How best can finite-time social distancing reduce epidemic final size?

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ABSTRACT

Given maximal social distancing duration and intensity, how can one minimize the epidemic final size, or equivalently the total number of individuals infected during the outbreak? A complete answer to this question is provided and demonstrated here for the SIR epidemic model. In this simplified setting, the optimal solution consists in enforcing the highest confinement level during the longest allowed period, beginning at a time instant that is the unique solution to certain 1D optimization problem. Based on this result, we present numerical essays showing the best possible performance for a large set of basic reproduction numbers and lockdown durations and intensities.

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1. Introduction

The current outbreak of Covid-19 and the entailed implementation of social distancing on an unprecedented scale, leads to a renewed interest in modelling and analysis of the non-pharmaceutical intervention strategies to control infectious diseases. In contrast to the removal of susceptible individuals (by vaccination) or infectious individuals (by isolation or quarantine) from the process of disease transmission, the term "social distancing" refers to attempts to directly reduce the infecting contacts within the population. Such actions may be obtained through voluntary actions, possibly fostered by government information campaigns, or by mandatory measures such as partial or total lockdown. Notice that, when no vaccine or therapy is available, such containment strategies constitute probably the only mid-term option.

Optimal control approaches have been abundantly explored in the past in the framework of control of transmissible diseases, see e.g. Lenhart and Workman, 2007; Sharomi and Malik, 2017 and bibliographical references in Bliman et al. (2020). Optimal control of social distancing (possibly coupled with vaccination, treatment or isolation) is usually considered through the minimization of a finite-time integral cost linear in the state, and quadratic in the input control variables or jointly bilinear in the two signals (Behncke, 2000; Yan et al., 2008; Lee et al., 2010; Lin et al., 2010; F. E. Alvarez et al., 2020; Djidjou-Demasse et al.,

2020). The authors of Morris et al. (2020) study the optimal control allowing to minimize the maximal value taken by the infected population. The integral of the deviation between the natural infection rate and its effective value due to confinement is used as a cost in Miclo et al. (2020), together with constraints on the maximal number of infected. In Angulo et al. (2020), the authors minimize the time needed to reach herd immunity, under the constraint of keeping the number of infected below a given value, in an attempt to preserve the public health system. Optimal public health interventions as a complement to vaccination campaigns have been studied in Buonomo et al. (2019a) and Buonomo et al. (2019b); see also (Manfredi and D'Onofrio, 2013) for more material on behavioral epidemiology.

The magnitude of the outbreak, usually called the epidemic *final size*, is another important characteristic. It is defined as the total number of initially susceptible individuals that become infected during the course of the epidemic. Abundant literature exists concerning this quantity, since Kermack and Mc Kendrick's paper from 1927 (Kermack and McKendrick, 1927); see Ma and Earn, 2006; Andreasen, 2011; Katriel, 2012; Miller, 2012 for important contributions to its computation in various deterministic settings. Recently, optimal control approach has been introduced to minimize the final size by temporary reduction of the contact rate on a given time interval $[0, D], D > 0$. This issue has been considered in Ketcheson (2020), with total lockdown and added integral term accounting for control cost; and in Bliman et al. (2020), where partial lockdown is considered as well. The corresponding optimal control is bang-bang, with maximal distancing intensity applied

on a subinterval $[T_0^*, D]$, for some unique $T_0^* \in [0, D)$ depending of the initial conditions, and no action otherwise.

In a population in which a large proportion of individuals is immune, either after vaccination or after having been infected, the infection is more likely to be disrupted. The *herd immunity* threshold is attained when the number of infected individuals begins to decrease over time. While the proportion of susceptible is asymptotically *always* smaller than this threshold, a significant proportion of initially susceptible individuals may still be infected until the epidemic is over. In this perspective, minimizing the epidemic final size can be seen as an attempt to stop the outbreak as close as possible after reaching the herd immunity.

While distancing enforcement cannot last for a long time, there is indeed no reason in practice why it should be restricted to start at a given date —typically "right now". Elaborating on Bliman et al. (2020), we consider in the present paper a more general optimal control problem, achieved through social distancing during a given maximal time duration $D > 0$, but without prescribing the onset of this measure. A key result below (Theorem 1) shows the existence of a unique time T^*, which depends upon the initial conditions, for which the optimal control corresponds to applying maximal distancing intensity on the interval $[T^*, T^* + D]$: this more natural setting yields a more efficient control strategy.

The paper is organized as follows. We introduce in Section 2 the precise setting of the problem under study and formulate the three main results: Theorem 1 demonstrates the existence and uniqueness of the optimal policy and provides a constructive characterization; Theorem 2 studies its dependence upon the lockdown intensity and duration; Theorem 3 shows that above a certain critical lockdown intensity, optimal social distancing on a sufficiently long period approaches herd immunity arbitrarily close. Section 3 provides illustrative numerical essays. The proof of Theorem 1 is the subject of Section 4. Concluding remarks are given in Section 5. Additional numerical results may be found in the Supplementary materials, as well as the corresponding algorithms and a link to the codes used for resolution.

2. Problem description and main results

Consider the system

$$
\begin{aligned}
\dot{S}(t) &= -u(t)\beta S(t)I(t), & t &\geqslant 0 \\
\dot{I}(t) &= u(t)\beta S(t)I(t) - \gamma I(t), & t &\geqslant 0
\end{aligned}
\tag{1}
$$

complemented with nonnegative initial data $S(0) = S_0, I(0) = I_0$ such that $S_0 + I_0 \leqslant 1$. The input u, taking on values in $[0, 1]$, models the effect of a social distancing policy: $u(t) = 1$ corresponds to absence of restrictions, while $u(t) = 0$, corresponding to complete lockdown, prohibits any contact and thus any transmission. In the sequel, we call *uncontrolled system* the system corresponding to $u \equiv 1$, and generally speaking restrict $u \in L^\infty(0, +\infty)$ to be *admissible*, that is by definition such that $\alpha \leqslant u(t) \leqslant 1$ for a given constant $\alpha \in [0, 1)$ and for almost any $t \geqslant 0$. The constant α, called here the *maximal lockdown intensity*[1] determines the most intense achievable social distancing.

We assume in all the sequel that the basic reproduction number $\mathscr{R}_0$ of the uncontrolled system fulfils (see e.g. Keeling and Rohani, 2008):

$$
\mathscr{R}_0 := \frac{\beta}{\gamma} > 1.
$$

This constant strongly characterizes the dynamics of this system. The effect of a *constant* input $u \in [0, 1]$ is obviously to change $\mathscr{R}_0$ in the *control reproduction number* (Brauer et al., 2008) $u\mathscr{R}_0$.

For any admissible u, one defines

$$
S_\infty(u) := \lim_{t \to \infty} S(t),
$$

for (S, I) the solution to (1). The quantity $S_0 - S_\infty(u)$ is the proportion of individuals initially susceptible, subsequently infected and finally removed, due to the outbreak and after completion of the latter. It is called the *attack ratio*, or the *epidemic final size* when numbers of individuals are considered instead of proportions. This notion plays a central role in the sequel.

For the uncontrolled model (1) (with $u \equiv 1$), the herd immunity is

$$
S_{\text{herd}} := \frac{\gamma}{\beta} = \frac{1}{\mathscr{R}_0}.
\tag{2}
$$

Any equilibrium $(S_{\text{equi}}, 0), 0 \leqslant S_{\text{equi}} \leqslant 1$, of this system is *stable* if $0 \leqslant S_{\text{equi}} \leqslant S_{\text{herd}}$ and *unstable* if $S_{\text{herd}} < S_{\text{equi}}$, so that the disease prospers if introduced in population where $\mathscr{R}_0 S_0 > 1$ (before it finally fades away), and dies out otherwise. Coherently with this observation, if $u(t)$ equals 1 after a finite time, then one has

$$
S_\infty(u) \leqslant S_{\text{herd}}.
$$

In this optic, attempting to reduce the epidemic final size by finite-time intervention is equivalent to try to stop it as closely as possible from the herd immunity threshold.

For any $0 < T \leqslant T'$ and $\alpha \in [0, 1)$, let $\mathscr{U}_{\alpha, T, T'}$ be the following subset of admissible inputs:

$$
\mathscr{U}_{\alpha, T, T'} := \{ u \in L^\infty(0, +\infty), \ \alpha \leqslant u(t) \leqslant 1 \text{ if } t \in [T, T'], u(t) = 1 \text{ otherwise} \}.
$$

We also consider the set of those functions $u_{T,T'}$ of $\mathscr{U}_{\alpha, T, T'}$ defined by

$$
u_{T,T'} = \mathbb{1}_{[0,T]} + \alpha \mathbb{1}_{[T,T']} + \mathbb{1}_{[T',+\infty)},
\tag{3}
$$

where the notation $\mathbb{1}$ denotes characteristic functions[2], and denote **1** the function of $L^\infty(0, +\infty)$ equal to 1 (almost) everywhere.

The main result of the paper is now given. It indicates how to optimally implement distancing measures, in order to minimize the epidemic final size. To state this result, introduce first the function ψ given by

$$
\psi : T \ni [0, \infty) \mapsto -\frac{I^T(T+D)}{I^T(T)} + (\alpha - 1)\gamma \int_T^{T+D} \frac{I^T(T+D)}{I^T(t)} dt + 1,
\tag{4}
$$

where $\left(S^T, I^T \right)$ denotes the solution to (1) with $u = u_{T,T+D}$ defined in (3).

Theorem 1. *For any $\alpha \in [0, 1)$ and $D > 0$, the optimal control problem*

$$
\sup_{T \geqslant 0} \ \sup_{u \in \mathscr{U}_{\alpha, T, T+D}} S_\infty(u) \quad (\mathscr{P}_{\alpha, D})
$$

admits a unique solution. The optimal control is equal to the function u_{T^*, T^*+D} defined in (3), where the value $T^* \geqslant 0$ is characterized by the fact that;

- *if $\psi(0) \geqslant 0$, then $T^* = 0$;*
- *if $\psi(0) < 0$, then T^* is the unique solution to*

$$
\psi(T^*) = 0.
\tag{5}
$$

---

[1] Therefore, a *smaller* value of the maximal lockdown intensity α may produce *more intense* lockdown.

[2] That is e.g. $\mathbb{1}_{[0,T]}(t) = 1$ if $t \in [0, T]$, 0 otherwise.

Moreover, if $T^* > 0$, then $S(T^*) > S_{\text{herd}}$ if $\alpha > 0$, and $S(T^*) = S_{\text{herd}}$ if $\alpha = 0$. Last, fixing $S_0 \in (S_{\text{herd}}, 1)$, it holds

$$\lim_{I_0 \searrow 0^+} T^* = +\infty.$$

For subsequent use, we denote (S^*, I^*) the optimal solution, and S_∞^* the value function of problem $(\mathscr{P}_{\alpha,D})$, that is by definition:

$$S_\infty^* = S_\infty^*(S_0, I_0) := \sup_{T \geqslant 0} \ \sup_{u \in \mathscr{U}_{\alpha,T,T+D}} S_\infty(u). \tag{6}$$

Theorem 1 establishes that, among all intervention strategies carried out on a time interval of length D with an intensity located at each time instant between α and 1, a single one minimizes the epidemic final size. The corresponding control is bang-bang and consists in enforcing the most intense social distancing level α on the time interval $[T^*, T^* + D]$, where $T^* \geqslant 0$ is uniquely assessed in the statement. The value of T^* depends upon the initial value (S_0, I_0) through the solution $\left(S^T, I^T\right)$ of System (1) appearing in the expression (4).

Assessing the value of $\psi(T)$ for given $T \geqslant 0$ amounts to solve the ordinary differential Eq. (1) and to evaluate the quantity in (4) — tasks routinely achieved by standard scientific computational environments. It is shown in the proof of Theorem 1 (Section 4.4) that, if $\psi(0) < 0$, then ψ is negative on $(0, T^*)$ and positive on (T^*, ∞). This remark permits implementation of an efficient bisection algorithm to assess the optimal value T^*. More details concerning the numerical methods may be found in the Supplementary materials.

We continue with some properties characterizing the dependence of the value function with respect to the parameters.

Theorem 2. *The value function S_∞^* is increasing with respect to the parameter $D > 0$ and decreasing with respect to the parameter $\alpha \in [0, 1)$.*

The statement of Theorem 2 corresponds to the intuition whereby longer or more intense interventions result in greater reduction of the epidemic final size.

Proof of Theorem 2. Let $0 < D \leqslant D'$ and $1 > \alpha \geqslant \alpha' \geqslant 0$, with $(D, \alpha) \neq (D', \alpha')$, and denote for short S_∞^* and $S_\infty'^*$ the corresponding optimal costs. From (6) and the observation that $\mathscr{U}_{\alpha,T,T+D} \subset \mathscr{U}_{\alpha',T,T+D'}$, one deduces easily that $S_\infty^* \leqslant S_\infty'^*$. Assume by contradiction that $S_\infty^* = S_\infty'^*$. Then the optimal value $S_\infty'^*$ is realized for two different optimal controls: one in $\mathscr{U}_{\alpha,T,T+D}$ and one in $\mathscr{U}_{\alpha',T,T+D'} \setminus \mathscr{U}_{\alpha,T,T+D}$. This contradicts the uniqueness of the optimal control, demonstrated in Theorem 1. One thus concludes that $S_\infty^* < S_\infty'^*$. $\square$

Theorem 2 leads to the following question: what is the benefit of increasing indefinitely the lockdown duration D, and is it possible by this mean to stop the disease spread arbitrarily close to the herd immunity? The next result answers tightly this issue.

Theorem 3. *For any $S_0 \in (S_{\text{herd}}, 1)$, define*

$$\overline{\alpha} := \frac{S_{\text{herd}}}{S_0 + I_0 - S_{\text{herd}}} (\ln S_0 - \ln S_{\text{herd}}). \tag{7}$$

Then $\overline{\alpha} \in (0, 1)$ and the following properties are fulfilled.

(i) *If $\alpha \in [0, \overline{\alpha}]$, then*

$$\lim_{D \to +\infty} S_\infty^* = S_{\text{herd}}. \tag{8}$$

(ii) *If $\alpha \in (\overline{\alpha}, 1]$, then*

$$\lim_{D \to +\infty} S_\infty^* = S_\infty(\alpha\mathbf{1}) < S_{\text{herd}}. \tag{9}$$

In accordance with the notations introduced before, $\alpha\mathbf{1} \equiv \alpha$ on $[0, +\infty)$, and $S_\infty(\alpha\mathbf{1})$ is the limit of $S(t)$ when $t \to +\infty$, for the solution of (1) corresponding to $u = \alpha\mathbf{1}$.

Theorem 3 establishes that, provided that the lockdown is sufficiently strong (more precisely, that $\alpha \leqslant \overline{\alpha}$), then long enough lockdown stops the disease propagation arbitrarily close after passing the herd immunity level. On the contrary, if the lockdown is too moderate ($\alpha > \overline{\alpha}$), the power of such an action is intrinsically limited. This phenomenon is clearly apparent in the simulations provided in Section 3.

Proof of Theorem 3. One sees easily that $\overline{\alpha} > 0$, due to the fact that $S_0 > S_{\text{herd}}$. On the other hand,

$$\overline{\alpha} < \frac{S_{\text{herd}}}{S_0 - S_{\text{herd}}} (\ln S_0 - \ln S_{\text{herd}}) = \frac{1}{S_0/S_{\text{herd}} - 1} \ln S_0/S_{\text{herd}} < 1.$$

Assume now that $\alpha \leqslant \overline{\alpha}$. Theorem 1 in Bliman et al., 2020 establishes that, for any $\varepsilon > 0$, there exist $D > 0$ and $u \in \mathscr{U}_{\alpha,0,D}$ such that $S_\infty(u) \in [S_{\text{herd}} - \varepsilon, S_{\text{herd}}]$. As $S_\infty^* \geqslant S_\infty(u)$, this shows that

$$\limsup_{D \to +\infty} S_\infty^* \geqslant S_{\text{herd}}.$$

Due to the fact that S_∞^* is increasing with respect to D, as demonstrated by Theorem 2, and that $S_\infty^* \leqslant S_{\text{herd}}$ for any D, one gets (8).

Suppose now $\alpha > \overline{\alpha}$. In such conditions, Bliman et al., 2020, Theorem 1 shows that, for any $D > 0$ and $u \in \mathscr{U}_{\alpha,0,D}$,

$$S_\infty(u) \leqslant S_\infty(\alpha\mathbf{1}) < S_{\text{herd}},$$

so that

$$\limsup_{D \to +\infty} S_\infty^* \leqslant S_\infty(\alpha\mathbf{1}).$$

On the other hand, the value of S_∞^* increases with D (Theorem 2), while $S_\infty(\alpha\mathbf{1})$ is the limit of $S_\infty(\alpha\mathbb{1}_{[0,D]})$ for $D \to +\infty$. This yields (9) and achieves the proof of Theorem 3. $\square$

3. Numerical illustrations

We show in this Section the results of several numerical tests. The algorithms designed to solve Problem $(\mathscr{P}_{\alpha,D})$ are provided in the Supplementary materials, as well as a link to the resolution codes. A case study is first presented in Section 3.1, based on estimated conditions of circulation of the SARS-CoV-2 in France before and during the confinement enforced between March 17th and May 11th, 2020. This example is chosen merely for its illustrative value, without claiming to a realistic description of the outburst.

The results provided and commented in Section 3.2 give a broader view. They show the maximal final size reduction that may be obtained for different basic reproduction numbers $\mathscr{R}_0$, and for various realistic values of the maximal lockdown intensity α and duration D. Results corresponding to a more comprehensive set of values of $\mathscr{R}_0$ are provided in the Supplementary materials.

3.1. Optimal lockdown in conditions of Covid-19 circulation in France, March–May 2020

The parameters used in the simulations of the present section are given in Table 1. We assume that, on the total number $N = 6.7 \times 10^7$ of individuals corresponding to the French population, there were initially no recovered individuals ($R_0 = 0$). The initial number of infected individuals is taken equal to 1000, a level crossed on March 8th (Worldometer, 2020), so that $I_0 = 1 \times 10^3/6.7 \times 10^7 \approx 1.49 \times \times 10^{-5}$. Estimates of the infection rate β, of the recovery rate γ and of the containment coefficient α_{lock} in France between March 17th and May 11th 2020, are bor-

Table 1
Value of the parameters used in the simulations for system (1) (see Salje et al., 2020).

Parameter	Name	Value
β	Infection rate	$0.29 \ \text{day}^{-1}$
γ	Recovery rate	$0.1 \ \text{day}^{-1}$
α_{lock}	Lockdown level (France, March-May 2020)	0.231
S_0	Initial proportion of susceptible cases	$1 - I_0$
I_0	Initial proportion of infected cases	1.49×10^{-5}
R_0	Initial proportion of removed cases	0

rowed from Salje et al. (2020). They yield the following values for the basic reproduction number and the herd immunity:

$$\mathscr{R}_0 \approx 2.9, \qquad S_{\text{herd}} \approx 0.34.$$

With the initial conditions chosen here, the critical lockdown intensity defined in (7) is

$$\overline{\alpha} \approx 0.56.$$

The optimal solution (S^*, I^*, R^*, u^*) of Problem $(\mathscr{P}_{\alpha,D})$ for a containment duration of 30 days (top), 60 days (middle) and 90 days (bottom) is shown in Fig. 1, when total lockdown is allowed ($\alpha = 0$). The evolution of the proportions of susceptible, infected and removed cases is shown on the left, the optimal control on the right. The optimal dates for starting the enforcement are given in Table 2, together with the optimal asymptotic proportion of susceptible cases and the peak value of the proportion of infected.

As unveiled by close observation, one recovers the fact, established in Theorem 1, that $S(T^*) = S_{\text{herd}}$: when $\alpha = 0$, the optimal confinement starts exactly when the herd immunity threshold is crossed. Also, the optimal value S^*_∞ is larger when D is larger (Theorem 2), and it is known from Theorem 3 that this value converges towards S_{herd} when D goes to infinity. It is indeed already indistinguishable graphically from this value for $D = 60$ and 90 days.

Fig. 2 shows the same numerical experiments than Fig. 1, with $\alpha = \alpha_{\text{lock}} \approx 0.231 < \overline{\alpha} \approx 0.56$. Optimal starting dates and asymptotic proportions of susceptible are given in Table 3. The results are qualitatively similar to Fig. 1. One sees that the lockdown begins earlier in the previous case, and the achieved S^*_∞ are smaller. An interesting feature is that the proportion of infected at the peak of the epidemic is *smaller* for $\alpha = \alpha_{\text{lock}}$ than for $\alpha = 0$. As a matter of fact, with a lockdown beginning earlier, the peak of the epidemic is lower. This phenomenon, which may seem paradoxical at first glance, clearly suggests that reducing the final size and the peak value constitutes two conflicting goals.

The optimal starting dates given by the numerical resolution constitute an evident difference with the effective implementation that took place during the Spring 2020 epidemic outburst: they are located in May, essentially at the time when, after two months of lockdown, first relaxation of the measures were introduced! This should not be a surprise: the rationale behind this policy was not aimed at reaching herd immunity, but at reducing infections, in order to avoid overwhelming the health system and to be able to implement contact tracing on a tractable scale. On the contrary, the results in Fig. 1 and 2 show a peak of infected cases almost equal to 30% of the population —about twenty million people—, demonstrating that the strategy consisting of reaching herd immunity without considering other factors would not be sustainable, even if achieved under the optimal policy analyzed here.

3.2. Maximal final size reduction under given epidemic and lockdown conditions

Once the optimal solution u^* is computed, one may easily determine numerically, thanks to Lemma 2 below, the optimal value S^*_∞, by solving the equation

$$S^*(T^* + D) + I^*(T^* + D) - \frac{\gamma}{\beta} \ln S^*(T^* + D) = S^*_\infty - \frac{\gamma}{\beta} \ln S^*_\infty,$$

where, as said before, (S^*, I^*) is the optimal solution.

Taking advantage of this principle, one may compute the optimal final size reduction corresponding to any epidemic and lockdown conditions. To fix the ideas, the value $\gamma = 0.1 \ \text{day}^{-1}$ is considered in this Section, corresponding to a mean recovery time of 10 days. The general case is obtained by scaling: for (1) with different γ, the values of S^*_∞ and T^* are obtained as

$$S^*_\infty = S'^*_\infty, \qquad T^* = \frac{0.1}{\gamma} T'^* \ \text{day},$$

where S'^*_∞, T'^* are the optimal cost and starting date obtained for the normalized system defined by the parameters $\gamma' := 0.1 \ \text{day}^{-1}$ and

$$\beta' := \frac{\beta}{\gamma} 0.1 \ \text{day}^{-1}, \qquad \alpha' := \alpha, \qquad D' := \frac{\gamma}{0.1} D \ \text{day}$$

(in such a way that $\mathscr{R}_0 = \beta'/\gamma' = \beta/\gamma$).

As an illustration, we present in the sequel computations obtained for $\mathscr{R}_0 = 2$ and $\mathscr{R}_0 = 6$. A more complete set of values is provided in the Supplementary materials. Fig. 3 shows the optimal final size value $S^*_\infty/S_{\text{herd}}$ as a function of the duration D, for several values of the lockdown intensity α ranging from 0 to 0.8. For D tending to 0, all curves meet at a common value that corresponds to the final size attained in absence of lockdown. One observes that the optimal value $S^*_\infty/S_{\text{herd}}$ increases as a function of the lockdown duration D, and decreases as a function of its maximal intensity α, as announced in Theorem 2. For $0 < \alpha < \overline{\alpha}$, the optimal value S^*_∞ converges towards S_{herd} (the best value one can expect) when D increases indefinitely; while for $\overline{\alpha} < \alpha \leqslant 1$, the optimal value is strictly smaller, and decreases with respect to α, as predicted by Theorem 3. Observe that the value of $\overline{\alpha}$ is larger for larger value of $\mathscr{R}_0$, making social distancing less efficient for diseases with larger basic reproduction number. This trend is corroborated by the inspection of wider assortment of values for $\mathscr{R}_0$, see the Supplementary materials. One sees that, depending upon the initial conditions, the optimal lockdown policy may induce a significant decrease of the final size.

For the same values of $\mathscr{R}_0$, we represent in Fig. 4 the dependency of $S^*_\infty/S_{\text{herd}}$ with respect to the parameter α, for values of D corresponding to 1, 2, 4 and 8 months. For α tending to 1, the value of $S^*_\infty/S_{\text{herd}}$ goes to the value achieved without control.

In the same way that Fig. 3 and 4 revealed the dependence of $S^*_\infty/S_{\text{herd}}$ respectively upon D and α, Fig. 5 and 6 show the dependence of T^* with respect to these parameters. Fig. 5 shows the variation of T^* with respect to D, for the same values of α than Fig. 3. The value of T^* decreases as a function of D and of α. One observes that for D close to 0, the optimal intervention begins at the time where herd immunity is crossed, for every value of α. It converges to a positive limit when $\alpha > \overline{\alpha}$, while it converges to 0 when $\alpha < \overline{\alpha}$. Notice that the value of T^* depends drastically upon $\mathscr{R}_0$.

Fig. 6 shows the variation of T^* with respect to α, for the same values of D than Fig. 4. The value of T^* also decreases with respect to α. When $\alpha = 0$, the optimal starting point is at the crossing of the immunity threshold.

For sake of space, the dependence of S^*_∞ and T^* with respect to the initial conditions (S_0, I_0) is not explored here. Partial results not provided suggest that both quantities increase with S_0 and
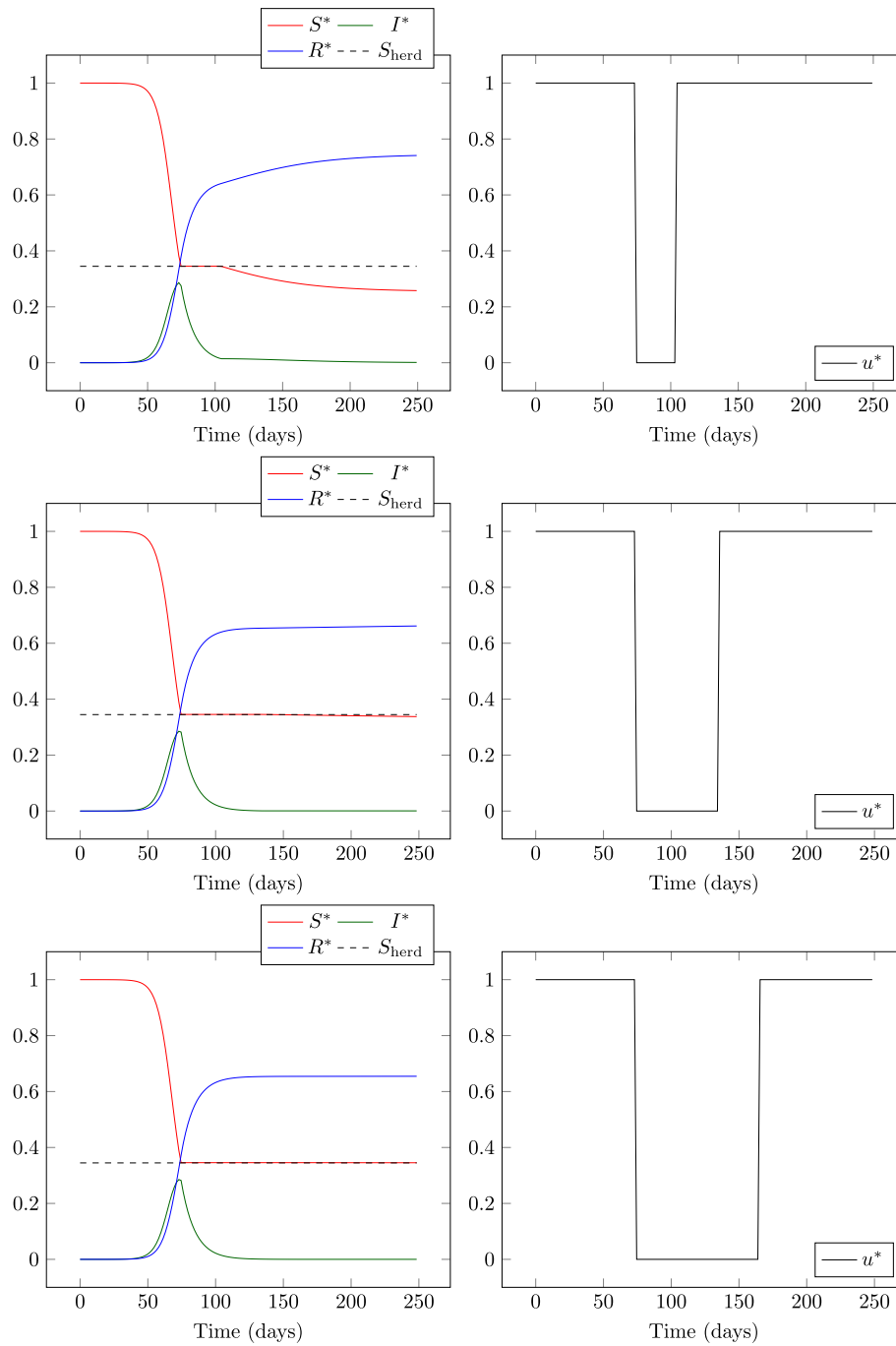
Fig. 1. Optimal solution (S^*, I^*, R^*, u^*) to Problem $(\mathcal{P}_{\alpha,D})$ computed for $\alpha = 0.0, D = 30, 60$ and 90 days. See numerical values in Table 2 and comments in the text.

Table 2
Characteristics of the optimal solutions computed with the parameters of Table 1, with lockdown intensity $\alpha = 0$ and duration $D = 0$ (no lockdown), $30, 60$ and 90 days. The starting dates are computed from the epidemic initial time on March 8th, where the cumulative number of infected exceeded 1000 cases. See the curves in Fig. 1, and explanations in text.

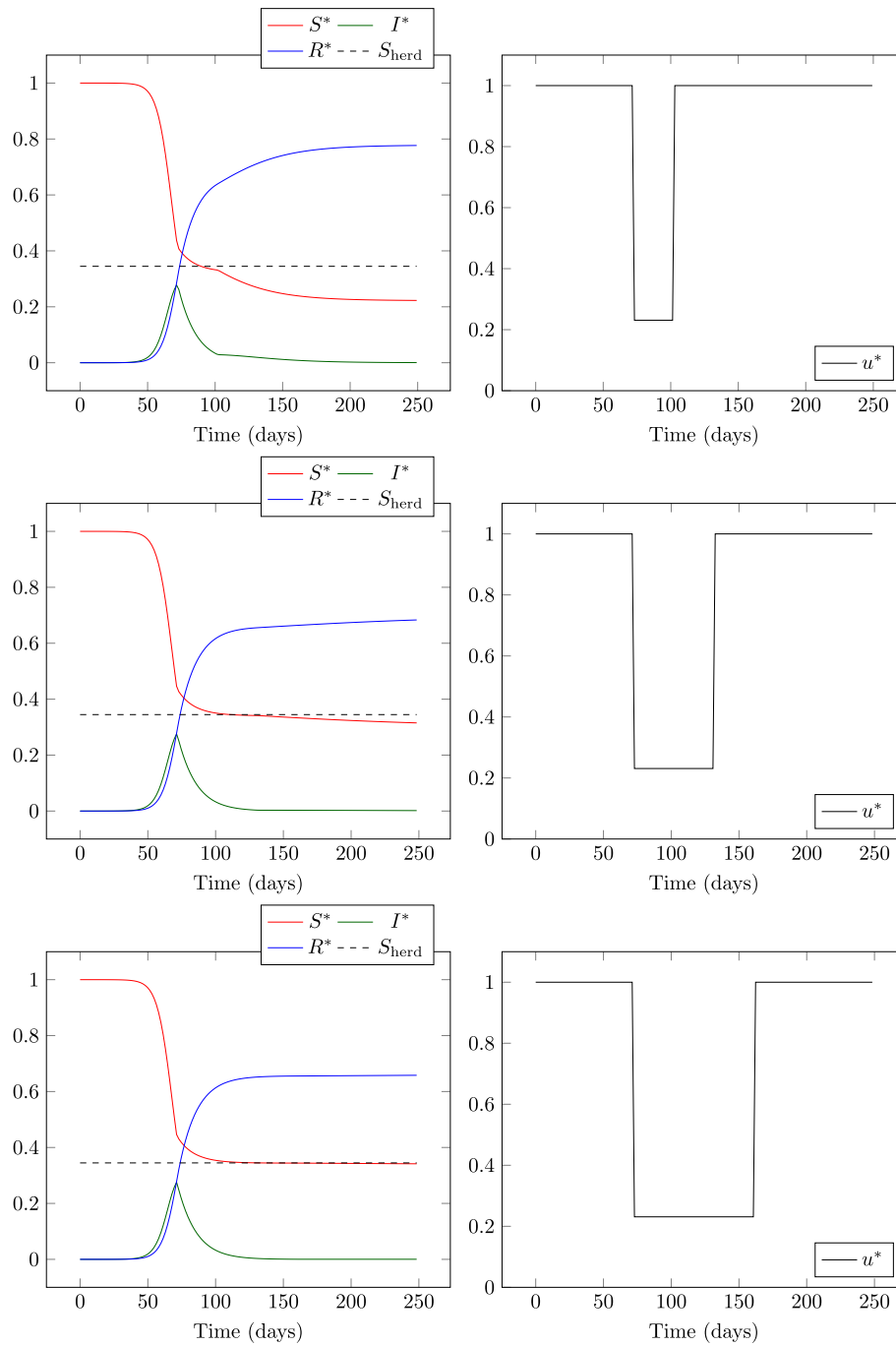| D	T^*	S_∞^*	$S_\infty^*/S_{\text{herd}}$	$\max_{t \geqslant 0} I(t)$
No lockdown	—	0.0668	0.194	0.288
30 days	$T^* = 74.3$ days (May 21st)	0.255	0.739	0.288
60 days	$T^* = 74.3$ days (May 21st)	0.323	0.937	0.288
90 days	$T^* = 74.3$ days (May 21st)	0.340	0.985	0.288

Fig. 2. Same than Fig. 1, with $\alpha = \alpha_{\text{lock}} \approx 0.231$. Numerical values are provided in Table 3.

Table 3
Similar to Table 2, with lockdown intensity $\alpha = \alpha_{\text{lock}} \approx 0.231$. See corresponding curves in Fig. 2.

D	T^*	S_∞^*	$S_\infty^*/S_{\text{herd}}$	$\max_{t \geqslant 0} I(t)$
No lockdown	—	0.0668	0.194	0.288
30 days	$T^* = 72.1$ days (May 19th)	0.222	0.644	0.282
60 days	$T^* = 71.5$ days (May 18th)	0.302	0.875	0.278
90 days	$T^* = 71.3$ days (May 18th)	0.331	0.959	0.277

decrease with I_0. Notice that this monotony property is at least true on the set $\{(S_0, I_0) : \ 0 \leqslant S_0 \leqslant S_{\text{herd}}, 0 \leqslant I_0, S_0 + I_0 \leqslant 1\}$, where the optimal mitigation must start immediately. Also, it is compatible

with the fact that T^* goes unbounded when I_0 converges to 0, see Theorem 1. Determining the set $\mathscr{S}$ of values of (S_0, I_0) for which it is optimal to start mitigation immediately is also an interesting
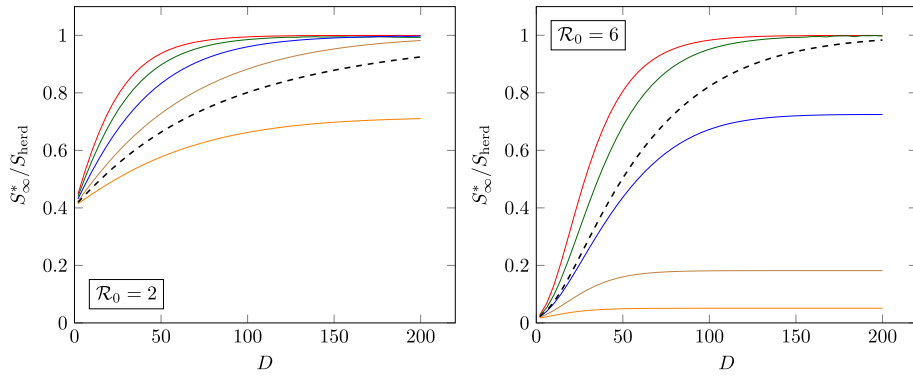
Fig. 3. Graph of $S^*_\infty/S_{\text{herd}}$ for Problem ($\mathcal{P}_{\alpha,D}$) as a function of D, for $\alpha \in \{0.0\ (\text{—}), 0.2\ (\text{—}), 0.4\ (\text{—}), 0.6\ (\text{—}), 0.8\ (\text{—}), \overline{\alpha}\ (\text{- -})\}$ and $\mathscr{R}_0 \in \{2, 6\}$.
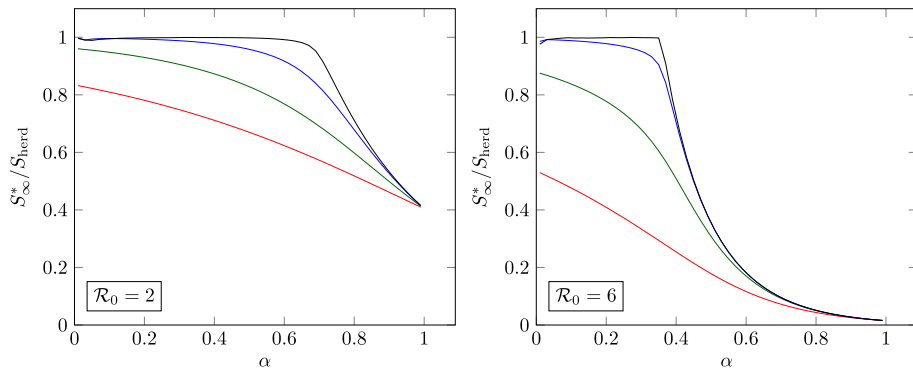
Fig. 4. Graph of $S^*_\infty/S_{\text{herd}}$ for Problem ($\mathcal{P}_{\alpha,D}$) as a function of α, for $D \in \{30\ (\text{—}), 60\ (\text{—}), 120\ (\text{—}), 240\ (\text{–})\}$ and $\mathscr{R}_0 \in \{2, 6\}$.
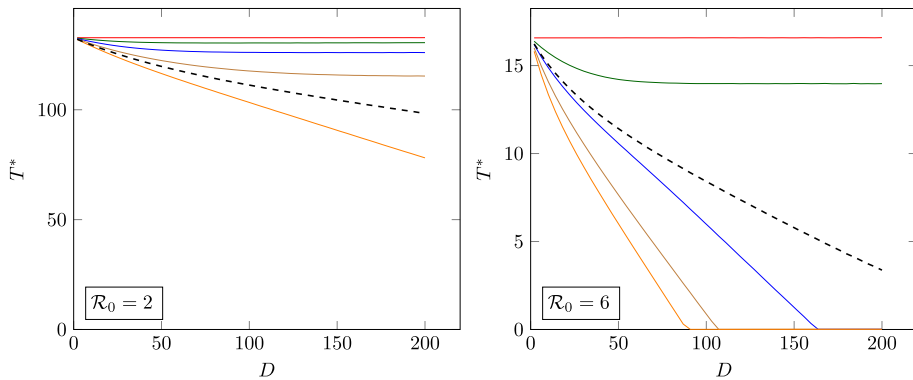
Fig. 5. Graph of T^* for Problem ($\mathcal{P}_{\alpha,D}$) as a function of D, for $\alpha \in \{0.0\ (\text{—}), 0.2\ (\text{—}), 0.4\ (\text{—}), 0.6\ (\text{—}), 0.8\ (\text{—}), \overline{\alpha}\ (\text{- -})\}$ and $\mathscr{R}_0 \in \{2, 6\}$.
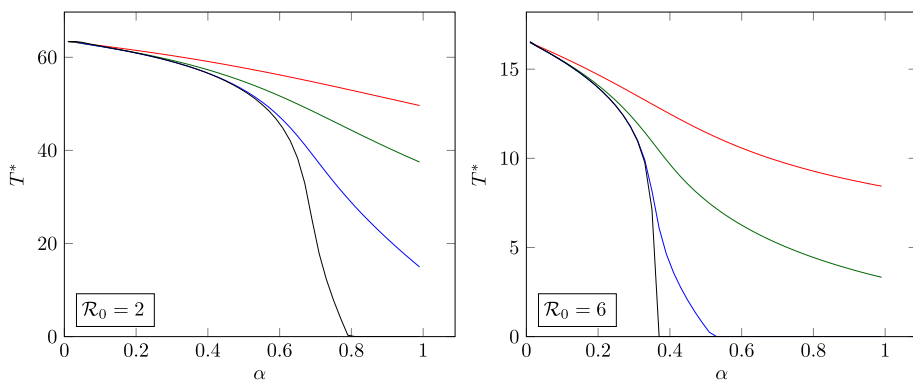
Fig. 6. Graph of T^* for Problem ($\mathcal{P}_{\alpha,D}$) as a function of α, for $D \in \{30\ (\text{—}), 60\ (\text{—}), 120\ (\text{—}), 240\ (\text{–})\}$ and $\mathscr{R}_0 \in \{2, 6\}$.

problem, not considered here. The set mentioned above is contained in $\mathscr{S}$, and one may conjecture that $(S_0, I_0) \in \mathscr{S}$ implies that $(S_0', I_0') \in \mathscr{S}$, whenever $S_0' \leqslant S_0, I_0' \geqslant I_0$.

4. Proof of Theorem 1

The proof is organized as follows. We first recall in Section 4.1 results obtained in Bliman et al. (2020) for the optimal control problem considered on intervention intervals of the type $[0, D], D > 0$. Using these results, one shows in Section 4.2 that any solution of problem $(\mathscr{P}_{\alpha,D})$ is of type (3) and may be determined by solving a 2D optimization problem. It is subsequently shown in Section 4.3 that the latter problem may be simplified to a 1D optimization problem, whose study is achieved in Section 4.4. Last, the property on the limit of T^* is demonstrated in Section 4.5.

4.1. Optimal control on a finite horizon $[0, D]$

After introducing some notations, we recall here optimal control results from Bliman et al. (2020).

For any $0 \leqslant t \leqslant t'$, any input u, and any positive initial condition $X_0 = (S_0, I_0)$ such that $S_0 + I_0 \leqslant 1$, one denotes

$$X(t', t; X_0; u) := (S(t', t; X_0; u), I(t', t; X_0; u))$$

the value at time t' of the solution of (1) departing at time t from X_0, with the control input u. This extended notation will be simplified when clear from the context.

Also, introduce the function $\Phi_{\mathscr{R}}$ defined for any $\mathscr{R} > 0$ by:

$$\Phi_{\mathscr{R}} : \mathbb{R}_+^* \times \mathbb{R}_+ \ni (S, I) \mapsto S + I - \frac{1}{\mathscr{R}} \ln S. \tag{10}$$

An important property is now given, which allows to define scalar quantities invariant along the trajectories. See details in Bliman et al., 2020, Lemma 3.1.

Lemma 1. *For any $u \in L^\infty([0, +\infty), [0, 1])$ and $\gamma \in \mathbb{R}$, one has*

$$\frac{d}{dt}[\Phi_{\mathscr{R}}(S(t), I(t))] = \left(\frac{\beta}{\mathscr{R}} u(t) - \gamma\right) I(t) \tag{11}$$

along any trajectory of system (1). In particular, if u is constant on a non-empty, possibly unbounded, interval, then the function $t \mapsto \Phi_{\mathscr{R}_0 u}(S(t), I(t))$ is constant on this interval along any trajectory of system (1).

The proof of Lemma 1 is straightforward and may be found in Bliman et al. (2020). This result allows to characterize the epidemic final size resulting from the use of an input control in $\mathscr{U}_{\alpha,T,T'}$, as stated now, see details and proof in Bliman et al., 2020, Lemma 3.2.

Lemma 2. *Let $0 \leqslant T < T'$ and $u \in \mathscr{U}_{\alpha,T,T'}$. For any trajectory of (1), $S_\infty(u)$ is the unique solution in $[0, S_{\text{herd}}]$ of the equation*

$$\Phi_{\mathscr{R}_0}(S_\infty(u), 0) = \Phi_{\mathscr{R}_0}(X(T', 0; X_0; u)),$$

where $\Phi_{\mathscr{R}_0}$ is given by (10).

Due to the fact that any control input $u \in \mathscr{U}_{\alpha,T,T+D}$ is equal to 1 on $[T + D, +\infty)$, the map $t \mapsto \Phi_{\mathscr{R}_0}(X(t, 0; X_0; u))$ is constant on that interval. On the other hand, the map $S \mapsto \Phi_{\mathscr{R}_0}(S, 0)$ is *decreasing* on the interval $[0, S_{\text{herd}}]$, so arguing as in Bliman et al. (2020), one deduces that solving problem $(\mathscr{P}_{\alpha,D})$ is equivalent to solve

$$\inf_{T \geqslant 0} \inf_{u \in \mathscr{U}_{\alpha,T,T+D}} \Phi_{\mathscr{R}_0}(X(T + D, 0; X_0; u)). \tag{12}$$

This property is central to our approach: it transforms $(\mathscr{P}_{\alpha,D})$, which consists in maximizing the limit of S at infinity, into an optimal control problem on a *finite time horizon*. This reduction proce-

dure is at the basis of the arguments in Ketcheson (2020) and Bliman et al. (2020). Based on the latter one obtains the following result. See details in Bliman et al., 2020, Theorem 2.3.

Theorem 4. *For any $\alpha \in [0, 1)$ and $D > 0$, the optimal control problem*

$$\sup_{u \in \mathscr{U}_{\alpha,0,D}} S_\infty(u) \tag{13}$$

admits a unique solution. Moreover the optimal control, denoted $u_{0,D}^*(X_0)$, is equal to the function $u_{T_0^*,D}$ defined in (3), for some uniquely defined $T_0^* \in [0, D)$.

Theorem 4 shows that the optimal control is bang-bang with at most two switches: a first one at some time $T_0^* \in [0, D)$, and a second one at time D. Moreover, the value of T_0^* appears as the result of a 1D optimization problem, as

$$\sup_{u \in \mathscr{U}_{\alpha,0,D}} S_\infty(u) = \sup_{T_0^* \in [0,D)} S_\infty\left(u_{T_0^*,D}\right).$$

4.2. Reduction to a 2D optimization problem

Consider now, for any $0 \leqslant T < T'$, the general problem

$$\inf_{u \in \mathscr{U}_{\alpha,T,T'}} \Phi_{\mathscr{R}_0}(X(T', 0; X_0; u)). \qquad (\mathscr{P}_{\alpha,T,T'})$$

The problem $(\mathscr{P}_{\alpha,0,D})$ is equivalent to (13), which is the subject of Theorem 4. The following result shows that problem $(\mathscr{P}_{\alpha,T,T'})$, under its equivalent form (12), benefits from this result.

Proposition 1. *Let $0 \leqslant T < T'$. There exists a unique optimal control $u_{T,T'}^*(X_0) \in \mathscr{U}_{\alpha,T,T'}$ for problem $(\mathscr{P}_{\alpha,T,T'})$ and it verifies:*

$$u_{T,T'}^*(X_0)(t) = u_{0,T'-T}^*(X(T, 0; X_0; \mathbf{1}))(t - T), \quad t \in [T, T']. \tag{14}$$

Formula (14) and the fact that $u_{T,T'}^*(X_0) \in \mathscr{U}_{\alpha,T,T'}$ imply that $u_{T,T'}^*(X_0)$ is equal to 1 on $[0, T] \cup [T', +\infty)$, so this function is uniquely defined on the whole $[0, +\infty)$ by the statement. Proposition 1 says that the optimal control for problem $(\mathscr{P}_{\alpha,T,T'})$ with initial value X_0 is equal to the optimal control for problem $(\mathscr{P}_{\alpha,0,T'-T})$ with initial value $X(T, 0; X_0; \mathbf{1})$, delayed from the time duration T and completed by 1 on the interval $[0, T]$. In turn, the point $X(T, 0; X_0; \mathbf{1})$ is the value at time T of the solution of (1) departing at time 0 from X_0 with input equal to $\mathbf{1}$. Therefore, solving $(\mathscr{P}_{\alpha,T,T'})$ with initial condition X_0 amounts to solve $(\mathscr{P}_{\alpha,0,T'-T})$ with initial condition $X(T, 0; X_0; \mathbf{1})$.

Before going further, let us prove the previous result.
Proof of Proposition 1. One may define a canonical bijection $\mathscr{C} : \mathscr{U}_{\alpha,0,T'-T} \to \mathscr{U}_{\alpha,T,T'}$ by

$$\mathscr{C}(u)(t) = 1 \text{ if } t \in [0, T], \qquad \mathscr{C}(u)(t) = u(t - T) \text{ if } t \in [T, +\infty)$$

for any $u \in \mathscr{U}_{\alpha,0,T'-T}$. By the semi-group property deduced from the fact that system (1) is stationary, one has for any $u \in \mathscr{U}_{\alpha,T,T'}$ and any $t \in [T, T']$,

$$X(t, 0; X_0; u) = X(t - T, 0; X(T, 0; X_0; \mathbf{1}); \mathscr{C}^{-1}(u)).$$

Applying this formula with $t = T'$ yields

$$X(T', 0; X_0; u) = X(T' - T, 0; X(T, 0; X_0; \mathbf{1}); \mathscr{C}^{-1}(u)).$$

Therefore, for any $u \in \mathscr{U}_{\alpha,T,T'}$,

$$\Phi_{\mathscr{R}_0}(X(T', 0; X_0; u)) = \Phi_{\mathscr{R}_0}(X(T' - T, 0; X(T, 0; X_0; \mathbf{1}); \mathscr{C}^{-1}(u))),$$

and this correspondence permits to achieve the demonstration. □

Using the qualitative properties of the solutions of problem (12) recalled above, we deduce from Proposition 1 that, for any $T \geqslant 0$, the problem

$$\inf_{u \in \mathscr{U}_{\alpha,T,T+D}} \Phi_{\mathscr{R}_0}(X(T+D,0;X_0;u))$$

admits a unique solution of the type $u_{T+\varepsilon_{T,D},T+D}$, for some $\varepsilon_{T,D} \in [0,D)$. We thus have proved so far that problem $(\mathscr{P}_{\alpha,D})$ is equivalent to the 2D optimization problem

$$\inf_{T \geqslant 0} \inf_{\varepsilon \in [0,D)} \Phi_{\mathscr{R}_0}(X(T+D,0;X_0;u_{T+\varepsilon,T+D})).$$

4.3. Reduction to a 1D optimization problem

In this section, we further reduce the complexity of the optimal control problem under study. We first show that the problem $(\mathscr{P}_{\alpha,D})$ admits a solution. This indeed amounts to show that no unbounded maximizing sequence of times $\{T_k\}_{k \in \mathbb{N}}$ is to be found.

Proposition 2. Problem $(\mathscr{P}_{\alpha,D})$ admits at least one solution.

Proof. Consider $\bar{X} := (\bar{S},\bar{I})$ the solution associated to $u = \mathbf{1}$, and let $\bar{T}$ be defined by $\bar{S}(\bar{T}) = S_{\mathrm{herd}}$. For this value $\bar{T}$, define $X^{\bar{T}} := (S^{\bar{T}}, I^{\bar{T}})$ as the solution to system (1) associated to $u_{\bar{T},\bar{T}+D}$. Lemma 1 shows that

- the map $t \mapsto \Phi_{\mathscr{R}_0}(\bar{X}(t))$ is constant on $[0,+\infty)$;
- the map $t \mapsto \Phi_{\mathscr{R}_0}(X^{\bar{T}}(t))$ is constant on $[0,\bar{T}]$ and on $[\bar{T}+D,+\infty)$;
- the value of $\Phi_{\mathscr{R}_0}(X^{\bar{T}}(t))$ on $[0,\bar{T}]$ is *smaller* than the value on $[\bar{T}+D,+\infty)$, because (11) implies that this map cannot increase on $[\bar{T},\bar{T}+D]$.

Therefore, the fact that $\Phi_{\mathscr{R}_0}(\bar{X}(0)) = \Phi_{\mathscr{R}_0}(X_0) = \Phi_{\mathscr{R}_0}(X^{\bar{T}}(0))$ implies

$$\Phi_{\mathscr{R}_0}(X^{\bar{T}}(\bar{T}+D)) < \Phi_{\mathscr{R}_0}(\bar{X}(\bar{T}+D)),$$

and thus

$$S_\infty(u_{\bar{T},\bar{T}+D}) - \frac{\gamma}{\beta} \ln(S_\infty(u_{\bar{T},\bar{T}+D})) < S_\infty(\mathbf{1}) - \frac{\gamma}{\beta} \ln(S_\infty(\mathbf{1})).$$

Since $\Phi_{\mathscr{R}_0}(\cdot,0)$ is decreasing on $(0, S_{\mathrm{herd}})$ and, by Lemma 2, $S_\infty(u_{\bar{T},\bar{T}+D}), S_\infty(\mathbf{1}) < S_{\mathrm{herd}}$, we deduce that

$$S_\infty(u_{\bar{T},\bar{T}+D}) > S_\infty(\mathbf{1}).$$

There thus exists T_1 sufficiently large, so that

$$S_\infty(\mathbf{1}) < \bar{S}(T_1) < S_\infty(u_{\bar{T},\bar{T}+D}).$$

Since S decreases along every trajectory, for each $T > T_1$ and $\varepsilon \in (0,D)$, one has

$$S_\infty(u_{T+\varepsilon,T+D}) < \bar{S}(T_1),$$

because $u_{T+\varepsilon,T+D} \equiv 1$ on $[0,T_1] \subset [0,T+\varepsilon]$. Therefore, one may thus restrict the search for optimal solutions of problem $(\mathscr{P}_{\alpha,D})$ to those (T,ε) that belong to the set $[0,T_1] \times [0,D]$. We conclude by observing that the problem

$$\inf_{T \in [0,T_1]} \inf_{\varepsilon \in [0,D]} \Phi_{\mathscr{R}_0}(X(T+D,0;X_0;u_{T+\varepsilon,T+D})), \qquad (15)$$

which consists in optimizing a continuous function on a finite-dimensional compact set, admits a non-void set of solutions. $\square$

We now show in the next result that every possible optimal solution for problem (15) corresponds to $\varepsilon = 0$. In other terms, any optimal policy consists in applying the more intense lockdown intensity during a duration exactly equal to D, not less.

Proposition 3. Any solution of problem $(\mathscr{P}_{\alpha,D})$ is of the type $u_{T,T+D}$ for some $T \geqslant 0$.

From Proposition 3 one deduces that problem $(\mathscr{P}_{\alpha,D})$ is equivalent to solving

$$\inf_{T \in [0,\bar{T}]} \Phi_{\mathscr{R}_0}(X(T+D,0;X_0;u_{T,T+D})). \qquad (16)$$

This achieves the announced reduction to a 1D optimization problem.

Proof of Proposition 3. Assume by contradiction that $u_{T+\varepsilon,T+D}$ is solution to problem $(\mathscr{P}_{\alpha,D})$ for some $\varepsilon > 0$. Then $u_{T+\varepsilon,T+D} \in \mathscr{U}_{\alpha,T+\varepsilon,T+D} \cap \mathscr{U}_{\alpha,T+\varepsilon,T+\varepsilon+D}$. We know from Proposition 2 and Theorem 4 that

$$\inf_{u \in \mathscr{U}_{\alpha,T+\varepsilon,T+\varepsilon+D}} \Phi_{\mathscr{R}_0}(X(T+D,0;X_0;u))$$

admits a unique solution, which writes $u_{T+\varepsilon+\delta,T+\varepsilon+D}$ for some $\delta \geqslant 0$. Since $u_{T+\varepsilon,T+D} \neq u_{T+\varepsilon+\delta,T+\varepsilon+D}$, one has

$$\Phi_{\mathscr{R}_0}(X(T+D,0;X_0;u_{T+\varepsilon+\delta,T+\varepsilon+D})) < \Phi_{\mathscr{R}_0}(X(T+D,0;X_0;u_{T+\varepsilon,T+D})).$$

This is in contradiction with the optimality of $u_{T+\varepsilon,T+D}$ for problem $(\mathscr{P}_{\alpha,D})$. Therefore, $\varepsilon = 0$ for any optimal control. $\square$

4.4. Solving the 1D optimization problem (16)

We now achieve the demonstration of Theorem 1, through the study of problem (16), establishing in particular uniqueness of the optimum control. Denote T^* an optimal solution of (16).

Step 1: necessary first order optimality conditions Let $u = u_{T,T+D}$ be an optimal control for problem $(\mathscr{P}_{\alpha,D})$. Let us introduce the criterion j given by

$$j(T) := \Phi_{\mathscr{R}_0}(S^T(T+D), I^T(T+D))$$
$$= I^T(T+D) + S^T(T+D) - \frac{\gamma}{\beta} \ln(S^T(T+D)),$$

where, as done in the definition of ψ in (4), (S^T, I^T) is the solution corresponding to the control $u_{T,T+D}$. For sake of simplicity, we will usually omit these subscripts in the sequel. With this notation, (16) simply writes

$$\inf_{T \geqslant 0} j(T). \qquad (17)$$

By using Lemma 2, one has for the solution (S^T, I^T):

$$I(t) + S(t) - \frac{\gamma}{\beta} \ln S(t) = c_0 \qquad \text{in} [0,T],$$
$$I(t) + S(t) - \frac{\gamma}{\alpha\beta} \ln S(t) = I(T) + S(T) - \frac{\gamma}{\alpha\beta} \ln S(T) \quad \text{in}[T,T+D],$$
$$(18)$$

where $c_0 := I_0 + S_0 - \frac{\gamma}{\beta} \ln S_0$. Eliminating $I(t)$ from (1) with $u = u_{T,T+D}$, thanks to (18), we infer that S solves the system

$$\dot{S} = -\beta S \left(c_0 - S + \frac{\gamma}{\beta} \ln S\right), \qquad \text{in}(0,T), \qquad (19a)$$

$$\dot{S} = -\alpha\beta S \left(c_0 + \frac{\gamma}{\beta}\left(1 - \frac{1}{\alpha}\right) \ln S(T) - S + \frac{\gamma}{\alpha\beta} \ln S\right), \quad \text{in}(T,T+D), \qquad (19b)$$

with the initial value $S(0) = S_0$. Using (18), one gets

$$\begin{aligned} j(T) &= I(T+D) + S(T+D) - \tfrac{\gamma}{\beta}\ln S(T+D) \\ &= I(T) + S(T) - \tfrac{\gamma}{\alpha\beta}\ln S(T) + \tfrac{\gamma}{\beta}\left(\tfrac{1}{\alpha}-1\right)\ln S(T+D) \\ &= c_0 + \tfrac{\gamma}{\beta}\ln S(T) - \tfrac{\gamma}{\alpha\beta}\ln S(T) + \tfrac{\gamma}{\beta}\left(\tfrac{1}{\alpha}-1\right)\ln S(T+D), \end{aligned}$$

so that the cost function reads

$$j(T) = c_0 + \frac{\gamma}{\beta}\left(\frac{1}{\alpha}-1\right)\ln\left(\frac{S^T(T+D)}{S^T(T)}\right).$$

We point out that this expression depends upon T through the arguments T and $T+D$ at which the function S^T is considered; but also through the value of the function S^T itself, which depends upon T through the input $u_{T,T+D}$. Special care is therefore needed to compute the derivative j' of j with respect to T. This constitutes the subject of the following technical lemma, whose proof is postponed to the end of the section, for sake of clarity. For simplicity, we denote in the sequel $S(\widehat{T+D}), \widehat{S(T)}$ and $\widehat{S(t)}$ the derivatives of the functions $S^T(T+D), S^T(T)$ and $S^T(t)$ with respect to T, that is:

$$S(\widehat{T+D}) := \frac{\partial[S^T(T+D)]}{\partial T}, \qquad \widehat{S(T)} := \frac{\partial[S^T(T)]}{\partial T},$$
$$\widehat{S(t)} := \frac{\partial[S^T(t)]}{\partial T}, \qquad t \in (T, T+D).$$

Lemma 3. *The following formulas hold.*

$$S(\widehat{T+D}) = \beta S^T(T+D)I^T(T+D)\left(-1 + (\alpha-1)\gamma I^T(T)\int_T^{T+D}\frac{ds}{I^T(s)}\right),$$
(20a)

$$\widehat{S(T)} = -\beta S^T(T)I^T(T),$$
(20b)

$$\widehat{S(t)} = (\alpha-1)\beta S^T(t)I^T(t)\left(1 + \gamma I^T(T)\int_T^t\frac{ds}{I^T(s)}\right), \quad t \in (T, T+D).$$
(20c)

Thanks to identities (20a)-(20b), one may compute

$$\begin{aligned} j'(T) &= \tfrac{\gamma}{\beta}\left(\tfrac{1}{\alpha}-1\right)\left(\frac{S(\widehat{T+D})}{S(T+D)} - \frac{\widehat{S(T)}}{S(T)}\right) \\ &= \gamma\left(\tfrac{1}{\alpha}-1\right)\left(I(T+D)\left(-1 + (\alpha-1)\gamma I(T)\int_T^{T+D}\frac{ds}{I(s)}\right) + I(T)\right) \\ &= \gamma\left(\tfrac{1}{\alpha}-1\right)I(T)\left(-\frac{I(T+D)}{I(T)} + (\alpha-1)\gamma\int_T^{T+D}\frac{I(T+D)}{I(s)}ds + 1\right). \end{aligned}$$

We deduce that $j'(T) = 0$ is equivalent to

$$\psi(T) = 0,$$
(21)

for the function ψ defined in (4).

Step 2: Zeros of j' and uniqueness of the optimal time T^* Integrating the second equation in (1) for $u = u_{T,T+D}$, one has for any $t \in (T, T+D)$, $I(t) = I(T+D)\exp\left(\int_{T+D}^t(\alpha\beta S(s)-\gamma)ds\right)$. Then, using the expression of ψ in (4), it follows that

$$\begin{aligned} \psi(T) = &-\exp\left(\int_T^{T+D}(\alpha\beta S(t)-\gamma)dt\right) \\ &+ (\alpha-1)\gamma\int_T^{T+D}\exp\left(\int_s^{T+D}(\alpha\beta S(t)-\gamma)dt\right)ds + 1. \end{aligned}$$

Introducing $\varphi(t) := \exp\left(\int_t^{T+D}(\alpha\beta S(s)-\gamma)ds\right)$ for $t \in [0, T+D]$, the last expression writes simply

$$\begin{aligned} \psi(T) &= -\varphi(T) + (\alpha-1)\gamma\int_T^{T+D}\varphi(t)\,dt + 1 \\ &= -\varphi(T) + (\alpha-1)\gamma\int_0^D\varphi(T+t)\,dt + 1. \end{aligned}$$

Differentiating the expressions in φ with respect to t and afterwards ψ with respect to T yields first

$$\varphi'(t) = \alpha\beta\left(S(T+D) - S(t) + \int_t^{T+D}\widehat{S(s)}\,ds\right)\varphi(t), \qquad t \geqslant 0,$$

and then

$$\begin{aligned} \psi'(T) = &-\alpha\beta\left(S(T+D) - S(T) + \int_T^{T+D}\widehat{S(s)}\,ds\right)\varphi(T) \\ &+ (\alpha-1)\gamma\alpha\beta\int_0^D\left(S(T+D) - S(T+t) + \int_{T+t}^{T+D}\widehat{S(s)}\,ds\right)\varphi(T+t)\,dt. \end{aligned}$$

On the one hand, S decreases along the trajectory, so $S(T+D) - S(T+t) < 0$ for any $t \in [0, D)$. On the other hand, $\widehat{S(t)} < 0$, see formula (20c). One then deduces that both terms in the addition in the previous formula are positive. The function ψ is thus increasing on $(0, \infty)$.

Step 3: The case $\alpha > 0$. Assume now that $\alpha > 0$. Then, for any T large enough in such a way that $S^T(T) < S_{\text{herd}}$, one has $\alpha\beta S^T(t) - \gamma < \alpha\beta S_{\text{herd}} - \gamma = \gamma(\alpha-1)$ for any $t \in (T, T+D)$, because the function S^T is decreasing along every trajectory. For such a sufficiently large T, one has

$$\varphi(t) = \exp\left(\int_t^{T+D}(\alpha\beta S(s)-\gamma)ds\right) < e^{\gamma(\alpha-1)(T+D-t)}, \quad t \in (T, T+D],$$
(22)

and thus

$$\begin{aligned} \psi(T) =\ &-\varphi(T) + (\alpha-1)\gamma\int_0^D\varphi(T+t)\,dt + 1 \\ >\ &-e^{\gamma(\alpha-1)D} + (\alpha-1)\gamma\int_0^D e^{(D-t)\gamma(\alpha-1)}dt + 1 = 0. \end{aligned}$$

Therefore, the function ψ being increasing, if $\psi(0) > 0$, then (21) has no solution. Equivalently there is no T such that $j'(T) = 0$, and thus $T^* = 0$. Conversely, if $\psi(0) \leq 0$, then (21) admits a unique solution T^*, which is the unique critical point of j. In the particular case where $\psi(0) = 0$, one has $T^* = 0$.

Remark 1. Notice that the function j is decreasing on $(0, T^*)$ and increasing on (T^*, ∞). This observation is useful for the numerical implementation.

The fact that $S(T^*) > S_{\text{herd}}$ if $T^* > 0$ comes as a byproduct of the previous considerations. Indeed, it has been shown that $\psi(T) > 0$ if $S^T(T) < S_{\text{herd}}$. Therefore, if $T^* > 0$, then $\psi(T^*) = 0$ and $S(T^*) \geqslant S_{\text{herd}}$. Noticing that the inequality in (22) is strict for any $t \in (T, T+D)$ yields the strict inequality $S(T^*) > S_{\text{herd}}$.

Step 4: The case $\alpha = 0$ In the case $\alpha = 0$, the solution S^T corresponding to $u_{T,T+D}$ is constant on $(T, T+D)$, and we deduce that

$$j(T) = I(T+D) + S(T) - \frac{\gamma}{\beta}\ln(S(T)) = \left(e^{-\gamma D}-1\right)I(T) + c_0.$$

We conclude using the fact that $I(T)$ is maximal when $S(T) = S_{\text{herd}}$, therefore T^* is such that $S(T^*) = S_{\text{herd}}$.

To terminate the work done in Section 4.4, it now remains to prove Lemma 3.

Proof of Lemma 3. Using the notation S^T previously defined, one has (see (19b)) on $(T, T+D)$

$$\dot{S}^T = -\alpha\beta S^T\left(c_0 + \frac{\gamma}{\beta}\left(1-\frac{1}{\alpha}\right)\ln\left(S^T(T)\right) - S^T + \frac{\gamma}{\alpha\beta}\ln S^T\right),$$

and at time $T, S^T(T)$ is defined thanks to (19a) by

$$\int_{S_0}^{S^T(T)} \frac{dv}{\beta v \left(c_0 - v + \frac{\gamma}{\beta} \ln v\right)} = -T. \tag{23}$$

By differentiating (23) with respect to T, one infers

$$\widehat{S^T(T)} = -\beta S^T(T)\left(c_0 - S^T(T) + \frac{\gamma}{\beta} \ln\left(S^T(T)\right)\right) = -\beta S^T(T) I^T(T),$$

that is (20b).

Furthermore, using (19b), one has

$$\int_{S^T(T)}^{S^T(T+D)} \frac{dv}{v\left(c_0 + \frac{\gamma}{\beta}\left(1 - \frac{1}{\alpha}\right)\ln\left(S^T(T)\right) - v + \frac{\gamma}{\alpha\beta} \ln v\right)} = -\alpha\beta D.$$

Differentiating this relation with respect to T yields

$$\frac{\widehat{S^T(T+D)}}{S^T(T+D)\left(c_0 + \frac{\gamma}{\beta}\left(1 - \frac{1}{\alpha}\right)\ln S^T(T) - S^T(T+D) + \frac{\gamma}{\alpha\beta}\ln S^T(T+D)\right)}$$

$$- \frac{\widehat{S^T(T)}}{S^T(T)\left(c_0 + \frac{\gamma}{\beta}\ln S^T(T) - S^T(T)\right)}$$

$$- \frac{\gamma}{\beta}\left(1 - \frac{1}{\alpha}\right)\frac{\widehat{S^T(T)}}{S^T(T)} \int_{S^T(T)}^{S^T(T+D)} \frac{dv}{v\left(c_0 + \frac{\gamma}{\beta}\left(1 - \frac{1}{\alpha}\right)\ln S^T(T) - v + \frac{\gamma}{\alpha\beta}\ln v\right)^2} = 0.$$

Let us simplify this latter identity. Observe first that, because of (18), one has

$$c_0 + \frac{\gamma}{\beta}\left(1 - \frac{1}{\alpha}\right)\ln S^T(T) = I^T(t) + S^T(t) - \frac{\gamma}{\alpha\beta}\ln S^T(t)$$

for each $t \in (T, T+D)$. By using at the same time the change of variable $v = S(t)$ and the expression of $I^T(t), t \in (T, T+D)$, extracted from this identity, we infer that

$$\int_{S^T(T)}^{S^T(T+D)} \frac{dv}{\beta v\left(c_0 + \frac{\gamma}{\beta}\left(1 - \frac{1}{\alpha}\right)\ln S^T(T) - v + \frac{\gamma}{\alpha\beta}\ln v\right)^2}$$

$$= \int_T^{T+D} \frac{1}{\beta S^T(s)\left(I^T(s)\right)^2}\dot{S}^T(s)\,ds = \int_T^{T+D} \frac{-\alpha S^T(s) I^T(s)}{S^T(s)\left(I^T(s)\right)^2}\,ds$$

$$= -\alpha \int_T^{T+D} \frac{ds}{I^T(s)}.$$

Combining all these facts leads to

$$0 = \frac{\widehat{S^T(T+D)}}{S^T(T+D) I^T(T+D)} + \beta + \gamma(1-\alpha)\beta I^T(T) \int_T^{T+D} \frac{ds}{I^T(s)},$$

and we arrive at (20a).

Similar arguments allow for the computation of $\widehat{S(t)}$. This achieves the proof of Lemma 3. □

4.5. Limit behaviour of T^* when I_0 vanishes

To complete the demonstration of Theorem 1, it now remains to prove the last property of the statement. The following result is instrumental for this purpose.

Lemma 4. *Assume $S_0 \in (S_{\text{herd}}, 1)$. For any $\bar{T} \geq 0$, there exist $c > 0$ and $\bar{I}_0 > 0$ such that*

$$\forall I_0 \in (0, \bar{I}_0), \quad \max_{T \in [0, \bar{T}]} \psi(T) < -c < 0, \tag{24}$$

where ψ defined in (4) depends upon I_0 through the initial value of $\left(S^T, I^T\right)$.

Using the characterization (already demonstrated above) of T^* given in Theorem 1, one deduces straightforwardly from Lemma 4 that

$$\lim_{I_0 \searrow 0^+} T^* = +\infty.$$

Proof of Lemma 4. Let $T \geq 0$. From the fact that $\dot{I}^T \leq (\beta S_0 - \gamma) I^T, I^T(0) = I_0$, one deduces that $0 \leq I^T(t) \leq I_0 e^{(\beta S_0 - \gamma)t}, t \in [0, T+D]$.

From this, one deduces that $|\dot{S}^T| = \beta u(t) S^T I^T \leq \eta(I_0) S^T$, where $\eta(I_0)$ represents, here and in the sequel, quantities that converge to 0 when I_0 vanishes, *uniformly on $[0, T+D]$ when they depend upon time t*. Therefore,

$$S^T(t) = S_0 + \eta(I_0), \qquad t \in [0, T+D].$$

Define now $\omega := \alpha\beta S_0 - \gamma$. From the foregoing, one has

$$\begin{cases} \dot{I}^T = (\beta S_0 - \gamma) I^T + I_0 \eta(I_0), & t \in [0, T], \\ \dot{I}^T = \omega I^T + I_0 \eta(I_0), & t \in [T, T+D]. \end{cases} \tag{25}$$

By integration one deduces from (25) that $I^T(t) = I_0 e^{(\beta S_0 - \gamma)t} + I_0 \eta(I_0)$ for any $t \in [0, T]$, and in particular that $I^T(T) = I_0\left(e^{(\beta S_0 - \gamma)T} + \eta(I_0)\right)$.

Assume first $\omega \neq 0$, then integration of the second formula in (25) yields

$$\begin{aligned} I^T(t) &= I^T(T) e^{\omega(t-T)} + I_0 \eta(I_0) \\ &= I_0\left(e^{(\beta S_0 - \gamma)T} e^{\omega(t-T)} + \eta(I_0)\right), \qquad t \in [T, T+D]. \end{aligned} \tag{26}$$

Using (26) to compute the value $\psi(T)$ in (4) then shows that

$$\psi(T) = -e^{\omega D} + (\alpha - 1)\gamma \frac{e^{\omega D} - 1}{\omega} + 1 + \eta(I_0)$$

$$= \left(\frac{(\alpha - 1)\gamma}{\omega} - 1\right)\left(e^{\omega D} - 1\right) + \eta(I_0). \tag{27}$$

If $\omega > 0$, then due to the fact that $\alpha - 1$ is negative, the first factor of the product is negative, while the second one is positive. If $\omega < 0$, then the second factor is negative, while the first one is positive, because

$$(\alpha - 1)\gamma - \omega = (\alpha - 1)\gamma - \alpha\beta S_0 + \gamma = \alpha\beta(S_{\text{herd}} - S_0) < 0,$$

with S_{herd} defined in (2). In any case, the zero-order term in (27) is negative when $\omega \neq 0$.

The case $\omega = 0$ is similar, with (27) replaced by

$$\psi(T) = -1 + (\alpha - 1)\gamma D + 1 + \eta(I_0) = (\alpha - 1)\gamma D + \eta(I_0). \tag{28}$$

As the higher-order terms $\eta(I_0)$ in (27) and (28) vanish when I_0 goes to 0 *uniformly on any compact of $[0, +\infty)$*, this demonstrates (24). □

5. Conclusion

Voluntarily ignoring many features important in the effective handling of a human epidemic (unmodeled sources of heterogeneity in the spread of the disease, limited hospital capacity, imprecise epidemiological data, partial respect of the enforcement measures…), we investigated here the effects of social distancing on a simple SIR model. In this simplified setting, we have shown that it is possible to exactly answer the following question: given maximal social distancing intensity and duration (but without prescribed starting date), how can one minimize the epidemic final size, that is the total number of individuals infected during the outbreak? Our contribution is threefold: we have proved the existence of a unique optimal policy, shown some of its key properties, and demonstrated how to determine it numerically by an easily tractable algorithm. As an outcome,

this provides the best possible policy, in the worst case where no vaccine or therapy exists. Numerical computations have been provided that exemplify the theoretical results and allowed to tabulate the maximal gain attainable in terms of cumulative number of infected during the outbreak, under various experimental conditions.

It is somewhat intuitive that the best policy achievable by imposing a lockdown of possibly time-varying, but limited, intensity on a time interval of which only the duration is restricted, is reached by enforcing the strictest distancing during the whole time interval. However, up to our knowledge this had not been proved or conjectured so far. Moreover, our results show that the onset of the lockdown is uniquely determined as the unique solution of a numerically tractable equation.

The fact that the optimal control does not begin from the earliest possible time is only an apparent paradox. As a matter of fact, epidemics behave somehow as wildfires —the propellant being the susceptible individuals. On the one hand, attempting to contain the spread too early is pointless, as essentially the same amount of propellant will be present after the end of the intervention, leading ultimately to the same epidemic final size. On the other hand, acting too late is also useless, as in this case most of the stock of propellant will have been already consumed at the time of the intervention. The best time to proceed lies in between, somewhere around the peak of the epidemic when the herd immunity threshold is crossed —typically some weeks after the beginning of the epidemics—, with larger or more intense intervention inducing larger mitigation effect. The results provided allow to determine precisely what is the best time to initiate social distancing.

As a last remark, notice that in the simplified setting considered here, limited hospital capacity or deaths caused by supplementary mortality are ignored. Among other extensions, we plan to address in the future the issue of minimization of the epidemic final size under adequate constraints.

Acknowledgments

Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at https://doi.org/10.1016/j.jtbi.2020.110557.

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