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Decision-making algorithm with complex hesitant fuzzy partitioned maclaurin symmetric mean aggregation operators and SWARA method

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Complex hesitant fuzzy sets (CHFSs) have emerged as a powerful tool for addressing uncertainty, especially in representing two-dimensional information through multiple possible values. This study addresses the limitations in existing methodologies by introducing two novel operators: the complex hesitant fuzzy partitioned Maclaurin symmetric mean (CHFPMSM) and the complex hesitant fuzzy weighted partitioned Maclaurin symmetric mean (CHFWPMSM). These operators enable effective aggregation of criteria by organizing them into independent partitions based on inherent properties, enhancing their applicability to real-world multi-criteria decision-making (MCDM) problems. To validate the reliability of these operators, essential properties such as idempotency, monotonicity, and boundedness are verified. The study further extends the stepwise weight assessment ratio analysis (SWARA) method to the CHFS framework, facilitating the derivation of attribute weights under uncertain and complex conditions. A robust MCDM methodology is then proposed, integrating the newly developed operators and the extended SWARA approach to address two-dimensional decision-making challenges effectively. The proposed methodology is applied to a practical case study of selecting the best supplier for electronic goods among five alternatives. Comprehensive sensitivity analysis is conducted to examine the stability of the decision-making process against variations in criteria weights. Additionally, a comparative analysis underscores the novelty and efficiency of the proposed methodology by benchmarking its results against existing approaches.

Keywords Complex hesitant fuzzy set, Partitioned operators, Supplier selection, Decision analysis.

The multiple criteria decision-making (MCDM) process involves evaluating a finite set of alternatives and ranking them based on their overall suitability for decision-makers (DnMs), while considering all criteria simultaneously. This evaluation typically incorporates both objective data and the subjective judgments of experts. In many cases, the available data is assumed to be precise. However, with the increasing complexity of modern systems, MCDM problems often involve information that is ambiguous, imprecise, or uncertain. To handle such situations, the concept of fuzzy sets (FS)[1] has been extensively studied. Since its introduction, numerous researchers have explored and expanded this concept in various domains[2]. In particular, distance and similarity measures have proven to be effective tools for assessing the degree of difference between pairs of FSs. These measures are widely applied in decision-making[3], medical diagnosis[4], and pattern recognition[5]. Wang[6] introduced similarity measures based on FSs, where membership grades are represented within the interval [0,1], providing a valuable means of capturing human opinions in a graded form. Over the past few decades, FS theory has attracted significant research attention[7–9].

Many researchers have explored the implications of extending the range of FS to include complex numbers within a unit disc on the complex plane, instead of being limited to real numbers. Ramot et al.[10] addressed this by introducing the concept of complex FS (CFS), where the grade of membership is represented as a complex number within a unit disc. CFS effectively handles two-dimensional information within a single set and has emerged as a

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significant tool for expressing human judgment in graded terms. In recent years, CFS has received considerable attention. Li and Chiang[11,12] investigated complex neuro-fuzzy systems and their function approximation. Yazdanbakhsh and Dick[13] conducted a systematic review of CFS and its logic, while Nguyen et al.[14] further developed the concept. However, the framework proposed by Ramot et al. remains the most comprehensive for DnMs. Building on the idea of CFS, Garg et al.[15] extended it to hesitant FS (HFS)[16], introducing the concept of complex HFS (CHFS) by associating multiple possible values from [0,1] as phase terms with hesitant fuzzy elements. This allows CHFS to express a richer set of human opinions, especially when hesitation or uncertainty exists in both magnitude and phase. Despite their structural complexity, CHFS retains the core advantage of HFS - enabling DnMs to flexibly express multiple evaluations under each criterion. Thus, CHFS provides a more nuanced and realistic representation of hesitation while preserving the interpretability and expressive power required in decision-making contexts. Garg et al.[15] further investigated various distance measures, analyzed their properties, and explored practical applications, enhancing the flexibility and applicability of CHFS in real-world decision-making. Khan et al.[17] explored priority degrees along with various types of distance measures, applying them effectively to medical diagnosis problems. Mahmood et al.[18] examined both exponential and non-exponential generalized similarity measures for CHFS, demonstrating their effectiveness through several numerical examples. Additionally, Talafha et al.[19] proposed an algorithm based on CHF operators and detailed its applications. Overall, CHFS offers a more robust and general framework than prevailing approaches such as FS, CFS, and HFS for handling complex and uncertain data in real-world decision-making, although research in this area remains relatively limited.

The significance of AOs in addressing decision-making problems has garnered considerable scholarly attention, particularly in the context of MCDM[20–25]. Among these, operators that consolidate multiple input variables into a single output are widely recognized. The literature encompasses a diverse range of aggregation techniques for MCDM applied across various fuzzy environments. Despite the distinct nature of these AOs, real-world problems often involve interrelated criteria that cannot be ignored. Consequently, certain operators account for correlations among the combined arguments. To explore these interdependencies, Bonferroni[26] introduced the Bonferroni Mean (BM) operator in 1950, which has since been applied to various decision-making challenges. Xu et al.[27] expanded the BM operator within an intuitionistic fuzzy framework, highlighting its beneficial properties. Liu et al.[28] further developed q-rung orthopair fuzzy BM operators and demonstrated their applicability to MCDM issues. Additionally, Liu et al.[29] introduced interaction partitioned Bonferroni mean (IFIPBM) operations for intuitionistic fuzzy (IF) numbers. In a parallel effort to capture correlations between pairs of arguments, Sykora et al.[30] introduced the Heronian Mean (HM) operator, with Liu et al.[31] adapting it to linguistic IFNs in MCDM contexts. Mo et al.[32] proposed Archimedean geometric HM aggregation operators based on dual hesitant fuzzy sets, which have also been applied to MCDM problems.

The Maclaurin symmetric mean (MSM), initially developed by Maclaurin[33], offers a broader representation of interrelationships among multiple input variables through its aggregation function, unlike BM and HM, which are limited to pairwise correlations. Detemple and Robertson[34] later refined the MSM operator, making it particularly effective in MCDM scenarios where attributes are independent. The MSM has gained increasing interest in recent years, leading to significant theoretical and practical advancements, particularly within the framework of FS theory. Researchers have extended the MSM concept to various fuzzy environments, including IF sets[35,36] and partitioned MSM operators[37,38]. Unlike traditional AOs, the MSM operator considers the relationships among multiple input factors, making it highly effective in providing flexible and reliable data combinations. This adaptability is especially valuable in MCDM situations where attributes vary significantly. Furthermore, partitioning data collections is essential for organizing and optimizing data storage and retrieval. Effective partitioning enhances data management by grouping related data, reducing redundancy, and improving query efficiency. It also boosts parallel processing capabilities, thereby enhancing system scalability and accelerating data processing.

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Furthermore, partitioning data collections is essential for organizing and optimizing data storage and retrieval. Effective partitioning enhances data management by grouping related data, reducing redundancy, and improving query efficiency. It also improves parallel processing capabilities, thereby enhancing system scalability and accelerating data processing.

In the MCDM procedure, the weighting of criteria is essential for DnMs. These weights are typically classified as either "objective" or "subjective"[39–41]. Objective weights are derived from decision matrices and are based on the data provided by DnMs[42,43]. In contrast, subjective weights reflect the opinions of DnMs regarding the relative importance of the criteria[40,44]. To calculate subjective weights,[39] introduced the "stepwise weight assessment ratio analysis (SWARA)" approach. This method is more straightforward and simpler compared to other weighting approaches such as the analytic hierarchy process and the best-worst method. Currently, Rani and Mishra[45] applied an MCDM methodology for selecting the ideal solar panel using Pythagorean FSs. Mishra et al.[46] employed a methodology combining SWARA and the complex proportional assessment (COPRAS) methods to evaluate bioenergy production methods using Intuitionistic FSs. Rani et al.[40] introduced a Pythagorean fuzzy SWARA model for assessing healthcare waste treatment methods. Furthermore, Hu et al.[47] integrated the SWARA approach with the ARAS method to analyze risks associated with the internet of things in supply chain management. Saeidi et al.[48] proposed an improved Pythagorean fuzzy SWARA technique to identify and prioritize sustainable human resource management factors in manufacturing companies. Despite these advancements, a significant limitation persists in current extensions of the SWARA method: they fail to adequately capture hesitation in both magnitude and phase, which are common in expert judgments. The CHF framework, which allows for the expression of multiple possible evaluations with associated phase terms, offers a solution to this limitation. This extension of SWARA provides a more precise and expressive decision-making framework by accounting for the richer nuances of human hesitation. Additionally, it enhances the effectiveness of decision-making by offering a flexible and robust approach that can handle varying levels of uncertainty, while maintaining interpretability and resilience in expert-driven evaluations.

In light of the aforementioned literature review, the present study is motivated by the following considerations.

i). While CHFSs provide a robust means of representing uncertain data in a two-dimensional format, existing CHF AOs primarily aggregate data without adequately correlating the input variables. This limitation undermines the reliability of decision-making, especially in complex MCDM problems where interdependencies between criteria are crucial.
ii). In practical scenarios, it's common to encounter situations where certain criteria are unrelated to others. For instance, in the context of company recruitment, various criteria such as professional capabilities ($\sqcup_1$) , professional quality ($\sqcup_2$), communication skills ($\sqcup_3$), moral character ($\sqcup_4$), and internship experience ($\sqcup_5$) may be considered. Typically, $\sqcup_1$ and $\sqcup_2$ are correlated with $\sqcup_5$, while $\sqcup_3$ and $\sqcup_4$ are independent of $\sqcup_1$, $\sqcup_2$, and $\sqcup_5$. This leads to a partitioning of criteria, where interrelationships exist within the same partition but not across different ones. However, this probe hasn't been explored in the context of CHFSs, and existing operators lack the capability to address such specific partitioning scenarios.
iii). Provision of criteria weight information through precise data in MCDM is often impractical. This is because uncertainty stems from subjective assessments made by DnMs, who may have varying opinions influenced by their unique experiences and educational backgrounds. In addressing such decision problems, CHFS proves to be a more effective approach than traditional fuzzy sets due to its adeptness at handling uncertain information. However, the existing literature falls short in developing subjective criteria weight determination approaches, particularly the SWARA method, which aims to derive weight vectors from DnMs' complex hesitant fuzzy assessments of criteria.

In response to the identified gaps, this study proposes the following key contributions:

i). The study introduces novel AOs using CHFSs that capture the interconnections between input data and effectively handle partitioned criteria in decision-making problems.
ii). The study develops and presents theorems, properties, and specific cases of the proposed AOs to validate their mathematical soundness.
iii). A new extension of the SWARA method, called CHF-SWARA, is introduced to calculate the weights of criteria in MCDM problems based on hesitant fuzzy assessments provided by DnMs.
iv). The study proposes a comprehensive decision-making methodology that combines the new AOs with the CHF-SWARA method to process complex decision matrices and identify the optimal solutions.
v). A case study on supplier selection demonstrates the practical applicability and effectiveness of the proposed methodology, showcasing its real-world relevance and utility.

In light of the above contributions, the present study seeks to answer the following research questions:

RQ1. How can CHFSs be effectively utilized to develop AOs that account for interrelationships among criteria in MCDM problems?
RQ2. Can the partitioning of criteria improve the reliability and interpretability of the aggregation process under uncertain and complex conditions?
RQ3. How can the SWARA method be adapted within the CHF environment to determine subjective weights when precise data is unavailable?
RQ4. Does the integration of the proposed CHFPMSM and CHFWPMSM operators with the extended CHF-SWARA method offer a more robust and practical framework for solving real-world decision-making problems like supplier selection?

The developed work is represented by the flowchart in Fig. 1.

The rest of the parts are arranged as follows: Section "Fundamental concepts" provides a review of fundamental concepts. In Section "Proposed partition MSM operators", the innovative theories behind the CHFPMSM and CHFWPMSM operators are introduced, along with a discussion of their specific cases. Section "Decision-making technique using CHFWPMSM operators" presents the proposed technique, which integrates the SWARA method with the newly devised operators. Section "Case study" applies this methodology to a case study focused on supplier selection, followed by a comprehensive analysis that underscores the necessity and effectiveness of the designed operators. Finally, Section "Conclusions" offers concluding remarks and suggests directions for future research.

Fundamental concepts

In this section, we establish the foundational concepts of HFS and CHFS.

Definition 1 [16] For a given universe of discourse U, a HFS B is defined as follows:

$$\mathrm{B} = \{(u, \mathscr{L}(u)) \,|u \in U\}, \tag{1}$$

$\mathscr{L}$ is a function that assigns a membership degree in the interval $[0,1]$ to each element $u \in U$, indicating its degree of membership in B.

For simplicity, $\mathscr{L} = \mathscr{L}(u)$ is denoted as a hesitant fuzzy element (HFE), and B represents the collection of all HFEs.

Definition 2 [49] For a given HFE $\mathscr{L}$, the score function is characterized as follows:

$$S(\mathscr{L}) = \sum_{l=1}^{\#\mathscr{L}} \mu_l / \#\mathscr{L}, \tag{2}$$

where $\#\mathscr{L}$ signifies the cardinality of $\mathscr{L}$.

For any two HFEs $\mathscr{L}_1$ and $\mathscr{L}_2$, if $S(\mathscr{L}_1) > S(\mathscr{L}_2)$, then $\mathscr{L}_1 > \mathscr{L}_2$; if $S(\mathscr{L}_1) = S(\mathscr{L}_2)$, then $\mathscr{L}_1 = \mathscr{L}_2$; if $S(\mathscr{L}_1) < S(\mathscr{L}_2)$, then $\mathscr{L}_1 < \mathscr{L}_2$.

For any two HFEs $\mathscr{L}_1$, and $\mathscr{L}_2$, Torra[16] and Xia and Xu[50] described the following operation rules:

Definition 3 [16,50] Let $\mathscr{L}_1$, and $\mathscr{L}_2$ be two HFEs and $\lambda > 0$, then

1. $\mathscr{L}_1 \oplus \mathscr{L}_2 = \bigcup_{\substack{k=1,2,\ldots,\#\mathscr{L}_1,\\ l=1,2,\ldots,\#\mathscr{L}_2}} \{\mu_k + \nu_l - \mu_k\mu_l\}$;

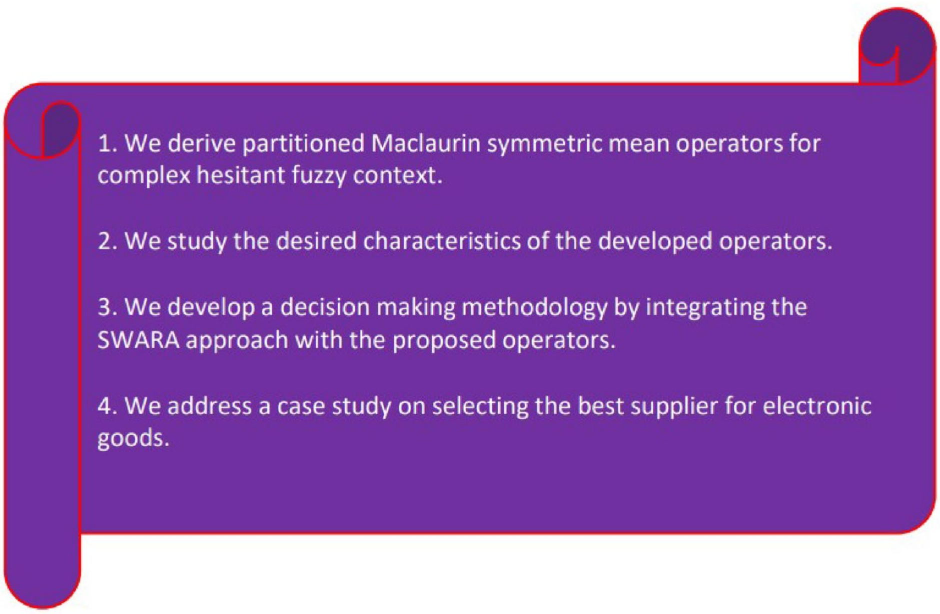

Fig. 1. Graphical illustration of the developed work.

2. $\mathscr{L}_1 \otimes \mathscr{L}_2 = \bigcup_{\substack{k=1,2,\ldots,\#\mathscr{L}_1,\\ l=1,2,\ldots,\#\mathscr{L}_2}} \{\mu_k \mu_l\}$;
3. $\lambda \mathscr{L}_1 = \bigcup_{k=1,2,\ldots,\#\mathscr{L}_1} \left\{1-(1-\mu_k)^{\lambda}\right\}$;
4. $\mathscr{L}_1^{\lambda} = \bigcup_{k=1,2,\ldots,\#\mathscr{L}_1} \left\{{\mu_k}^{\lambda}\right\}$;
5. $(\mathscr{L}_1)^c = \bigcup_{k=1,2,\ldots,\#\mathscr{L}_1} \{1-\mu_k\}$.

Definition 4 [15] For a given universe of discourse U, a CHFS $\mathcal{S}$ is defined as follows:

$$\mathcal{S} = \left\{\left(s, \mu(s) = \mu_r(s)\, \mathrm{e}^{\iota 2\pi \mu_c(s)}\right) | s \in \mathcal{S}\right\}, \tag{3}$$

where μ_r and μ_c are functions taking values in $[0, 1]$, which represent the membership grade of amplitude term and phase term of the element $s \in U$ to the set $\mathcal{S}$.

For convenience, $\mathscr{H} = \left(s, \mu(s) = \mu_r(s)\, \mathrm{e}^{\iota 2\pi \mu_c(s)}\right)$ is named as a CHF element (CHFE), and $\mathcal{S}$ is the set of all CHFEs.

Definition 5 [19] The score function for a CHFE $\mathscr{H}$ is given by

$$S(\mathscr{H}) = \frac{1}{2}\left(\frac{1}{\#\mu_r}\sum_{h\in\mu_r} h + \frac{1}{\#\mu_c}\sum_{ß\in\mu_c} ß\right), \tag{4}$$

where $\#\mu_r$ and $\#\mu_c$ denote the cardinality of μ_r and μ_c, respectively.

Definition 6 [19] For any two CHFEs $\mathscr{H}_1$ and $\mathscr{H}_2$, denoted by $\mathscr{H}_1 > \mathscr{H}_2$ rely on the criteria that $S(\mathscr{H}_1) > S(\mathscr{H}_2)$.

Definition 7 [15] Let $\mathscr{H}_1$, and $\mathscr{H}_2$ be any two CHFEs and $\lambda > 0$, then

1. $\mathscr{H}_1 \oplus \mathscr{H}_2 = \bigcup_{h_1 \in \mu_{r_1}, h_2 \in \mu_{r_2} ß_1 \in \mu_{c_1}, ß_2 \in \mu_{c_2}} \left\{(h_1 + h_2 - h_1 h_2)\, \mathrm{e}^{\iota 2\pi(ß_1 + ß_2 - ß_1 ß_2)}\right\}$;
2. $\mathscr{H}_1 \otimes \mathscr{H}_2 = \bigcup_{h_1 \in \mu_{r_1}, h_2 \in \mu_{r_2} ß_1 \in \mu_{c_1}, ß_2 \in \mu_{c_2}} \left\{(h_1 h_2)\, \mathrm{e}^{\iota 2\pi(ß_1 ß_2)}\right\}$;
3. $\lambda \mathscr{H}_1 = \bigcup_{h_1 \in \mu_{r_1}, ß_1 \in \mu_{c_1}} \left\{\left(1-(1-h_1)^{\lambda}\right) \mathrm{e}^{\iota 2\pi\left(1-(1-ß_1)^{\lambda}\right)}\right\}$;
4. $\mathscr{H}_1^{\lambda} = \bigcup_{h_1 \in \mu_{r_1}, ß_1 \in \mu_{c_1}} \left\{\left(h_1^{\lambda}\right) \mathrm{e}^{\iota 2\pi\left(ß_1^{\lambda}\right)}\right\}$;
5. $\mathscr{H}_1^{c} = \bigcup_{h_1 \in \mu_{r_1}, ß_1 \in \mu_{c_1}} \left\{(1-h_1)\, \mathrm{e}^{\iota 2\pi(1-ß_1)}\right\}$.

Proposed partition MSM operators

This section presents the implementation of the PMSM operator within the CHF model, introducing two operators-CHFPMSM and CHFWPMSM-and highlighting their key properties.

CHFPMSM aggregation operator

Definition 8 Let $\mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_n$ be a range of *n* CHFEs. Then, CHFPMSM operator is characterized as

$$CHFPMSM^{(\Lambda)}(\mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_n) = \frac{1}{\flat} \oplus_{\Upsilon=1}^{\flat} \left(\frac{\oplus_{1\le \mathrm{s}_1 < \ldots < \mathrm{s}_\varrho \le o_\Upsilon} \left(\otimes_{\varrho=1}^{\Lambda} \mathscr{H}_{\mathrm{s}_\varrho}\right)}{C_{o_\Upsilon}^{\Lambda}}\right)^{\frac{1}{\Lambda}}, \tag{5}$$

where $\flat$ denotes the number of categories, Λ is a parameter, $\Lambda = 1, 2, \ldots, o_\Upsilon$, o_Υ denotes the number of criteria in category $\flat_\Upsilon$, $(\mathrm{s}_1, \mathrm{s}_2, \ldots, \mathrm{s}_\Lambda)$ includes all the Λ-tuples of $(1, 2, \ldots, o_\Upsilon)$, $C_{o_\Upsilon}^{\Lambda}$ expresses the binomial coefficient, whose expression is $C_{o_\Upsilon}^{\Lambda} = \frac{o_\Upsilon!}{\Lambda!(o_\Upsilon - \Lambda)!}$.

Theorem 1 *For given CHFEs $\mathscr{H}_{\mathrm{s}}$ $(\mathrm{s} = 1, 2, \ldots, n)$. The aggregated result of Formula (5) is also a CHFE, described as in Eq. (6):*

$$CHFPMSM^{(\Lambda)}\left(\mathscr{H}_1,\mathscr{H}_2,...,\mathscr{H}_n\right)= \bigcup_{\substack{h_{S_\varrho}\in\mu_{r_{S_\varrho}},\\ \text{ß}_{S_\varrho}\in\mu_{c_{S_\varrho}}}} \left\{\begin{array}{l} 1-\left(\prod\limits_{\Upsilon=1}^{\flat}\left(1-\left(1-\left(\prod\limits_{1\le S_1<\ldots\le o_\Upsilon}\left(1-\prod\limits_{\varrho=1}^{\Lambda}h_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\Upsilon}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}},\\ e^{\iota 2\pi\left(1-\left(\prod\limits_{\Upsilon=1}^{\flat}\left(1-\left(1-\left(\prod\limits_{1\le S_1<\ldots\le o_\Upsilon}\left(1-\prod\limits_{\varrho=1}^{\Lambda}\text{ß}_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\Upsilon}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}}\right)} \end{array}\right\}. \quad (6)$$

Proof By the operational laws of *CHFEs*, we can write

$$\otimes_{\varrho=1}^{\Lambda}\mathscr{H}_{S_\varrho}= \bigcup_{\substack{h_{S_\varrho}\in\mu_{r_{S_\varrho}},\\ \text{ß}_{S_\varrho}\in\mu_{c_{S_\varrho}}}} \left\{\prod_{\varrho=1}^{\Lambda}h_{S_\varrho}, e^{\iota 2\pi\left(\prod\limits_{\varrho=1}^{\Lambda}\text{ß}_{S_\varrho}\right)}\right\}$$

$$\underset{1\le S_1<\ldots<S_k\le o_\Upsilon}{\oplus}\left(\otimes_{\varrho=1}^{\Lambda}\mathscr{H}_{S_\varrho}\right)= \bigcup_{\substack{h_{S_\varrho}\in\mu_{r_{S_\varrho}},\\ \text{ß}_{S_\varrho}\in\mu_{c_{S_\varrho}}}} \left\{\begin{array}{l} 1-\prod\limits_{1\le S_1<\ldots\le o_\Upsilon}\left(1-\prod\limits_{\varrho=1}^{\Lambda}h_{S_\varrho}\right),\\ e^{\iota 2\pi\left(1-\prod\limits_{1\le S_1<\ldots\le o_\Upsilon}\left(1-\prod\limits_{\varrho=1}^{\Lambda}\text{ß}_{S_\varrho}\right)\right)} \end{array}\right\}$$

$$\left(\frac{\oplus_{1\le S_1<\ldots<S_k\le 0_\Upsilon}\left(\otimes_{l=1}^{\Lambda}\mathscr{H}_{S_\varrho}\right)}{C^{\Lambda}_{o_\Upsilon}}\right)^{\frac{1}{\Lambda}}= \bigcup_{\substack{h_{S_\varrho}\in\mu_{r_{S_\varrho}},\\ \text{ß}_{S_\varrho}\in\mu_{c_{S_\varrho}}}} \left\{\begin{array}{l} \left(1-\left(\prod\limits_{1\le S_1<\ldots\le o_\Upsilon}\left(1-\prod\limits_{\varrho=1}^{\Lambda}h_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\Upsilon}}}\right)^{\frac{1}{\Lambda}},\\ e^{\iota 2\pi\left(1-\left(\prod\limits_{1\le S_1<\ldots\le o_\Upsilon}\left(1-\prod\limits_{\varrho=1}^{\Lambda}\text{ß}_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\Upsilon}}}\right)^{\frac{1}{\Lambda}}} \end{array}\right\}$$

$$\oplus_{\Upsilon=1}^{\flat}\left(\frac{\oplus_{1\le S_1<\ldots<S_k\le 0_\Upsilon}\left(\otimes_{\varrho=1}^{\Lambda}\mathscr{H}_{S_\varrho}\right)}{C^{\Lambda}_{o_\Upsilon}}\right)^{\frac{1}{\Lambda}}= \bigcup_{\substack{h_{S_\varrho}\in\mu_{r_{S_\varrho}},\\ \text{ß}_{S_\varrho}\in\mu_{c_{S_\varrho}}}} \left\{\begin{array}{l} 1-\prod\limits_{\Upsilon=1}^{\flat}\left(1-\left(1-\left(\prod\limits_{1\le S_1<\ldots\le o_\Upsilon}\left(1-\prod\limits_{\varrho=1}^{\Lambda}h_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\Upsilon}}}\right)^{\frac{1}{\Lambda}}\right),\\ e^{\iota 2\pi\left(1-\prod\limits_{\Upsilon=1}^{\flat}\left(1-\left(1-\left(\prod\limits_{1\le S_1<\ldots\le o_\Upsilon}\left(1-\prod\limits_{\varrho=1}^{\Lambda}\text{ß}_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\Upsilon}}}\right)^{\frac{1}{\Lambda}}\right)\right)} \end{array}\right\}$$

$$\frac{1}{\flat}\oplus_{\Upsilon=1}^{\flat}\left(\frac{\oplus_{1\le S_1<\ldots<S_k\le 0_\Upsilon}\left(\otimes_{\varrho=1}^{\Lambda}\mathscr{H}_{S_\varrho}\right)}{C^{\Lambda}_{o_\Upsilon}}\right)^{\frac{1}{\Lambda}}= \bigcup_{\substack{h_{S_\varrho}\in\mu_{r_{S_\varrho}},\\ \text{ß}_{S_\varrho}\in\mu_{c_{S_\varrho}}}} \left\{\begin{array}{l} 1-\left(\prod\limits_{\Upsilon=1}^{\flat}\left(1-\left(1-\left(\prod\limits_{1\le S_1<\ldots\le o_\Upsilon}\left(1-\prod\limits_{\varrho=1}^{\Lambda}h_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\Upsilon}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}},\\ e^{\iota 2\pi\left(1-\left(\prod\limits_{\Upsilon=1}^{\flat}\left(1-\left(1-\left(\prod\limits_{1\le S_1<\ldots\le o_\Upsilon}\left(1-\prod\limits_{\varrho=1}^{\Lambda}\text{ß}_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\Upsilon}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}}\right)} \end{array}\right\}.$$

This completes the verification.

□

Similarly, we can observe that the CHFPMSM operator possesses several key properties, such as idempotency, monotonicity, and boundedness.

Theorem 2 (Idempotency) *If the given CHFEs* $\mathscr{H}_{\mathrm{S}}(\mathrm{s}=1,2,3,...,n)$ *are all equal, that is,* $\mathscr{H}_{\mathrm{S}}=\mathscr{H}\,(\mathrm{s}=1,2,...,n)$ *. Then*

$$CHFPMSM^{(\Lambda)}\left(\mathscr{H}_1,\mathscr{H}_2,...,\mathscr{H}_n\right)=\mathscr{H}. \tag{7}$$

Proof By Eq. (6), we have

$$\begin{aligned}
&CHFPMSM^{(\Lambda)}\left(\mathscr{H}_1,\mathscr{H}_2,...,\mathscr{H}_n\right)\\
&=\bigcup_{\substack{h_{\mathrm{S}_\varrho}\in\mu_{r_{\mathrm{S}_\varrho}},\\ \text{ß}_{\mathrm{S}_\varrho}\in\mu_{c_{\mathrm{S}_\varrho}}}}\left\{\begin{array}{l}1-\left(\prod_{\Upsilon=1}^{\flat}\left(1-\left(1-\left(\prod_{1\le \mathrm{S}_1<\ldots\le o_\Upsilon}\left(1-\prod_{\varrho=1}^{\Lambda}h_{\mathrm{S}_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\Upsilon}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}},\\
\mathrm{e}^{\iota 2\pi\left(1-\left(\prod_{\Upsilon=1}^{\flat}\left(1-\left(1-\left(\prod_{1\le \mathrm{S}_1<\ldots\le o_\Upsilon}\left(1-\prod_{\varrho=1}^{\Lambda}\text{ß}_{\mathrm{S}_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\Upsilon}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}}\right)}\end{array}\right\}\\
&=\bigcup_{\substack{h_{\mathrm{S}_\varrho}\in\mu_{r_{\mathrm{S}_\varrho}},\\ \text{ß}_{\mathrm{S}_\varrho}\in\mu_{c_{\mathrm{S}_\varrho}}}}\left\{\begin{array}{l}1-\left(\prod_{\Upsilon=1}^{\flat}\left(1-\left(1-\left(\prod_{1\le \mathrm{S}_1<\ldots\le o_\Upsilon}\left(1-h^{\Lambda}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\Upsilon}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}},\\
\mathrm{e}^{\iota 2\pi\left(1-\left(\prod_{\Upsilon=1}^{\flat}\left(1-\left(1-\left(\prod_{1\le \mathrm{S}_1<\ldots\le o_\Upsilon}\left(1-\text{ß}^{\Lambda}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\Upsilon}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}}\right)}\end{array}\right\}\\
&=\bigcup_{\substack{h_{\mathrm{S}_\varrho}\in\mu_{r_{\mathrm{S}_\varrho}},\\ \text{ß}_{\mathrm{S}_\varrho}\in\mu_{c_{\mathrm{S}_\varrho}}}}\left\{\begin{array}{l}1-\left(\prod_{\Upsilon=1}^{\flat}\left(1-\left(1-\left(\left(1-h^{\Lambda}\right)^{C^{\Lambda}_{o_\Upsilon}}\right)^{\frac{1}{C^{\Lambda}_{o_\Upsilon}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}},\\
\mathrm{e}^{\iota 2\pi\left(1-\left(\prod_{\Upsilon=1}^{\flat}\left(1-\left(1-\left(\left(1-\text{ß}^{\Lambda}\right)^{C^{\Lambda}_{o_\Upsilon}}\right)^{\frac{1}{C^{\Lambda}_{o_\Upsilon}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}}\right)}\end{array}\right\}\\
&=\bigcup_{\substack{h_{\mathrm{S}_\varrho}\in\mu_{r_{\mathrm{S}_\varrho}},\\ \text{ß}_{\mathrm{S}_\varrho}\in\mu_{c_{\mathrm{S}_\varrho}}}}\left\{\begin{array}{l}1-\left(\prod_{\Upsilon=1}^{\flat}\left(1-\left(1-\left(1-h^{\Lambda}\right)\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}},\\
\mathrm{e}^{\iota 2\pi\left(1-\left(\prod_{\Upsilon=1}^{\flat}\left(1-\left(1-\left(1-\text{ß}^{\Lambda}\right)\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}}\right)}\end{array}\right\}\\
&=\bigcup_{\substack{h_{\mathrm{S}_\varrho}\in\mu_{r_{\mathrm{S}_\varrho}},\\ \text{ß}_{\mathrm{S}_\varrho}\in\mu_{c_{\mathrm{S}_\varrho}}}}\left\{1-\left(\prod_{\Upsilon=1}^{\flat}\left(1-h\right)\right)^{\frac{1}{\flat}},\mathrm{e}^{\iota 2\pi\left(1-\left(\prod_{\Upsilon=1}^{\flat}\left(1-\text{ß}\right)\right)^{\frac{1}{\flat}}\right)}\right\}\\
&=\bigcup_{h\in\mathscr{H},\text{ß}\in\mathscr{H}}\left(h,\mathrm{e}^{\iota 2\pi \text{ß}}\right)=\mathscr{H}.
\end{aligned}$$

□

Theorem 3 (Monotonicity) *Let* $\mathscr{H}_{\mathrm{S}}=\left(h_{\mathrm{S}},\mathrm{e}^{\iota 2\pi \text{ß}_{\mathrm{S}}}\right)$ *and* $\mathscr{H}^{*}_{\mathrm{S}}=\left(h^{*}_{\mathrm{S}},\mathrm{e}^{\iota 2\pi \text{ß}^{*}_{\mathrm{S}}}\right)$ *be two ranges of CHFEs, for* $\mathrm{s}=1,2,...,n$ *such that* $h_{\mathrm{S}}\ge h^{*}_{\mathrm{S}}$ *and* $\text{ß}_{\mathrm{S}}\ge \text{ß}^{*}_{\mathrm{S}}$*, then*

$$CHFPMSM^{(\Lambda)}\left(\mathscr{H}_1,\mathscr{H}_2,...,\mathscr{H}_n\right)\ge CHFPMSM^{(\Lambda)}\left(\mathscr{H}^{*}_1,\mathscr{H}^{*}_2,...,\mathscr{H}^{*}_n\right). \tag{8}$$

Proof As $h_{\mathrm{S}}\ge h^{*}_{\mathrm{S}}$ and $\text{ß}_{\mathrm{S}}\ge \text{ß}^{*}_{\mathrm{S}}$ for all s, we have $\prod_{\varrho=1}^{\Lambda}h_{\mathrm{S}_\varrho}\ge\prod_{\varrho=1}^{\Lambda}h^{*}_{\mathrm{S}_\varrho}$

$$\Longrightarrow 1-\prod_{\varrho=1}^{\Lambda}h_{\mathrm{S}_\varrho}\le 1-\prod_{\varrho=1}^{\Lambda}h^{*}_{\mathrm{S}_\varrho}$$

$$\Longrightarrow \bigcup_{h_{S_\varrho}\in\mu_{r_{S_\varrho}}} \prod_{1\le S_1<\ldots\le o_\curlyvee} \left(1-\prod_{\varrho=1}^{\Lambda} h_{S_\varrho}\right) \le \bigcup_{h^*_{S_\varrho}\in\mathscr{H}^*_{S_\varrho}} \prod_{1\le S_1<\ldots\le o_\curlyvee} \left(1-\prod_{\varrho=1}^{\Lambda} h^*_{S_\varrho}\right)$$

$$\Longrightarrow \bigcup_{h_{S_\varrho}\in\mu_{r_{S_\varrho}}} \left(\prod_{1\le S_1<\ldots\le o_\curlyvee} \left(1-\prod_{\varrho=1}^{\Lambda} h_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\curlyvee}}} \le \bigcup_{h^*_{S_\varrho}\in\mathscr{H}^*_{S_\varrho}} \left(\prod_{1\le S_1<\ldots\le o_\curlyvee} \left(1-\prod_{\varrho=1}^{\Lambda} h^*_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\curlyvee}}}$$

$$\Longrightarrow \bigcup_{h_{S_\varrho}\in\mu_{r_{S_\varrho}}} \left(1-\left(\prod_{1\le S_1<\ldots\le o_\curlyvee} \left(1-\prod_{\varrho=1}^{\Lambda} h_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\curlyvee}}}\right)^{\frac{1}{\Lambda}}$$

$$\ge \bigcup_{h^*_{S_\varrho}\in\mathscr{H}^*_{S_\varrho}} \left(1-\left(\prod_{1\le S_1<\ldots\le o_\curlyvee} \left(1-\prod_{\varrho=1}^{\Lambda} h^*_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\curlyvee}}}\right)^{\frac{1}{\Lambda}}$$

$$\Longrightarrow \bigcup_{h_{S_\varrho}\in\mu_{r_{S_\varrho}}} \prod_{o_\curlyvee}^{\flat} \left(1-\left(1-\left(\prod_{1\le S_1<\ldots\le o_\curlyvee} \left(1-\prod_{\varrho=1}^{\Lambda} h_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\curlyvee}}}\right)^{\frac{1}{\Lambda}}\right)$$

$$\le \bigcup_{h^*_{S_\varrho}\in\mathscr{H}^*_{S_\varrho}} \prod_{o_\curlyvee}^{\flat} \left(1-\left(1-\left(\prod_{1\le S_1<\ldots\le o_\curlyvee} \left(1-\prod_{\varrho=1}^{\Lambda} h^*_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\curlyvee}}}\right)^{\frac{1}{\Lambda}}\right)$$

$$\Longrightarrow \bigcup_{h_{S_\varrho}\in\mu_{r_{S_\varrho}}} \left(1-\left(\prod_{o_\curlyvee}^{\flat}\left(1-\left(1-\left(\prod_{1\le S_1<\ldots\le o_\curlyvee} \left(1-\prod_{\varrho=1}^{\Lambda} h_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\curlyvee}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}}\right) \ge$$

$$\bigcup_{h^*_{S_\varrho}\in\mathscr{H}^*_{S_\varrho}} \left(1-\left(\prod_{o_\curlyvee}^{\flat}\left(1-\left(1-\left(\prod_{1\le S_1<\ldots\le o_\curlyvee} \left(1-\prod_{\varrho=1}^{\Lambda} h^*_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\curlyvee}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}}\right).$$

Similarly, we can get $\bigcup_{ß_{S_\varrho}\in\mu_{c_{S_\varrho}}} \left(1-\left(\prod_{o_\curlyvee}^{\flat}\left(1-\left(1-\left(\prod_{1\le S_1<\ldots\le o_\curlyvee} \left(1-\prod_{\varrho=1}^{\Lambda} ß_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\curlyvee}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}}\right)$

$$\ge \bigcup_{ß^*_{S_\varrho}\in\mathscr{H}^*_{S_\varrho}} \left(1-\left(\prod_{o_\curlyvee}^{\flat}\left(1-\left(1-\left(\prod_{1\le S_1<\ldots\le o_\curlyvee} \left(1-\prod_{\varrho=1}^{\Lambda} ß^*_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\curlyvee}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}}\right).$$

Now compare the values of $CHFPMSM^{(\Lambda)}(\mathscr{H}_1,\mathscr{H}_2,\ldots,\mathscr{H}_n)$ with the value of $CHFPMSM^{(\Lambda)}(\mathscr{H}_1^*,\mathscr{H}_2^*,\ldots,\mathscr{H}_n^*)$. Let $\mathscr{H}=\left(h,\mathrm{e}^{\iota 2\pi ß}\right)=CHFPMSM^{(\Lambda)}(\mathscr{H}_1,\mathscr{H}_2,\ldots,\mathscr{H}_n)$ and let $\mathscr{H}^*=(h^*,\mathrm{e}^{\iota 2\pi ß^*})=CHFPMSM^{(\Lambda)}(\mathscr{H}_1^*,\mathscr{H}_2^*,\ldots,\mathscr{H}_n^*)$. Then, we can get

$$h=\bigcup_{h_{S_\varrho}\in\mu_{r_{S_\varrho}}} \left(1-\left(\prod_{o_\curlyvee}^{\flat}\left(1-\left(1-\left(\prod_{1\le S_1<\ldots\le o_\curlyvee} \left(1-\prod_{\varrho=1}^{\Lambda} h_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\curlyvee}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}}\right)$$

$$h^*=\bigcup_{h^*_{S_\varrho}\in\mu^*_{r_{S_\varrho}}} \left(1-\left(\prod_{o_\curlyvee}^{\flat}\left(1-\left(1-\left(\prod_{1\le S_1<\ldots\le o_\curlyvee} \left(1-\prod_{\varrho=1}^{\Lambda} h^*_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\curlyvee}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}}\right)$$

$$ß=\bigcup_{ß_{S_\varrho}\in\mu_{r_{S_\varrho}}} \left(1-\left(\prod_{o_\curlyvee}^{\flat}\left(1-\left(1-\left(\prod_{1\le S_1<\ldots\le o_\curlyvee} \left(1-\prod_{\varrho=1}^{\Lambda} ß_{S_\varrho}\right)\right)^{\frac{1}{C^{\Lambda}_{o_\curlyvee}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}}\right)$$

$$\text{ß}^{*}=\bigcup_{\text{ß}^{*}_{\text{S}_{\varrho}}\in\mu^{*}_{r\text{S}_{\varrho}}}\left(1-\left(\prod_{o_{\curlyvee}}^{\flat}\left(1-\left(1-\left(\prod_{1\leq\text{S}_1<\ldots\leq o_{\curlyvee}}\left(1-\prod_{\varrho=1}^{\Lambda}\text{ß}^{*}_{\text{S}_{\varrho}}\right)\right)^{\frac{1}{C^{\Lambda}_{o_{\curlyvee}}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}}\right).$$

As a result to $h\geq h^{*}$ and $\text{ß}\geq\text{ß}^{*}$ in the aforementioned analysis, then $\mathscr{H}\geq\mathscr{H}^{*}$, that is $CHFPMSM^{(\Lambda)}\left(\mathscr{H}_1,\mathscr{H}_2,...,\mathscr{H}_n\right)\geq CHFPMSM^{(\Lambda)}\left(\mathscr{H}^{*}_1,\mathscr{H}^{*}_2,...,\mathscr{H}^{*}_n\right)$,which complete the verification of this characteristic. □

Theorem 4 (Boundedness) *For a given CHFEs suppose that* $\mathscr{H}^{-}=\min_{\text{S}}\mathscr{H}_{\text{S}}$ *and* $\mathscr{H}^{+}=\max_{\text{S}}\mathscr{H}_{\text{S}}$ *and* $(\text{s}=1,2,...,n)$*, then*

$$\mathscr{H}^{-}\leq CHFPMSM^{(\Lambda)}\left(\mathscr{H}_1,\mathscr{H}_2,...,\mathscr{H}_n\right)\leq\mathscr{H}^{+}. \tag{9}$$

Proof Given that $\mathscr{H}^{-}=\min_{\text{S}}\mathscr{H}_{\text{S}}\leq\mathscr{H}_{\text{S}}$ according to Theorems 2 and 3, we can express it as follows:

$\mathscr{H}^{-}=CHFPMSM^{(\Lambda)}\left(\mathscr{H}^{-},\mathscr{H}^{-},...,\mathscr{H}^{-}\right)\leq CHFPMSM^{(\Lambda)}\left(\mathscr{H}_1,\mathscr{H}_2,...,\mathscr{H}_n\right).$ Similarly, $CHFPMSM^{(\Lambda)}\left(\mathscr{H}_1,\mathscr{H}_2,...,\mathscr{H}_n\right)\leq CHFPMSM^{(\Lambda)}\left(\mathscr{H}^{+},\mathscr{H}^{+},...,\mathscr{H}^{+}\right)=\mathscr{H}^{+}$. Thus, we have $\mathscr{H}^{-}\leq CHFPMSM^{(\Lambda)}\left(\mathscr{H}_1,\mathscr{H}_2,...,\mathscr{H}_n\right)\leq\mathscr{H}^{+}$. □

Lemma 1 [51] *Let* $\varphi_{\text{S}}\geq 0$ *and* $\wp_{\text{S}}>0$ *meeting* $\sum_{\text{S}=1}^{n}\wp_{\text{S}}=1$*, then*

$$\prod_{\text{S}=1}^{n}\varphi_{\text{S}}^{\wp_{\text{S}}}\leq\sum_{\text{S}=1}^{n}\varphi_{\text{S}}\wp_{\text{S}}\ , \tag{10}$$

with equality if and only if $\varphi_1=\wp_1$, $\varphi_2=\wp_2=$,...,$\varphi_n=\wp_n$.

Theorem 5 (Parameter monotonicity) *For given range of CHFEs* $\mathscr{H}_{\text{S}}\left(\text{s}=1,2,...,n\right)$*, and* $\Lambda=1,2,...,\min_{\curlyvee}\{o_{\curlyvee}\}$*. The HFPMSM operator decreases as the parameter* Λ *increases.*

Proof On the basis of Eq. (6), we get

$$HFPMSM^{(\Lambda)}\left(\mathscr{H}_1,\mathscr{H}_2,...,\mathscr{H}_n\right)=$$
$$\bigcup_{\substack{h_{\text{S}_{\varrho}}\in\mu_{r\text{S}_{\varrho}},\\ \text{ß}_{\text{S}_{\varrho}}\in\mu_{c\text{S}_{\varrho}}}}\left\{\begin{array}{c}\left(1-\prod_{\curlyvee=1}^{\flat}\left(1-\left(1-\prod_{1\leq\text{S}_1<\ldots<\text{S}_k\leq o_{\curlyvee}}\left(1-\prod_{\varrho=1}^{\Lambda}h_{\text{S}_{\varrho}}\right)^{\frac{1}{C^{\Lambda}_{o_{\curlyvee}}}}\right)^{\frac{1}{\Lambda}}\right)^{\frac{1}{\flat}}\right),\\ e^{\iota 2\pi\left(1-\left(\prod_{\curlyvee=1}^{\flat}\left(1-\left(1-\left(\prod_{1\leq\text{S}_1<\ldots\leq o_{\curlyvee}}\left(1-\prod_{\varrho=1}^{\Lambda}\text{ß}_{\text{S}_{\varrho}}\right)\right)^{\frac{1}{C^{\Lambda}_{o_{\curlyvee}}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}}\right)}\end{array}\right\}.$$

For simplicity, let

$$\mathcal{G}_{\nu r}\left(\Lambda\right)=\left(1-\prod_{\curlyvee=1}^{\flat}\left(1-\left(1-\prod_{1\leq\text{S}_1<\ldots<\text{S}_k\leq o_{\curlyvee}}\left(1-\prod_{\varrho=1}^{\Lambda}h_{\text{S}_{\varrho}}\right)^{\frac{1}{C^{\Lambda}_{o_{\curlyvee}}}}\right)^{\frac{1}{\Lambda}}\right)^{\frac{1}{\flat}}\right),$$

and

$$\mathcal{G}_{\nu c}\left(\Lambda\right)=\left(1-\left(\prod_{\curlyvee=1}^{\flat}\left(1-\left(1-\left(\prod_{1\leq\text{S}_1<\ldots\leq o_{\curlyvee}}\left(1-\prod_{\varrho=1}^{\Lambda}\text{ß}_{\text{S}_{\varrho}}\right)\right)^{\frac{1}{C^{\Lambda}_{o_{\curlyvee}}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}}\right).$$

Utilizing Lemma 1, we have

$$\prod_{1\le S_1<\ldots<S_k\le o_\Upsilon}\left(1-\prod_{\varrho=1}^{\Lambda}h_{S_\varrho}\right)^{\frac{1}{C_{o_\Upsilon}^{\Lambda}}}\le\sum_{1\le S_1<\ldots<S_k\le o_\Upsilon}\frac{\left(1-\prod_{\varrho=1}^{\Lambda}h_{S_\varrho}\right)}{C_{o_\Upsilon}^{\Lambda}}$$

$$\Rightarrow 1-\prod_{1\le S_1<\ldots<S_k\le o_\Upsilon}\left(1-\prod_{\varrho=1}^{\Lambda}h_{S_\varrho}\right)^{\frac{1}{C_{o_\Upsilon}^{\Lambda}}}\ge 1-\sum_{1\le S_1<\ldots<S_k\le o_\Upsilon}\frac{\left(1-\prod_{\varrho=1}^{\Lambda}h_{S_\varrho}\right)}{C_{o_\Upsilon}^{\Lambda}}=\sum_{1\le S_1<\ldots<S_k\le o_\Upsilon}\frac{\prod_{\varrho=1}^{\Lambda}h_{S_\varrho}}{C_{o_\Upsilon}^{\Lambda}}$$

$$\Rightarrow\prod_{\Upsilon=1}^{\flat}\left(1-\left(1-\prod_{1\le S_1<\ldots<S_k\le o_\Upsilon}\left(1-\prod_{\varrho=1}^{\Lambda}h_{S_\varrho}\right)^{\frac{1}{C_{o_\Upsilon}^{\Lambda}}}\right)\right)^{\frac{1}{\flat}}\le\prod_{\Upsilon=1}^{\flat}\left(1-\left(\sum_{1\le S_1<\ldots<S_k\le o_r}\frac{\prod_{\varrho=1}^{\Lambda}h_{S_\varrho}}{C_{o_\Upsilon}^{\Lambda}}\right)\right)^{\frac{1}{\flat}}.$$

Therefore

$$\mathcal{G}_{\nu r}(\Lambda)=\left(1-\prod_{\Upsilon=1}^{\flat}\left(1-\left(1-\prod_{1\le S_1<\ldots<S_k\le o_\Upsilon}\left(1-\prod_{\varrho=1}^{\Lambda}h_{S_\varrho}\right)^{\frac{1}{C_{o_\Upsilon}^{\Lambda}}}\right)^{\frac{1}{\Lambda}}\right)^{\frac{1}{\flat}}\right)$$

$$\ge\left(1-\prod_{\Upsilon=1}^{\flat}\left(1-\left(\sum_{1\le S_1<\ldots<S_k\le o_\Upsilon}\frac{\prod_{\varrho=1}^{\Lambda}h_{S_\varrho}}{C_{o_\Upsilon}^{\Lambda}}\right)\right)^{\frac{1}{\flat}}\right).$$

In what follows, we take the proof by contradiction process. Let $\mathcal{G}_{\nu r}(\Lambda)$ monotonically increases as Λ increases, then

$$\mathcal{G}_{\nu r}\left(\min_{\Upsilon}\{o_\Upsilon\}\right)>\ldots>\mathcal{G}_{\nu r}(2)>\mathcal{G}_{\nu r}(1)$$

and we can have

$$\mathcal{G}_{\nu r}(1)\ge\left(1-\prod_{\Upsilon=1}^{\flat}\left(1-\left(\sum_{1\le S_1<\ldots<S_k\le o_\Upsilon}\frac{\prod_{\varrho=1}^{1}h_{S_\varrho}}{C_{o_\Upsilon}^{1}}\right)\right)^{\frac{1}{\flat}}\right)=$$

$$\left(1-\prod_{\Upsilon=1}^{\flat}\left(1-\left(\sum_{S_\varrho=1}^{o_\Upsilon}\frac{h_{S_\varrho}}{o_\Upsilon}\right)\right)^{\frac{1}{\flat}}\right).$$

Further, we let $o_\Upsilon=o$, then we have

$$\mathcal{G}_{\nu r}\left(\min_{\Upsilon}\{o_\Upsilon\}\right)=G_{\nu r}(o)=\left(1-\prod_{\Upsilon=1}^{\flat}\left(1-\left(1-\prod_{1\le S_1<\ldots<S_k\le o}\left(1-\prod_{\varrho=1}^{o}h_{S_\varrho}\right)^{\frac{1}{C_o^o}}\right)^{\frac{1}{o}}\right)^{\frac{1}{\flat}}\right)$$

$$=\left(1-\prod_{\Upsilon=1}^{\flat}\left(1-\left(\prod_{\varrho=1}^{o}h_{S_\varrho}\right)^{\frac{1}{o}}\right)^{\frac{1}{\flat}}\right).$$

According to our assumption

$$\mathcal{G}_{\nu r}(o) = \left(1 - \prod_{\Upsilon=1}^{\flat}\left(1 - \left(\prod_{\varrho=1}^{o} h_{\mathrm{S}_\varrho}\right)^{\frac{1}{o}}\right)^{\frac{1}{\flat}}\right) > \mathcal{G}_{\nu r}(1)$$

$$\geq \left(1 - \prod_{\Upsilon=1}^{\flat}\left(1 - \left(\sum_{\mathrm{S}_\varrho=1}^{o_\Upsilon} \frac{h_{\mathrm{S}_\varrho}}{o_\Upsilon}\right)\right)^{\frac{1}{\flat}}\right)$$

$$= \left(1 - \prod_{\Upsilon=1}^{\flat}\left(1 - \left(\sum_{\mathrm{S}_\varrho=1}^{o} \frac{h_{\mathrm{S}_\varrho}}{o}\right)\right)^{\frac{1}{\flat}}\right)$$

$$\Rightarrow \left(\prod_{\varrho=1}^{o} h_{\mathrm{S}_\varrho}\right)^{\frac{1}{o}} > \sum_{\mathrm{S}_\varrho=1}^{o} \frac{h_{\mathrm{S}_\varrho}}{o}.$$

Clearly, $\left(\prod_{\varrho=1}^{o} h_{\mathrm{S}_\varrho}\right)^{\frac{1}{o}} > \sum_{\mathrm{S}_\varrho=1}^{o} \frac{h_{\mathrm{S}_\varrho}}{o}$ is contradiction to the Lemma 1. Thus, as the Λ increases, function $\mathcal{G}_{\nu r}(\Lambda)$ is monotonically decreasing.

Similarly, we can show that as the Λ increases, function $\mathcal{G}_{\nu c}(\Lambda)$ is also monotonically decreasing.

In accordance with the above analysis, we have

$HFPMSM^{(\Lambda)}(\mathscr{H}_1, \mathscr{H}_2, ..., \mathscr{H}_n) > HFPMSM^{(\Lambda+1)}(\mathscr{H}_1, \mathscr{H}_2, ..., \mathscr{H}_n)$.

Thereby, the HFPMSM operator monotonically decreases with respect to the parameter Λ. □

Theorem 6 *For given range of CHFEs* $\mathscr{H}_{\mathrm{S}} = (\mathrm{s} = 1, 2, ..., n)$, $\Lambda = 1, 2, ..., \min_{\Upsilon} o_\Upsilon$. *Then*

$$\min\left\{CHFPMSM^{(\Lambda)}(\mathscr{H}_1, \mathscr{H}_2, ..., \mathscr{H}_n)\right\} = CHFPMSM^{(\min_{\Upsilon}\{o_\Upsilon\})}(\mathscr{H}_1, \mathscr{H}_2, ..., \mathscr{H}_n)$$

and

$$\max\left\{CHFPMSM^{(\Lambda)}(\mathscr{H}_1, \mathscr{H}_2, ..., \mathscr{H}_n)\right\} = CHFPMSM^{(1)}(\mathscr{H}_1, \mathscr{H}_2, ..., \mathscr{H}_n) =$$

$$= \bigcup_{\substack{h_{\mathrm{S}_\varrho} \in \mu_{r\mathrm{S}_\varrho},\\ \text{ß}_{\mathrm{S}_\varrho} \in \mu_{c\mathrm{S}_\varrho}}} \left\{ 1 - \left(\prod_{\Upsilon=1}^{\flat}\left(\prod_{1<\mathrm{S}_1<\ldots<o_\Upsilon}\left(1 - h_{\mathrm{S}_\varrho}\right)\right)^{\frac{1}{o_\Upsilon}}\right)^{\frac{1}{\flat}}, e^{\iota 2\pi\left(1 - \left(\prod_{\Upsilon=1}^{\flat}\left(\prod_{1<\mathrm{S}_1<\ldots<o_\Upsilon}\left(1-\text{ß}_{\mathrm{S}_\varrho}\right)\right)^{\frac{1}{o_\Upsilon}}\right)^{\frac{1}{\flat}}\right)} \right\}.$$

Proof The proof of Theorem 6 is excluded as it can be readily derived in line with Theorem 5. □

We now examine numerous peculiar instances of the *CHFPMSM* operator concerning various parameter values.

Case 1: When there is only one category, i.e., $\flat = 1$, then according to the developed *CHFPMSM* operator, we have

$$CHFPMSM^{(\Lambda)}(\mathscr{H}_1, \mathscr{H}_2, ..., \mathscr{H}_n) =$$

$$\bigcup_{\substack{u_{\mathrm{S}_\varrho} \in \mu_{r\mathrm{S}_\varrho},\\ \text{ß}_{\mathrm{S}_\varrho} \in \mu_{c\mathrm{S}_\varrho}}} \left\{ \begin{array}{l} 1 - \left(\prod_{\Upsilon=1}^{1}\left(1 - \left(1 - \left(\prod_{1\leq\mathrm{S}_1<\ldots\leq o_\Upsilon}\left(1 - \prod_{\varrho=1}^{\Lambda} h_{\mathrm{S}_\varrho}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\Lambda}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{1}}, \\ e^{\iota 2\pi\left(1 - \left(\prod_{\Upsilon=1}^{1}\left(1 - \left(1 - \left(\prod_{1\leq\mathrm{S}_1<\ldots\leq o_\Upsilon}\left(1 - \prod_{\varrho=1}^{\Lambda} \text{ß}_{\mathrm{S}_\varrho}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\Lambda}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{1}}\right)} \end{array} \right\}$$

$$= \bigcup_{\substack{h_{S_\varrho} \in \mu_{r_{S_\varrho}}, \\ ß_{S_\varrho} \in \mu_{c_{S_\varrho}}}} \left\{ \begin{array}{c} \left(1-\left(\prod_{1\le S_1<\ldots\le o_\curlyvee}\left(1-\prod_{\varrho=1}^{\Lambda} h_{S_\varrho}\right)\right)^{\frac{1}{C_{o_\curlyvee}^{\Lambda}}}\right)^{\frac{1}{\Lambda}}, \\ e^{\iota 2\pi\left(1-\left(\prod_{1\le S_1<\ldots\le o_\curlyvee}\left(1-\prod_{\varrho=1}^{\Lambda} ß_{S_\varrho}\right)\right)^{\frac{1}{C_{o_\curlyvee}^{\Lambda}}}\right)^{\frac{1}{\Lambda}}} \end{array} \right\}. \tag{11}$$

This is the CHF MSM operator.

Case 2: When $\flat = 1$ and $\Lambda = 1$, then according to the developed *CHFPMSMS* operator, we have

$$CHFPMSM^{(1)}\left(\mathscr{H}_1, \mathscr{H}_2, ..., \mathscr{H}_n\right) =$$

$$\bigcup_{\substack{h_{S_\varrho} \in \mu_{r_{S_\varrho}}, \\ ß_{S_\varrho} \in \mu_{c_{S_\varrho}}}} \left\{ \begin{array}{c} 1-\left(\prod_{\curlyvee=1}^{1}\left(1-\left(1-\left(\prod_{1\le S_1<\ldots\le o_\curlyvee}\left(1-\prod_{\varrho=1}^{1} h_{S_\varrho}\right)\right)^{\frac{1}{C_{o_\curlyvee}^{1}}}\right)^{\frac{1}{1}}\right)\right)^{\frac{1}{1}}, \\ e^{\iota 2\pi\left(1-\left(\prod_{\curlyvee=1}^{1}\left(1-\left(1-\left(\prod_{1\le S_1<\ldots\le o_\curlyvee}\left(1-\prod_{\varrho=1}^{1} ß_{S_\varrho}\right)\right)^{\frac{1}{C_{o_\curlyvee}^{1}}}\right)^{\frac{1}{1}}\right)\right)^{\frac{1}{1}}\right)} \end{array} \right\}$$

$$= \bigcup_{\substack{h_{S_1} \in \mu_{r_{S_1}}, \\ ß_{S_1} \in \mu_{c_{S_1}}}} \left\{ \begin{array}{c} 1-\left(\prod_{1\le S_1\le o_\curlyvee}\left(1-h_{S_1}\right)\right)^{\frac{1}{C_{o_\curlyvee}^{1}}}, \\ e^{\iota 2\pi\left(1-\left(\prod_{1\le S_1\le o_\curlyvee}\left(1-ß_{S_1}\right)\right)^{\frac{1}{C_{o_\curlyvee}^{1}}}\right)} \end{array} \right\} \text{let } (s_1 = k)$$

$$= \bigcup_{\substack{h_k \in \mu_{r_k}, \\ ß_k \in \mu_{c_k}}} \left\{ \begin{array}{c} 1-\left(\prod_{k=1}^{o_\curlyvee}\left(1-h_k\right)\right)^{\frac{1}{o_\curlyvee}}, \\ e^{\iota 2\pi\left(1-\left(\prod_{k=1}^{o_\curlyvee}\left(1-ß_k\right)\right)^{\frac{1}{o_\curlyvee}}\right)} \end{array} \right\}, \tag{12}$$

which is reduced to CHF averaging operator.

Case 3: When $\flat = 1$ and $\Lambda = 2$, then the definition of *CHFPMSM* operator, we have

$$CHFPMSM^{(2)}\left(\mathscr{H}_1, \mathscr{H}_2, ..., \mathscr{H}_n\right) =$$

$$\bigcup_{\substack{h_{S_\varrho} \in \mu_{r_{S_\varrho}}, \\ ß_{S_\varrho} \in \mu_{c_{S_\varrho}}}} \left\{ \begin{array}{c} 1-\left(\prod_{\curlyvee=1}^{1}\left(1-\left(1-\left(\prod_{1\le S_1<\ldots\le o_\curlyvee}\left(1-\prod_{\varrho=1}^{2} h_{S_\varrho}\right)\right)^{\frac{1}{C_{o_\curlyvee}^{2}}}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{1}}, \\ e^{\iota 2\pi\left(1-\left(\prod_{\curlyvee=1}^{1}\left(1-\left(1-\left(\prod_{1\le S_1<\ldots\le o_\curlyvee}\left(1-\prod_{\varrho=1}^{2} ß_{S_\varrho}\right)\right)^{\frac{1}{C_{o_\curlyvee}^{2}}}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{1}}\right)} \end{array} \right\}$$

$$= \bigcup_{\substack{h_{S_\varrho} \in \mu_{r_{S_\varrho}}, \\ ß_{S_\varrho} \in \mu_{c_{S_\varrho}}}} \left\{ \begin{array}{c} \left(1-\left(\prod_{1\le S_1<S_2\le o_\curlyvee}\left(1-h_{S_1}h_{S_2}\right)\right)^{\frac{1}{C_{o_\curlyvee}^{2}}}\right)^{\frac{1}{2}}, \\ e^{\iota 2\pi\left(1-\left(\prod_{1\le S_1<S_2\le o_\curlyvee}\left(1-ß_{S_1}ß_{S_2}\right)\right)^{\frac{1}{C_{o_\curlyvee}^{2}}}\right)^{\frac{1}{2}}} \end{array} \right\}$$

$$= \bigcup_{\substack{h_{S_\varrho} \in \mu_{r_{S_\varrho}}, \\ ß_{S_\varrho} \in \mu_{c_{S_\varrho}}}} \left\{ \begin{array}{l} \left(1-\left(\prod_{1 \le S_1 < S_2 \le o_\Upsilon} (1-h_{S_1} h_{S_2})\right)^{\frac{2}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}}, \\ e^{\iota 2\pi \left(1-\left(\prod_{1 \le S_1 < S_2 \le o_\Upsilon} (1-ß_{S_1} ß_{S_2})\right)^{\frac{2}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}}} \end{array} \right\}$$

$$= \bigcup_{\substack{h_{S_\varrho} \in \mu_{r_{S_\varrho}}, \\ ß_{S_\varrho} \in \mu_{c_{S_\varrho}}}} \left\{ \begin{array}{l} \left(1-\left(\prod_{S_1,S_2=1;S_1 \ne S_2}^{o_\Upsilon} (1-h_{S_1} h_{S_2})\right)^{\frac{1}{2}\cdot\frac{2}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}}, \\ e^{\iota 2\pi \left(1-\left(\prod_{S_1,S_2=1;S_1 \ne S_2}^{o_\Upsilon} (1-ß_{S_1} ß_{S_2})\right)^{\frac{1}{2}\cdot\frac{2}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}}} \end{array} \right\}$$

$$= \bigcup_{\substack{h_{S_\varrho} \in \mu_{r_{S_\varrho}}, \\ ß_{S_\varrho} \in \mu_{c_{S_\varrho}}}} \left\{ \begin{array}{l} \left(1-\left(\prod_{S_1,S_2=1;S_1 \ne S_2}^{o_\Upsilon} (1-h_{S_1} h_{S_2})\right)^{\frac{1}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}}, \\ e^{\iota 2\pi \left(1-\left(\prod_{S_1,S_2=1;S_1 \ne S_2}^{o_\Upsilon} (1-ß_{S_1} ß_{S_2})\right)^{\frac{1}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}}} \end{array} \right\}, \tag{13}$$

which is reduced to CHF Bonferroni mean operator $CHFBM^{(1,1)}(\mathscr{H}_1, \mathscr{H}_2, ..., \mathscr{H}_n)$.

Case 4: When $\flat = 1$ and $\Lambda = n$, then according to the definition of *CHFPMSM* operator, we have

$$CHFPMSM^{(n)}(\mathscr{H}_1, \mathscr{H}_2, ..., \mathscr{H}_n) =$$

$$\bigcup_{\substack{h_{S_\varrho} \in \mu_{r_{S_\varrho}}, \\ ß_{S_\varrho} \in \mu_{c_{S_\varrho}}}} \left\{ \begin{array}{l} 1-\left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1 \le S_1 < \ldots S_\Lambda \le n}\left(1-\prod_{\varrho=1}^{n} h_{S_\varrho}\right)\right)^{\frac{1}{C_n^n}}\right)^{\frac{1}{n}}\right)\right)^{\frac{1}{1}}, \\ e^{\iota 2\pi \left(1-\left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1 \le S_1 < \ldots S_\Lambda \le n}\left(1-\prod_{\varrho=1}^{n} ß_{S_\varrho}\right)\right)^{\frac{1}{C_n^n}}\right)^{\frac{1}{n}}\right)\right)^{\frac{1}{1}}\right)} \end{array} \right\}$$

$$= \bigcup_{\substack{h_{S_\varrho} \in \mu_{r_{S_\varrho}}, \\ ß_{S_\varrho} \in \mu_{c_{S_\varrho}}}} \left\{ \left(1-\left(1-\prod_{\varrho=1}^{n} h_{S_\varrho}\right)\right)^{\frac{1}{n}}, e^{\iota 2\pi \left(1-\left(1-\prod_{\varrho=1}^{n} ß_{S_\varrho}\right)\right)^{\frac{1}{n}}} \right\}. \tag{14}$$

This is the CHF geometric mean operator.

CHFWPMSM aggregation operator

In this section, we introduce the CHFWPMSM operator. While it is assumed that all criteria are equally important, practical decision-making scenarios often involve varying weights for each criterion. Therefore, it is essential to assign a specific weight to each criterion. Let the weight of each criterion, denoted as $\mathscr{H}$s($s = 1, 2, \ldots, n$), be represented by $\mathfrak{w}$s, where $0 \le \mathfrak{w}s \le 1$ and $\sum^{n} s = 1\mathfrak{w}_S = 1$. The *CHFWPMSM* operator for *n CHFEs* is then defined as follows:

Definition 9 Let $\mathscr{H}_1, \mathscr{H}_2, ..., \mathscr{H}_n$ be the set of *n CHFEs*. Then, the *CHFWPMSM* operator is characterized as

$$CHFPMSM^{(\Lambda)}(\mathscr{H}_1, \mathscr{H}_2, ..., \mathscr{H}_n) = \frac{1}{\flat} \oplus_{\Upsilon=1}^{\flat} \left(\frac{\bigoplus_{1 \le S_1 < \ldots < S_l \le o_\Upsilon} \left(\otimes_{\varrho=1}^{\Lambda} (\mathscr{H}_{S_\varrho})^{\mathfrak{w}_{S_\varrho}} \right)}{C_{o_\Upsilon}^{\Lambda}} \right)^{\frac{1}{\Lambda}}, \tag{15}$$

where $\flat$ symbolizes the number of categories, Λ is a parameter, $\Lambda = 1, 2, ..., o_\Upsilon$, o_Υ denotes the number of criteria in category $\flat_\Upsilon$, $(\mathrm{s}_1, \mathrm{s}_2, ..., \mathrm{s}_\Lambda)$ includes all the Λ-tuples of $(1, 2, ..., o_\Upsilon)$, $C^\Lambda_{o_\Upsilon}$ expresses the binomial coefficient, whose expression is $C^\Lambda_{o_\Upsilon} = \frac{o_\Upsilon!}{\Lambda!(o_\Upsilon-\Lambda)!}$ and $\mathfrak{w}_\mathrm{S} > 0$ such that the weight satisfying $\sum_{\mathrm{S}=1}^{n} \mathfrak{w}_\mathrm{S} = 1$.

Theorem 7 *For given CHFEs* $\mathscr{H}_\mathrm{S}$ $(\mathrm{s} = 1, 2, ..., n)$. *The aggregated result of Formula* (15) *is also a CHFE, characterized as below:*

$$CHFWPMSM^{(\Lambda)}(\mathscr{H}_1, \mathscr{H}_2, ..., \mathscr{H}_n) = \bigcup_{\substack{h_{\mathrm{S}_\varrho} \in \mu_{r_{\mathrm{S}_\varrho}}, \\ \text{ß}_{\mathrm{S}_\varrho} \in \mu_{c_{\mathrm{S}_\varrho}}}} \left\{ \begin{array}{l} 1 - \left(\prod_{\Upsilon=1}^{\flat} \left(1 - \left(1 - \left(\prod_{1 \le \mathrm{S}_1 < \ldots \le o_\Upsilon} \left(1 - \prod_{\varrho=1}^{\Lambda} h_{\mathrm{S}_\varrho}^{\mathfrak{w}_{\mathrm{S}_\varrho}} \right) \right)^{\frac{1}{C^\Lambda_{o_\Upsilon}}} \right)^{\frac{1}{\Lambda}} \right) \right)^{\frac{1}{\flat}}, \\ e^{\iota 2\pi \left(1 - \left(\prod_{\Upsilon=1}^{\flat} \left(1 - \left(1 - \left(\prod_{1 \le \mathrm{S}_1 < \ldots \le o_\Upsilon} \left(1 - \prod_{\varrho=1}^{\Lambda} \text{ß}_{\mathrm{S}_\varrho}^{\mathfrak{w}_{\mathrm{S}_\varrho}} \right) \right)^{\frac{1}{C^\Lambda_{o_\Upsilon}}} \right)^{\frac{1}{\Lambda}} \right) \right)^{\frac{1}{\flat}} \right)} \end{array} \right\}. \tag{16}$$

Proof By the operational laws of *CHFEs*, we can write

$$\otimes_{\varrho=1}^{\Lambda} \left(\mathscr{H}_{\mathrm{S}_\varrho} \right)^{\mathfrak{w}_{\mathrm{S}_\varrho}} = \bigcup_{\substack{h_{\mathrm{S}_\varrho} \in \mu_{r_{\mathrm{S}_\varrho}}, \\ \text{ß}_{\mathrm{S}_\varrho} \in \mu_{c_{\mathrm{S}_\varrho}}}} \left\{ \prod_{\varrho=1}^{\Lambda} h_{\mathrm{S}_\varrho}^{\mathfrak{w}_{\mathrm{S}_\varrho}}, e^{\iota 2\pi \prod_{\varrho=1}^{\Lambda} \text{ß}_{\mathrm{S}_\varrho}^{\mathfrak{w}_{\mathrm{S}_\varrho}}} \right\}$$

$$\oplus_{1 \le \mathrm{S}_1 < \ldots < \mathrm{S}_k \le o_\Upsilon} \left(\otimes_{\varrho=1}^{\Lambda} \left(\mathscr{H}_{\mathrm{S}_\varrho} \right)^{\mathfrak{w}_{\mathrm{S}_\varrho}} \right) = \bigcup_{\substack{h_{\mathrm{S}_\varrho} \in \mu_{r_{\mathrm{S}_\varrho}}, \\ \text{ß}_{\mathrm{S}_\varrho} \in \mu_{c_{\mathrm{S}_\varrho}}}} \left\{ \begin{array}{l} 1 - \prod_{1 \le \mathrm{S}_1 < \ldots \le o_\Upsilon} \left(1 - \prod_{\varrho=1}^{\Lambda} h_{\mathrm{S}_\varrho}^{\mathfrak{w}_{\mathrm{S}_\varrho}} \right), \\ e^{\iota 2\pi \left(1 - \prod_{1 \le \mathrm{S}_1 < \ldots \le o_\Upsilon} \left(1 - \prod_{\varrho=1}^{\Lambda} \text{ß}_{\mathrm{S}_\varrho}^{\mathfrak{w}_{\mathrm{S}_\varrho}} \right) \right)} \end{array} \right\}$$

$$\left(\frac{\oplus_{1 \le \mathrm{S}_1 < \ldots < \mathrm{S}_k \le 0_\Upsilon} \left(\otimes_{\varrho=1}^{\Lambda} \left(\mathscr{H}_{\mathrm{S}_\varrho} \right)^{\mathfrak{w}_{\mathrm{S}_\varrho}} \right)}{C^\Lambda_{o_\Upsilon}} \right)^{\frac{1}{\Lambda}} = \bigcup_{\substack{\mathscr{H}_{\mathrm{S}_\varrho} \in \mu_{r_{\mathrm{S}_\varrho}}, \\ \text{ß}_{\mathrm{S}_\varrho} \in \mu_{c_{\mathrm{S}_\varrho}}}} \left\{ \begin{array}{l} \left(1 - \left(\prod_{1 \le \mathrm{S}_1 < \ldots \le o_\Upsilon} \left(1 - \prod_{\varrho=1}^{\Lambda} h_{\mathrm{S}_\varrho}^{\mathfrak{w}_{\mathrm{S}_\varrho}} \right) \right)^{\frac{1}{C^\Lambda_{o_\Upsilon}}} \right)^{\frac{1}{\Lambda}}, \\ e^{\iota 2\pi \left(1 - \left(\prod_{1 \le \mathrm{S}_1 < \ldots \le o_\Upsilon} \left(1 - \prod_{\varrho=1}^{\Lambda} \text{ß}_{\mathrm{S}_\varrho}^{\mathfrak{w}_{\mathrm{S}_\varrho}} \right) \right)^{\frac{1}{C^\Lambda_{o_\Upsilon}}} \right)^{\frac{1}{\Lambda}}} \end{array} \right\}$$

$$\otimes_{\Upsilon=1}^{\flat} \left(\frac{\oplus_{1 \le \mathrm{S}_1 < \ldots < \mathrm{S}_k \le 0_\Upsilon} \left(\otimes_{\varrho=1}^{\Lambda} \left(\mathscr{H}_{\mathrm{S}_\varrho} \right)^{\mathfrak{w}_{\mathrm{S}_\varrho}} \right)}{C^\Lambda_{o_\Upsilon}} \right)^{\frac{1}{\Lambda}} = \bigcup_{\substack{h_{\mathrm{S}_\varrho} \in \mu_{r_{\mathrm{S}_\varrho}}, \\ \text{ß}_{\mathrm{S}_\varrho} \in \mu_{c_{\mathrm{S}_\varrho}}}} \left\{ \begin{array}{l} 1 - \prod_{\Upsilon=1}^{\flat} \left(1 - \left(1 - \left(\prod_{1 \le \mathrm{S}_1 < \ldots \le o_\Upsilon} \left(1 - \prod_{\varrho=1}^{\Lambda} h_{\mathrm{S}_\varrho}^{\mathfrak{w}_{\mathrm{S}_\varrho}} \right) \right)^{\frac{1}{C^\Lambda_{o_\Upsilon}}} \right)^{\frac{1}{\Lambda}} \right), \\ e^{\iota 2\pi \left(1 - \prod_{\Upsilon=1}^{\flat} \left(1 - \left(1 - \left(\prod_{1 \le \mathrm{S}_1 < \ldots \le o_\Upsilon} \left(1 - \prod_{\varrho=1}^{\Lambda} \text{ß}_{\mathrm{S}_\varrho}^{\mathfrak{w}_{\mathrm{S}_\varrho}} \right) \right)^{\frac{1}{C^\Lambda_{o_\Upsilon}}} \right)^{\frac{1}{\Lambda}} \right) \right)} \end{array} \right\}$$

$$\frac{1}{\flat}\otimes_{\Upsilon=1}^{\flat}\left(\frac{\bigoplus_{1\le \mathrm{s}_1\le\ldots\le \mathrm{s}_k\le o_\Upsilon}\left(\otimes_{\varrho=1}^{\Lambda}\left(\mathscr{H}_{\mathrm{s}_\varrho}\right)^{\mathfrak{w}_{\mathrm{s}_\varrho}}\right)}{C_{o_\Upsilon}^{\Lambda}}\right)^{\frac{1}{\Lambda}}=$$

$$\bigcup_{\substack{h_{\mathrm{s}_\varrho}\in\mu_{r_{\mathrm{s}_\varrho}},\\ \text{ß}_{\mathrm{s}_\varrho}\in\mu_{c_{\mathrm{s}_\varrho}}}}\left\{\begin{array}{l}1-\left(\prod_{\Upsilon=1}^{\flat}\left(1-\left(1-\left(\prod_{1\le i_1<\ldots\le o_\Upsilon}\left(1-\prod_{\varrho=1}^{\Lambda}h_{\mathrm{s}_\varrho}^{\mathfrak{w}_{\mathrm{s}_\varrho}}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\Lambda}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}},\\ \mathrm{e}^{\iota 2\pi\left(1-\left(\prod_{\Upsilon=1}^{\flat}\left(1-\left(1-\left(\prod_{1\le \mathrm{s}_1<\ldots\le o_\Upsilon}\left(1-\prod_{\varrho=1}^{\Lambda}\text{ß}_{\mathrm{s}_\varrho}^{\mathfrak{w}_{\mathrm{s}_\varrho}}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\Lambda}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{\flat}}\right)}\end{array}\right\}.$$

This completes the verification.

□

We now explore several specific cases of the *CHFPMSM* operator, focusing on different values of the parameter ϱ.

Case 1: When there is only one category, i.e., $\flat=1$, then according to the developed CHFWPMSM operator, we have

$$CHFWPMSM^{(\Lambda)}\left(\mathscr{H}_1,\mathscr{H}_2,\ldots,\mathscr{H}_n\right)=$$

$$\bigcup_{\substack{h_{\mathrm{s}_\varrho}\in\mu_{r_{\mathrm{s}_\varrho}},\\ \text{ß}_{\mathrm{s}_\varrho}\in\mu_{c_{\mathrm{s}_\varrho}}}}\left\{\begin{array}{l}1-\left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1\le \mathrm{s}_1<\ldots\le o_\Upsilon}\left(1-\prod_{\varrho=1}^{\Lambda}h_{\mathrm{s}_\varrho}^{\mathfrak{w}_{\mathrm{s}_\varrho}}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\Lambda}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{1}},\\ \mathrm{e}^{\iota 2\pi\left(1-\left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1\le \mathrm{s}_1<\ldots\le o_\Upsilon}\left(1-\prod_{\varrho=1}^{\Lambda}\text{ß}_{\mathrm{s}_\varrho}^{\mathfrak{w}_{\mathrm{s}_\varrho}}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\Lambda}}}\right)^{\frac{1}{\Lambda}}\right)\right)^{\frac{1}{1}}\right)}\end{array}\right\}.$$

$$=\bigcup_{\substack{h_{\mathrm{s}_\varrho}\in\mu_{r_{\mathrm{s}_\varrho}},\\ \text{ß}_{\mathrm{s}_\varrho}\in\mu_{c_{\mathrm{s}_\varrho}}}}\left\{\begin{array}{l}\left(1-\left(\prod_{1\le \mathrm{s}_1<\ldots\le o_\Upsilon}\left(1-\prod_{\varrho=1}^{\Lambda}h_{\mathrm{s}_\varrho}^{\mathfrak{w}_{\mathrm{s}_\varrho}}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\Lambda}}}\right)^{\frac{1}{\Lambda}},\\ \mathrm{e}^{\iota 2\pi\left(1-\left(\prod_{1\le \mathrm{s}_1<\ldots\le o_\Upsilon}\left(1-\prod_{\varrho=1}^{\Lambda}\text{ß}_{\mathrm{s}_\varrho}^{\mathfrak{w}_{\mathrm{s}_\varrho}}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\Lambda}}}\right)^{\frac{1}{\Lambda}}}\end{array}\right\}, \tag{17}$$

which is reduced to the CHF WMSM operator.

Case 2: When $\flat=1$ and $\Lambda=1$, then based on the developed *CHFWPMSM* operator, we have

$$CHFWPMSM^{(\Lambda)}\left(\mathscr{H}_1,\mathscr{H}_2,\ldots,\mathscr{H}_n\right)=$$

$$\bigcup_{\substack{h_{\mathrm{s}_\varrho}\in\mu_{r_{\mathrm{s}_\varrho}},\\ \text{ß}_{\mathrm{s}_\varrho}\in\mu_{c_{\mathrm{s}_\varrho}}}}\left\{\begin{array}{l}1-\left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1\le \mathrm{s}_1<\ldots\le o_\Upsilon}\left(1-\prod_{\varrho=1}^{1}h_{\mathrm{s}_\varrho}^{\mathfrak{w}_{\mathrm{s}_\varrho}}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{1}}}\right)^{\frac{1}{1}}\right)\right)^{\frac{1}{1}},\\ \mathrm{e}^{\iota 2\pi\left(1-\left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1\le \mathrm{s}_1<\ldots\le o_\Upsilon}\left(1-\prod_{\varrho=1}^{1}\text{ß}_{\mathrm{s}_\varrho}^{\mathfrak{w}_{\mathrm{s}_\varrho}}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{1}}}\right)^{\frac{1}{1}}\right)\right)^{\frac{1}{1}}\right)}\end{array}\right\}.$$

$$=\bigcup_{\substack{h_{\mathrm{s}_\varrho}\in\mu_{r_{\mathrm{s}_\varrho}},\\ \text{ß}_{\mathrm{s}_\varrho}\in\mu_{c_{\mathrm{s}_\varrho}}}}\left\{\begin{array}{l}\left(1-\left(\prod_{1\le \mathrm{s}_1\le o_\Upsilon}\left(1-h_{\mathrm{s}_1}^{\mathfrak{w}_{\mathrm{s}_1}}\right)\right)^{\frac{1}{o_\Upsilon}}\right),\\ \mathrm{e}^{\iota 2\pi\left(1-\left(\prod_{1\le \mathrm{s}_1\le o_\Upsilon}\left(1-\text{ß}_{\mathrm{s}_1}^{\mathfrak{w}_{\mathrm{s}_1}}\right)\right)^{\frac{1}{o_\Upsilon}}\right)}\end{array}\right\}\text{let } \mathrm{s}_1=k$$

$$=\bigcup_{\substack{h_k\in\mu_{r_k},\\ \text{ß}_k\in\mu_{c_k}}}\left\{\begin{array}{l}\left(1-\left(\prod_{k=1}^{o_\Upsilon}\left(1-h_k^{\mathfrak{w}_k}\right)\right)^{\frac{1}{o_\Upsilon}}\right),\\ \mathrm{e}^{\iota 2\pi\left(1-\left(\prod_{k=1}^{o_\Upsilon}\left(1-\text{ß}_k^{\mathfrak{w}_k}\right)\right)^{\frac{1}{o_\Upsilon}}\right)}\end{array}\right\}, \tag{18}$$

which is reduced to CHF weighted averaging operator.

Case 3: When $\flat = 1$ and $\Lambda = 2$, then according to the definition of *CHFWPMSM* operator, we have

$$CHFWPMSM^{(2)}\left(\mathscr{H}_1, \mathscr{H}_2, ..., \mathscr{H}_n\right) = \bigcup_{\substack{h_{S_\varrho} \in \mu_{r_{S_\varrho}},\\ ß_{S_\varrho} \in \mu_{c_{S_\varrho}}}} \left\{ \begin{array}{l} 1-\left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1\le S_1<\ldots\le o_\Upsilon}\left(1-\prod_{\varrho=1}^{2} h_{S_\varrho}^{\mathfrak{w}_{S_\varrho}}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{2}}}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{1}}, \\ e^{\iota 2\pi\left(1-\left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1\le S_1<\ldots\le o_\Upsilon}\left(1-\prod_{\varrho=1}^{2} ß_{S_\varrho}^{\mathfrak{w}_{S_\varrho}}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{2}}}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{1}}\right)} \end{array} \right\}$$

$$= \bigcup_{\substack{h_{S_\varrho} \in \mu_{r_{S_\varrho}},\\ ß_{S_\varrho} \in \mu_{c_{S_\varrho}}}} \left\{ \begin{array}{l} \left(1-\left(\prod_{1\le S_1<S_2\le o_\Upsilon}\left(1-h_{S_1}^{\mathfrak{w}_{S_1}} h_{S_2}^{\mathfrak{w}_{S_2}}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{2}}}\right)^{\frac{1}{2}}, \\ e^{\iota 2\pi\left(1-\left(\prod_{1\le S_1<S_2\le o_\Upsilon}\left(1-ß_{S_1}^{\mathfrak{w}_{S_1}} ß_{S_2}^{\mathfrak{w}_{S_2}}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{2}}}\right)^{\frac{1}{2}}} \end{array} \right\}$$

$$= \bigcup_{\substack{h_{S_\varrho} \in \mu_{r_{S_\varrho}},\\ ß_{S_\varrho} \in \mu_{c_{S_\varrho}}}} \left\{ \begin{array}{l} \left(1-\left(\prod_{1\le S_1<S_2\le o_\Upsilon}\left(1-h_{S_1}^{\mathfrak{w}_{S_1}} h_{S_2}^{\mathfrak{w}_{S_2}}\right)\right)^{\frac{2}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}}, \\ e^{\iota 2\pi\left(1-\left(\prod_{1\le S_1<S_2\le o_\Upsilon}\left(1-ß_{S_1}^{\mathfrak{w}_{S_1}} ß_{S_2}^{\mathfrak{w}_{S_2}}\right)\right)^{\frac{2}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}}} \end{array} \right\}$$

$$= \bigcup_{\substack{h_{S_\varrho} \in \mu_{r_{S_\varrho}},\\ ß_{S_\varrho} \in \mu_{c_{S_\varrho}}}} \left\{ \begin{array}{l} \left(1-\left(\prod_{S_1,S_2=1;S_1\neq S_2}^{o_\Upsilon}\left(1-h_{S_1}^{\mathfrak{w}_{S_1}} h_{S_2}^{\mathfrak{w}_{S_2}}\right)\right)^{\frac{1}{2}\cdot\frac{2}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}}, \\ e^{\iota 2\pi\left(1-\left(\prod_{S_1,S_2=1;S_1\neq S_2}^{o_\Upsilon}\left(1-ß_{S_1}^{\mathfrak{w}_{S_1}} ß_{S_2}^{\mathfrak{w}_{S_2}}\right)\right)^{\frac{1}{2}\cdot\frac{2}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}}} \end{array} \right\}$$

$$= \bigcup_{\substack{h_{S_\varrho} \in \mu_{r_{S_\varrho}},\\ ß_{S_\varrho} \in \mu_{c_{S_\varrho}}}} \left\{ \begin{array}{l} \left(1-\left(\prod_{S_1,S_2=1;S_1\neq S_2}^{o_\Upsilon}\left(1-h_{S_1}^{\mathfrak{w}_{S_1}} h_{S_2}^{\mathfrak{w}_{S_2}}\right)\right)^{\frac{1}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}}, \\ e^{\iota 2\pi\left(1-\left(\prod_{S_1,S_2=1;S_1\neq S_2}^{o_\Upsilon}\left(1-ß_{S_1}^{\mathfrak{w}_{S_1}} ß_{S_2}^{\mathfrak{w}_{S_2}}\right)\right)^{\frac{1}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}}} \end{array} \right\}, \tag{19}$$

which is reduced to CHF weighted Bonferroni mean operator $CHFWBM^{(1,1)}\left(\mathscr{H}_1, \mathscr{H}_2, ..., \mathscr{H}_n\right)$.

Case 4: When $\flat = 1$ and $\Lambda = n$, then according to the definition of *CHFPMSM* operator, we have

$$CHFPMSM^{(n)}\left(\mathscr{H}_1, \mathscr{H}_2, ..., \mathscr{H}_n\right) = \bigcup_{\substack{h_{S_\varrho} \in \mu_{r_{S_\varrho}},\\ ß_{S_\varrho} \in \mu_{c_{S_\varrho}}}} \left\{ \begin{array}{l} 1-\left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1\le S_1<\ldots S_\Lambda\le n}\left(1-\prod_{\varrho=1}^{n} h_{S_\varrho}^{\mathfrak{w}_{S_\varrho}}\right)\right)^{\frac{1}{C_{n}^{n}}}\right)^{\frac{1}{n}}\right)\right)^{\frac{1}{1}}, \\ e^{\iota 2\pi\left(1-\left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1\le S_1<\ldots S_\Lambda\le n}\left(1-\prod_{\varrho=1}^{n} ß_{S_\varrho}^{\mathfrak{w}_{S_\varrho}}\right)\right)^{\frac{1}{C_{n}^{n}}}\right)^{\frac{1}{n}}\right)\right)^{\frac{1}{1}}\right)} \end{array} \right\}$$

$$= \bigcup_{\substack{h_{\mathrm{S}_{\varrho}} \in \mu_{r_{\mathrm{S}_{\varrho}}}, \\ ß_{\mathrm{S}_{\varrho}} \in \mu_{c_{\mathrm{S}_{\varrho}}}}} \left\{ \left(1-\left(1-\prod_{\varrho=1}^{n} h_{\mathrm{S}_{\varrho}}^{\mathfrak{w}_{\mathrm{S}_{\varrho}}}\right)\right)^{\frac{1}{n}}, e^{\iota 2\pi\left(1-\left(1-\prod_{\varrho=1}^{n} ß_{\mathrm{S}_{\varrho}}^{\mathfrak{w}_{\mathrm{S}_{\varrho}}}\right)\right)^{\frac{1}{n}}} \right\}. \tag{20}$$

This is the CHF weighted geometric mean operator.

Decision-making technique using CHFWPMSM operators

MCDM techniques are vital across multiple domains, such as engineering, networking, and industrial processes. Researchers employ these methods to make informed decisions by considering multiple criteria. These techniques are versatile tools that contribute to enhanced decision-making in various fields.This section aims to structure the MCDM technique around the defined CHF operators.

Let $\bigcirc = \{\bigcirc_1, \bigcirc_2, ..., \bigcirc_m\}$ represent the set of alternatives, $\sqcup = \{\sqcup_1, \sqcup_2, ..., \sqcup_n\}$ represent the set of criteria, and $\mathfrak{w}$=($\mathfrak{w}_1, \mathfrak{w}_2, ..., \mathfrak{w}_n$) be the weight vector associated with these criteria. The weight vector $\mathfrak{w}$ where $\sum_{g=1}^{n} \mathfrak{w}_g$=1 and $\mathfrak{w}_g \in [0, 1]$, reflects the relative importance of each criterion in the decision-making process. DnMs evaluate each alternative $\bigcirc_i$ against the criteria $\sqcup_g$ using CHF elements $\mathscr{H}_{i_g}$. Assume the criteria set $\sqcup = \{\sqcup_1, \sqcup_2, ..., \sqcup_n\}$ is divided into $\flat$ distinct categories $\{P_1, P_2, ..., P_\flat\}$. Criteria within the same category are interrelated, while no connections exist between criteria in different categories. The following steps outline the key stages of the proposed approach.

Step 1: Formation of the CHF Matrix: In light of the previously mentioned situation, the MCDM problem can be structured by creating the following decision matrix:

$$\mathrm{K}_{m\times n} = \begin{pmatrix} \mathscr{H}_{11} & \cdots & \mathscr{H}_{1g} & \cdots & \mathscr{H}_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathscr{H}_{i1} & \cdots & \mathscr{H}_{ig} & \cdots & \mathscr{H}_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathscr{H}_{m1} & \cdots & \mathscr{H}_{mg} & \cdots & \mathscr{H}_{mn} \end{pmatrix}. \tag{21}$$

Step 2: Normalization: At this stage, the CHF evaluation matrix $K_{p\times r}$ is adjusted to account for both benefit and cost criteria. These two types of criteria behave oppositely-benefit criteria perform better as their values increase, whereas cost criteria perform worse under the same conditions. Therefore, it is necessary to convert the cost criteria into benefit criteria. This is achieved by applying the following normalization method to ensure that all criteria are aligned accordingly.

$$\mathscr{H}_{ig}^{*} = \begin{cases} \mathscr{H}_{ig}, & \sqcup_g \text{ is a befit criterion,} \\ \left(\mathscr{H}_{ig}\right)^{c}, & \sqcup_g \text{ is cost criteria.} \end{cases} \tag{22}$$

Step 3: Weights determination by the SWARA method: The initial stage of the SWARA method involves ranking the criteria and comparing each criterion in pairs from highest to lowest. Subsequently, a relative coefficient needs to be computed. Then, the weight necessary for addressing MADM problems is assessed. Below are the steps outlining the evaluation of criteria weights using SWARA.

Step 3-A: Calculate the score values. The score values $S(\mathscr{H})$ of CHFEs are determined using Eq. (4).

Step 3-B: Arranging criteria according to DnMs's opinions. The most important criteria are ranked highest based on DnMs's preference, followed by less significant criteria in subsequent ranks.

Step 3-C: Determine the degree of comparative significance (s_g). Evaluate the relative significance of each criteria concerning the preceding criteria.

Step 3-D: Compute the comparative coefficient (k_g) utilizing

$$k_g = \begin{cases} 1, & \text{if } g = 1 \\ s_{g+1}, & \text{if } g > 1. \end{cases} \tag{23}$$

Step 3-E: Determine the recalculated weight (p_g) via Eq. (24)

$$p_g = \begin{cases} 1, & \text{if } g = 1 \\ \frac{p_{g-1}}{k_g}, & \text{if } g > 1. \end{cases} \tag{24}$$

Step 3-F: Calculate the final weight of each criteria utilizing the following formula:

$$\mathfrak{w}_g = \frac{p_g}{\sum\limits_{g=1}^{n} p_g}. \tag{25}$$

Step 4: Aggregation: The proposed CHFWPMSM operator is employed to calculate the aggregated assessment score for each alternative $\bigcirc_i (i = 1, 2, ..., m)$, as demonstrated below:

$$CHFWPMSM^{(\Lambda)}\left(\mathscr{H}_{i1}, \mathscr{H}_{i2}, ..., \mathscr{H}_{in}\right) = \bigcup_{\substack{h_{ig_\varrho} \in \mu_{r_{ig_\varrho}}, \\ \text{ß}_{ig_\varrho} \in \mu_{c_{ig_\varrho}}}} \left\{ \begin{array}{l} 1 - \left(\prod\limits_{\Upsilon=1}^{\flat} \left(1 - \left(1 - \left(\prod\limits_{1 \le ig_1 < ... \le o_\Upsilon} \left(1 - \prod\limits_{\varrho=1}^{\Lambda} h_{ig_\varrho}^{\mathfrak{w}_{ig_\varrho}} \right) \right)^{\frac{1}{C_{o_\Upsilon}^{\Lambda}}} \right)^{\frac{1}{\Lambda}} \right) \right)^{\frac{1}{\flat}}, \\ \mathrm{e}^{\iota 2\pi \left(1 - \left(\prod\limits_{\Upsilon=1}^{\flat} \left(1 - \left(1 - \left(\prod\limits_{1 \le ig_1 < ... \le o_\Upsilon} \left(1 - \prod\limits_{\varrho=1}^{\Lambda} \text{ß}_{ig_\varrho}^{\mathfrak{w}_{ig_\varrho}} \right) \right)^{\frac{1}{C_{o_\Upsilon}^{\Lambda}}} \right)^{\frac{1}{\Lambda}} \right) \right)^{\frac{1}{\flat}} \right)} \end{array} \right\}. \tag{26}$$

where $\flat$ denotes the number of categories, Λ is a parameter, $\Lambda = 1, 2, ..., o_\Upsilon$, o_Υ represents the number of criteria in category $\flat_\Upsilon$, $(k_1, k_2, ..., k_\Lambda)$ includes all the Λ-tuples of $(1, 2, ..., o_\Upsilon)$, $C_{o_\Upsilon}^{\Lambda}$ expresses the binomial coefficient, whose expression is $C_{o_\Upsilon}^{\Lambda} = \frac{o_\Upsilon!}{\Lambda!(o_\Upsilon - \Lambda)!}$ and $\mathfrak{w}_g \geq 0$ symbolize the weight of the criteria $\sqcup_g$(g=1,2,...,n) and $\mathscr{H}_{ig}^{*}$ normalized value of $\mathscr{H}_{ig}$(g=1,2,...,n) obtained from Step 2.

Step 5: Score calculation: Using Eq. (4), the score values for the aggregated CHFEs $\mathscr{H}_{ig}$(g=1,2,...,n) are determined.

Step 6: Ranking of alternatives: In this concluding phase, the options are arranged according to their score rankings, and the most suitable solution is selected.

The ranking of alternatives from the decision process helps guide the final choice. DnMs rely on this ranking to select the option that best fits the strategic goals and constraints of the situation. The chosen alternative represents the most suitable solution, ensuring that the decision is based on a well-structured and thoughtful approach. These results provide a foundation for practical implementation, helping stakeholders make informed choices backed by thorough evaluation.

To better understand the methodology, a schematic representation is shown in Fig. 2.

Case study

This section presents a practical example of selecting the best supplier for electronic goods, followed by key strategies for optimization.

Problem Statement: As Thailand's electronics industry evolves with Industry 4.0, choosing the right suppliers to meet the growing demands for advanced technologies, sustainability, and supply chain efficiency has become more challenging. Traditional methods for supplier selection are no longer sufficient to address these new, complex criteria, leaving a gap in the decision-making process. This study introduces an enhanced MCDM framework designed to evaluate suppliers more effectively, ensuring alignment with Industry 4.0 requirements and improving decision-making precision in line with strategic business objectives.

A case study of Thailand's electronics manufacturing industry

Thailand's electronics manufacturing industry isn't just a cornerstone of the country's economy; it's also a big player on the global stage. As one of Southeast Asia's top exporters of electronic goods, this industry has fueled Thailand's GDP, sparked innovation, driven technological advancements, and created millions of jobs. However, with the fast-paced changes in global manufacturing, driven by Industry 4.0 (I4.0), the industry faces both exciting opportunities and daunting challenges.

So, what's Industry 4.0 all about? Well, it's essentially the fourth industrial revolution, where advanced technologies like artificial intelligence (AI), machine learning (ML), the Internet of Things (IoT), cloud computing, big data analytics, and smart robotics come together. These innovations create 'smart factories' where machines aren't just working hard-they're working smart. They talk to each other, adapt to new demands, optimize operations, and cut costs, all in real time.

For Thailand's electronics manufacturing sector, embracing these technologies isn't just a choice; it's a necessity. To keep up with the global competition, the industry needs to move away from traditional, labour-heavy processes and step into the world of automated, intelligent production. This shift is vital, especially when considering rising labour costs, unpredictable raw material prices, and the increasing demand for high-quality, customized products. But there's more on the line than just staying competitive. The push for greener, more sustainable practices is stronger than ever, and global regulations are tightening. Adopting I4.0 technologies can help manufacturers reduce waste, lower energy consumption, and comply with these regulations. This not only boosts efficiency but also bolsters Thailand's reputation as a responsible global player. This study zooms in on a crucial part of this transformation: picking the right suppliers to support the industry's move towards smart manufacturing. Supplier selection has always been critical, but in the I4.0 era, it's become even more complex.

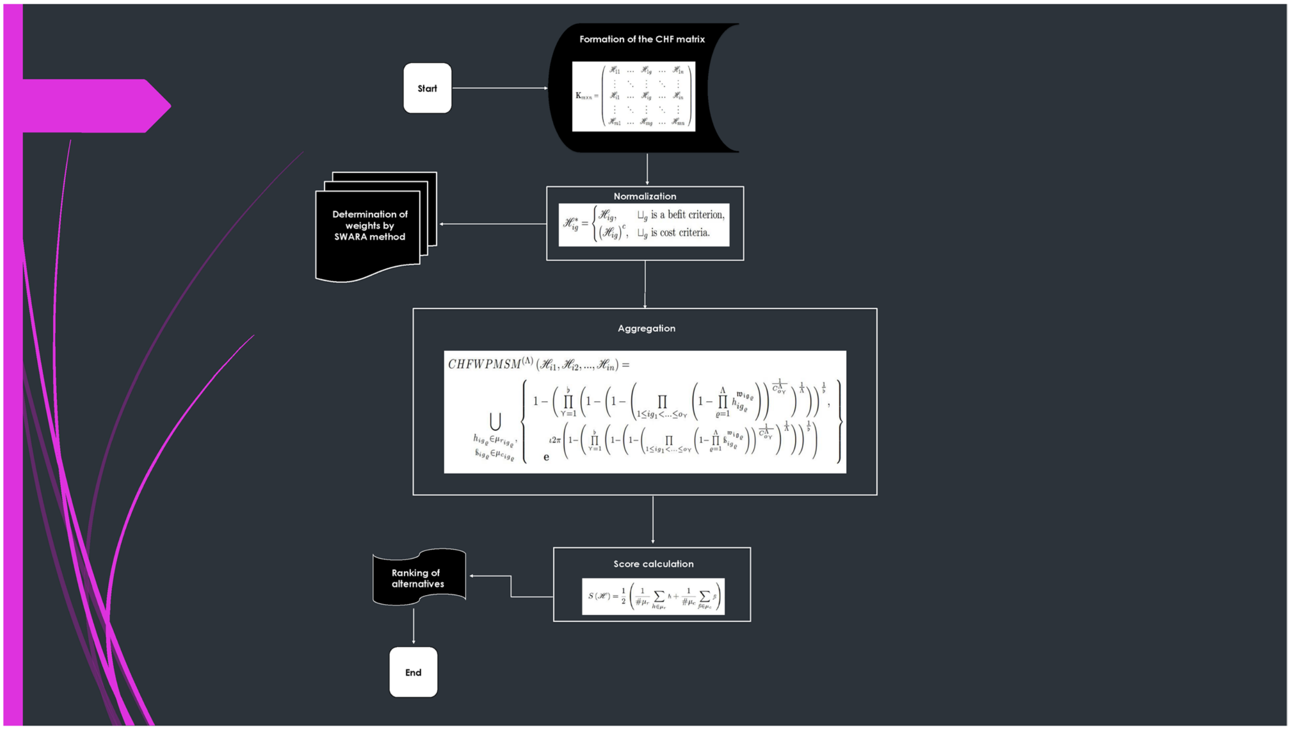

Fig. 2. Flowchart of the proposed method.

Now, manufacturers need to consider a supplier's tech capabilities, flexibility, sustainability practices, and how well they integrate into digital supply chains.

To tackle this complexity, we've proposed an MCDM model. This model helps manufacturers evaluate and choose the best suppliers by considering all these factors. We tested this model with a real-world case study of Company XYZ, a leading electronics manufacturer in Thailand, which is currently upgrading its production facilities to meet I4.0 standards. Company XYZ, knowing how crucial it is to align its supply chain with its strategic goals, identified five potential suppliers for electronic components: SUP-A, SUP-B, SUP-C, SUP-D, and SUP-E. These suppliers were shortlisted based on their quality, cost, delivery time, and innovation. However, choosing the final supplier required a deeper dive into criteria that are essential for I4.0 success. Through thorough research and expert consultations, we identified three main criteria groups: technology integration, supply chain management, and sustainability. These groups cover seven specific criteria that are key to supplier performance in smart manufacturing. We carefully chose these criteria to align with Company XYZ's goals, like boosting operational efficiency, cutting environmental impact, and maintaining top-notch product quality.

The technology integration group looks at how well suppliers adopt and implement advanced manufacturing technologies, like AI, ML, smart sensors, and robotics. Suppliers who excel here can significantly help Company XYZ optimize production, improve quality, and reduce costs-vital for creating a smart factory where machines handle complex tasks with minimal human intervention. Supply chain management focuses on how efficiently suppliers can deliver goods and services. This includes their ability to adapt to demand changes, manage logistics, and sync their operations with Company XYZ's production schedules. Flexibility and responsiveness are crucial here, directly affecting the company's ability to meet customer demands and stay competitive.

Finally, the sustainability group examines the environmental and social responsibilities of the suppliers. This includes how well they comply with environmental regulations, their commitment to reducing waste and emissions, and their involvement in corporate social responsibility (CSR) initiatives. In today's world, where sustainability is a growing concern for consumers and regulators, having suppliers with strong environmental and social credentials is invaluable for any company looking to enhance its brand image and meet global standards. By applying the MCDM model to these criteria, Company XYZ was able to systematically assess each supplier's strengths and weaknesses. The model provided a clear decision-making framework, ensuring that all relevant factors were considered and that the final selection aligned perfectly with the company's strategic goals. The specific criteria used for this evaluation, as detailed in Table 1, encompass critical aspects such as technology integration, supply chain management, and sustainability, which are essential for the company's successful transition towards Industry 4.0. The importance values assigned to the considered criteria $\sqcup_g$ $(g = 1, 2, ..., 6)$ in terms of CHF numbers are $\mathscr{H}_1 = \{0.4, 0.8\}\, e^{\iota 2\pi\{0.5, 0.6, 0.7\}}$, $\mathscr{H}_2 = \{0.2, 0.4\}\, e^{\iota 2\pi\{0.3, 0.5\}}$, $\mathscr{H}_3 = \{0.4, 0.6\}\, e^{\iota 2\pi\{0.5\}}$, $\mathscr{H}_4 = \{0.5\}\, e^{\iota 2\pi\{0.3, 0.8\}}$, $\mathscr{H}_5 = \{0.1, 0.2, 0.4\}\, e^{\iota 2\pi\{0.2, 0.6\}}$ and $\mathscr{H}_6 = \{0.1, 0.3\}\, e^{\iota 2\pi\{0.6, 0.8\}}$, respectively. This approach not only facilitated the selection of the most suitable supplier but also contributed to the company's broader efforts to maintain its competitive edge in the global electronics manufacturing industry.

The following outlines the sequential phases constituting the presented methodology.

	$\sqcup_1$	$\sqcup_2$	$\sqcup_3$	$\sqcup_4$	$\sqcup_5$	$\sqcup_6$
$\bigcirc_1$	$\{0.4, 0.6\}\, e^{\iota 2\pi\{0.1,0.3\}}$	$\{0.7, 0.4, 0.3\}\, e^{\iota 2\pi\{0.1,0.2\}}$	$\{0.2, 0.3\}\, e^{\iota 2\pi\{0.5\}}$	$\{0.2, 0.7\}\, e^{\iota 2\pi\{0.1,0.3\}}$	$\{0.1, 0.3\}\, e^{\iota 2\pi\{0.2,0.6,0.5\}}$	$\{0.6, 0.8\}\, e^{\iota 2\pi\{0.1,0.2\}}$
$\bigcirc_2$	$\{0.1, 0.4, 0.5\}\, e^{\iota 2\pi\{0.1,0.2\}}$	$\{0.2, 0.3\}\, e^{\iota 2\pi\{0.2,0.3\}}$	$\{0.1, 0.4\}\, e^{\iota 2\pi\{0.2,0.6\}}$	$\{0.3, 0.6\}\, e^{\iota 2\pi\{0.2,0.4\}}$	$\{0.2, 0.3\}\, e^{\iota 2\pi\{0.6,0.7\}}$	$\{0.7\}\, e^{\iota 2\pi\{0.4,0.5\}}$
$\bigcirc_3$	$\{0.5, 0.6\}\, e^{\iota 2\pi\{0.1,0.2,0.3\}}$	$\{0.1, 0.2\}\, e^{\iota 2\pi\{0.4,0.5,0.6\}}$	$\{0.4\}\, e^{\iota 2\pi\{0.3,0.6\}}$	$\{0.5, 0.6, 0.7\}\, e^{\iota 2\pi\{0.1,0.2\}}$	$\{0.2, 0.3, 0.5\}\, e^{\iota 2\pi\{0.6,0.7\}}$	$\{0.3, 0.4\}\, e^{\iota 2\pi\{0.6\}}$
$\bigcirc_4$	$\{0.8\}\, e^{\iota 2\pi\{0.4,0.7\}}$	$\{0.3, 0.4\}\, e^{\iota 2\pi\{0.2,0.5\}}$	$\{0.1, 0.3\}\, e^{\iota 2\pi\{0.2,0.6\}}$	$\{0.6, 0.8\}\, e^{\iota 2\pi\{0.1,0.2\}}$	$\{0.4\}\, e^{\iota 2\pi\{0.4,0.7\}}$	$\{0.5, 0.7\}\, e^{\iota 2\pi\{0.4,0.5\}}$
$\bigcirc_5$	$\{0.2, 0.3\}\, e^{\iota 2\pi\{0.3,0.5\}}$	$\{0.4, 0.6\}\, e^{\iota 2\pi\{0.3,0.7\}}$	$\{0.5, 0.7\}\, e^{\iota 2\pi\{0.6\}}$	$\{0.6\}\, e^{\iota 2\pi\{0.3,0.4,0.5\}}$	$\{0.1, 0.3, 0.4\}\, e^{\iota 2\pi\{0.4,0.5\}}$	$\{0.5, 0.6\}\, e^{\iota 2\pi\{0.8\}}$

Table 1. CHF evaluation matrix.

Criteria	Score values	Comparative significance	Comparative coefficient	Recalculated weights	Final weights
$\sqcup_1$	0.6000	0.0000	1.0000	1.0000	0.1905
$\sqcup_4$	0.5250	0.0750	1.0750	0.9302	0.1772
$\sqcup_3$	0.5000	0.0250	1.0250	0.9075	0.1729
$\sqcup_6$	0.4500	0.0500	1.0500	0.8643	0.1647
$\sqcup_2$	0.3500	0.1000	1.1000	0.7858	0.1497
$\sqcup_5$	0.3167	0.0333	1.0333	0.7604	0.1450

Table 2. Evaluation of criteria weights using SWARA method.

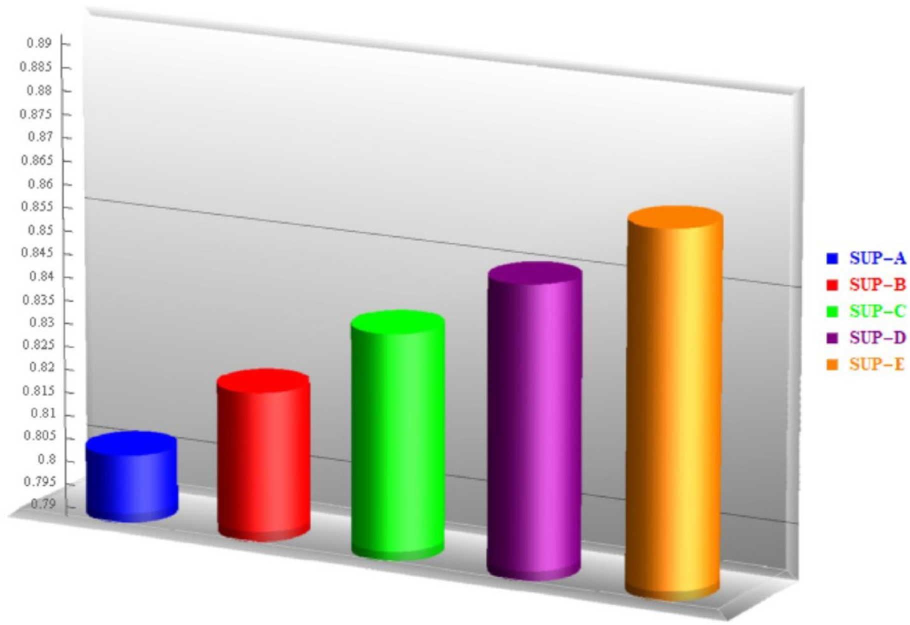

Fig. 3. Bar graph representation of the suppliers' ranking.

Step 1: The CHF assessment data is presented in Table 1.

Step 2: Since all the criteria are benefit type, so there is no need to do normalization.

Step 3: We apply the developed formulations of the SWARA method as outlined in Step 3. The results derived from this implementation have been systematically calculated and are tabulated in Table 2.

Step 4: Now, in Eq. (26), by setting $\Lambda = 2$, the aggregated values for each supplier $\bigcirc_i (i = 1, 2, ..., m)$ can be computed as follows:

$\bigcirc_1 = \{\ 0.8291, 0.8502, 0.8117, ..., 0.8726, 0.8819\ \}^{\iota 2\pi\{0.7293, 0.7519, 0.7397, ..., 0.8053, 0.8249\}},$

$\bigcirc_2 = \{\ 0.7824, 0.8114, 0.8174, ..., 0.8721, 0.8770\ \}^{\iota 2\pi\{0.8040, 0.8157, 0.8091, ..., 0.8644, 0.8735\}},$

$\bigcirc_3 = \{\ 0.8246, 0.8284, 0.8370, ..., 0.8718, 0.8755\ \}^{\iota 2\pi\{0.8285, 0.8411, 0.8500, ..., 0.8751, 0.8834\}},$

$\bigcirc_4 = \{\ 0.8615, 0.8707, 0.8807, ..., 0.8961, 0.9029\ \}^{\iota 2\pi\{0.8068, 0.8235, 0.8291, ..., 0.8792, 0.8963\}},$

$\bigcirc_5 = \{\ 0.8414, 0.8503, 0.8483, ..., 0.8905, 0.8978\ \}^{\iota 2\pi\{0.8718, 0.8819, 0.8855, ..., 0.9043, 0.9162\}}.$

Step 5: Under Eq. (4), the score value of the suppliers $\bigcirc_i (i = 1, 2, ..., 5)$ can be fined as shown below: $S(\bigcirc_1) = 0.8189,\ S(\bigcirc_2) = 0.8368, S(\bigcirc_3) = 0.8536, S(\bigcirc_4) = 0.8680, S(\bigcirc_5) = 0.8837.$

Step 6: Based on the calculated values, the final ranking can be established $\bigcirc_5 > \bigcirc_4 > \bigcirc_3 > \bigcirc_2 > \bigcirc_1$. Thus, the best supplier is SUP-E.

The resulting rankings are visually represented in Fig. 3.

Sensitivity analysis

The validation of the results and the accuracy of the decision-making process can be achieved through sensitivity analysis. This analysis involves making specific changes to the primary model. In this study, we considered the exchange of weights among criteria. According to the selected approach, the weight of each criterion is swapped with the weights of the other criteria. For example, the possible weights for the most influential criteria might be 0.1772, 0.1729, 0.1647, 0.1497, and 0.1450. The weights for the remaining criteria can then be calculated using Eq. (27)[52]:

$$\mathfrak{w}_{c'_g} = \frac{1-\mathfrak{w}_e}{1-\mathfrak{w}_o}\mathfrak{w}_g. \tag{27}$$

Here, $\mathfrak{w}_o$ represents the weight assigned to a criterion, $\mathfrak{w}_e$ is the exchanged weight, $\mathfrak{w}_g$ is the given weight of criteria and $\mathfrak{w}_{c'_g}$ is the calculated weight of the criteria. It is important to ensure that the sum of all criteria weights equals 1 when implementing the weight replacement strategy.

For example, if $\mathfrak{w}_o$ for the most influential criterion is 0.1905 and is exchanged with a weight of 0.1772, the weights for the other criteria will be determined as follows: $\mathfrak{w}_{c'_2} = 0.1473$, $\mathfrak{w}_{c'_3} = 0.1701$, $\mathfrak{w}_{c'_4} = 0.2007$, $\mathfrak{w}_{c'_5} = 0.1427$,$\mathfrak{w}_{c'_6} = 0.1620$ Similarly, a total of 30 weight combinations are presented in Table 3.

The ranking outcomes for the 30 sets of criteria were generated using a specifically designed algorithm and are presented in Table 5. As observed from the table, although variations in the weight components lead to slight changes in the score values of the alternatives, the overall ranking of suppliers remains consistent across all 30 scenarios. This analysis suggests that the proposed approach is robust, demonstrating the ability to maintain the overall ranking even with minor adjustments to the weight vector.

Comparative analysis

The following section provides a comparative analysis with other existing AOs, including the complex hesitant fuzzy weighted average (CHFWA)[19], complex hesitant fuzzy weighted geometric (CHFWG)[19], hesitant fuzzy weighted partition Maclaurin symmetric mean (HFWPMSM)[38], hesitant fuzzy weighted Maclaurin symmetric mean (HFWMSM)[53], hesitant fuzzy Aczel-Alsina weighted average (HFAAWA)[54], hesitant fuzzy Aczel-Alsina weighted geometric (HFAAWG)[54], hesitant fuzzy Frank weighted average (HFFWA)[55], and hesitant fuzzy Frank weighted geometric (HFFWG)[55]. This comparison aims to highlight the advantages of the proposed approach. The final results derived from applying these AOs to the previous case study, using weight information derived from the proposed SWARA technique, along with the final ranking outcomes, are tabulated in Table 6 and Fig. 4.

The results presented in Table 5 indicate that the method proposed in this article, along with the approaches based on existing AOs, except for HFWMSM[53], consistently identifies the same best supplier. Moreover,

Tests	$\mathfrak{w}_1$	$\mathfrak{w}_2$	$\mathfrak{w}_3$	$\mathfrak{w}_4$	$\mathfrak{w}_5$	$\mathfrak{w}_6$
Test 1	0.1497	0.2222	0.1646	0.1687	0.1380	0.1568
Test 2	0.1729	0.1465	0.2040	0.1734	0.1419	0.1613
Test 3	0.1772	0.1473	0.1701	0.2007	0.1427	0.1620
Test 4	0.1450	0.1417	0.1637	0.1678	0.2259	0.1559
Test 5	0.1647	0.1451	0.1676	0.1717	0.1405	0.2104
Test 6	0.1164	0.1905	0.1816	0.1861	0.1523	0.1731
Test 7	0.1958	0.1729	0.1307	0.1822	0.1491	0.1693
Test 8	0.1969	0.1772	0.1787	0.1272	0.1498	0.1702
Test 9	0.1895	0.1450	0.1719	0.1762	0.1536	0.1638
Test 10	0.1939	0.1647	0.1760	0.1804	0.1476	0.1374
Test 11	0.1591	0.1530	0.1905	0.1811	0.1481	0.1682
Test 12	0.1853	0.1914	0.1497	0.1724	0.1410	0.1602
Test 13	0.1915	0.1505	0.1772	0.1695	0.1458	0.1655
Test 14	0.1843	0.1448	0.1450	0.1714	0.1952	0.1593
Test 15	0.1886	0.1482	0.1647	0.1755	0.1436	0.1794
Test 16	0.1668	0.1522	0.1757	0.1905	0.1474	0.1674
Test 17	0.1843	0.1990	0.1673	0.1497	0.1403	0.1594
Test 18	0.1895	0.1489	0.1806	0.1729	0.1442	0.1639
Test 19	0.1833	0.1441	0.1664	0.1450	0.2027	0.1585
Test 20	0.1876	0.1475	0.1703	0.1647	0.1428	0.1871
Test 21	0.1077	0.1581	0.1826	0.1872	0.1905	0.1739
Test 22	0.1916	0.1411	0.1739	0.1782	0.1497	0.1655
Test 23	0.1969	0.1547	0.1220	0.1832	0.1729	0.1703
Test 24	0.1980	0.1556	0.1797	0.1185	0.1772	0.1710
Test 25	0.1950	0.1532	0.1770	0.1814	0.1647	0.1287
Test 26	0.1441	0.1545	0.1784	0.1828	0.1497	0.1905
Test 27	0.1871	0.1768	0.1698	0.1741	0.1425	0.1497
Test 28	0.1924	0.1512	0.1581	0.1790	0.1464	0.1729
Test 29	0.1934	0.1520	0.1755	0.1547	0.1472	0.1772
Test 30	0.1861	0.1463	0.1689	0.1731	0.1806	0.1450

Table 3. Various test scenarios for weight sensitivity.

Scenario	$S(\bigcirc_1)$	$S(\bigcirc_2)$	$S(\bigcirc_3)$	$S(\bigcirc_4)$	$S(\bigcirc_5)$	Ranking
Test 1	0.8186	0.8380	0.8532	0.8625	0.8868	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 2	0.8200	0.8369	0.8540	0.8674	0.8847	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 3	0.8186	0.8374	0.8537	0.8681	0.8837	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 4	0.8190	0.8354	0.8540	0.8660	0.8811	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 5	0.8192	0.8377	0.8530	0.8671	0.8865	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 6	0.8180	0.8374	0.8527	0.8617	0.8870	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 7	0.8179	0.8378	0.8534	0.8676	0.8837	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 8	0.8207	0.8367	0.8541	0.8660	0.8855	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 9	0.8187	0.8363	0.8536	0.8681	0.8829	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 10	0.8192	0.8369	0.8543	0.8676	0.8828	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 11	0.8191	0.8369	0.8535	0.8662	0.8848	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 12	0.8186	0.8379	0.8536	0.8659	0.8850	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 13	0.8191	0.8366	0.8536	0.8678	0.8838	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 14	0.8179	0.8355	0.8535	0.8678	0.8805	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 15	0.8187	0.8370	0.8533	0.8679	0.8843	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 16	0.8185	0.8370	0.8534	0.8669	0.8842	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 17	0.8198	0.8377	0.8539	0.8650	0.8860	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 18	0.8192	0.8367	0.8537	0.8679	0.8839	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 19	0.8192	0.8348	0.8539	0.8671	0.8811	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 20	0.8191	0.8369	0.8533	0.8677	0.8849	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 21	0.8193	0.8374	0.8539	0.8634	0.8855	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 22	0.8189	0.8365	0.8537	0.8685	0.8831	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 23	0.8175	0.8370	0.8535	0.8687	0.8817	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 24	0.8209	0.8354	0.8544	0.8669	0.8837	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 25	0.8189	0.8359	0.8543	0.8680	0.8808	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 26	0.8188	0.8374	0.8530	0.8654	0.8861	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 27	0.8190	0.8373	0.8538	0.8665	0.8842	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 28	0.8184	0.8370	0.8534	0.8680	0.8837	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 29	0.8194	0.8365	0.8535	0.8674	0.8845	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$
Test 30	0.8183	0.8352	0.8536	0.8674	0.8806	$\bigcirc_5>\bigcirc_4>\bigcirc_3>\bigcirc_2>\bigcirc_1$

Table 4. Ranking results by the proposed method with various weight vectors.

Aggregation operator	$S(\bigcirc_1)$	$S(\bigcirc_2)$	$S(\bigcirc_3)$	$S(\bigcirc_4)$	$S(\bigcirc_5)$	Ranking
Proposed CHFWPMSM	0.8189	0.8368	0.8536	0.8680	0.8837	$\bigcirc_5 > \bigcirc_4 > \bigcirc_3 > \bigcirc_2 > \bigcirc_1$
CHFWA[19]	0.3811	0.4175	0.4359	0.4466	0.5173	$\bigcirc_5 > \bigcirc_4 > \bigcirc_3 > \bigcirc_2 > \bigcirc_1$
CHFWG[19]	0.2974	0.3411	0.3531	0.3644	0.4550	$\bigcirc_5 > \bigcirc_4 > \bigcirc_3 > \bigcirc_2 > \bigcirc_1$
HFWPMSM[38]	0.8519	0.8540	0.8481	0.8548	0.8731	$\bigcirc_5 > \bigcirc_4 > \bigcirc_2 > \bigcirc_1 > \bigcirc_3$
HFWMSM[53]	0.1537	0.1514	0.1582	0.1526	0.1359	$\bigcirc_3 > \bigcirc_1 > \bigcirc_4 > \bigcirc_2 > \bigcirc_5$
HFAAWA[54]	0.5087	0.5375	0.5303	0.5649	0.6063	$\bigcirc_5 > \bigcirc_4 > \bigcirc_2 > \bigcirc_3 > \bigcirc_1$
HFAAWG[54]	0.2161	0.2551	0.2453	0.2676	0.3660	$\bigcirc_5 > \bigcirc_4 > \bigcirc_2 > \bigcirc_3 > \bigcirc_1$
HFFWA[55]	0.3699	0.4070	0.4259	0.4352	0.5086	$\bigcirc_5 > \bigcirc_4 > \bigcirc_3 > \bigcirc_2 > \bigcirc_1$
HFFWG[55]	0.3076	0.3511	0.3651	0.3754	0.4641	$\bigcirc_5 > \bigcirc_4 > \bigcirc_3 > \bigcirc_2 > \bigcirc_1$

Table 5. Comparative outcomes.

the overall ranking observed in previous studies by Talafha et al.[19] and Qin et al.[55], where the order is $\bigcirc_5 > \bigcirc_4 > \bigcirc_3 > \bigcirc_2 > \bigcirc_1$, perfectly aligns with the findings derived from our proposed method. This congruence serves as a strong validation of the approach we have developed, demonstrating its accuracy and reliability. The alignment of our results with established rankings underscores the robustness of the operator designed in this research. Based on these derived results, we proceed with a detailed discussion and analysis to highlight the key advantages and strengths of the framework implemented in this study.

i). According to Table 5, the results obtained using the operators from Talafha et al.[19] match with those generated by our proposed operator. However, our formulated operator has a distinct advantage: it effectively captures interrelationships between criteria and allows for the division of criteria into multiple groups, an

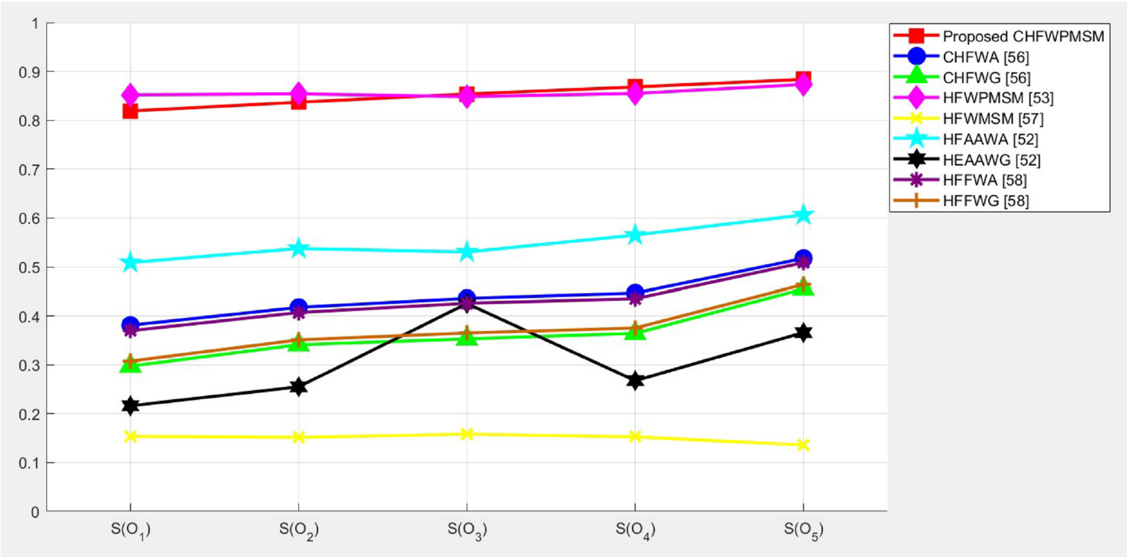

Fig. 4. Ranking results with various aggregation techniques.

Criteria group	Criteria	Explanation
Technology integration	$\sqcup_1$ Smart Manufacturing Technologies	Smart manufacturing integrates AI and ML to optimize production processes, reducing downtime and improving product quality.
	$\sqcup_2$ Advanced Robotics	Advanced robotics involves the use of highly automated, intelligent robots capable of performing complex tasks with minimal human intervention.
Supply Chain Management	$\sqcup_3$ Supply Chain Flexibility	Supply chain flexibility measures the ability to adapt to sudden changes in demand, supply disruptions, and market conditions.
	$\sqcup_4$ Logistics Efficiency	Logistics efficiency assesses how effectively a company manages the flow of goods from suppliers to the production line, minimizing delays.
Sustainability	$\sqcup_5$ Environmental Compliance	Environmental compliance refers to adherence to environmental regulations, including waste management, emissions control, and resource efficiency.
	$\sqcup_6$ Corporate Social Responsibility (CSR)	CSR reflects a company's commitment to ethical practices, sustainability, and community engagement.

Table 6. Criteria explanation.

aspect that existing approaches do not address. Additionally, these basic operators fail to account for correlations between criteria, which limits their ability to fully represent the complex interactions inherent in decision-making processes.

ii). The data presented in Table 1 is inherently complex, comprising both amplitude and phase terms. However, the existing operators[38,53–55] are designed to work exclusively with amplitude terms. To apply these operators to the data in question, we removed the phase terms, and some of these results align with those obtained when considering both amplitude and phase terms, it is important to note that this method is not without limitations. The omission of phase terms results in data loss, which can lead to erroneous outcomes. In contrast, the proposed approach retains the ability to handle complex data in its entirety, ensuring no data loss and thereby reducing the risk of inaccuracies.

iii). According to Table 5, the supplier rankings obtained using the approaches by Ali[38] and Qin et al.[53] are $\bigcirc_5 > \bigcirc_4 > \bigcirc_2 > \bigcirc_1 > \bigcirc_3$ and $\bigcirc_3 > \bigcirc_1 > \bigcirc_4 > \bigcirc_2 > \bigcirc_5$, respectively. It is evident that Ali's ranking is somewhat different, while Qin et al.'s ranking deviates more significantly from the results produced by the proposed operator. Although Ali's[38] method accounts for the relationships among criteria and allows for partitioning them into groups, it only considers the amplitude terms, thereby neglecting the phase component, which leads to substantial information loss. Similarly, Qin et al.[53] also neglect the phase terms, with an additional drawback of not classifying criteria into different categories-a step that is particularly necessary in the context of the example considered, as evidenced in Table 6. Both of these methods[38,53] can be viewed as special cases of the structured operator introduced in this study, making the designed approach more comprehensive and adaptable to various scenarios.

In light of the above comparative analysis, the key strengths of the proposed method can be summarized as follows:

1. The proposed CHFWPMSM operator effectively captures interrelationships between criteria and supports partitioning into multiple groups, unlike traditional operators such as CHFWA or CHFWG[19], which overlook these aspects.
2. Unlike existing methods[38,53–55] that discard phase terms and rely only on amplitude, the proposed method retains both, ensuring full data integrity and improving result accuracy.
3. The rankings produced by the proposed operator align closely with those from established methods, validating its reliability while offering improved robustness in handling complex data structures.
4. Methods by Ali[38] and Qin et al.[53] can be seen as special cases of the proposed approach. While Ali[38] includes grouping but omits phase, and Qin et al.[53] lacks both, the proposed method integrates these features for a more comprehensive evaluation.
5. By handling fuzzy, complex, and interrelated data, the proposed method provides enhanced support for real-world decision-making, particularly in industrial applications like supplier selection.

Managerial implications

This study offers critical insights for managers in Thailand's electronics manufacturing sector:

i. Strategic supplier selection: The MCDM model provides a framework for selecting suppliers based on essential factors such as technology adoption, supply chain adaptability, and sustainability, ensuring alignment with long-term business goals.
ii. Supply chain and cost optimization: The model helps identify suppliers with strong supply chain performance, enabling quicker response times and cost reductions, key for maintaining competitiveness.
iii. Sustainability and compliance: By evaluating suppliers' sustainability efforts, the model assists in meeting regulatory requirements while boosting the company's reputation for responsible practices.
iv. Boosting competitive edge: Leveraging advanced technologies allows managers to streamline operations, minimize waste, and enhance product quality, fostering sustained success in Industry 4.0.
v. Risk management: The model helps assess potential risks, ensuring that supplier relationships remain resilient and adaptable in the face of future changes.

In summary, this study presents a practical tool for improving supplier selection and strengthening strategic decision-making in a rapidly evolving industry.

Conclusions

This section presents the study's key findings, highlights its limitations, and provides recommendations for future research to expand the applicability and scope of the framed methodology.

Findings:

This study demonstrated the efficacy of CHFS in addressing complex and uncertain data within MCDM frameworks. The CHFPMSM and CHFWPMSM operators introduced in this research were specifically designed to aggregate CHF data with precision. These operators not only account for interrelationships between criteria but also incorporate their partitioned relationships, addressing a significant gap in the literature. Additionally, we developed an MCDM methodology integrating the framed SWARA technique with these operators. This approach enables the partitioning of criteria into distinct groups, facilitating the measurement of relationships within each group and effectively handling situations where criteria weights are unavailable in advance. A numerical example involving supplier selection validated the proposed method, and comparative analysis highlighted its superior accuracy and robustness in managing uncertainty and employing multiple partitions among criteria.

Research limitations:

Despite its contributions, the study is subject to the following limitations:

i. Exclusion of non-membership grading: The proposed method focuses solely on membership grading, neglecting non-membership grading, which may limit its applicability in scenarios where DnMs lack complete information about the problem.
ii. Computational complexity: While enhancing decision-making processes, the complex structure of the method imposes a higher computational burden compared to existing approaches, potentially limiting its scalability.
iii. Handling extreme values: The proposed operators do not account for power aggregation characteristics, making them less suitable for scenarios involving extreme data values, where power AOs may be more effective.

Recommendations for future research:

Future research could explore the inclusion of non-membership grading to address incomplete or imprecise data and develop more efficient algorithms to reduce computational complexity. Incorporating power aggregation characteristics would enhance suitability for scenarios involving extreme values. Additionally, the proposed method could be extended to other practical applications[56–61] for further validation.

Data availability

All data generated or analysed during this study are included in this published article.

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All the authors contributed equally in this article.

Declarations

Competing interests
The authors declare no competing interests.

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