since the aim is to extremize the value of (40), then we require to express  $S$  in terms of just one parameter. For that purpose we note that the extremal and boundary curves must intersect at the point  $x_0$ .

Thus, from (32) and (37) we get

$$cx_0 = \frac{1}{x_0}, \quad (41)$$

or

$$c = \frac{1}{x_0^2}. \quad (42)$$

Substituting (42) into (40) it is obtained

$$S(x_0) = \left(1 + \frac{1}{x_0^4}\right)x_0. \quad (43)$$

The value of  $x_0$  which extremizes the value of (43) is obtained, differentiating (43) and solving the equation  $S' = 0$ , to obtain

$$S'(x_0) = 1 - \frac{3}{x_0^4} = 0, \quad (44)$$

or

$$x_0 = 3^{1/4}. \quad (45)$$

Besides, from the substitution of (45) into second derivative of (43)  $S''(x_0) = 12/x_0^5$ , we conclude that

$$S''(3^{1/4}) > 0. \quad (46)$$

For the elementary criterion of second derivative for functions of one independent variable, we conclude the critical value (45) corresponds to a relative minimum of (43).

Substitution of (45) into (42) yields in

$$c = \frac{1}{\sqrt{3}}. \quad (47)$$

Thus, from (37) and (47), the extremal function sought is

$$y(x) = \frac{x}{\sqrt{3}}. \quad (48)$$

Finally, the extremum value of  $S$  is obtained from the substitution of (45) into (43)

$$S = \frac{4}{3^{3/4}}. \quad (49)$$

What is more, since (43) has only one critical point, then (49) is also an absolute minimum [19].

[5] solved this example by using Euler equation (8) and transversality condition (11) obtaining (48), but without determining if this maximizes or minimizes (30).

**Example 2.** This case study analyzes from the point of view of calculus of variations with variable endpoint conditions, a typical problem of mechanics. A cannon fires its projectiles with speed  $V_0$  directly on the slope of a hill with angle of inclination  $\beta$ , as shown in Fig. 1. Neglecting air resistance, what angle  $\theta$  should the cannon form with respect to the horizontal so that its shots have the greatest distance traveled  $R$  on the slope of the hill?

In this case study, we will use the first method already explained in Section 4.

With the end of emphasize the usefulness of GBM, in place of writing directly the equation of motion, which is simple because in accordance with the hypothesis, the only force acting on the particle is the force of gravity, we will resort to the Lagrangian formulation for the conservative mechanical system, consisting of a particle of mass  $m$  and speed  $V = \sqrt{(\dot{x})^2 + (\dot{y})^2}$  where we denote  $\dot{x} = dx/dt$ ,  $\dot{y} = dy/dt$ , and subject to the action of the force of gravity.

The functional for this case is the action integral, which is expressed in terms of the Lagrangian function

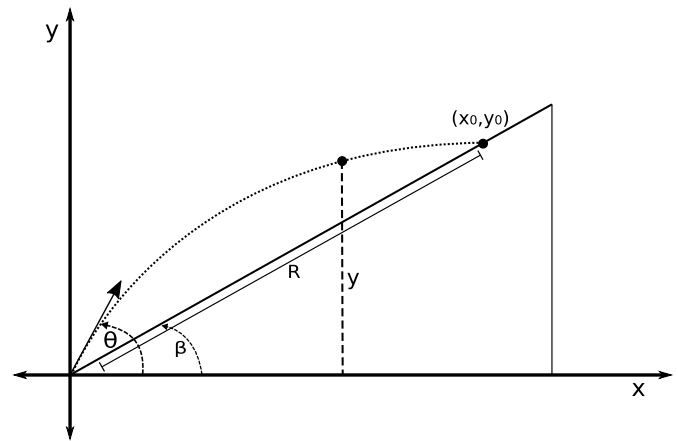


Fig. 1. A cannon fires its projectiles with speed  $V_0$ , with angle  $\theta$ , directly on the slope of a hill with angle of inclination  $\beta$ .

$$L = m/2 (\dot{x}^2 + \dot{y}^2) - V(y), \quad (50)$$

as follows [18].

$$S[y] = \int_0^{t_0} \left( \frac{m}{2} (\dot{x}^2 + \dot{y}^2) - V(y) \right) dt, \quad (51)$$

where  $V(y)$  is identified as the potential energy, which for this problem is given by

$$V(y) = mgy \quad (52)$$

and  $T = m/2 (\dot{x}^2 + \dot{y}^2)$  is the kinetic energy (see Fig. 1).

Since Lagrangian function is not an explicit function of  $t$ , then GBM provides a known first integral of motion as follows [17, 18].

Step 1. We express the integrand of (51) (that is, the Lagrangian) in terms of increments (including the differential of  $t$ )

$$\frac{m}{2} \left( \left( \frac{\delta x}{\delta t} \right)^2 + \left( \frac{\delta y}{\delta t} \right)^2 \right) \delta t - mgy \delta t \rightarrow \frac{m}{2} \left( \left( \frac{\delta x}{\delta t} \right)^2 + \left( \frac{\delta y}{\delta t} \right)^2 \right) - mgy \delta t \quad (53)$$

Step 2. Since the integrand does not depend explicitly on  $t$ , then we differentiate (53) respect to  $\delta t$ , to obtain, after some elementary algebraic arrangements, and equating to a constant  $c$ .

$$\frac{m}{2} \left( \left( \frac{\delta x}{\delta t} \right)^2 + \left( \frac{\delta y}{\delta t} \right)^2 \right) + mgy = -c = E. \quad (54)$$

Step 3. Considering the limit  $\delta t \rightarrow 0$ , (54) adopts the form

$$\frac{m}{2} \left[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \right] + mgy = E, \quad (55)$$

or

$$\frac{mV^2}{2} + mgy = E. \quad (56)$$

In the last expression  $E$  denotes the mechanical energy of the system, which for this case is constant.

On the other hand, since the integrand of (51) does not depend explicitly on  $x$ , we obtain in the same way, after differentiating (53) respect to  $\delta x$ , and equating to a constant  $P$ .

$$m \left( \frac{\delta x}{\delta t} \right) = P. \quad (57)$$

Considering the limit  $\delta t \rightarrow 0$ , (57) adopts the form

$$m \frac{dx}{dt} = P. \quad (58)$$

(58) expresses the conservation of  $x$  component of momentum, and for the same reason, that  $dx/dt$  is a constant.

In order to obtain the equation of motion, we differentiate (55) with respect to time, to get

$$m \left( \frac{dy}{dt} \right) \left( \frac{d^2y}{dt^2} \right) + mg \left( \frac{dy}{dt} \right) = 0, \quad (59)$$

or

$$\left( \frac{d^2y}{dt^2} \right) = -g, \quad (60)$$

after we take into account that  $dx/dt$  is a constant (see (58)).

The integration of (58) yields immediately to

$$x(t) = V_0 \cos \theta t, \quad (61)$$

where

$$V_0 \cos \theta = P/m. \quad (62)$$

In the same way, integrating twice (60)

$$y(t) = V_0 \sin \theta t - \frac{1}{2} g t^2. \quad (63)$$

By eliminating the time between (61) and (63) we get immediately

$$y = x \tan \theta - \frac{g x^2}{2 V_0^2 \cos^2 \theta}. \quad (64)$$

It is the well-known equation of the parabola, for this problem [20, 21, 22].

Since, we are seeking the greatest traveled distance on the slope of the hill, then our strategy will be to find the maximum value of distance on the slope.

As it is well known, the problem of calculating the distance between two points is equivalent to find the shortest curve joining the mentioned points. For it is necessary to minimize the functional integral.

$$R[y] = \int_0^{x_0} \sqrt{1 + y'^2(x)} dx. \quad (65)$$

Next we will employ GBM to get the differential equation that has to be satisfied for a function  $y(x)$  in order to extremize (65).

Step 1. We express the integrand of (65) in terms of increments.

$$\sqrt{\left( 1 + \left( \frac{\delta y}{\delta x} \right)^2 \right)} \delta x \rightarrow \sqrt{\delta x^2 + \delta y^2}. \quad (66)$$

Step 2. Again, as in the case of the previous example, the integrand does not depend explicitly neither on  $x$  nor on  $y$ , thus we could differentiate (66) irrespectively respect to  $\delta x$  or  $\delta y$ .

Differentiating (66) respect to  $\delta y$ , rewriting in terms of  $\delta y/\delta x$ , and equating to a constant  $k$ , we obtain

$$\frac{\delta y/\delta x}{\sqrt{\left( 1 + \left( \frac{\delta y}{\delta x} \right)^2 \right)}} = k \quad (67)$$

Step 3. Considering  $\delta x \rightarrow 0$ , (67) adopts the form

$$\frac{y'(x)}{\sqrt{1 + y'(x)^2}} = k, \quad (68)$$

this is the differential equation for extrema.

It is clear that (68) can be expressed as

$$y' = a, \quad (69)$$

where  $a = \tan \beta$  (see Fig. 1).

In order to solve (69), we separate variables and integrate; to obtain

$$y = ax + d, \quad (70)$$

where  $d$ , is an integration constant; thus we get the expected result; the extrema are straight lines.

From the Fig. 1, it is clear that  $d = 0$ , and (70) adopts the form:

$$y = ax, \quad (71)$$

(the straight lines from the origin to  $(x_0, y_0)$  on the hill).

Besides, the derivative of (71) results in

$$y' = a. \quad (72)$$

Next, MWTC method will be applied.

Following the algorithm MWTC, we substitute (72) into (65) to get

$$R(x_0) = \int_0^{x_0} \sqrt{1 + a^2} dx, \quad (73)$$

where we have explicitly recognized that the functional  $R$  is from here on a function of parameter  $a$  and the abscissa of the right end point  $x_0$ .

After evaluating (73) it is obtained

$$R(x_0) = \sqrt{1 + a^2} x_0. \quad (74)$$

Since the parabola and the slope of the hill must intersect at the above mentioned point  $x_0$ , from (64) and (71) we obtain

$$\alpha x_0 - \beta x_0^2 = ax_0, \quad (75)$$

where

$$\alpha = \tan \theta, \quad \beta = \frac{g}{2 V_0^2 \cos^2 \theta}. \quad (76)$$

Solving for  $x_0$  we get

$$x_0 = \frac{\alpha - a}{\beta}, \quad (77)$$

thus, from (74) and (77), it is obtained

$$R(\theta) = \sqrt{1 + a^2} \left( \frac{\alpha - a}{\beta} \right) = \sqrt{1 + a^2} \left( \frac{2 V_0^2 \cos^2 \theta \tan \theta - 2 a V_0^2 \cos^2 \theta}{g} \right). \quad (78)$$

The value of  $\theta$  which extremizes the value of (78) is obtained differentiating (78) and solving the equation  $R' = 0$ , to obtain.

$$1 - 2 \sin^2 \theta + 2 a \sin \theta \cos \theta = 0. \quad (79)$$

In order to get an equation which contains only  $\sin \theta$ , we employ the known identity  $\cos^2 \theta = 1 - \sin^2 \theta$ .

Thus, (79) acquires the form

$$(4 + 4a^2) \sin^4 \theta - (4 + 4a^2) \sin^2 \theta + 1 = 0, \quad (80)$$

utilizing the substitution  $u = \sin^2 \theta$ , the solution of (80) can be expressed as

$$\sin \theta = \sqrt{\frac{\sqrt{4 + 4a^2} + 2a}{2\sqrt{4 + 4a^2}}}. \quad (81)$$

Applying again the criterion of second derivative of the elementary calculus with the purpose of determining the character of the obtained critical value would be rather cumbersome for this case; nevertheless, it is clear that (81) corresponds to a maximum. We note from (78) that the minimum would have corresponded to  $\theta = \pi/2$  and  $\theta = \beta$  (since  $a = \tan \beta$ ). Furthermore (81) adopts known value for the largest traveled distance corresponding to a horizontal terrain in the limit  $a \rightarrow 0$ ;  $\theta \rightarrow \pi/4$ . Thus, the result obtained is consistent for this limit case [21]. As the interval of interest  $\theta \in [\beta, \pi/2]$ , then, strictly this problem is one of the absolute extrema [20]. Nevertheless, since (78) is a non-negative function and  $R(\beta) = R(\pi/2) = 0$ , then the critical value (81) corresponds

to extrema of (78), which is indeed, a maximum. Therefore, the use of the criterion of second derivative resulted equivalent to consider the open interval  $(\beta, \pi/2)$  since the mentioned criterion is applied for relative extrema, and as discussed above, this is indeed correct for this case.

**Example 3.** This case study presents an example of economics, the Ramsey growth model. Next, it is provided a basic explanation of this model [23]. We assume the production of a homogeneous product  $Y$  requires the investments of an amount of capital  $K$ . Disregarding the dependence of  $Y$  on other values, then the production is only a function of the capital, that is to say.

$$Y = \phi(K). \quad (82)$$

Of the production  $Y$  a part  $D$  is consumed, and the remainder  $I = Y - D$ , is invested in such a way that it gives place to a change in capital, therefore.

$$I = dK/dt. \quad (83)$$

From the above we deduced that

$$Y = \frac{dK}{dt} + D = \phi(K). \quad (84)$$

Next, we propose to determine the distribution of consumption and investment as a function of time  $t$ , in order that total product  $v$  results maximum.

The total product is defined by the integral of the instantaneous product  $N(t)$ , which in turn depends of the consumption

$$N(t) = N(D(t)), \quad (85)$$

that is to say

$$v = \int_{t_0}^{t_1} N(D(t))dt. \quad (86)$$

From (84) and (86) we obtain the variational problem

$$v = \int_{t_0}^{t_1} N \left( \phi(K(t)) - \frac{dK(t)}{dt} \right) dt \rightarrow \max. \quad (87)$$

With the purpose to exemplify, we propose the following simplified possibility

$$\phi(K) = bK, \quad (88)$$

i.e. the production is assumed proportional to capital, and

$$N(D) = -a(D - C^*)^2, \quad (89)$$

where  $C^*$ ,  $a$  and  $b$ , are constants ( $a > 0$ ,  $b > 0$ ).

Thus, after substituting (88) into (87), and considering (89), we get

$$-v(K) = \int_{t_0}^{t_1} a \left( bK(t) - \frac{dK(t)}{dt} - C^* \right)^2 dt \rightarrow \min. \quad (90)$$

Next, we will express (90) in the form of the following variational problem

$$J[K] = \int_0^T a \left( bK(t) - \frac{dK(t)}{dt} - C^* \right)^2 dt, \quad (91)$$

where we select  $t_0 = 0$ .

The capital stock  $K(0)$  at the initial time of the planning period is assumed known and is given by  $K(0) = K_0$ . On the other hand, the planner will not want to determine how big the capital will be at the time  $t = T$ ,

since he is only concerned with maximizing the total benefit. Thus, this is an example of a variational problem with variable right end point.

Next, we pick out the following values employed in [24]. That article provided an approximate solution for this variational problem by employing Adomian Decomposition Method.  $a = b = C^* = 1$ ,  $T = 1$ ,  $y_0 = 2$ . Thus, (91) adopts the form

$$J[K] = \int_0^1 (K(t) - K'(t) - 1)^2 dt. \quad (92)$$

This problem has the following exact solution [24]

$$K(t) = 1 + e^t. \quad (93)$$

This is relevant because it will allow us to know the accuracy of the approximate solutions that we will obtain later (see Table 1).

Next, we will find the corresponding Euler equation of the problem, which is necessary for the two proposed methods. This case study will get a precise approximate analytical solution for this problem. For that purpose, GBM is employed to integrand of (92) (including as usual the differential of  $t$ ).

**Step 1.** First, we perform the indicated algebraic operation and after it is expressed the integrand in terms of increments.

$$\begin{aligned} & \left[ K^2 + \left( \frac{\delta K}{\delta t} \right)^2 + 1 - 2K \left( \frac{\delta K}{\delta t} \right) - 2K + 2 \left( \frac{\delta K}{\delta t} \right) \right] \delta t \\ & \rightarrow K^2 \delta t + \left( \frac{\delta K}{\delta t} \right)^2 \delta t + \delta t - 2K \delta K - 2K \delta t + 2 \delta K. \end{aligned} \quad (94)$$

**Step 2.** Since the integrand of (92) does not depend explicitly on  $t$ , then in accordance with GBM, we differentiate (94) respect to  $\delta t$ . Thus, differentiating the expression to the right of the arrow, rewriting in terms of  $\delta K/\delta t$ , and equating the resulting expression to a constant  $c$  we obtain

$$K^2 - \left( \frac{\delta K}{\delta t} \right)^2 + 1 - 2K = c. \quad (95)$$

**Step 3.** Considering  $\delta t \rightarrow 0$ , (95) adopts the form

$$K^2 - (K')^2 + 1 - 2K = c. \quad (96)$$

This is the differential equation for extrema.

With the purpose to work with a second order differential equation, we differentiate the above equation respect to  $t$  to get.

$$K'' - K + 1 = 0, \quad (97)$$

subject to  $K(0) = 2$  (for the case of METC method, it is still required to add the conditions of transversality, see below).

We note that unlike (96), differential equation (97) is linear.

**MWTC method.** Although (97) admits the exact solution (93), the goal of this case study is to provide an accurate analytical approximate solution by using a methodology based only in basic integrals with a minimum effort. For this purpose we will use Boundary Value Problems Picard Method (BVPP) [25], which is a modification of classic Picard method employed to obtain approximate solutions for both, linear and nonlinear ordinary differential equations, defined with boundary conditions. Other important aspect of this example is that it shows a possible way to aboard problems which really don't possess an exact solution.

In accordance with BVPP [25, 26], we express (97) in terms of the following integral equation.

$$K = 2 + \alpha t + \iint (K - 1) dx dx', \quad (98)$$

where  $\alpha$  denotes the value of  $K'(0)$ , unknown until now.

The corresponding iterative equation, derived from (98) is expressed as [25].



$$K_n = 2 + \alpha t + \iint (K_{n-1} - 1) dt dt', \quad (99)$$

for  $n = 1, 2, 3, \dots$ .

It is important at this time to clarify the following. Despite one of the advantages of BVPP (unlike the classic Picard method, which is employed to solve problems with initial conditions) is that it allows to select as trial function polynomials provided with parameters to be determined for the same method in order to accelerate the convergence of boundary value problems [25]; for sake of simplicity we choose as trial function the initial condition of the problem  $y_0 = 2$ .

Thus for  $n = 1$ , (99) adopts the form

$$K_1 = 2 + \alpha t + \iint dt dt',$$

after integrating twice, we get

$$K_1 = 2 + \alpha t + \frac{t^2}{2}. \quad (100)$$

In the same way for  $n = 2$ , we obtain from (99) and (100).

$$K_2 = 2 + \alpha t + \iint \left( \alpha t + \frac{t^2}{2} + 1 \right) dt dt',$$

after performing the indicated elementary integrations, we get

$$K_2(t) = 2 + \alpha t + \frac{\alpha t^3}{6} + \frac{t^4}{24} + \frac{t^2}{2}. \quad (101)$$

Assuming that second iteration is sufficient (note the ease of the process that led to (101)) then in accordance with the first method, we have to substitute (101) into (92) with the purpose to determine the value of  $\alpha$  that extremizes (92). In order to ease this procedure we note by simple differentiation that the integrand of (92) can be rewritten as follows.

$$(K(t) - K'(t) - 1)^2 = K'^2(t) + 2K^2(t) - 2K(t) + \frac{d(2K(t) + t - K^2(t))}{dt}, \quad (102)$$

so, after substituting (102) into (92), we obtain.

$$J[K] = \int_0^1 (K'^2(t) + K^2(t) - 2K(t)) dt + [2K(t) + t - K^2(t)]_0^1, \quad (103)$$

or

$$J[K] = \int_0^1 (K'^2(t) + K^2(t) - 2K(t)) dt + [2K(1) + 1 - K^2(1)], \quad (104)$$

where we employed the boundary conditions and from (101)

$$K(1) = K_2(1) = \frac{61}{24} + \frac{7\alpha}{6}. \quad (105)$$

The substitution of (101) and (105) into (104) involves a set of elementary operations that lead to the result

$$J(\alpha) = 1.78730159\alpha^2 + 5.071180556\alpha + 6.89344687 - \left( \frac{61}{24} + \frac{7\alpha}{6} \right)^2, \quad (106)$$

where we have already considered that functional integral  $J$  is from here on a function of parameter  $\alpha$ .

The value of  $\alpha$  which extremizes the value of (106) is obtained as usual, differentiating (106) respect to  $\alpha$  and solving the equation  $J' = 0$ , to obtain

$$J'(\alpha) = 0.85238096\alpha - 0.85937498 = 0, \quad (107)$$

or

$$\alpha = 1.00820527. \quad (108)$$

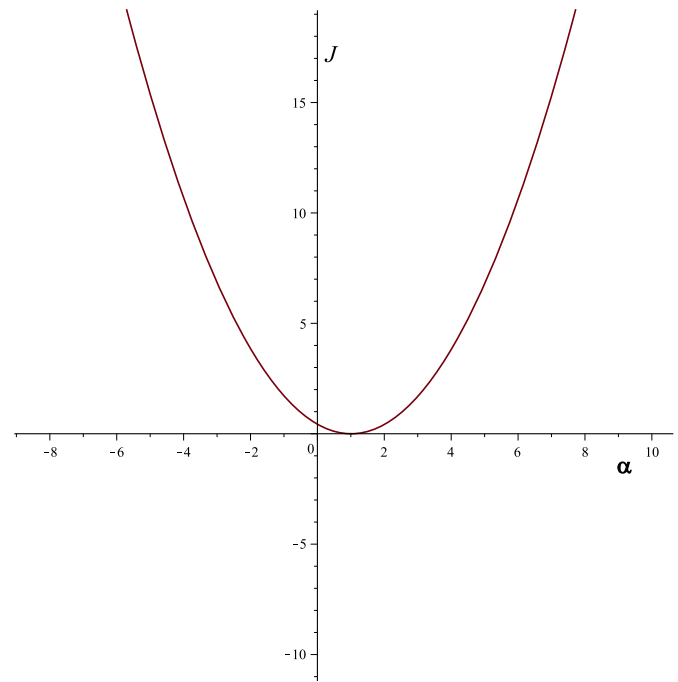


Fig. 2. Graphic for function (106).

Besides, by substituting (108) into second derivative of (106), we conclude that

$$J''(1.00820527) = 0.85238096 > 0. \quad (109)$$

For the elementary criterion of second derivative for functions with one independent variable, we conclude the critical value (108) corresponds indeed to a relative minimum of (106).

In a sequence, since (106) has only one critical point, then (108) corresponds also to an absolute minimum as it should be (see (90), (91) and Fig. 2).

Thus, the sought solution is obtained, substituting (108) into (101) to get

$$K_2(t) = 2 + 1.00820527t + 0.16803421t^3 + \frac{t^4}{24} + \frac{t^2}{2}. \quad (110)$$

**METC method.** Such as it was mentioned, it is relevant to remark that GBM also provides the transversality conditions following essentially the same procedure employed to find the Euler-Lagrange equations in the cases already mentioned [17] (see Subsection 3.3 and Section 4).

Since that the end point can move along the straight line  $t = 1$ , the transversality condition is obtained employing GBM, differentiating (94) respect to  $\delta K$  (see (29)).

$$\frac{d}{d(\delta K)} \left[ K^2 \delta t + \frac{(\delta K)^2}{\delta t} + \delta t - 2K \delta K - 2K \delta t + 2\delta K \right] = \frac{2\delta K}{\delta t} - 2K + 2. \quad (111)$$

In accordance with GBM methodology, next we take the limit  $\delta t \rightarrow 0$ , in such a way that (111) adopts the form

$$\left[ \frac{2dK}{dt} - 2K + 2 \right]_{t=1} = 0 \quad (112)$$

(see (29)) or

$$K(1) - K'(1) - 1 = 0 \quad (113)$$

(note that (113) is a Robin-like boundary condition).

Thus, the problem to solve is one with boundary conditions, which is expressed as a whole, as follows

**Table 1**

Comparison of the proposed solutions (110) and (116) with exact solution (93). (A. Error means Absolute Error.)

t	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	1
1 + exp(t)	2.105170	2.221402	2.349858	2.491824	2.648721	2.82211	3.013752	3.225540	3.718281
(110)	2.105992	2.223051	2.352336	2.495102	2.652711	2.826618	3.018383	3.229664	3.717906
(116)	2.117691	2.246566	2.3879	2.543066	2.713541	2.9009	3.106816	3.333066	3.854166
A. Error (110)	$8.21 \times 10^{-4}$	$1.64 \times 10^{-3}$	$2.47 \times 10^{-3}$	$3.27 \times 10^{-3}$	$3.98 \times 10^{-3}$	$4.5 \times 10^{-3}$	$4.63 \times 10^{-3}$	$4.12 \times 10^{-3}$	$3.75 \times 10^{-4}$
A. Error (116)	$1.25 \times 10^{-2}$	$2.51 \times 10^{-2}$	$3.8 \times 10^{-2}$	$5.12 \times 10^{-2}$	$6.48 \times 10^{-2}$	$7.87 \times 10^{-2}$	$9.3 \times 10^{-2}$	$1.07 \times 10^{-1}$	$1.35 \times 10^{-1}$

$$K'' - K + 1 = 0, \quad (114)$$

$$K(0) = 2, K(1) - K'(1) - 1 = 0.$$

Of course, we will use the solution already obtained (101) in order to recalculate  $\alpha$ , by solving (114).

We note that (101) already satisfies the condition  $K(0) = 2$ , so what remains is to replace (101) into (113) to obtain an elementary algebraic equation to  $\alpha$ ; the value obtained is

$$\alpha = 1.125. \quad (115)$$

Thus, substituting (115) into (101) we obtain the following approximate solution

$$K(t) = 2 + 1.125t + 0.1875t^3 + \frac{t^4}{24} + \frac{t^2}{2}. \quad (116)$$

The table shows the comparison between the proposed solutions (110) and (116). Again, we emphasize the ease to obtain (113) and (116).

From Table 1 is appreciated that (110) is more accurate than (116), although the procedure that led to (110) be more extensive. In this regard, we note that the second method calculated the value of the unknown  $\alpha$  optimizing its value to ensure the proposed solution satisfies the transversality condition. Since both methods adjusted an approximate solution (see (101)) it seems reasonable that the first method, which optimizes  $S$ , through the complete domain of the problem has been more accurate. If more accurate approximate solutions are required, more BVPP iterations should be carried out than those performed to obtain (101). Although this article suggested BVPP, there are other methods that could be used to find approximate solutions to variational problems, such as the Adomian Decomposition Method [24], HPM [27], LTHPM [28, 29, 30], among others. Finally, unlike METC, the proposed MWTC method, once again employed elementary criterion of elementary calculus in order to show that the only critical value (108) corresponded indeed to a minimum of (106) (indeed an absolute minimum, see Fig. 2). We noted that the proposed solutions (110) and (116) are polynomial functions of fourth degree which only contain five terms, ideal for applications.

**Example 4.** Determine the extremum for the functional.

$$S[y] = \int_0^{x_0} (y'^2 + xy') dx, \quad (117)$$

subject to boundary condition  $y(0) = 0$ , and the right end point is constrained to lie on the line  $y(x) = -x$ .

This example is relevant because its solution determines the existence of two extremal functions and it will be solved by using the two methods presented in this work.

First we will employ GBM in order to obtain the differential equation that has to be satisfied for a function  $y(x)$  in order to extremize (117).

Step 1. We express the integrand of (117) in terms of increments.

$$\left( \left( \frac{\delta y}{\delta x} \right)^2 + x \left( \frac{\delta y}{\delta x} \right) \right) \delta x \rightarrow \frac{(\delta y)^2}{\delta x} + x \delta y. \quad (118)$$

Step 2. We note the integrand does not depend explicitly on  $y$ , thus we differentiate (118) respect to  $\delta y$ .

After differentiating (118) respect to  $\delta y$ , rewriting in terms of  $\delta y/\delta x$ , and equating to a constant  $k$ , we obtain

$$2\delta y/\delta x + x = k. \quad (119)$$

Step 3. Considering  $\delta x \rightarrow 0$ , (119) adopts the form

$$2y'(x) + x = k. \quad (120)$$

This is the differential equation for extrema.

In order to solve (120), we separate variables and integrate; to obtain immediately

$$y = \frac{kx}{2} - \frac{x^2}{4}, \quad (121)$$

where, we have already considered the boundary condition  $y(0) = 0$ .

Next, we complete the solution of the proposed problem by using two methodologies.

**MWTC method.** In accordance with MWTC, we substitute (121) into (117) to get

$$S(k, x_0) = \int_0^{x_0} \left( \left( \frac{k}{2} - \frac{x}{2} \right)^2 + \frac{k}{2}x - \frac{x^2}{2} \right) dx, \quad (122)$$

where as usual, we have already considered that functional  $S$  is from here on a function of parameters  $x_0$  and  $k$ .

After evaluating (122) it is obtained

$$S(k, x_0) = \frac{k^2 x_0}{4} - \frac{x_0^3}{12}. \quad (123)$$

Since the aim is to extremize the value of  $S$ ; then, we require expressing it in terms of just one parameter. For that purpose we note that the extremal and boundary curve must intersect at the point  $x_0$ .

Thus, from (121) and  $y(x) = -x$  we get

$$-x_0 = \frac{kx_0}{2} - \frac{x_0^2}{4}, \quad (124)$$

or

$$k = \frac{x_0}{2} - 2. \quad (125)$$

Substituting (125) into (123) it is obtained

$$S(x_0) = \frac{1}{4} \left( \frac{x_0}{2} - 2 \right)^2 x_0 - \frac{x_0^3}{12}. \quad (126)$$

The value of  $x_0$  which extremizes the value of (126) is obtained differentiating (126) respect to  $x_0$  and solving the equation  $S' = 0$ , to get

$$S'(x_0) = -\frac{1}{16}x_0^2 - x_0 + 1 = 0, \quad (127)$$

or

$$x_0 = \frac{-16 \pm 17.8885438}{2} \quad (128)$$

Thus, we have two values of  $x_0$ .

$$x_0^+ = 0.9442719, \quad (129)$$

where  $x_0^+$  denotes the upper solution of (128).

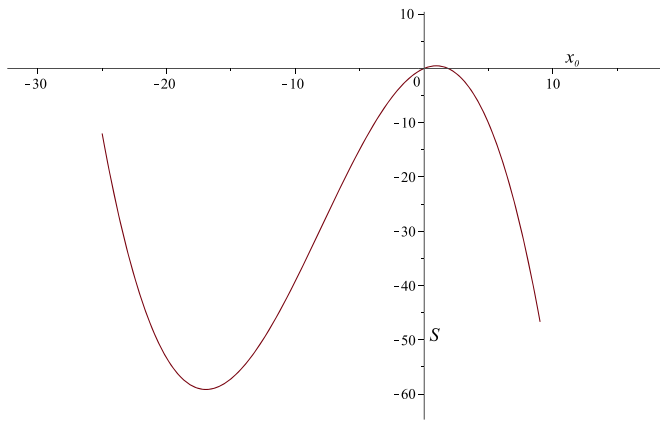


Fig. 3. Graphic for function (126).

From (125), to this corresponds the value of

$$k = -1.527864. \quad (130)$$

In the same way, the lower solution of (128) is

$$x_0^- = -16.9442719, \quad (131)$$

and from (125) we obtain

$$k = -10.472136. \quad (132)$$

On the other hand it is immediate that

$$S''(x_0) = -\frac{1}{8}x_0 - 1. \quad (133)$$

We note that

$$S''(x_0^+) < 0, \quad (134)$$

and

$$S''(x_0^-) > 0. \quad (135)$$

Thus, we conclude from (130) and (132) the existence of two extremal functions (see (121)):

$$y = -0.763932x - \frac{x^2}{4}, \quad (136)$$

which corresponds to  $x_0(+) = 0.9442719$ , while

$$y = -5.236068x - \frac{x^2}{4}, \quad (137)$$

corresponds to  $x_0(-) = -16.9442719$ .

For the elementary criterion of second derivative, we conclude that in accordance with (134), (136) maximizes (117), and from (135) we conclude that (137) minimizes (117) (see Fig. 3 and Section 6).

**METC method.** In accordance with this method, we start from (121). As the right end point is constrained to lie on the line  $y(x) = -x$ , then the transversality condition is obtained employing GBM, substituting (118) into (27) and performing the elementary derivatives respect to increments indicated, to obtain.

$$\frac{(\delta y)^2}{\delta x} + x \delta y \rightarrow \left( -\left( \frac{\delta y}{\delta x} \right)^2 \delta_v x + \left( 2 \frac{\delta y}{\delta x} + x \right) \delta_v y \right)_{x=x_0}. \quad (138)$$

As usual, we take the limit  $\delta x \rightarrow 0$ , in such a way that (138) adopts the form

$$[-y'^2(x) \delta_v x + (2y'(x) + x) \delta_v y]_{x=x_0} = 0. \quad (139)$$

Since, the terminal curve is  $y = -x$ , then  $\delta_v y = -\delta_v x$ , and the above equation is rewritten as

$$[-y'^2(x) - 2y'(x) - x]_{x=x_0} \delta_v x = 0. \quad (140)$$

Since  $\delta_v x$  varies arbitrarily, then

$$[-y'^2(x) - 2y'(x) - x]_{x=x_0} = 0, \quad (141)$$

thus, the transversality condition sought is

$$y'^2(x_0) + 2y'(x_0) + x_0 = 0. \quad (142)$$

From the condition that the extremal and terminal curves are intersected at  $x = x_0$ , then we get equation (125) again.

Substituting (121) into (142) result in

$$\frac{1}{4}(k - x_0)^2 + k = 0, \quad (143)$$

and from (125), we obtain

$$x_0 = 2k + 4. \quad (144)$$

Thus, from (143) and (144) yield immediately in

$$k^2 + 12k + 16 = 0. \quad (145)$$

The solutions of the above equation are

$$k_1 = -1.527864, \quad (146)$$

$$k_2 = -10.472136. \quad (147)$$

The substitution of (146) and (147) into (144), result in the values.

$$x_{01} = 0.9442719, \quad (148)$$

and

$$x_{02} = -16.9442719, \quad (149)$$

respectively.

We note that (146)–(149) are the same results already obtained by using the first methodology ((129), (131) and (130), (132)). As a consequence, we obtain again the extremal functions (136) and (137). We emphasize that GBM allowed us to know the differential equation of the problem and its transversality condition essentially in the same way, which systematizes the procedure for this kind of problems. From the point of view of the known procedure of Euler, it is clear that this is less systematic because it requires memorizing formulas, besides it results sometimes cumbersome. We noted that, again, unlike MWTC, our second procedure only provided the character extreme of its solutions. Going further, from the formal point of view of the variational calculus in order to determinate if a solution corresponds to a maximum or a minimum, it would entail deepening the theory of the subject, and the present work has not any proposals in this regard.

**Example 5.** Determine the distance between the curves  $y = x^2$  and  $y = x - 5$  [31].

The relevance of this final example is that in this case the two boundaries are unspecified.

As occurred with Example 2 in this problem we require minimize the functional.

$$d[y] = \int_{x_1}^{x_2} \sqrt{1 + y'^2(x)} dx. \quad (150)$$

This example will be solved by using both methods presented in this work.

The Euler equation for (150) was already obtained by using GBM in the example (2), taking into account that the functions (65) and (150)

coincide (see (66)–(70)), the expected answer is of course a straight line.

$$y = A + Bx, \quad (151)$$

for some integration constants  $A$  and  $B$ .

**MWTC method.** In accordance with this method, we substitute (151) into (150) to obtain

$$d(x_1, x_2, B) = \int_{x_1}^{x_2} \sqrt{1 + B^2} dx, \quad (152)$$

where we have explicitly recognized, as in another previous examples that at this stage functional integral  $d$  is from here on a function of parameter  $B$  and the abscissa of the end points  $x_1$  and  $x_2$ .

After evaluating (152) it is obtained

$$d(x_1, x_2, B) = \sqrt{1 + B^2} (x_2 - x_1) \quad (153)$$

We note that the extremal and boundary curves must intersect at the points corresponding to  $x_1$  and  $x_2$ . Thus, from (151) and boundary curves we obtain

$$A + Bx_1 = x_1^2, \quad (154)$$

$$A + Bx_2 = x_2 - 5. \quad (155)$$

Since the previous equations (153)–(155) involve four independent variables, then the procedure will consist of expressing  $d$  as a function of two of such these variables.

For this purpose it is directly subtracted (154) from (155) to obtain

$$x_2 = \frac{Bx_1 - x_1^2 - 5}{B - 1}. \quad (156)$$

Thus, after substituting (156) into (153), we get

$$d(x_1, B) = \sqrt{1 + B^2} \left[ \frac{Bx_1 - x_1^2 - 5}{B - 1} - x_1 \right]. \quad (157)$$

Following the usual procedure to determine the relative maxima and minima of a function of two variables, we partially differentiate (157) respect to  $x_1$  and  $B$  in order to obtain the critical values from the system of equations.

$$\frac{\partial d}{\partial x_1} = 0, \quad (158)$$

$$\frac{\partial d}{\partial B} = 0. \quad (159)$$

The substitution of (157) into (158) and (159) respectively, yield immediately to the following values.

$$B = -1, \quad (160)$$

$$x_1 = \frac{1}{2}. \quad (161)$$

In order to know the values of  $x_2$  and  $A$ , we substitute the values of (160) and (161) into (156), and (160) into (154) respectively, to get the values

$$x_2 = \frac{23}{8}, \quad (162)$$

$$A = \frac{3}{4}. \quad (163)$$

Thus, the straight line that extremizes (157) results of substituting (160) and (163) into (151)

$$y = \frac{3}{4} - x. \quad (164)$$

On the other hand, in the same way as it happened with the previous examples, it is possible to determine if the extremal function (164)

maximizes or minimizes (150). For this purpose, we will employ a basic result of differential calculus of two variables; the criterion of the second partial derivatives for relative extrema. Next, we will consider only the necessary results for this work (for more details see [19, 32]).

Let  $(a, b)$  be a critical point of  $z = f(x, y)$ , then the criterion depends to a large extent on the function

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2. \quad (165)$$

So that if  $D(A, B) > 0$ , and  $f_{xx}(a, b) > 0$ , then  $f(a, b)$  is a relative minimum, while if  $D(A, B) > 0$ , and  $f_{xx}(a, b) < 0$ , then  $f(a, b)$  is a relative maximum.

Next we will adapt (165) to our problem (157)

$$D(x_1, B) = d_{x_1 x_1}(x_1, B)d_{BB}(x_1, B) - [d_{x_1 B}(x_1, B)]^2. \quad (166)$$

The obtained partial results are the following

$$d_{BB}\left(\frac{1}{2}, -1\right) = \frac{19}{16\sqrt{2}} > 0, \quad (167)$$

$$d_{x_1 x_1}\left(\frac{1}{2}, -1\right) = \sqrt{2} > 0, \quad (168)$$

$$d_{x_1 B}\left(\frac{1}{2}, -1\right) = 0. \quad (169)$$

Therefore, substituting equations (167)–(169) into (166) it is obtained

$$D\left(\frac{1}{2}, -1\right) = \frac{19}{16} > 0. \quad (170)$$

Thus from (168) and (170) we concluded that  $d(1/2, -1)$  is a relative minimum. In a sequence, since (157) has only one critical point then  $d(1/2, -1)$  is also an absolute minimum as it should be. After implementing this criterion, (168) and (170) showed that (164) indeed minimizes (150). We emphasize again the simplicity of the method and the information it provides.

**METC method.** In accordance with METC, we start from (26) and (27) for this variational problem of two moving boundaries.

Such as it was already mentioned, it is easy to recall the following: the result of differentiating with respect to  $\delta x$  is multiplied by  $\delta_v x$ , and what results from deriving with respect to  $\delta y$  is multiplied by  $\delta_v y$ .

Explaining in detail the above procedure.

At first place we express the integrand of (150) in terms of increments.

$$\sqrt{\left(1 + \left(\frac{\delta y}{\delta x}\right)^2\right)} \delta x \rightarrow \sqrt{\delta x^2 + \delta y^2}. \quad (171)$$

Differentiating (171) respect to  $\delta y$ , and rewriting in terms of  $\delta y/\delta x$  we obtain

$$\frac{\delta y/\delta x}{\sqrt{\left(1 + \left(\frac{\delta y}{\delta x}\right)^2\right)}} \quad (172)$$

Considering  $\delta x \rightarrow 0$ , (172) adopts the form

$$\frac{y'(x)}{\sqrt{1 + y'(x)^2}}. \quad (173)$$

Next, we differentiate (171) respect to  $\delta x$  and rewriting the result in terms of  $\delta y/\delta x$ , we obtain

$$\frac{1}{\sqrt{\left(1 + \left(\frac{\delta y}{\delta x}\right)^2\right)}}. \quad (174)$$

In the limit  $\delta x \rightarrow 0$  (174) takes the form

$$\frac{1}{\sqrt{1 + y'(x)^2}}. \quad (175)$$

Thus, by substituting (173) and (175) into (26) and (27), we get

$$\left[ \frac{1}{\sqrt{1+y'^2(x)}} \delta_v x + \frac{y'(x)}{\sqrt{1+y'^2(x)}} \delta_v y \right]_{x=x_1} = 0, \quad (176)$$

$$\left[ \frac{1}{\sqrt{1+y'^2(x)}} \delta_v x + \frac{y'(x)}{\sqrt{1+y'^2(x)}} \delta_v y \right]_{x=x_2} = 0. \quad (177)$$

Note how GBM provides both, the Euler equation (see (66)–(68)) and transversality conditions in a simple and systematic way, essentially following the same mathematical steps for both.

Since the boundary curves are  $y = x^2$  and  $y = x - 5$  then

$$\delta_v y = 2x \delta_v x, \quad (178)$$

$$\delta_v y = \delta_v x \quad (179)$$

After substituting (178) and (179) into (176) and (177) respectively we obtain

$$\left[ \frac{1}{\sqrt{1+y'^2(x)}} + \frac{2xy'(x)}{\sqrt{1+y'^2(x)}} \right]_{x=x_1} \delta_v x = 0, \quad (180)$$

$$\left[ \frac{1}{\sqrt{1+y'^2(x)}} + \frac{y'(x)}{\sqrt{1+y'^2(x)}} \right]_{x=x_2} \delta_v x = 0. \quad (181)$$

Since  $\delta_v x$  varies arbitrarily, then

$$\left[ \frac{1}{\sqrt{1+y'^2(x)}} + \frac{2xy'(x)}{\sqrt{1+y'^2(x)}} \right]_{x=x_1} = 0, \quad (182)$$

$$\left[ \frac{1}{\sqrt{1+y'^2(x)}} + \frac{y'(x)}{\sqrt{1+y'^2(x)}} \right]_{x=x_2} = 0. \quad (183)$$

Taking into account that extrema function is (151), then the above conditions can be rewritten as

$$\frac{1}{\sqrt{1+B^2}} + \frac{2x_1 B}{\sqrt{1+B^2}} = 0, \quad (184)$$

$$\frac{1}{\sqrt{1+B^2}} + \frac{B}{\sqrt{1+B^2}} = 0. \quad (185)$$

In short, it is necessary to solve the non-linear system of four equations and four unknowns (154), (155), (184), and (185).

From (184) and (185) the values of  $x_1$  and  $B$  are deduced with ease; with that information, we get the following results from this system

$$A = 3/4, \quad B = -1, \quad x_1 = 1/2, \quad x_2 = 23/8. \quad (186)$$

Thus, the straight line that extremizes (157) results of substituting the first two results of (186) into (151) to get (164). Of course, the results obtained by the two methods coincide. It is worthwhile retaking here the matter about the ease with which GBM provided Euler equation and transversality condition without resorting at all to Euler formalism; as a matter of fact it is sufficient to recall the simple rules of Subsection 3.3, but essentially, the same mathematical steps are followed for both.

## 6. Discussion

The main objective of this work was to offer a methodology based only on elementary calculus and algebra, in order to analyze and solve some variational problems with moving boundaries, without resorting to the Euler-Lagrange formalism.

To achieve the above objective, we mainly proposed to use GBM method (see Section 2). [17] showed that by following a procedure which generalizes the one employed for Bernoulli to solve the Brachistochrone problem it is possible to provide the Euler-Lagrange equations for the case where one of the variables does not appear explicitly in the

integrand of the variational problem to solve. This article proposed five case studies where GBM was employed to determine the Euler equations without employing the known formalism; as a matter of fact, it was enough to perform elementary derivatives respect to the increment ( $\delta x$  or  $\delta y$ ) corresponding to cyclic variable and afterwards applying some straightforward algebra. Nevertheless one of the relevant contributions of this work was to extend the application of GBM for the case of problems with moving boundaries.

The comparison of the transversality conditions, derived from Euler's formalism (see (10)–(13)), with the equations ((19) and (25)), showed that for practical purposes, transversality conditions can be derived from the GBM formalism. This extension of GBM method allows to solve a variational problem of this nature completely without the help of Euler formalism for the case where the integrand of (1) has some cyclic variable. However, it is necessary to note that (10) and (11) can always be used to get the conditions of transversality, regardless of whether the corresponding Euler equation can be determined or not from the GBM formalism. In brief, GBM provides both the Euler-Lagrange equations and conditions of transversality for a wide class of variational problems using a direct, elementary, and above all, systematic methodology. In our case studies, we identify this procedure as method two. The first method starts again finding the corresponding Euler-Lagrange equation using GBM, but from that moment it differs. As mentioned, the idea of this procedure is to conceive that a problem of moving boundaries can be visualized as a succession of two optimization processes. The first one is the use of GBM in order to find the corresponding Euler equation. The solution of this provides an extremum curve that connects the starting point to a point, in principle arbitrary, on the boundary curve. On the other hand, since the previous solution depends on the coordinates of the final point, then the next optimization is to find at what point on the boundary curve should end the curve so that the value of the functional integral proposed reaches an extreme value among all the points of the boundary curve.

From the above, we proposed to substitute the solution that emanates from the Euler equation in the functional integral, in such a way that it results being a function of some parameters and of some of the coordinates of the extreme point. After expressing the mentioned functional in terms of a single variable (for the case of one unspecified boundary), it is possible to use the elementary calculus to determine the coordinates of the final point and thereby obtaining the extreme value of the functional integral. Note that this procedure replaces the conditions of transversality, by a single systematic procedure, which at the same time to determine the unknown point, and the whole solution of the problem. Given the spirit of GBM of expressing a variational problem as one of elementary calculus [17], then it seems natural to highlight the unity of the whole procedure above described. To clarify the aforementioned, consider that as well as GBM provides the Euler equations in a systematic manner based on elementary calculus, the procedure explained allows determining the final end point using elementary methods without needing to keep in mind the different conditions of transversality to be used depending on each problem, as it was seen in the proposed cases.

It is necessary to clarify that the utility of this method is conditioned to the solubility of the integral that results of substituting in the functional integral, the solution that emanates from the Euler equation. However, there are important cases of applications where this methodology is successfully applied; in our case studies we used it extensively (as MWTC method). The question of finding solutions to the Euler-Lagrange equation is of great importance, but the subject is too broad to provide infallible methods in its solution, except in cases where the equation admits an exact solution. Following the objective of this work, in order to show the proposed methodology as best as possible, we selected examples with Euler equations accessible to solve, since that ODES solution methods depend too much on the nature of the problem to be solved. We emphasized in the solution of the linear equation



(97), such as it was mentioned, although this admits the exact solution (93), the objective was to show a potentially useful procedure, even in cases of differential equations where an exact analytical solution cannot be provided. Therefore this work proposed Boundary Value Problems Picard Method (BVPP) [25] as a tool, indeed easy to use, based on an iterative process, which in general employs some adjustment parameters as well as elementary integrals to find precise analytical approximate solutions (the simplicity and good precision of BVPP is in accordance with the objectives of this work and was the reason to use this method instead of others, see [25]). This method is as easy to use as the classic Picard method; but it applies to the case of problems of boundary conditions, while the known Picard method is used for problems of initial conditions. It is also common that few iterations of this method are sufficient to get good approximations, just like it happened in this article (see Section 5) [25]. We note that, it is possible to obtain solutions through the use of BVPP, not only in the case of ODES without an exact solution, but also for the case of problems that, even having an exact solution, its obtention is difficult, cumbersome to evaluate or for the case of problems where the dependent variable remains implicit in an exact solution.

We employed more frequently the first procedure, because it is more direct and seeks express functional integral  $S$  as a simple function of one variable (or two for the case of problems with two unspecified boundary points) for which the methods of elementary calculus can be applied. As a matter of fact, MWTC does not employ the concept of transversality at all.

In order to show the practical usefulness of the proposed method, we presented two examples of application for the areas of physics and economics. As a matter of fact, the Ramsey growth model was studied by MWTC and METC methods and the obtained results were compared and commented in some detail. In general we employed MWTC in all the proposed examples, while METC in three problems, not only for the above mentioned reasons, but for the following; it is well known, from the view of the formalism of variational calculus, that it is a rather difficult question to determine the maximum or minimum character of a functional integral [2, 5]. On the contrary, MWTC is able to determine it, by using elementary procedures, such as the criterion of the second derivative from the elementary calculus, which is advantageous. Thus, for instance, [5], resolved the same problem that was proposed as Example 1 by using Euler's known formalism that involves the knowledge of the transversality condition (11). Naturally [5] obtained the same results of this work. For instance, it determined that (45) corresponds to the abscissa of the point on the boundary curve (32) for which the value of (30) is extremum but it did not determine that also minimizes it.

The second case study (which is about a typical problem of mechanics) is particularly relevant because it shows the versatility of GBM. First it was employed in the context of Lagrangian formulation in order to deduce the equation of motion (60) [18, 19, 20, 21]. It is convenient to mention here that GBM was able to provide the mathematical expressions for some physical constants of motion. The mechanical energy (56) and the  $x$  component of momentum (58) as well as the above mentioned equation of motion; thus, from obtaining these last relevant results together with all the previous information make the techniques presented in this article attractive for practical applications.

## 7. Conclusions

The objective of this work was to provide a practical methodology with the purpose of solving variational problems with moving boundaries using only elementary methods of calculus and algebra, without resorting to the known Euler-Lagrange formalism. With the end to emphasize the practical character of this work, the examples were solved step by step. One of the main tools of this work was the GBM method [17] because it allowed us to find the Euler-Lagrange equations by mere differentiation in the way already explained along this work; since this

procedure is systematic, it is not necessary to recall the Euler equations (Subsection 1.2). Once the differential equation is solved, two methods or procedures were proposed to complement the solution of the variational problem.

MWTC method, was the most employed throughout this work, because following the idea of GBM, it proposed a simple and systematic procedure, different from the usual methodology of the Euler method. Once a solution is known (exact or approximate), the method calculates the unknown parameters by substituting it directly in the functional integral  $S$  with the purpose to express  $S$  as a function of one or two of the parameters, so that is easy to extremize it, and even determine if we obtain a maximum or a minimum. Thus, the proposed method expresses a variational problem with moving boundaries in terms of an elementary calculus problem.

The disadvantage of this procedure is that it is not always possible to evaluate the integral that emanates once the solution is substituted in  $S$ . As a matter of fact, for these cases it is possible to resort to METC as an alternative method. However, this work presented several examples where MWTC worked satisfactorily.

On the other hand, METC also starts from the solution of the Euler equation of the problem. This method deduces the usual transversality conditions for these problems but without resorting to the Euler-Lagrange formalism. It was remarkable that after comparing the conditions of transversality expressed from the point of view of the Euler formalism (12) and (13) with the corresponding results (19) and (25) obtained for GBM in order to obtain Euler equations, we noted that GBM method also provides, from a practical point of view, the conditions of transversality in addition to Euler equation. Nevertheless, the application of GBM to find the conditions of transversality is general, since it is not required that one of the variables be cyclic in  $S$ . It is relevant to note that GBM finds the Euler equation and transversality conditions in the same systematic way by using essentially the same systematic elementary procedure (recalling that for the Euler equation case, one of the variables should not appear explicitly in the functional). Of course, once the Euler formalism or METC method was used, what remains is to solve the same differential equation with the boundary conditions proposed. It is usual that after employing conditions of transversality, only is determined the extreme character of the solution obtained because from the point of view of the variational calculus, to determine if an extremum corresponds to a maximum or minimum of a functional is not a straightforward task. In fact, sometimes it is possible to say something on this subject, appealing to arguments of plausibility. For example physical arguments, if the functional in question is related to some problem of physics and so on. This contrasts with the first proposed method, where it is possible to answer the previous question by using analytical elementary methods.

Regarding the question of the solution methods to the Euler equation; unfortunately this issue depends too much on the details of the equation to be solved. Sometimes it admits an exact solution, but it is common that this does not happen. For this case, and also when we require to find some approximate solution with little effort and good precision for those cases of the Euler equation, which, even having an exact solution, it turns out to be it too extensive, cumbersome, or it is presented implicitly; it was suggested the BVPP method, which in particular was used to find approximate analytical solutions to the problem (97). Since this method allows finding frequently good approximations by means of elementary integrals and few iterations; it resulted useful with the objectives of this work, where the goal was showing how to resolve with ease, certain variational problems without resorting to the formal procedures known. In fact, Example 3 showed that BVPP is potentially useful to provide handy accurate analytical approximate solutions. At this point, we suggested other techniques with potential to find solutions to variational problems, and that would be a good alternative for BVPP.

Finally, given the utility demonstrated by GBM, a future work should enlarge the use of the method for the general case of problems where

it is required finding the Euler Lagrange equation for functionals that depend explicitly on  $x$ ,  $y$  and  $y'$ , which is equivalent to generalizing the GBM method in order to find the Euler Lagrange equation for the more general case.

## Declarations

### Author contribution statement

U. Filobello-Nino, H. Vazquez-Leal: Conceived and designed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

J. Huerta-Chua: Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

R.A. Callejas Molina, M.A. Sandoval-Hernandez: Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data.

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The authors declare no conflict of interest.

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