Heliyon 7 (2021) e08083

Contents lists available at ScienceDirect

Heliyon

journal homepage: www.cell.com/heliyon

Research article

Stochastic modeling of mortality rates and Mortality-at-Risk forecast by taking conditional heteroscedasticity effect into account

Khreshna Syuhada *, Arief Hakim

Statistics Research Division, Institut Teknologi Bandung, Jalan Ganesa 10, Bandung 40132, Indonesia

ARTICLE INFO

Keywords:
Autoregressive
ARCH
Life insurance
Pension fund
Time series forecasting
Stochastic volatility
Value-at-Risk

ABSTRACT

Mortality and mortality rate have become the major issues in insurance industries, for instance, life insurance and pension fund. Such industries will, in particular, be concerned with the quantification of risk attached, say longevity risk, to insurance products that may receive severe impacts from the fall of mortality rate. In this paper, we model the mortality rate by using an Autoregressive (AR) model with a conditional heteroscedasticity effect. This effect is accommodated by a stochastic model of Autoregressive Conditional Heteroscedastic (ARCH) as well as a Stochastic Volatility Autoregressive (SVAR) model. Furthermore, we do forecasting of what so-called Mortality-at-Risk (MaR) by adopting the Value-at-Risk framework and its improvement. The calculation of the MaR forecast for those two models is conducted with significantly different approaches.

1. Introduction

For insurance industries, life insurance and pension fund in particular, mortality and forecasting accurate mortality rate are very important subjects to discuss and develop continuously. Their concern is mainly on whether or not they are able to provide capital allocation or reserve when they are exposed to high longevity risk due to the fall of mortality rate.

The nature of longevity risk is long-term forecasting. It is, however, still useful and challenging to seek a one-step-ahead forecast as the dynamic change in mortality rates may occur over a short-term horizon, say in one year; see, e.g., Plat (2011) and Richards et al. (2013). Besides, we may also consider that the short-term forecasting will give more benefits to the insurance companies and the policymakers since they may keep "flexibility" in adjusting mortality rate models.

The mortality rate has reached its breakthrough since the famous model of Lee and Carter (1992). The Lee–Carter model brings us attention of having mortality index and stochastic term assumed to be normally distributed. This model aimed to describe the natural logarithm of the mortality rate, rather than the mortality rate directly. In fact, the mortality index is basically the mean of the log mortality rate, given previous information.

The improvements in modeling the mortality rate have been carried out by several authors. For instance, Giacometti et al. (2012) and Lin et al. (2015) proposed to model the log mortality rate by considering dynamic conditional mean as well as dynamic conditional variance or volatility. The latter consideration is basically adopted from modeling asset returns. Specifically, the proposed models incorporated the previous log mortality rates in the conditional mean of the current log mortality rate via an Autoregressive (AR) model. The conditional heteroscedasticity effect was involved through (Generalized) Autoregressive Conditional Heteroscedastic or (G)ARCH model.

In this paper, we aim at adopting a stochastic mortality rate model of AR-ARCH as in Giacometti et al. (2012) and Lin et al. (2015). As for the comparison to ARCH, a Stochastic Volatility Autoregressive (SVAR) of Taylor (1986) is also proposed to accommodate the conditional heteroscedasticity effect. Due to the stationarity reason, both the AR-ARCH and AR-SVAR models are applied to the change in the log mortality rate, instead of the log mortality rate itself. We, specifically, utilize the data of the log mortality rate changes for the United States population. Note that the need for having the above stochastic models with time-varying volatility comes from the fact that the data tend to be serially correlated and volatile over time. We show this fact by using the serial correlation test as well as the ARCH effect test of Engle (1982).

In addition, we introduce a risk measure called Mortality-at-Risk (MaR) to determine the greatest risk or fall of mortality rates for a fixed period of future years that can be tolerated at a certain level of confidence. Calculating MaR is carried out by adopting the risk measure of Value-at-Risk (VaR), well known in quantitative risk management. In order to obtain a MaR forecast with a better coverage probability,

* Corresponding author.
E-mail address: khreshna@math.itb.ac.id (K. Syuhada).

https://doi.org/10.1016/j.heliyon.2021.e08083
Received 23 January 2021; Received in revised form 14 July 2021; Accepted 24 September 2021

the forecasting method is improved. This improvement is motivated by the works of Vidoni (2004), Kabaila and Syuhada (2008, 2010), and Syuhada (2020).

2. Material and methods

2.1. Data

This study utilizes data for both genders of the United States population extracted from the Human Mortality Database's website (mortality.org). The data we choose consist of the number of deaths and the number of individuals exposed to the death risk at age 1 to 90 in the calendar year 1933 to 2018.

2.2. Mortality rates

By $D_{x,t}$ we denote the number of deaths. As noted by Wilmoth et al. (2021), it is basically calculated from the sum of (a) $D^L_{x,t}$: the number of individuals of age x in year t who die before year $t+1$ and (b) $D^U_{x,t}$: the number of individuals aged x in the beginning of year t but they die before reaching age $x+1$. Meanwhile, the number of individuals exposed to the death risk at age x in year t, we denote by $E_{x,t}$, is the sum of $E^L_{x,t}$ and $E^U_{x,t}$. They are formulated by

$$E^L_{x,t} = \frac{1}{2}P_{x,t+1} + \frac{1}{6}D^L_{x,t},$$
$$E^U_{x,t} = \frac{1}{2}P_{x,t} - \frac{1}{6}D^U_{x,t},$$

respectively, where $P_{x,t}$ ($P_{x,t+1}$) refers to the number of individuals aged x in the beginning of year t ($t+1$).

We then represent by $M_{x,t}$ the mortality rate at age x in year t. This rate is defined by dividing $D_{x,t}$ by $E_{x,t}$, that is

$$M_{x,t} = \frac{D_{x,t}}{E_{x,t}}.$$

This means that the above mortality rate quantifies the number of deaths per certain size of population exposed to the death risk. For example, in the US population, the number of males, say at age $x=60$ in the beginning of year $t=2010$, who die before year 2011 (source (a)) is $D^L_{60,2010} = 9{,}326$. Meanwhile, the number of males of age 60 in year 2010 but they die before they reach age 61 is $D^U_{60,2010} = 9{,}776$ (source (b)). Thus, the number of deaths is 19,102. Furthermore, it is known that $E_{60,2010} = 1{,}724{,}924$. As a result, we obtain a mortality rate $M_{60,2010} = 0.011$, eleven deaths over population of a thousand. Note that the natural logarithm transformation is commonly taken in order to avoid working with small mortality rate; for the above case, $\ln(M_{60,2010})$ is equal to -4.503.

2.3. Stochastic mortality rate models with conditional heteroscedasticity effect

At a specified age $x \in \{x_0, x_0+1, \ldots, X\}$ for given positive integers x_0 and X with $x_0 < X$, the yearly log mortality rate data $\{\ln(M_{x,t})\}_{t=t_0,\ldots,T}$ may be modeled as a linear function of mortality index. This has been provided in the famous mortality rate model of Lee and Carter (1992) by

$$\mathrm{E}\left[\ln(M_{x,t})\middle|\mathcal{I}_{x,t-1}\right] + \varepsilon_{x,t}, \tag{1}$$

for $t \in \{t_0, t_0+1, \ldots, T\}$, where t_0 and T are fixed positive integers with $t_0 < T$. The first term in Eq. (1), the conditional mean of the log mortality rate in year t given previous information $\mathcal{I}_{x,t-1}$, is $a_x + b_x k_t$, where a_x denotes the average mortality at age x, k_t represents the time-varying mortality level, and b_x describes the response at age x to the time-varying factor. Meanwhile, $\varepsilon_{x,t}$ is a stochastic term assumed to be normally distributed with a mean of zero and a constant variance, say $\sigma^2_{\varepsilon_x}$.

The above model, however, may be unsuitable. The reason is that it only models the conditional mean of the log mortality rate. Also, the constant conditional variance (volatility) assumption does not meet the empirical behavior of volatility. As demonstrated by Chai et al. (2013), the mortality data for most ages significantly exhibit a conditional heteroscedasticity effect, also known as Engle's ARCH effect. This feature indicates that the squared mortality data not only vary over time but also are serially correlated. Consequently, an alternative model is required to describe the conditional heteroscedastic mortality rate by incorporating the previous squared observation in the current volatility equation.

Giacometti et al. (2012) and Lin et al. (2015), for instance, first proposed to model the conditional mean of $\ln(M_{x,t})$ as

$$a_x + b_x \ln(M_{x,t-1}), \tag{2}$$

where $a_x, b_x \in \mathbb{R}$, so that we can observe the mortality level k_t, that is modeled as a random walk in the Lee–Carter model. Note that a model whose conditional mean is given in Eq. (2) is a first-order Autoregressive or AR(1) model of $\{\ln(M_{x,t})\}_{t=t_0,\ldots,T}$. Giacometti et al. (2012) further set up the error process $\{\varepsilon_{x,t}\}_{t=t_0,\ldots,T}$ for this model to follow a conditional heteroscedastic time series model of ARCH type with conditional mean and variance as below:

$$\mathrm{E}\left(\varepsilon_{x,t}\middle|\mathcal{I}_{x,t-1}\right) = 0,$$
$$\mathrm{Var}\left(\varepsilon_{x,t}\middle|\mathcal{I}_{x,t-1}\right) = \gamma_x + \delta_x\,\varepsilon^2_{x,t-1},$$

respectively, where $\gamma_x \in (0,\infty)$ and $\delta_x \in [0,\infty)$. The vector $(a_x, b_x, \gamma_x, \delta_x)^\top$ of parameters needs to be restricted such that the stationarity condition applies to the above process. However, it is important to note that the stationarity condition for the time series data $\{\ln(M_{x,t})\}_{t=t_0,\ldots,T}$ is not achieved. This may be unfortunate for time series modeling.

We instead work with the change in the log mortality rates. For each fixed age $x \in \{x_0, x_0+1, \ldots, X\}$, it refers to the difference between the log of the current mortality rate and the log of the previous one, that is

$$Y_{x,t} = \ln(M_{x,t}) - \ln(M_{x,t-1}) = \ln\left(\frac{M_{x,t}}{M_{x,t-1}}\right), \tag{3}$$

where $t \in \{t_0+1, t_0+2, \ldots, T\}$. An AR(1) model is employed to describe the conditional mean of $Y_{x,t}$, given $\mathcal{Y}_{x,t-1} = \{Y_{x,s}\}_{s=t_0+1,\ldots,t-1}$. We keep the conditional heteroscedasticity effect for the error term accommodated by two different models: ARCH(1) and first-order Stochastic Volatility Autoregressive, SVAR(1). The former assumes observable volatility whilst, in contrast, the latter takes unobserved (latent) volatility into consideration.

2.3.1. AR(1)-ARCH(1) model

The first assumption is that, for each age $x \in \{x_0, x_0+1, \ldots, X\}$, the process $\{Y_{x,t}\}_{t=t_0+1,\ldots,T}$ of the changes in the log mortality rates follows an AR(1)-ARCH(1) model given by

$$Y_{x,t} = a_x + b_x Y_{x,t-1} + \varepsilon_{x,t}, \tag{4a}$$
$$\varepsilon_{x,t} = \sigma_{x,t}\,\xi_{x,t}, \tag{4b}$$
$$\sigma^2_{x,t} = \gamma_x + \delta_x\,\varepsilon^2_{x,t-1}. \tag{4c}$$

The parameter vector $\theta_x = (a_x, b_x, \gamma_x, \delta_x)^\top$ of the above model is assumed to belong to $\mathbb{R} \times (-1,1) \times (0,\infty) \times [0,1)$ in order to ensure that the process is stationary. The term $\xi_{x,t}$ is an innovation following a standard normal distribution, $N(0,1)$. As the stochastic term, the innovation $\xi_{x,t}$ determines the statistical properties of $\{\varepsilon_{x,t}\}_{t=t_0+1,\ldots,T}$ as well as $\{Y_{x,t}\}_{t=t_0+1,\ldots,T}$. Specifically, the error process has a conditional distribution as below:

$$\varepsilon_{x,t}\middle|\mathcal{Y}_{x,t-1} \sim N\left(0, \sigma^2_{x,t}\right), \tag{5}$$

where $\sigma^2_{x,t} = \gamma_x + \delta_x (Y_{x,t-1} - a_x - b_x Y_{x,t-2})^2$ that varies over years but remains observable. Meanwhile, $Y_{x,t}$, given $\mathcal{Y}_{x,t-1}$, is normally distributed with a conditional mean of $\mathrm{E}\left(Y_{x,t} \middle| \mathcal{Y}_{x,t-1}\right) = a_x + b_x Y_{x,t-1}$ and a conditional variance of $\mathrm{Var}\left(Y_{x,t} \middle| \mathcal{Y}_{x,t-1}\right) = \sigma^2_{x,t}$.

2.3.2. AR(1)-SVAR(1) model

We further assume to have a latent volatility $V_{x,t} = \ln(\sigma^2_{x,t})$ that nonlinearly enters our model formulated by

$$Y_{x,t} = a_x + b_x Y_{x,t-1} + \varepsilon_{x,t}, \tag{6a}$$

$$\varepsilon_{x,t} = \exp(V_{x,t}/2)\, \xi_{x,t}, \tag{6b}$$

$$V_{x,t} = \gamma_x + \delta_x V_{x,t-1} + \eta_{x,t}, \tag{6c}$$

with $(a_x, \gamma_x, b_x, \delta_x)^\top \in \mathbb{R}^2 \times (-1,1)^2$. The latent volatility process $\{V_{x,t}\}_{t=t_0+1,\ldots,T}$ follows a stochastic model of AR(1) with an innovation term $\eta_{x,t}$. The process $\{Y_{x,t}\}_{t=t_0+1,\ldots,T}$ of the log mortality rate changes is, therefore, said to follow a stationary AR(1)-SVAR(1) model. We assume the innovations $\xi_{x,t}$ and $\eta_{x,t}$ to be independent and normally distributed with a parameter $(0,1)^\top$ and $\left(0, \sigma^2_{\eta_x}\right)^\top$, respectively, where $\sigma^2_{\eta_x} \in (0,\infty)$. This assumption implies that $Y_{x,t}$, conditional on $(V_{x,t}, \mathcal{Y}_{x,t-1})^\top$, is normally distributed with a conditional mean and variance equal to $a_x + b_x Y_{x,t-1}$ and $\exp(V_{x,t})$, respectively. Meanwhile, the latent volatility $V_{x,t}$, given $(\mathcal{V}_{x,t-1}, \mathcal{Y}_{x,t-1})^\top$, has a conditional distribution of normal with a mean of $\gamma_x + \delta_x V_{x,t-1}$ and a variance of $\sigma^2_{\eta_x}$, where $\mathcal{V}_{x,t-1} = \{V_{x,s}\}_{s=t_0+1,\ldots,t-1}$. This means that $\sigma^2_{\eta_x}$ measures the volatility of the volatility shocks. Furthermore, its unconditional distribution is derived as below:

$$V_{x,t} \sim N\left(\frac{\gamma_x}{1-\delta_x}, \frac{\sigma^2_{\eta_x}}{1-\delta_x^2}\right). \tag{7}$$

2.4. Mortality-at-Risk (MaR) forecasts

We now aim to forecast a risk measure that quantifies a risk arising from the mortality rate process $\{M_{x,t}\}_{t=t_0,\ldots,T}$ for a fixed age $x \in \{x_0, x_0+1, \ldots, X\}$. The risk refers to the fall of mortality rates for a given time horizon and the risk measure is called Mortality-at-Risk (MaR). We define MaR as the greatest fall of mortality rates for a time horizon of τ years at a given confidence level of $1-\alpha$:

$$\mathrm{P}_{\theta_x}\left(M_{x,T} - M_{x,T+\tau} \le \mathrm{MaR}^{1-\alpha}_{x,T+\tau}(\theta_x) \middle| \mathcal{Y}_{x,T}\right) = 1-\alpha, \tag{8}$$

for all θ_x, where τ is a fixed positive integer and $\alpha \in (0,1)$. Specifically, we are interested in forecasting a one-year MaR. Thus, we take $\tau = 1$ and aim to determine a one-step-ahead forecast of $\mathrm{MaR}^{1-\alpha}_{x,T+1}(\theta_x)$.

From Eq. (3), we see that $Y_{x,T+1} = \ln(M_{x,T+1}/M_{x,T})$. We further suppose $Q^\alpha_{x,T+1}(\theta_x)$ to be the α-quantile of the conditional distribution of the future error $\varepsilon_{x,T+1}$, given $\mathcal{Y}_{x,T}$. This means that it satisfies $\mathrm{P}_{\theta_x}\left(\varepsilon_{x,T+1} \le Q^\alpha_{x,T+1}(\theta_x) \middle| \mathcal{Y}_{x,T}\right) = \alpha$. By manipulating $Y_{x,T+1}$, we obtain

$$\mathrm{MaR}^{1-\alpha}_{x,T+1}(\theta_x) = \left[1 - \exp\left(a_x + b_x Y_{x,T} + Q^\alpha_{x,T+1}(\theta_x)\right)\right] M_{x,T}. \tag{9}$$

Since the parameter θ_x is unknown, it needs to be estimated from the available data. Let $\hat{\theta}_x$ denote its estimator obtained through a certain method. We then have an "estimative" one-year MaR forecast, $\mathrm{MaR}^{1-\alpha}_{x,T+1}(\hat{\theta}_x)$, by replacing θ_x by $\hat{\theta}_x$. The coverage probability of this estimative MaR forecast may be computed to assess its accuracy. We observe that

$$\begin{aligned}
&\mathrm{P}_{\theta_x}\left(M_{x,T} - M_{x,T+1} \le \mathrm{MaR}^{1-\alpha}_{x,T+1}(\hat{\theta}_x) \middle| \mathcal{Y}_{x,T}\right) \\
&= \mathrm{P}_{\theta_x}\left[1 - \frac{M_{x,T+1}}{M_{x,T}} \le 1 - \exp\left(\hat{a}_x + \hat{b}_x Y_{x,T} + Q^\alpha_{x,T+1}(\hat{\theta}_x)\right) \middle| \mathcal{Y}_{x,T}\right] \\
&= 1 - \mathrm{P}_{\theta_x}\left[\frac{M_{x,T+1}}{M_{x,T}} \le \exp\left(\hat{a}_x + \hat{b}_x Y_{x,T} + Q^\alpha_{x,T+1}(\hat{\theta}_x)\right) \middle| \mathcal{Y}_{x,T}\right] \\
&= 1 - \mathrm{P}_{\theta_x}\left(Y_{x,T+1} \le \hat{a}_x + \hat{b}_x Y_{x,T} + Q^\alpha_{x,T+1}(\hat{\theta}_x) \middle| \mathcal{Y}_{x,T}\right) \\
&= 1 - \mathrm{P}_{\theta_x}\left(\varepsilon_{x,T+1} \le Q^\alpha_{x,T+1}(\hat{\theta}_x) \middle| \mathcal{Y}_{x,T}\right)
\end{aligned}$$

that is equal to

$$1 - \mathrm{E}_{\theta_x}\left[F_{\varepsilon_{x,T+1}|\mathcal{Y}_{x,T}}\left(Q^\alpha_{x,T+1}(\hat{\theta}_x); \theta_x\right) \middle| \mathcal{Y}_{x,T}\right], \tag{10}$$

where $F_{\varepsilon_{x,T+1}|\mathcal{Y}_{x,T}}(\cdot;\theta_x)$ is the conditional distribution function of $\varepsilon_{x,T+1}$, given $\mathcal{Y}_{x,T}$. The expectation term in Eq. (10), computed with respect to the distribution of $\varepsilon_{x,T+1} \middle| \mathcal{Y}_{x,T}$, may be shown to be equal to $\alpha + O(n^{-1})$, where $n = T - t_0$ refers to the size of the data. This implies that the above conditional coverage probability differs from the $1-\alpha$ level of confidence by $O(n^{-1})$.

Vidoni (2004) and Kabaila and Syuhada (2008, 2010) have suggested so-called "improved" forecasting limits with better coverage properties. Their approach is basically to correct the $O(n^{-1})$ term to become $O(n^{-3/2})$. We aim to adopt this approach to derive an improved MaR forecast in addition to the estimative MaR forecast.

2.4.1. MaR forecast for AR(1)-ARCH(1) model

When the AR(1)-ARCH(1) model with a normal innovation is assumed for the changes in the log mortality rates, we may employ the conditional normal distribution of the error term previously stated in Eq. (5) to obtain the MaR forecast. We first forecast the estimative quantile $Q^\alpha_{x,T+1}(\hat{\theta}_x)$ as below:

$$Q^\alpha_{x,T+1}(\hat{\theta}_x) = \Phi^{-1}(\alpha)\sqrt{\hat{\gamma}_x + \hat{\delta}_x (Y_{x,T} - \hat{a}_x - \hat{b}_x Y_{x,T-1})^2},$$

where $\theta_x = (a_x, b_x, \gamma_x, \delta_x)^\top$ is a conditional maximum likelihood estimator for the parameter $\hat{\theta}_x = (\hat{a}_x, \hat{b}_x, \hat{\gamma}_x, \hat{\delta}_x)^\top$ whilst $\Phi(\cdot)$ and $\Phi^{-1}(\cdot)$ denote the standard normal distribution function and its inverse, respectively. The estimative $\mathrm{MaR}^{1-\alpha}_{x,T+1}(\hat{\theta}_x)$ forecast is then equal to

$$\left[1 - \exp\left(\hat{a}_x + \hat{b}_x Y_{x,T} + \Phi^{-1}(\alpha)\sqrt{\hat{\gamma}_x + \hat{\delta}_x (Y_{x,T} - \hat{a}_x - \hat{b}_x Y_{x,T-1})^2}\right)\right] M_{x,T}. \tag{11}$$

In order to determine the forecast of the improved one-year MaR denoted by ${}^{+}\mathrm{MaR}^{1-\alpha}_{x,T+1}(\hat{\theta}_x)$, we need to find the improved version of the $Q^\alpha_{x,T+1}(\hat{\theta}_x)$ forecast represented by ${}^{+}Q^\alpha_{x,T+1}(\hat{\theta}_x)$. Firstly, we observe that the conditional coverage probability $\mathrm{P}_{\theta_x}\left(\varepsilon_{x,T+1} \le Q^\alpha_{x,T+1}(\hat{\theta}_x) \middle| \mathcal{Y}_{x,T}\right)$ of the estimative $Q^\alpha_{x,T+1}(\hat{\theta}_x)$ forecast may be expressed as

$$\mathrm{E}_{\theta_x}\left[\Phi\left(\Phi^{-1}(\alpha)\sqrt{\frac{\hat{\gamma}_x + \hat{\delta}_x (Y_{x,T} - \hat{a}_x - \hat{b}_x Y_{x,T-1})^2}{\gamma_x + \delta_x (Y_{x,T} - a_x - b_x Y_{x,T-1})^2}}\right) \middle| \mathcal{Y}_{x,T}\right]$$

that differs from α by $O(n^{-1})$. Then, the improved ${}^{+}Q^\alpha_{x,T+1}(\hat{\theta}_x)$ forecast defined by

$${}^{+}Q^\alpha_{x,T+1}(\hat{\theta}_x) = Q^\alpha_{x,T+1}(\hat{\theta}_x) - \frac{\mathrm{P}_{\theta_x}\left(\varepsilon_{x,T+1} \le Q^\alpha_{x,T+1}(\hat{\theta}_x) \middle| \mathcal{Y}_{x,T}\right) - \alpha}{f_{\varepsilon_{x,T+1}|\mathcal{Y}_{x,T}}\left(Q^\alpha_{x,T+1}(\hat{\theta}_x); \theta_x\right)} \tag{12}$$

has a conditional coverage probability equal to $\alpha + O(n^{-3/2})$, where $f_{\varepsilon_{x,T+1}|\mathcal{Y}_{x,T}}(\cdot;\theta_x)$ denotes the probability function of the conditional distribution of $\varepsilon_{x,T+1}$, given $\mathcal{Y}_{x,T}$. We, finally, define the improved MaR forecast as follows:

$${}^{+}\mathrm{MaR}^{1-\alpha}_{x,T+1}(\hat{\theta}_x) = \left[1 - \exp\left(\hat{a}_x + \hat{b}_x Y_{x,T} + {}^{+}Q^\alpha_{x,T+1}(\hat{\theta}_x)\right)\right] M_{x,T}. \tag{13}$$

2.4.2. MaR forecast for AR(1)-SVAR(1) model

For the case of the AR(1)-SVAR(1) model, we need to forecast the unconditional α-quantile of the error $\varepsilon_{x,T+1}$ rather than its conditional one. The forecasting is carried out by using the unconditional distribution of the latent volatility $V_{x,T+1}$. We have already stated in Eq. (7) that

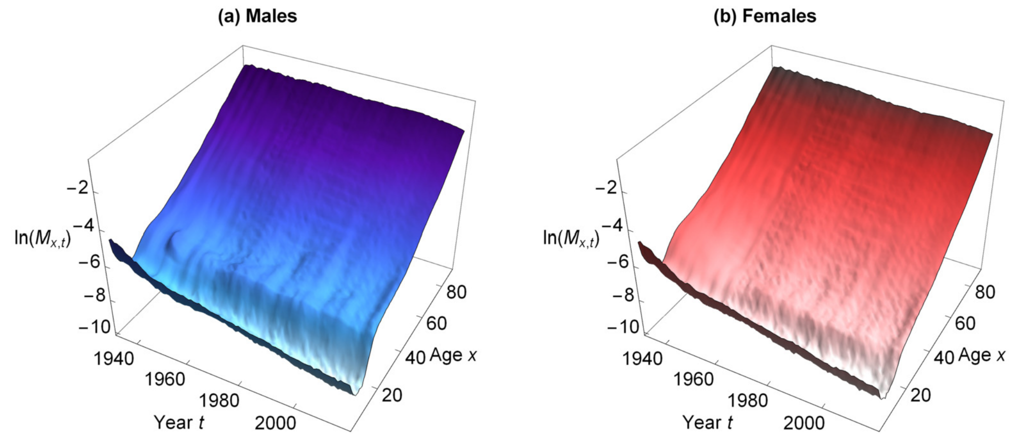

Fig. 1. The log mortality rates for (a) males and (b) females.

$V_{x,T+1}$ has a normal distribution with a mean of $\mu_{V_x} = \gamma_x/(1-\delta_x)$ and a variance of $\sigma^2_{V_x} = \sigma^2_{\eta_x}/(1-\delta^2_x)$. For all $z \in \mathbb{R}$, we observe

$$\begin{aligned}\mathrm{P}_{\theta_x}(\varepsilon_{x,T+1} \le z) &= \mathrm{P}_{\theta_x}\left(\xi_{x,t} \le \frac{z}{\exp(V_{x,T+1}/2)}\right) \\ &= \mathrm{E}_{\theta_x}\left[\Phi\left(\frac{z}{\exp(V_{x,T+1}/2)}\right)\right]\end{aligned}$$

that is equal to the following integral:

$$\int_{-\infty}^{\infty} \Phi\left(\frac{z}{\exp(v/2)}\right)\phi\left(\frac{v-\mu_{V_x}}{\sigma_{V_x}}\right)\mathrm{d}v, \tag{14}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$, respectively, denote the distribution function and the probability function of the standard normal distribution. We then suppose $Q^{\alpha}_{x,T+1}(\theta_x)$ to be a solution for z in the equality $\mathrm{P}_{\theta_x}(\varepsilon_{x,T+1} \le z) = \alpha$.

The integral in Eq. (14), however, can not be evaluated analytically. To overcome the difficulty, we conduct a numerical method by firstly truncating this integral as in Syuhada (2020). For a given positive constant c, we consider $\mu_{V_x} - c\,\sigma_{V_x}$ and $\mu_{V_x} + c\,\sigma_{V_x}$ as its lower and upper bounds, respectively. Thus, the truncation produces an error bounded above by

$$\begin{aligned}&\int_{-\infty}^{\mu_{V_x}-c\,\sigma_{V_x}} \Phi\left(\frac{z}{\exp(v/2)}\right)\phi\left(\frac{v-\mu_{V_x}}{\sigma_{V_x}}\right)\mathrm{d}v \\ &+ \int_{\mu_{V_x}+c\,\sigma_{V_x}}^{\infty} \Phi\left(\frac{z}{\exp(v/2)}\right)\phi\left(\frac{v-\mu_{V_x}}{\sigma_{V_x}}\right)\mathrm{d}v \\ &\le 2\int_{\mu_{V_x}+c\,\sigma_{V_x}}^{\infty} \phi\left(\frac{v-\mu_{V_x}}{\sigma_{V_x}}\right)\mathrm{d}v = 2\left[1-\Phi(c)\right].\end{aligned}$$

In this study, we set $c = 8$, so that the truncation error obtained is bounded above by 1.33×10^{-15}. This truncation is then taken into account in finding a solution $Q^{\alpha}_{x,T+1}(\tilde{\theta}_x)$, where $\tilde{\theta}_x = (\tilde{a}_x, \tilde{b}_x, \tilde{\gamma}_x, \tilde{\delta}_x, \tilde{\sigma}^2_{\eta_x})^{\top}$ denotes an estimator for $\theta_x = (a_x, b_x, \gamma_x, \delta_x, \sigma^2_{\eta_x})^{\top}$. Once such $Q^{\alpha}_{x,T+1}(\tilde{\theta}_x)$ is derived, we compute the estimative MaR forecast as below:

$$\mathrm{MaR}^{1-\alpha}_{x,T+1}(\tilde{\theta}_x) = \left[1 - \exp\left(\tilde{a}_x + \tilde{b}_x\, Y_{x,T} + Q^{\alpha}_{x,T+1}(\tilde{\theta}_x)\right)\right] M_{x,T}. \tag{15}$$

Now, the improved one-year MaR forecast, ${}^{+}\mathrm{MaR}^{1-\alpha}_{x,T+1}(\tilde{\theta}_x)$, for the AR(1)-SVAR(1) model is defined by

$${}^{+}\mathrm{MaR}^{1-\alpha}_{x,T+1}(\tilde{\theta}_x) = \left[1 - \exp\left(\tilde{a}_x + \tilde{b}_x\, Y_{x,T} + {}^{+}Q^{\alpha}_{x,T+1}(\tilde{\theta}_x)\right)\right] M_{x,T} \tag{16}$$

having an unconditional coverage probability of $1 - \alpha + O(n^{-3/2})$, where

$${}^{+}Q^{\alpha}_{x,T+1}(\tilde{\theta}_x) = Q^{\alpha}_{x,T+1}(\tilde{\theta}_x) - \frac{\mathrm{P}_{\theta_x}\left(\varepsilon_{x,T+1} \le Q^{\alpha}_{x,T+1}(\tilde{\theta}_x)\right) - \alpha}{f_{\varepsilon_{x,T+1}}\left(Q^{\alpha}_{x,T+1}(\tilde{\theta}_x); \theta_x\right)}. \tag{17}$$

The unconditional probability function $f_{\varepsilon_{x,T+1}}(\cdot;\theta_x)$ of $\varepsilon_{x,T+1}$ in Eq. (17) evaluated at $Q^{\alpha}_{x,T+1}(\tilde{\theta}_x)$ is given by

$$f_{\varepsilon_{x,T+1}}\left(Q^{\alpha}_{x,T+1}(\tilde{\theta}_x); \theta_x\right) = \int_{-\infty}^{\infty} \phi\left(\frac{Q^{\alpha}_{x,T+1}(\tilde{\theta}_x)}{\exp(v/2)}\right)\phi\left(\frac{v-\mu_{V_x}}{\sigma_{V_x}}\right)\mathrm{d}v.$$

3. Result and discussion

An empirical study is conducted on the log mortality rate data for the United States population. The data spanning from the year 1933 to 2018 include both males and females aged 1–90. Fig. 1 displays their three-dimensional surface plots that show a common "hump" achieving the lowest level for around the age of 10. In Fig. 2, we further report the summary of statistics for the yearly log mortality rates at each age. This figure also summarizes the resulting p-values of several statistical tests. The first one is the Jarque–Bera test rejecting the null hypothesis of normality at a 5% level of significance for several ages. The ADF test provides evidence that the data of the yearly log mortality rates are not stationary for all ages since the p-values are all above the significance level under consideration. This result leads us to note that we may not carry out time series modeling of the log mortality rates although both the data and their volatility appear to be time-varying and serially correlated based on the Ljung–Box and Engle's ARCH tests, respectively.

We now move forward to the data of the changes in the log mortality rates whose surfaces are plotted in Fig. 3. It may be observed that these data are more volatile than the previous data of the log mortality rates, but their stationarity condition is achieved for all ages. This is because both the Engle's ARCH and ADF tests produce low p-values (see Fig. 4). Hence, a conditional heteroscedastic time series model is appropriately applied to these data. The use of an autoregressive term is important since the data have a negative mean and exhibit a serial correlation. Note that this negative mean of the yearly log mortality rate changes indicates that the fall of the log mortality rate, as well as that of the mortality rate, occurs almost every year. Furthermore, a normality assumption also seems suitable due to a low value of kurtosis and a high p-value of the Jarque–Bera test for almost all ages.

Time series modeling is then conducted to the data of the yearly log mortality rate changes. For each fixed age, we first apply the AR(1) model for the conditional mean. After employing the conditional maximum likelihood method to estimate its parameters, we compute an error as the difference between the actual value of the observation and the fitted value in the same year for each age.

The ARCH(1) and SVAR(1) models are then fitted to the resulting error data to capture the conditional heteroscedasticity effect. To find

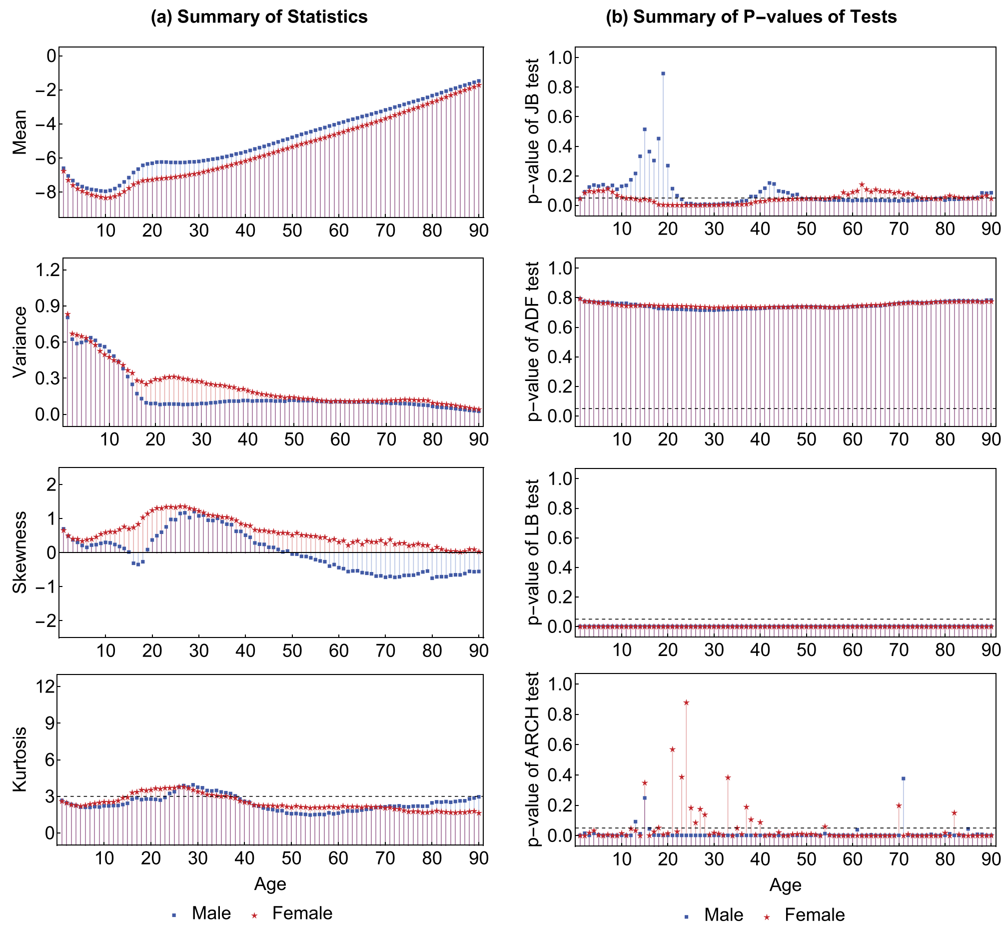

Fig. 2. The summary of (a) statistics and (b) p-values of hypothesis tests for the yearly log mortality rates. The statistics include mean, variance, skewness, and kurtosis. Meanwhile, the tests are the Jarque–Bera (JB) test of normality, the augmented Dickey–Fuller (ADF) test of stationarity, the Ljung–Box (LB) test of serial correlation, and the Engle's ARCH test of conditional heteroscedasticity effect. The dashed line on the right column represents the 5% level of significance.

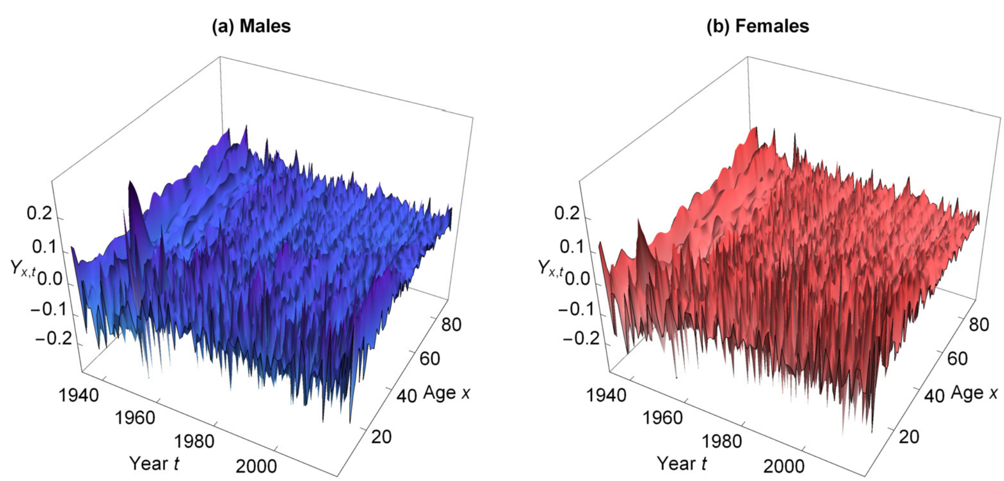

Fig. 3. The changes in the log mortality rates for (a) males and (b) females. Their values are computed according to Eq. (3).

an estimator for the model parameter vector, the maximum likelihood method is also taken into consideration. For the former model, we, specifically, carry out maximization of the conditional log-likelihood function. Meanwhile, the log-likelihood function of each error process assumed to follow the SVAR(1) model is estimated through the Importance Sampling procedure before being maximized. To compare the performance between those two models and to determine the one having a better fit, we adopt the most common tool, namely Akaike Information Criterion (AIC), whose values are demonstrated in Fig. 5. From Table 1, we find that the AIC of the ARCH(1) model attains a lower value than that of the SVAR(1) model on 66 of 90 datasets for males and on 67.78% datasets for females. This result implies that, in terms of AIC, the ARCH(1) model is superior to the SVAR(1) model in accommodating the conditional heteroscedasticity effect in the error processes. The latter performs worse with an extremely high value of AIC for several datasets.

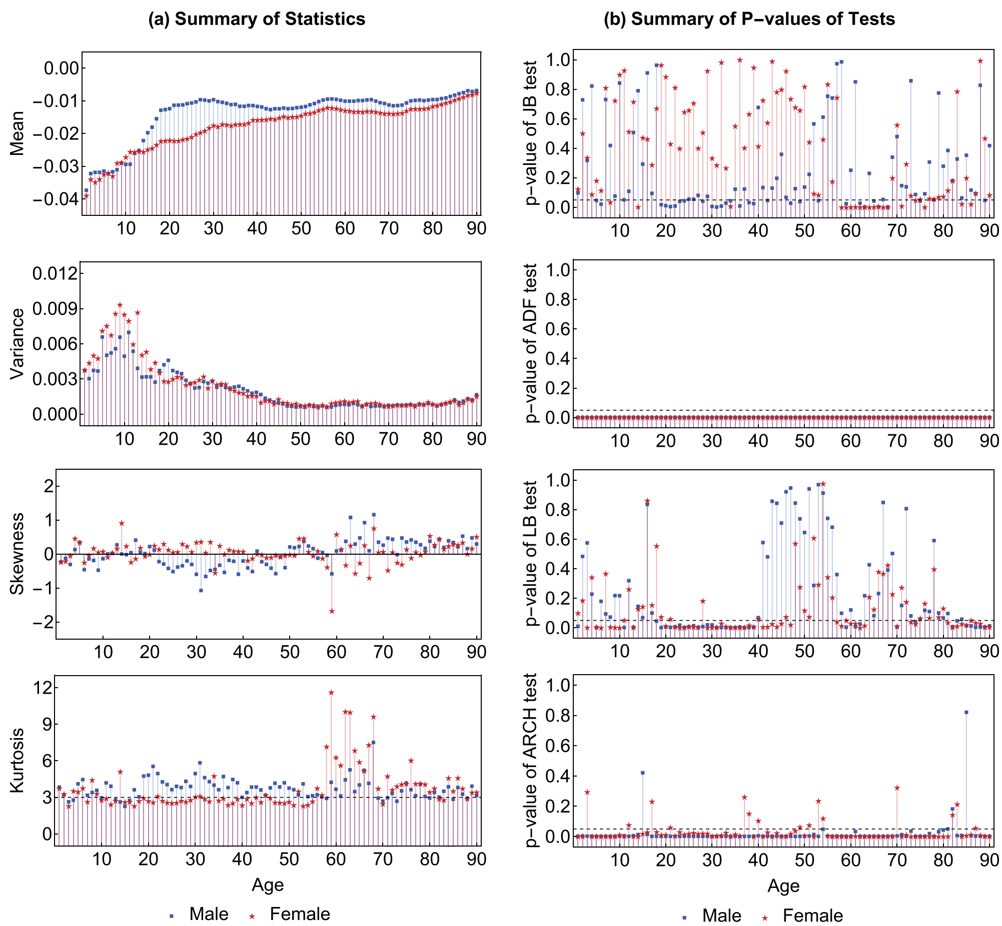

Fig. 4. The summary of (a) statistics and (b) p-values of hypothesis tests for the yearly log mortality rate changes. The statistics include mean, variance, skewness, and kurtosis. Meanwhile, the tests are the Jarque–Bera (JB) test of normality, the augmented Dickey–Fuller (ADF) test of stationarity, the Ljung–Box (LB) test of serial correlation, and the Engle's ARCH test of conditional heteroscedasticity effect. The dashed line on the right column represents the 5% level of significance.

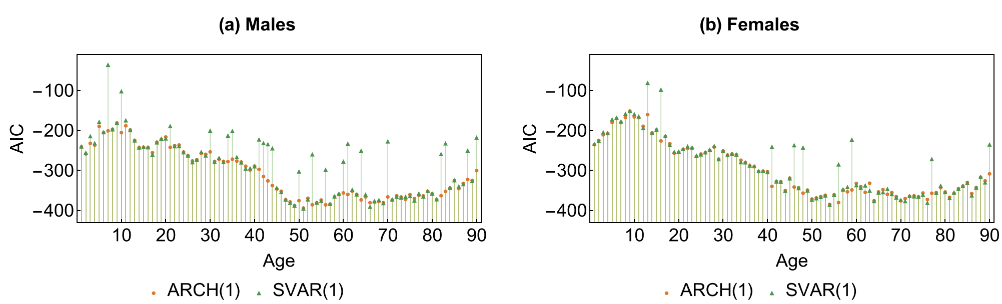

Fig. 5. The values of AIC of the ARCH(1) and SVAR(1) models for the error processes for (a) males and (b) females. The AIC value of each model is computed as follows: $\text{AIC} = -2\ell(\hat{\theta}_x) + 2p$, where $\ell(\hat{\theta}_x)$ denotes the maximized (conditional) log-likelihood function evaluated at the estimated parameter $\hat{\theta}_x$ and p is the number of the components of such $\hat{\theta}_x$.

Table 1. The number and the percentage (in parentheses) of lower values of AIC between the ARCH(1) and SVAR(1) models. The higher percentage is in boldface.

Model	Male	Female
ARCH(1)	66 (**73.33%**)	61 (**67.78%**)
SVAR(1)	24 (26.67%)	29 (32.22%)

The above proposed models of AR(1)-ARCH(1) and AR(1)-SVAR(1) are then employed to forecast the one-year MaR. The former produces the conditional MaR forecast whilst the latter accounts for its unconditional distribution. The MaR forecast for each model is found through two approaches. The first is substituting the estimated parameter vector to the MaR formula, so that we have an estimative MaR forecast. Meanwhile, through the second approach, an improved MaR forecast is obtained by modifying the resulting estimative MaR forecast. The computation is conducted for each fixed age at a 99.50% level of confidence.

The resulting estimative and improved MaR forecasts for both genders are transformed through the natural logarithm, as shown in Fig. 6. It is found that the log of the estimative MaR forecast does not obviously differ from the log of the improved one computed based on the AR(1)-ARCH(1) model. Meanwhile, when the AR(1)-SVAR(1) model is considered, the estimative and improved MaR forecasts exhibit significantly different values at the same age. The log of the MaR forecast is low for younger males and females at the age around 10 and below 20. As the age increases, the log of the MaR forecast goes up, suggest-

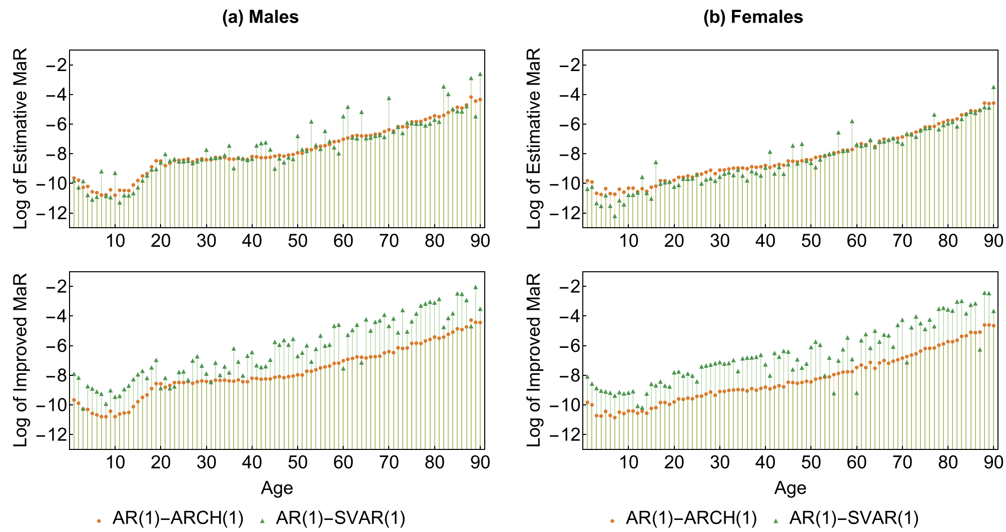

Fig. 6. The log of the estimative and improved MaR forecasts at a 99.50% level of confidence based on the AR(1) model plus the ARCH(1) and SVAR(1) terms for (a) males and (b) females. The estimative and improved MaR forecasts for the former heteroscedastic model are computed according to Eqs. (11) and (13). Meanwhile, for the latter, we compute such forecasts based on Eqs. (15) and (16).

ing that a greater fall of mortality rates is experienced by older people. This result is in line with the behavior of the (log) mortality rates themselves.

4. Concluding remark

Stochastic models for the changes in the log mortality rates have been constructed by accounting for the conditional heteroscedasticity effect in the conditional variance or volatility. Other stylized facts in the mortality rate data may be interesting and important to investigate and, hence, available models may also be extended to capture such facts. Furthermore, the (vine) copula method perhaps will provide a better goodness-of-fit of the models to the data by involving mortality rate dependence. The dependence may be analyzed between male and female individuals, among different years, among different ages, or among different states/countries in a certain region; see, e.g., Zhou and Ji (2021).

Mortality-at-Risk (MaR) we propose is basically a probability-based risk measure since it is determined by using the quantile of the fall of mortality rates at a certain level of confidence. This risk measure may be modified by employing (i) the expected-based framework, that accounts for the potential fall of mortality rates beyond the MaR, or (ii) an expectile, that may be more sensitive to the magnitude of extreme falls; see, e.g., Syuhada et al. (2021).

Declarations

Author contribution statement

K. Syuhada: Conceived and designed the experiments; Analyzed and interpreted the data; Wrote the paper.

A. Hakim: Performed the experiments; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Funding statement

This work was supported by Institut Teknologi Bandung/Kemenristek BRIN research grant (PDUPT).

Data availability statement

Data will be made available on request.

Declaration of interests statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

Acknowledgements

We are indebted to Prof. Steven Haberman (City, University of London) and Salsabila (Institut Teknologi Bandung) for a thoughtful discussion.

References

Chai, C.M.H., Siu, T.K., Zhou, X., 2013. A double-exponential GARCH model for stochastic mortality. Eur. Actuar. J. 3, 385–406.
Engle, R.F., 1982. Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. Econometrica 50 (4), 987–1007.
Giacometti, R., Betocchi, M., Rachev, S.T., Fabozzi, F.J., 2012. A comparison of the Lee-Carter model and AR-ARCH model for forecasting mortality rates. Insur. Math. Econ. 50 (1), 85–93.
Kabaila, P., Syuhada, K., 2008. Improved prediction limits for AR(p) and ARCH(p) processes. J. Time Ser. Anal. 29 (2), 213–223.
Kabaila, P., Syuhada, K., 2010. The asymptotic efficiency of improved prediction intervals. Stat. Probab. Lett. 80 (17–18), 1348–1353.
Lee, R.D., Carter, L.R., 1992. Modeling and forecasting US mortality. J. Am. Stat. Assoc. 87 (419), 659–671.
Lin, T., Wang, C.-W., Tsai, C.C.-L., 2015. Age-specific copula-AR-GARCH mortality models. Insur. Math. Econ. 61, 110–124.
Plat, R., 2011. One-year Value-at-Risk for longevity and mortality. Insur. Math. Econ. 49, 462–470.
Richards, S.J., Currie, I.D., Ritchie, G.P., 2013. A Value-at-Risk framework for longevity trend risk. Br. Actuar. J. 19 (1), 116–139.
Syuhada, K., 2020. The improved Value-at-Risk for heteroscedastic processes and their coverage probability. J. Probab. Stat. 2020, 7638517.
Syuhada, K., Hakim, A., Nur'aini, R., 2021. The expected-based value-at-risk and expected shortfall using quantile and expectile with application to electricity market data. Commun. Stat., Simul. Comput. https://doi.org/10.1080/03610918.2021.1928191.
Taylor, S.J., 1986. Modeling Financial Time Series. Wiley, Chichester.
Vidoni, P., 2004. Improved prediction intervals for stochastic process models. J. Time Ser. Anal. 25 (1), 137–154.
Wilmoth, J.R., Andreev, K., Jdanov, D., Glei, D.A., Riffe, T., Boe, C., et al., 2021. Methods protocol for the human mortality database (version 6). Available at: https://www.mortality.org/Public/Docs/MethodsProtocol.pdf.
Zhou, R., Ji, M., 2021. Modelling mortality dependence: an application of dynamic vine copula. Insur. Math. Econ. 99, 241–255.