



Common Coherence Witnesses and Common Coherent States

Bang-Hai Wang ^{1,*} , Zi-Heng Ding ^{2,3}, Zhihao Ma ^{2,3,*} and Shao-Ming Fei ^{4,5,*} 

- ¹ School of Computer Science and Technology, Guangdong University of Technology, Guangzhou 510006, China
- ² School of Mathematical Sciences, MOE-LSC, Shanghai Jiao Tong University, Shanghai 200240, China; dzh713618@126.com
- ³ Shenzhen Institute for Quantum Science and Engineering, Southern University of Science and Technology, Shenzhen 518055, China
- ⁴ School of Mathematical Sciences, Capital Normal University, Beijing 100048, China
- ⁵ Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany
- * Correspondence: bhwang@gdut.edu.cn (B.-H.W.); mazhihao@sjtu.edu.cn (Z.M.); feishm@cnu.edu.cn (S.-M.F.)

Abstract: We show the properties and characterization of coherence witnesses. We show methods for constructing coherence witnesses for an arbitrary coherent state. We investigate the problem of finding common coherence witnesses for certain class of states. We show that finitely many different witnesses W_1, W_2, \dots, W_n can detect some common coherent states if and only if $\sum_{i=1}^n t_i W_i$ is still a witnesses for any nonnegative numbers $t_i (i = 1, 2, \dots, n)$. We show coherent states play the role of high-level witnesses. Thus, the common state problem is changed into the question of when different high-level witnesses (coherent states) can detect the same coherence witnesses. Moreover, we show a coherent state and its robust state have no common coherence witness and give a general way to construct optimal coherence witnesses for any comparable states.

Keywords: common coherence witnesses; high-level witnesses; common coherent states

PACS: 03.65.Ud; 03.65.Ca; 03.67.Mn; 03.67.-a



Citation: Wang, B.-H.; Ding, Z.-H.; Ma, Z.; Fei, S.-M. Common Coherence Witnesses and Common Coherent States. *Entropy* **2021**, *23*, 1136. <https://doi.org/10.3390/e23091136>

Academic Editors: Leong Chuan Kwek

Received: 30 July 2021
Accepted: 30 August 2021
Published: 31 August 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Originating from the fundamental superposition principle in quantum mechanics, quantum coherence [1,2] plays a crucial role in quantum metrology [3,4], quantum algorithms [5], nanoscale thermodynamics [6–10] and energy transportation in the biological systems [11–14]. Detecting and quantifying quantum coherence, therefore, become fundamental problems in the emerging quantum areas. Numerous impressive schemes on measures of quantum coherence have been presented [15–26].

The coherence witness, inspired by entanglement witnesses, is arguably a powerful tool for coherence detection in experiments [26–32] and coherence quantification in theory [33,34]. It directly detects any coherent states and gives rise to measures of quantum coherence without state tomography. Compared with the entanglement witness, the coherence witness has many different characteristics deserving to be investigated extensively.

Two natural questions arise that when different coherence witnesses can detect some common coherent states and when different coherent states can be detected by some common coherence witnesses in finite-dimensional systems. Although these two similar questions related to entanglement witnesses have been well solved, separately [35–37], the problems of common coherence witnesses and common coherent states remain unsolved.

In this paper we systematically investigate and solve the problems of common coherence witnesses and common coherent states. This paper is organized as follows. In Section 2, we review the concept of coherence witnesses and the methods of constructing coherence witnesses. In Section 3 we show sufficient and necessary conditions for any given two or many coherence witnesses to be incomparable, and deal with problem of common

coherence witnesses. In Section 4, we characterize coherent states based on high-level witnesses and solve the problem when different coherent states can be detected by common coherence witnesses. Summary and discussions are given in Section 4.

2. Common Coherence Witnesses

With respect to a fixed basis $\{|i\rangle\}_{i=1,2,\dots,d}$ of the d -dimensional Hilbert Space \mathcal{H} , a state is called incoherent if it is diagonal in this basis. Denote \mathcal{I} the set of incoherent states. The density operator of an arbitrary incoherent state $\delta \in \mathcal{I}$ is of the form,

$$\delta = \sum_{i=1}^d \delta_i |i\rangle\langle i|. \tag{1}$$

Clearly, the set of incoherent states \mathcal{I} is convex and compact. Since the set of all incoherent states is convex and compact, there must exist a hyperplane which separates an arbitrary given coherent state from the set of all incoherent states by the Hahn-Banach theorem [38]. We call this hyperplane a coherence witness [26,27]. A coherence witness is an Hermitian operator, $W = W^\dagger$, such that (i) $\text{tr}(W\delta) \geq 0$ for all incoherent states $\delta \in \mathcal{I}$, and (ii) there exists a coherent state π such that $\text{tr}(W\pi) < 0$. More precisely, an Hermitian operator W on \mathcal{H} is a coherence witness if (i') its diagonal elements are all non-negative, and (ii') there is at least one negative eigenvalue. Following the definition of incoherent states and the Hahn-Banach theorem, we can restrict the condition (i) to $\text{tr}(W\delta) = 0$ and relax (ii) to $\text{tr}(W\pi) \neq 0$ [26,33,39]. As coherence witnesses are hermitian quantum mechanical observables, they can be experimentally implemented [28–32].

Since the density matrix of an entangled quantum state can not be diagonal, from the Definition (1) an entangled quantum state must be a coherent state. Therefore, the entanglement witnesses are also kinds of coherence witnesses with respect to a fixed basis. We denote \mathbf{S} the set of all separable states, \mathbf{E} the set of all entangled states, \mathbf{I} the set of all incoherent states and \mathbf{C} the set of all coherent states. Figure 1a illustrates the schematic picture of the relations between entanglement and coherence. Therefore, we can construct coherence witnesses in a similar way of constructing entanglement witnesses [40,41].

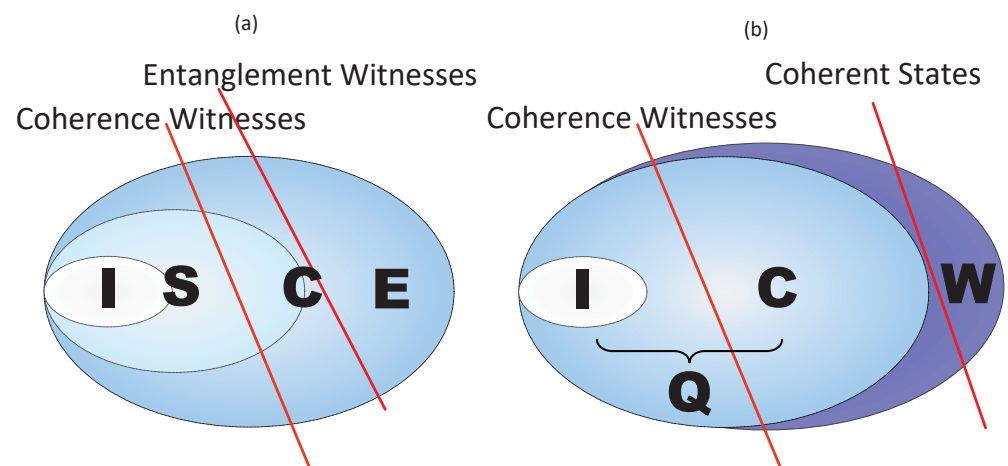


Figure 1. (Color online) (a) With respect to a fixed basis, all entanglement witnesses are also coherence witnesses. (b) We denote \mathbf{Q} the set of all quantum states and \mathbf{W} the set of all coherence witnesses. Coherent states play the role of high-level witnesses witnessing coherence witnesses.

For a given coherent state $|\psi\rangle\langle\psi|$, one has coherence witness,

$$W = \alpha I - |\psi\rangle\langle\psi|, \tag{2}$$

where I is the identity matrix and $\alpha = \max \text{Tr}(\delta|\psi\rangle\langle\psi|)$ with the maximal running over all incoherent state δ . Coherence witnesses can also be constructed from geometrical methods,

$$W = \frac{1}{N}(\delta - \rho + \text{Tr}(\delta(\rho - \delta))I), \tag{3}$$

where δ is the closest incoherent state to ρ , $N = \|\rho - \delta\|$ and $\|A\| \equiv \sqrt{\text{Tr}(A^\dagger A)}$. Recently, a general way of constructing a coherence witness for an arbitrary state has been provided [33,34]: $W_\rho = -\rho + \Delta(\rho)$ is an optimal coherence witness to detect the coherence of ρ , where $\Delta(\rho) = \sum_{i=0}^{d-1} \langle i|\rho|i\rangle |i\rangle\langle i|$ is the dephasing operation in the reference basis $\{|i\rangle\}_{i=0}^{d-1}$. More general constructions of coherence witnesses are also given in [33,34].

For a coherence witness W , we define $D_W = \{\rho \mid \text{tr}(\rho W) < 0\}$, namely, the set of all coherent states “witnessed” by W . Give two coherence witnesses W_1 and W_2 , we say that W_2 is finer than W_1 if $D_{W_1} \subseteq D_{W_2}$, that is, if all the coherent states “witnessed” by W_1 are also “witnessed” by W_2 . We call W optimal if there exists no other coherence witness which is finer than it. It is shown that a coherent witness is optimal if and only if its diagonal elements are all zero [33]. For normalization we set $\|W\|_\infty = 1$ as there exist traceless coherence witnesses.

Moreover, given two coherence witnesses W_1 and W_2 , we say that W_2 and W_1 are incomparable if $D_{W_1} \cap D_{W_2} = \emptyset$. Two coherence witnesses W_1 and W_2 can detect some common coherent states if $D_{W_1} \cap D_{W_2} \neq \emptyset$. To proceed, we need the following lemma.

Lemma 1. *If W_2 and W_1 are incomparable, i.e., $D_{W_1} \cap D_{W_2} = \emptyset$ and if $D_W \subset D_{W_1} \cup D_{W_2}$, then either $D_W \subset D_{W_1}$ or $D_W \subset D_{W_2}$.*

Proof. On the contrary, suppose that both $D_{W_1} \cap D_W$ and $D_{W_2} \cap D_W$ are nonempty. Take $\rho_i \in D_{W_i} \cap D_W, i = 1, 2$. Consider the segment $[\rho_1, \rho_2]$ consisting of $\rho_t = (1 - t)\rho_1 + t\rho_2$, where $0 \leq t \leq 1$. As D_W is convex, we obtain

$$[\rho_1, \rho_2] \subset D_W \subset D_{W_1} \cup D_{W_2}. \tag{4}$$

Thus

$$[\rho_1, \rho_2] = (D_{W_1} \cap [\rho_1, \rho_2]) \cup (D_{W_2} \cap [\rho_1, \rho_2]), \tag{5}$$

which means that $[\rho_1, \rho_2]$ can be divided into two convex parts. It follows that there is $0 < t_0 < 1$ such that $\{\rho_t : 0 \leq t < t_0\} \subset D_{W_1}, \{\rho_t : t_0 < t \leq 1\} \subset D_{W_2}$ and either $\rho_{t_0} \in D_{W_1}$ or $\rho_{t_0} \in D_{W_2}$.

Assume that $\rho_{t_0} \in D_{W_1}$; then $\text{tr}(W_1\rho_{t_0}) < 0$. Thus, for sufficiently small $\varepsilon > 0$ with $t_0 + \varepsilon \leq 1$, we have

$$\begin{aligned} 0 &\leq \text{tr}(\rho_{t_0+\varepsilon}W_1) \\ &= \text{tr}(\rho_{t_0}W_1) + \varepsilon[\text{tr}(\rho_2W_1) - \text{tr}(\rho_1W_1)] < 0 \end{aligned}$$

which leads to a contradiction. Similarly, $\rho_{t_0} \in D_{W_2}$ leads to a contradiction as well. This completes the proof. \square

Theorem 1. *W_2 and W_1 are incomparable (no common coherent states can be detected) if and only if there exist $a > 0$ and $b > 0$ such that $W_{a,b} = aW_1 + bW_2$ is positive.*

Proof. Obviously, if W_2 is finer than W_1 , then W_2 is finer than $W_{a,b}$ and $W_{a,b}$ is finer than W_1 for positive a and b . Hence, $D_{W_1} \cap D_{W_2} \subseteq D_W = \emptyset$ since $W_{a,b} = aW_1 + bW_2$ for some $a > 0$ and $b > 0$. Take $t = \frac{a}{b}$.

By Lemma 1, we have $D_{W_{a,b}} \subset D_{W_1}$ or $D_{W_{a,b}} \subset D_{W_2}$ for all $a > 0$ and $b > 0$. Then $D_{W_{a,b}} = D_{\frac{1}{b}W_{a,b}} = D_{tW_1+W_2} = D_{\frac{t}{1+t}W_1+\frac{1}{1+t}W_2}$. Hence, we obtain $D_{W_{a,b}} = D_{\lambda W_1+(1-\lambda)W_2} \doteq W_\lambda$ by taking $\lambda = \frac{t}{1+t}$, where $\lambda \in (0, 1)$. We now can consider W_λ as $W_{a,b}$. When t varies

from 0 to ∞ continuously, then λ varies from 0 to 1 continuously, which means that D_{W_λ} also varies from D_{W_2} to D_{W_1} continuously. Take $\lambda_0 = \sup\{\lambda : D_{W_\lambda} \subset D_{W_2}\}$.

We claim that if $D_{W_{\lambda_0}} \subset D_{W_2}$ then there exist $0 < \varepsilon < 1 - \lambda_0$ such that $W_{\lambda_0+\varepsilon}$ is a positive operator. Otherwise, if for all $0 < \varepsilon < 1 - \lambda_0$, $D_{W_{\lambda_0+\varepsilon}} \neq \emptyset$, then we have $D_{W_{\lambda_0}} \subset D_{W_2}, D_{W_{\lambda_0+\varepsilon}} \subset D_{W_1}$, and for all $\rho \in D_{W_{\lambda_0}}$

$$\text{tr}(W_{\lambda_0}\rho) < 0, \text{tr}(W_{\lambda_0}\rho) + \varepsilon(\text{tr}(W_1\rho) - \text{tr}(W_2\rho)) \geq 0. \tag{6}$$

Note that $\text{tr}(W_1\rho) \geq 0$ and $\text{tr}(W_2\rho) < 0$, the second part of the last inequality is positive, and ε is any small positive number, so the last inequality is impossible.

On the other hand, if $D_{W_{\lambda_0}} \subset D_{W_1}$ then there exist $0 < \varepsilon < \lambda_0$ such that $D_{W_{\lambda_0-\varepsilon}}$ is a positive operator. Otherwise, if for all $0 < \varepsilon < \lambda_0$, $D_{W_{\lambda_0-\varepsilon}} \neq \emptyset$, then we have $D_{W_{\lambda_0}} \subset D_{W_1}, D_{W_{\lambda_0-\varepsilon}} \subset D_{W_2}$, and for all $\rho \in D_{W_{\lambda_0}}$, we have

$$\text{tr}(W_{\lambda_0}\rho) < 0, \text{tr}(W_{\lambda_0}\rho) + \varepsilon(\text{tr}(W_2\rho) - \text{tr}(W_1\rho)) \geq 0. \tag{7}$$

For the similar reason of Equation (6), Equation (7) is impossible as well.

To sum up the previous discussion, no matter $D_{W_{\lambda_0}} \subset D_{W_1}$ or $D_{W_{\lambda_0}} \subset D_{W_2}$, there exists $\lambda \geq 0$, or equivalently $t > 0$ ($a > 0$ and $b > 0$) such that W_λ ($W_{a,b}$) is a positive operator, which completes the proof of the theorem. \square

Corollary 1. W_2 and W_1 are not incomparable if and only if $W_{a,b} = aW_1 + bW_2$ are witnesses for all $a > 0$ and $b > 0$.

Theorem 1 can be generalized to the case of finitely many witnesses. We have the following result.

Theorem 2. W_1, W_2, \dots, W_n are incomparable if and only if there exist $t_i > 0$ ($i = 1, 2, \dots, n$) such that $W = \sum_{i=1}^n t_i W_i$ is positive.

Proof. (i) The “if” part. If $W = \sum_{i=1}^n t_i W_i \geq 0$ for $t_i \geq 0$, then $D_W = \emptyset$. Let $S = \{W_i | 1 \leq i \leq n\}$ and the convex hull of $\text{cov}(S) = \{\sum_{i=1}^K t_i W_i | t_i \geq 0, \sum_{i=1}^K t_i = 1, W_i \in S, K \in \mathbb{N}\}$. Without loss of generality we assume that any subset of S can detect some coherent states simultaneously. For $n = 2$, Theorem 2 holds as it reduces to the Theorem 1. Now assume that the Theorem 2 holds for $K \leq n - 1$. We prove that Theorem 2 holds for $K = n$. Indeed we only need to prove the case of $n = 3$. The case of arbitrary n can be proved in a similar way.

By the assumption, we have $D_{W_1} \neq \emptyset, D_{W_1} \cap D_{W_2} \neq \emptyset$ and $D_{W_1} \cap D_{W_3} \neq \emptyset$. But $D_{W_1} \cap D_{W_2} \cap D_{W_3} = \emptyset$, that is, $(D_{W_1} \cap D_{W_2}) \cap D_{W_1} \cap D_{W_3} = \emptyset$. Let $W_{b,c} = bW_2 + cW_3$, where $b > 0$ and $c > 0$. We have $D_{W_1} \cap D_{W_{b,c}} \subset (D_{W_1} \cap D_{W_2}) \cup (D_{W_1} \cap D_{W_3})$. Since $(D_{W_1} \cap D_{W_2})$ and $(D_{W_1} \cap D_{W_3})$ are disjoint and $D_{W_1} \cap D_{W_{b,c}}$ is convex, $D_{W_1} \cap D_{W_{b,c}}$ varies from $(D_{W_1} \cap D_{W_3})$ to $(D_{W_1} \cap D_{W_2})$, whenever $\frac{b}{c}$ varies from 0 to ∞ . By the similar argument to that in the proof of Theorem 1, we conclude that there exist $\frac{b_0}{c_0} > 0$ such that $D_{W_1} \cap D_{W_{b,c}} = \emptyset$. Therefore, $W = a'W_1 + b'W_{b,c} = a'W_1 + b'bW_2 + b'cW_3 \geq 0$ for some $a' > 0$ and $b' > 0$. By induction on n we complete the proof of (i).

(ii) The “only if” part is clear. If $D_W = \emptyset$, then there exist W such that $W \geq 0$ ($W \in \text{cov}(S)$) from the proof in (i). It follows that W is not a witness, which gives a contraction. \square

3. Common Coherent States

A framework which assembles hierarchies of “witnesses” has been proposed in [42]. In this framework, a coherence witness can witness coherent states, and on the other hand, a coherent state can also act as a “high-level-witness” of coherence witnesses which witnesses coherence witnesses. Concretely, when a coherence witness W detects a coherent state ρ , we say that W “witnesses” the coherence of the state ρ . A question naturally arises. What “witnesses” coherence witnesses. It is known that the set of quantum states (incoherent

states and coherent states) is also convex and compact. Thus, by the Hahn-Banach theorem, there is at least one “high-level” witness “witnessing” a coherence witness, see Figure 1b.

For a high-level witness of coherence witnesses Π , one has (i'') $\text{tr}(\Pi\rho) \geq 0$ for all quantum states ρ , and (ii'') there exists at least one coherence witness W such that $\text{tr}(\Pi W) < 0$. Coherence witnesses “witness” coherent states and coherent states “witness” coherence witnesses. Coherent states play the role of witnesses. Since coherent states are also (high-level) witnesses, the question when different coherent states can be detected by some common coherence witnesses can be transformed into the question when different high-level witnesses (coherent states) can detect the same coherence witnesses. From the high-level-witness role played by coherent states and the Theorem 1, we have the following result.

Theorem 3. *Two coherent states ρ_1 and ρ_2 are incomparable, i.e., $D_{\rho_1} \cap D_{\rho_2} = \emptyset$, if and only if there exists $0 < t < 1$ such that $\rho_t = t\rho_1 + (1-t)\rho_2$ is an incoherent state.*

The robust of coherence $\mathcal{C}_{\mathcal{R}}(\rho)$ [26,27] of a coherent state $\rho \in \mathcal{D}(\mathbb{C}^d)$ is defined as

$$\mathcal{C}_{\mathcal{R}}(\rho) = \min_{\tau \in \mathcal{D}(\mathbb{C}^d)} \left\{ s \geq 0 \mid \frac{\rho + s\tau}{1+s} = \delta \in \mathcal{I} \right\}, \quad (8)$$

where $\mathcal{D}(\mathbb{C}^d)$ stands for the convex set of density operators acting on a d -dimensional Hilbert space. We have the following conclusions.

Corollary 2. *Any coherent state ρ and the state minimizing s in (8) τ have no common coherence witnesses.*

Corollary 3. *Two coherent states ρ_1 and ρ_2 are not incomparable if and only if there does not exist $0 < t < 1$ such that $\rho_t = t\rho_1 + (1-t)\rho_2$ is an incoherent state.*

From the general construction of optimal coherence witnesses for an arbitrary coherent state [33,34] and Corollary 3, there also exists a general way of constructing a common optimal coherence witness for different coherent states.

Corollary 4. *For two given not incomparable coherent states ρ_1 and ρ_2 , the optimal coherence witness $W = aW_{\rho_1} + bW_{\rho_2}$ detects both the coherence of ρ_1 and ρ_2 , where $a > 0$, $b > 0$ and $W_{\rho_i} = -\rho_i + \Delta(\rho_i)$ ($i = 1, 2$).*

It is also not difficult to generalize Theorem 3 to the case for finitely many coherent states.

Theorem 4. *The coherent states $\rho_1, \rho_2, \dots, \rho_n$ are incomparable if and only if there exist $\sum_{i=1}^n t_i = 1$, $t_i > 0$ ($i = 1, 2, \dots, n$) such that $\rho = \sum_{i=1}^n t_i \rho_i$ is an incoherent state.*

4. Summary and Discussion

To summarize, we have investigated the properties of coherent witnesses and the methods of constructing coherence witnesses for any arbitrarily given coherent states. We have presented the conditions for different witnesses to detect the same coherent states, as well as the conditions for a set of different coherent states whose coherence can be detected by a common set of coherence witnesses. Here, we mainly considered the case of discrete quantum systems in finite-dimensional Hilbert spaces. In fact our results hold also for infinite-dimensional cases, since our main results are proved without the additional assumption $\text{tr}(W_1) = \text{tr}(W_2)$. However, the coherence in continuous variable systems (such as light modes) is significantly different from the case of the discrete systems. For instance, the set of Gaussian states must be closed and convex, but not necessarily bounded by the Hahn-Banach theorem [43]). Our investigations may highlight further researches on these related problems.

Author Contributions: Conceptualization, B.-H.W.; methodology, B.-H.W., Z.-H.D. and S.-M.F.; validation, Z.-H.D., Z.M. and S.-M.F.; writing—original draft preparation, B.-H.W.; writing—review and editing, S.-M.F.; supervision, B.-H.W., Z.M. and S.-M.F.; funding acquisition, B.-H.W. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by the National Natural Science Foundation of China under Grant Nos. 62072119, 61672007 and 12075159, Guangdong Basic and Applied Basic Research Foundation under Grant No. 2020A1515011180, Shenzhen Institute for Quantum Science and Engineering, Southern University of Science and Technology (Grant Nos. SIQSE202005, SIQSE202001), Natural Science Foundation of Shanghai (Grant No. 20ZR1426400), Beijing Natural Science Foundation (Z190005), the Academician Innovation Platform of Hainan Province, and Academy for Multidisciplinary Studies, Capital Normal University.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Streltsov, A.; Adesso, G.; Plenio, M.B. Colloquium: Quantum coherence as a resource. *Rev. Mod. Phys.* **2017**, *89*, 041003, arXiv:quant-ph/1609.02439.
2. Hu, M.-L.; Hu, X.; Wang, J.-C.; Peng, Y.; Zhang, Y.-R.; Fan, H. Quantum coherence and geometric quantum discord. *Phys. Rep.* **2018**, *1*, 762–764. [[CrossRef](#)]
3. Giovannetti, V.; Lloyd, S.; Maccone, L. Quantum-Enhanced Measurements: Beating the Standard Quantum Limit. *Science* **2004**, *306*, 1330. [[CrossRef](#)]
4. Giovannetti, V.; Lloyd, S.; Maccone, L. Advances in quantum metrology. *Nat. Photonics* **2011**, *5*, 222. [[CrossRef](#)]
5. Hillery, M. Coherence as a resource in decision problems: The Deutsch-Jozsa algorithm and a variation. *Phys. Rev. A* **2016**, *93*, 012111. [[CrossRef](#)]
6. Åberg, J. Catalytic coherence. *Phys. Rev. Lett.* **2014**, *113*, 150402. [[CrossRef](#)]
7. Gour, G.; Müller, M.P.; Narasimhachar, V.; Spekkens, R.W.; Halpern, N.Y. The Resource Theory of Informational Nonequilibrium in Thermodynamics. *Phys. Rep.*, **2015**, *583*, 1–58. [[CrossRef](#)]
8. Lostaglio, M.; Jennings, D.; Rudolph, T. Description of quantum coherence in thermodynamic processes requires constraints beyond free energy. *Nat. Commun.* **2015**, *6*, 6383. [[CrossRef](#)]
9. Narasimhachar, V.; Gour, G. Low-temperature thermodynamics with quantum coherence. *Nat. Commun.* **2015**, *6*, 7689. [[CrossRef](#)]
10. Francica, G.; Goold, J.; Plastina, F. Role of coherence in the nonequilibrium thermodynamics of quantum systems. *Phys. Rev. E* **2019**, *99*, 042105. [[CrossRef](#)]
11. Lloyd, S. Quantum coherence in biological systems. *J. Phys. Conf. Ser.* **2011**, *302*, 012037. [[CrossRef](#)]
12. Huelga, S.F.; Plenio, M.B. Vibrations, quanta and biology. *Contemp. Phys.* **2013**, *54*, 181. [[CrossRef](#)]
13. Lambert, N.; Chen, Y.-N.; Cheng, Y.-C.; Li, C.-M.; Chen, G.-Y.; Nori, F. Quantum biology. *Nat. Phys.* **2013**, *9*, 10. [[CrossRef](#)]
14. Romero, E.; Augulis, R.; Novoderezhkin, V.I.; Ferretti, M.; Thieme, J.; Zigmantas, D.; van Grondelle, R. Quantum coherence in photosynthesis for efficient solar-energy conversion. *Nat. Phys.* **2014**, *10*, 676. [[CrossRef](#)] [[PubMed](#)]
15. Baumgratz, T.; Cramer, M.; Plenio, M.B. Quantifying Coherence. *Phys. Rev. Lett.* **2014**, *113*, 140401. [[CrossRef](#)]
16. Yuan, X.; Zhou, H.; Cao, Z.; Ma, X. Intrinsic randomness as a measure of quantum coherence. *Phys. Rev. A* **2015**, *92*, 022124. [[CrossRef](#)]
17. Fang, K.; Wang, X.; Lami, L.; Regula, B.; Adesso, G. Probabilistic Distillation of Quantum Coherence. *Phys. Rev. Lett.* **2018**, *121*, 070404. [[CrossRef](#)]
18. Winter, A.; Yang, D. Operational Resource Theory of Coherence. *Phys. Rev. Lett.* **2016**, *116*, 120404. [[CrossRef](#)] [[PubMed](#)]
19. Streltsov, A.; Singh, U.; Dhar, H.S.; Bera, M.N.; Adesso, G. Measuring Quantum Coherence with Entanglement. *Phys. Rev. Lett.* **2015**, *115*, 020403. [[CrossRef](#)]
20. Du, S.; Bai, S.; Qi, X. Coherence measures and optimal conversion for coherent states. *Quantum Inf. Comput.* **2015**, *15*, 1307.
21. Qi, X.; Gao, T.; Yan, F. Measuring coherence with entanglement concurrence. *J. Phys. A Math. Theor.* **2017**, *50*, 285301. [[CrossRef](#)]
22. Bu, K.F.; Singh, U.; Fei, S.M.; Pati, A.K.; Wu, J.D. Maximum Relative Entropy of Coherence: An Operational Coherence Measure. *Phys. Rev. Lett.* **2017**, *119*, 150405. [[CrossRef](#)] [[PubMed](#)]
23. Huang, Z.; Situ, H. Dynamics of quantum correlation and coherence for two atoms coupled with a bath of fluctuating massless scalar field. *Ann. Phys.* **2017**, *377*, 484. [[CrossRef](#)]
24. Jin, Z.-X.; Fei, S.-M. Quantifying quantum coherence and nonclassical correlation based on Hellinger distance. *Phys. Rev. A* **2018**, *97*, 062342. [[CrossRef](#)]
25. Xi, Z.; Yuwen, S. Coherence measure: Logarithmic coherence number. *Phys. Rev. A* **2019**, *99*, 022340. [[CrossRef](#)]
26. Napoli, C.; Bromley, T.R.; Cianciaruso, M.; Piani, M.; Johnston, N.; Adesso, G. Robustness of Coherence: An Operational and Observable Measure of Quantum Coherence. *Phys. Rev. Lett.* **2016**, *116*, 150502. [[CrossRef](#)] [[PubMed](#)]

27. Piani, M.; Cianciaruso, M.; Bromley, T.R.; Napoli, C.; Johnston, N.; Adesso, G. Robustness of asymmetry and coherence of quantum states. *Phys. Rev. A* **2016**, *93*, 042107. [[CrossRef](#)]
28. Nie, Y.; Zhou, H.; Guan, J.-Y.; Zhang, Q.; Ma, X.; Zhang, J.; Pan, J.-W. Quantum Coherence Witness with Untrusted Measurement Devices. *Phys. Rev. Lett.* **2019**, *123*, 090502. [[CrossRef](#)]
29. Wang, Y.-T.; Tang, J.-S.; Wei, Z.-Y.; Yu, S.; Ke, Z.-J.; Xu, X.; Li, C.-F.; Guo, G.-C. Directly Measuring the Degree of Quantum Coherence using Interference Fringes. *Phys. Rev. Lett.* **2017**, *118*, 020403. [[CrossRef](#)] [[PubMed](#)]
30. Zheng, W.; Ma, Z.; Wang, H.; Fei, S.-M.; Peng, X. Experimental Demonstration of Observability and Operability of Robustness of Coherence. *Phys. Rev. Lett.* **2018**, *120*, 230504. [[CrossRef](#)] [[PubMed](#)]
31. Ringbauer, M.; Bromley, T.R.; Cianciaruso, M.; Lami, L.; Lau, W.Y.S.; Adesso, G.; White, A.G.; Fedrizzi, A.; Piani, M. Certification and Quantification of Multilevel Quantum Coherence. *Phys. Rev. X* **2018**, *8*, 041007. [[CrossRef](#)]
32. Ma, J.; Hakande, A.; Yuan, X.; Ma, X. Coherence as a resource for source-independent quantum random-number generation. *Phys. Rev. A* **2019**, *99*, 022328. [[CrossRef](#)]
33. Wang, B.-H.; Zhou, S.-Q.; Ma, Z.; Fei, S.-M. Tomographic Witnessing and Holographic Quantifying of Coherence. *Quant. Inform. Proc.* **2021**, *20*, 181.
34. Ma, Z.; Zhang, Z.; Dai, Y.; Dong, Y.; Zhang, C. Detecting and estimating coherence based on coherence witnesses. *Phys. Rev. A* **2021**, *103*, 012409. [[CrossRef](#)]
35. Wu, Y.; Han, Y.; Guo, G. Tetrahedral-octahedral network structure transition in simulated vitreous SiO₂. *Phys. Lett. A* **2006**, *356*, 402. [[CrossRef](#)]
36. Wu, Y.-C.; Guo, G.-C. The hierarchies of “witnesses” and the properties and characterization of entangled states as super entanglement witnesses. *Phys. Rev. A* **2007**, *75*, 052333. [[CrossRef](#)]
37. Hou, J.; Guo, Y. When different entanglement witnesses detect the same entangled states. *Phys. Rev. A* **2010**, *82*, 052301. [[CrossRef](#)]
38. Edwards, R.E. *Functional Analysis, Theory and Application*; Holt, Rinehart and Winston: New York, NY, USA, 1965.
39. Ren, H.; Lin, A.; He, S.; Hu, X. Quantitative coherence witness for finite dimensional states. *Ann. Phys.* **2017**, *387*, 281–289. [[CrossRef](#)]
40. Gühne, O.; Tóth, G. Entanglement detection. *Phys. Rep.* **2009**, *474*, 1–75. [[CrossRef](#)]
41. R. Horodecki, Horodecki, P.; Horodecki, M.; Horodecki, K. Quantum entanglement. *Rev. Mod. Phys.* **2009**, *81*, 865. [[CrossRef](#)]
42. Wang, B.-H. Entangled states in the role of witnesses. *Phys. Rev. A* **2018**, *97*, 050302(R). [[CrossRef](#)]
43. Bruß, D.; Cirac, J.I.; Horodecki, P.; Hulpke, F.; Kraus, B.; Lewenstein, M.; Sanpera, A. Reflections upon separability and distillability. *J. Mod. Opt.* **2002**, *49*, 1399. [[CrossRef](#)]