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Dynamics of fractional order nonlinear system: A realistic perception with neutrosophic fuzzy number and Allee effect

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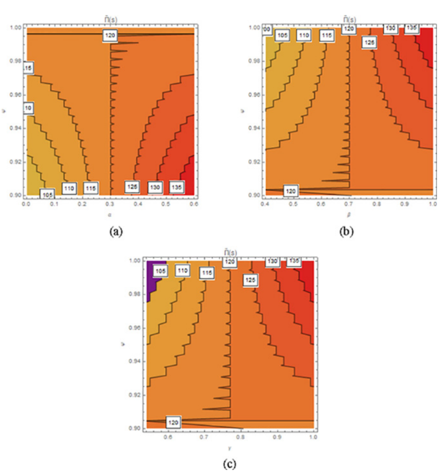
GRAPHICAL ABSTRACT



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ABSTRACT

Introduction: The fusion of fractional order differential equations and fuzzy numbers has been widely used in modelling different engineering and applied sciences problems. In addition to these, the Allee effect, which is of high importance in field of biology and ecology, has also shown great contribution among other fields of sciences to study the correlation between density and the mean fitness of the subject.
Objectives: The present paper is intended to measure uncertain dynamics of an economy by restructuring the Cobb-Douglas paradigm of the renowned Solow-Swan model. The purpose of study is further boosted innovatively by subsuming the perception of logistic growth with Allee effect in the dynamics of physical capital and labor force.
Methods: Fractional order derivative and neutrosophic fuzzy (NF) theory are applied on the parameters of the Cobb-Douglas equation. Distinctively, cogitating fractional order derivative to study the change at each fractional stage; single-valued triangular neutrosophic fuzzy numbers (SVTNFN) to cope the uncertain situations; logistic growth function with Allee effect to analyze the factors in natural way, are the significant and novel features of this endeavor.

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Results: The incorporation of the aforementioned theories and effects in the Cobb-Douglas equation, resulted in producing maximum sustainable capital investment and maximum capacity of labor force. The solutions in intervals located different possible solutions for different membership degrees, which accumulated the uncertain circumstances of a country.

Conclusion: Explicitly, these notions add new facts and figures not only in the dynamical study of capital and labor, which has been overlooked in classical models, but also left the door open for discussion and implementation on classical models of different fields.

Introduction

The economic growth theory proves to be the most constructive theory for the economists, businessmen and industrialists. It provides the most fundamental principles of economics, which explains different production activities, such as decisions about the quantity of a commodity to be produced and to be sold. The neo-classical theory of economic growth was firstly established by Robert Solow [1] as an alternative to Harrod-Domar model of growth [2–4]. Following their basic concepts and introducing a two way production function, the distribution of the national income with the help of labor-capital input and total factor productivity was described by Cobb and Douglas [4]. In continuation of this, many enhancements have been made in the production function by considering different concepts and economic factors. For instance, the substitution production function with constant elasticity [5] models the constant elasticity of substitution between the parameters in a particular domain.

Using the Harrod definition of technological change, Eltis [6] derived a mathematical equation for the physical and decision making structures in the mechanism of changes occurring in capital-labor ratio, which effects the economic growth. Razzaq et al. [7] defined dynamics of economic growth by relating capital-labor equation with ecological system of equations and illustrated different steady-states of economy. Thus, according to a country's development strategies and requirements, many economists and researchers had brought diversities in the classical models of economic growth [8–11].

As evaluation level of human's knowledge cannot be carried out with precise measurements, so the perspectives of economic growth in studying economy of any country will be inadequate if the parameters are taken in crisp form. For instance, economic growth of any country is generally represented with the help of nonlinear curves that show the rise and fall of GDP. Monitoring these curves at each minor level will make the observer to use the statements such as, "A very little rise has been noticed in the GDP, it's probably due to the slight increase in the capital investment." or "The economy of country A growing more rapidly than that of country B". The phrases "very little", "more" and "slight" are imprecise terms with different levels of correctness. Therefore, nowadays mathematical modelling is carried out more realistically by involving theories of uncertainty. These concepts enable to consider imprecise factors of real-world in the model as well, despite taking it constant as done previously. Due to the positive impact and growing usage of these theories, many advancements have been made according to the recent requirements. Among many of the new generalized concepts [12–16], these days, neutrosophic fuzzy (NF) theory [17–22] has been used extensively as an alternative of intuitionistic fuzzy numbers. It has the capability to articulate another state of human intelligence, which is termed as indeterminacy. In NF, the grading of the imprecise parameters is divided into three categories; truth, falsity and indeterminacy and so it consists of three membership functions. All these degrees of uncertainty play important role in modelling, as these have the proficiency to deal with the impreciseness levels of human intelligence.

Henceforth, in contrast to the existing economic models, this endeavor features the key factors of economy i.e. physical capital and labor force as generalized neutrosophic fuzzy numbers. We have extended the applications of fractional calculus and NF theory in the study of economic growth model. Recently, countless fields of science and technology have been found in the literature that has implemented fractional notion in the models to study the sensitivities. Regardless of being computationally difficult, the property of calculating the rate of change of any function at a fractional stage is the attention-grabbing characteristic of fractional calculus. Flow behaviors of fluids [23], analysis of chaos of various dynamical systems [24], bifurcation and limit cycles of any ecological system [25], study of epidemic diseases in biomedical areas [26] and many others [27–31], have been found with fractional order derivatives. The novelties in the governing study can be collectively pointed out as:

- Caputo type fractional derivative of system of nonlinear differential equations.
- The logistic growth and Allee effect on capital and labor.
- SVTNFN that reflect the economic uncertain circumstances.

The logistic growth [32,33] is incorporated on the labor force, which describes the self-limiting growth i.e. it considers the assumption that rate of reproduction is proportional to the existing population and the available resources. Hence, the increasing path of the labor population and capital investment will not keep on increasing, but according to available resources of any country it will reach a limiting level. In addition, Allee effect [32,33] is another meaningful biological phenomenon that is widely used in the study of ecological system. This effect further defines the correlation between the individual fitness of the population and population density. Its inclusion in the governing model will locate the effects of increase and extinction of labors occurring due to the high or low population of labors. To be precise, the aforementioned objectives of this endeavor will definitely be advantageous in different aspects, for instance, the constructed model will assist to,

- Study the fractional order change in capital and population growth of an economy.
- Analyze maximum sustainable capital investment and maximum capacity of labor force.
- Evaluate the results in intervals, despite a crisp solution, which will accumulate the uncertain circumstances of a country.
- Conclude the parameters in intervals, which will locate different possible solutions for different membership degrees.
- Visualize the outcomes through different dimensional plots.

Furthermore, the approach adopted in the remaining paper is summarized as: section 2 comprises the preliminaries of fractional calculus and Section 3 consists of fundamentals of neutrosophic fuzzy theory with some of its arithmetic operations. Section 4 illustrates the implementation of the above theories on re-modelling Cobb-Douglas production function along with the equations of physical capital and labor force. In Section 5, graphical solutions are deliberated that are developed through *Mathematica 11.0*, whereas Section 6 wrap-ups the whole discussion by postulating the findings, deduced from the study.

Preliminaries

Here the preambles are presented that are pertinent for the remaining paper. In addition, Table 1 describes the symbols, which are used in definitions and for mathematically modelling the assumptions for the proposed application. It includes certain prologues of fractional calculus**.**

Definition 1. *The Caputo fractional derivative of order $m-1<\psi\leqslant m$ of any arbitrary function, $y:[\varepsilon,\varsigma]\to\Re$, where m is any positive integer, is described as [34]:*

$$\varepsilon CD_s^{\psi}y(s)=\frac{1}{\Gamma(m-\psi)}\int_{\varepsilon}^{s}(s-\tau)^{m-\psi-1}y^{(m)}(\tau)d\tau\,, \tag{1}$$

as the left Caputo fractional derivative and

$$sCD_{\varsigma}^{\psi}y(s)=\frac{(-1)^m}{\Gamma(m-\psi)}\int_{s}^{\varsigma}(\tau-s)^{m-\psi-1}y^{(m)}(\tau)d\tau\,, \tag{2}$$

as the right Caputo fractional derivative. Furthermore, the operator ${}^{C}D_s^{\psi}$also satisfies the following properties:

Table 1
Nomenclature.

Symbols	Description
ψ	Order of fractional derivative
$\aleph$	Natural numbers
$\Re$	Real numbers
ϖ	Laplace parameter
$\hat{g}(\varpi)$	Laplace transform ofg(s)
ℓ	Laplace transform
${}^{C}D_s^{\psi}$	Caputo fractional operator
$T_{\tilde{A}}(y)$	Truth membership function
$I_{\tilde{A}}(y)$	Indeterminacy membership function
$F_{\tilde{A}}(y)$	Falsity membership function
α	Truth membership degree
β	Indeterminacy membership degree
γ	Falsity membership degree
λ	Maximum truth membership degree
ω	Minimum indeterminacy membership degree
δ	Minimum falsity membership degree
$\tilde{\Delta}$	Single valued triangular neutrosophic fuzzy number
Z_{NF}	Set of generalized single-valued neutrosophic fuzzy numbers
$\Pi(s)$	Physical capital in crisp
$\tilde{\Pi}(s)$	Physical capital in SVTNFF
$\Omega(s)$	Amount of labour in crisp
$\tilde{\Omega}(s)$	Amount of labour in SVTNFF
$\tilde{\Lambda}$	Total factor productivity
χ	Output elasticity
$\tilde{t}$	Saving
$\tilde{n}$	Depreciation of capital
$\tilde{f}$	Maximum sustainable amount
$\tilde{\xi}$	Intrinsic growth
$\tilde{p}$	Carrying capacity
$\tilde{q}$	Allee threshold
$L_{(\cdot)}(\cdot;\cdot)$	Left branch of SVTNFF
$R_{(\cdot)}(\cdot;\cdot)$	Right branch of SVTNFF

$${}^{C}D_s^{\psi}u=0\,,\ (u\text{ is constant}), \tag{3}$$

$$\mathrm{I}^{\psi}\,{}^{C}D_s^{\psi}y(s)=y(s)-\sum_{i=0}^{n-1}\frac{y^{(i)}(0)}{i\,!}s^i, \tag{4}$$

where $\mathrm{I}^{\psi}$is the well-known Riemann-Liouville fractional integral.

$${}^{C}D_s^{\psi}s^{\hbar}=\frac{\Gamma(\hbar+1)}{\Gamma(\hbar+1-\psi)}s^{\hbar-\psi}\,,\qquad \hbar\in\Re \tag{5}$$

and

$${}^{C}D_s^{\psi}(u\;y(s)+v\,y(s))=u\,{}^{C}D_s^{\psi}y(s)+v\,{}^{C}D_s^{\psi}\,y(s),\qquad u,v\in\Re \tag{6}$$

Definition 2. *Let $\tilde{g}(\varpi)$ denotes the Laplace transform of $g(s)$ i.e., $\ell[g(s)]=\tilde{g}(\varpi)=\int_0^{\infty}e^{-\varpi t}g(s)ds$, then the Laplace transform of $g(s)$ in Caputo fractional derivative sense of order ψ is defined as [35]:*

$$\ell\left[{}^{C}D_s^{\psi}g(s)\right]=\varpi^{\psi}\,\tilde{g}(\varpi)-\sum_{\eta=0}^{m-1}\varpi^{\psi-\eta-1}g^{(\eta)}(0),\ m-1\leqslant\psi \leqslant m, m\in\aleph \tag{7}$$

On considering $m=1$, Eq. (7) becomes,

$$\ell\left[{}^{C}D_s^{\psi}g(s)\right]=\varpi^{\psi}\,\tilde{g}(\varpi)-\varpi^{\psi-1}g(0),\ 0<\psi\leqslant 1 \tag{8}$$

After the linearization of ϖ^{ψ} i.e.,

$$\varpi^{\psi}\approx\psi\,\varpi^1+(1-\psi)\,\varpi^0=\psi\,\varpi+(1-\psi) \tag{9}$$

Eq. (8) converts into,

$$\ell\left[{}^{C}D_s^{\psi}g(s)\right]=\psi\,\dot{g}(s)+(1-\psi)(g(s)-g(0)),\ 0<\psi\leqslant 1 \tag{10}$$

$$\dot{g}(s)=\frac{dg(s)}{ds}$$

Generalized neutrosophic fuzzy numbers

Neutrosophic fuzzy numbers are the generalized form of intuitionistic fuzzy numbers. It encompasses three states of human intelligence; truth, hesitant and falsity by means of three membership degrees. This section gives detailed elaboration of several properties along with the algebraic operations of generalized NF [21,22].

Definition 3. *Let a set X be fixed, then a single-valued neutrosophic set $\tilde{A}$ in $X$is defined with truth-membership function $T_{\tilde{A}}(y)$, indeterminacy-membership function $I_{\tilde{A}}(y)$ and falsity membership function $F_{\tilde{A}}(y)$as:*

$$\tilde{A}=[\{y;T_{A^{\sim}}(y),\ I_{A^{\sim}}(y),\ F_{A^{\sim}}(y)\}:y\in X] \tag{11}$$

where $T_{\tilde{A}}(y):X\to[0,\ 1]$, $I_{\tilde{A}}(y):X\to[0,\ 1]$ and $F_{\tilde{A}}(y):X\to[0,\ 1]$, respectively under the condition,

$$0\leqslant T_{A^{\sim}}(y)+I_{A^{\sim}}(y)+F_{A^{\sim}}(y)\leqslant 3 \tag{12}$$

Definition 4. *Let $a_i,b_i,c_i\in\Re$be such that $a_i\leqslant b_i\leqslant c_i$for $i=1,2,3$, and suppose $\lambda,\omega,\delta\in[0,1]$be any real numbers, then a SVTN set on $\Re$, is defined as:*

$$\tilde{A} = [\{(a_1, b_1, c_1; \lambda),\ (a_2, b_2, c_2; \omega),\ (a_3, b_3, c_3; \delta)\}] \quad (13)$$

Let $\mu_{\tilde{A}}$, $\vartheta_{\tilde{A}}$ and $\psi_{\tilde{A}}$ be the truth, indeterminacy and falsity membership functions respectively, defined by

$$\mu_{A^\sim}(y) = \begin{cases} g^l_\mu(y), & a_1 \leqslant y \leqslant b_1 \\ g^r_\mu(y), & b_1 \leqslant y \leqslant c_1 \ \ \mu_{A^\sim} : \Re \to [0, \lambda], \\ 0, & otherwise \end{cases} \quad (14)$$

$$\vartheta_{A^\sim}(y) = \begin{cases} g^l_\vartheta(y), & a_2 \leqslant y \leqslant b_2 \\ g^r_\vartheta(y), & b_2 \leqslant y \leqslant c_2\ ,\ \vartheta_{A^\sim} : \Re \to [\omega,\ 1] \\ 0, & otherwise \end{cases} \quad (15)$$

and

$$\psi_{A^\sim}(y) = \begin{cases} g^l_\psi(y), & a_3 \leqslant y \leqslant b_3 \\ g^r_\psi(y), & b_3 \leqslant y \leqslant c_3\ ,\ \psi_{A^\sim} : \Re \to [\delta,\ 1] \\ 0, & otherwise \end{cases} \quad (16)$$

where,

- $\lambda$is the maximum truth-membership degree, ω is minimum indeterminacy-membership degree and δ is minimum falsity-membership degree.
- The functions $g^l_\mu : [a_1, b_1] \to [0, \lambda]$, $g^r_\vartheta : [c_2, d_2] \to [\omega, 1]$, $g^r_\psi : [c_3, d_3] \to [\delta,\ 1]$ are continuous and non-decreasing.
- The functions $g^r_\mu : [c_1, d_1] \to [0, \lambda]$, $g^l_\vartheta : [a_2, b_2] \to [\omega, 1]$, $g^l_\psi : [a_3, b_3] \to [\delta,\ 1]$ are continuous and non-increasing.

For a SVTNFN, $\tilde{\Delta} = \{(a, b, c); \lambda, \omega, \delta\}$, the functions are defined as:

$$g^l_\mu = \frac{(y-a)\lambda}{b-a},\ g^l_\vartheta = \frac{(b - y + \omega(y-a))}{b-a},\ g^l_\psi = \frac{(b - y + \delta(y-a))}{b-a},\ \text{for } a \leqslant y \leqslant b \quad (17)$$

and

$$g^r_\mu = \frac{(c-y)\lambda}{c-b},\ g^r_\vartheta = \frac{(y - b + \omega(c-y))}{c-b},\ g^r_\psi = \frac{(y - b + \delta(c-y))}{c-b},\ \text{for } b \leqslant y \leqslant c \quad (18)$$

Definition 5. *Let Z_{NF} be the set of generalized single-valued neutrosophic fuzzy numbers and suppose $\tilde{\Delta} \in Z_{NF}$, then (α, β, γ)–cuts of SVTNFN, signified by $\tilde{\Delta}^{(\alpha,\beta,\gamma)}$, is expressed as*:

$$\tilde{\Delta}^{(\alpha,\beta,\gamma)} = \{y | \mu_{\Delta^\sim}(y) \geqslant \alpha, \vartheta_{\Delta^\sim}(y) \leqslant \beta, \psi_{\Delta^\sim}(y) \leqslant \gamma, y \in \Re\} \quad (19)$$

which satisfies the conditions $0 \leqslant \alpha \leqslant \lambda$, $\omega \leqslant \beta \leqslant 1$, $\delta \leqslant \gamma \leqslant 1$, and $0 \leqslant \alpha + \beta + \gamma \leqslant 1$.

Definition 6. *Let $\tilde{\Delta} = \{(a, b, c); \lambda, \omega, \delta\}$ be a SVTNFN, then α–cut of the $\tilde{\Delta}$, signified by $\tilde{\Delta}^{\alpha}$, is described as*:

$$\tilde{\Delta}^{\alpha} = \{y | \mu_{\Delta^\sim}(y) \geqslant \alpha, y \in \Re\},\ \alpha \in [0, \lambda] \quad (20)$$

which can also be described as closed intervals i.e.,

$$\tilde{\Delta}^{\alpha} = [L_{\Delta^\sim}(\alpha), R_{\Delta^\sim}(\alpha)] \quad (21)$$

where $L_{\tilde{\Delta}}(\alpha) = \frac{(\lambda-\alpha)a+\alpha b}{\lambda}$and $R_{\tilde{\Delta}}(\alpha) = \frac{(\lambda-\alpha)c+\alpha b}{\lambda}$.

Definition 7. *Suppose $\tilde{\Delta} \in Z_{NF}$, then β–cut of $\tilde{\Delta}$, represented by $\tilde{\Delta}^{\beta}$, is outlined as*:

$$\tilde{\Delta}^{\beta} = \{y | \vartheta_{\Delta^\sim}(y) \leqslant \beta, y \in \Re\},\ \beta \in [\omega, 1] \quad (22)$$

with its closed interval form,

$$\tilde{\Delta}^{\beta} = [L_{\Delta^\sim}(\beta), R_{\Delta^\sim}(\beta)] \quad (23)$$

where $L_{\tilde{\Delta}}(\beta) = \frac{(1-\beta)b+(\beta-\omega)a}{1-\omega}$and $R_{\tilde{\Delta}}(\beta) = \frac{(1-\beta)b+(\beta-\omega)c}{1-\omega}$.

Definition 8. *Let $\tilde{\Delta} = \{(a, b, c); \lambda, \omega, \delta\}$ be a SVTNFN, then γ–cut of $\tilde{\Delta}$, symbolized by $\tilde{\Delta}^{\gamma}$, is explained as*:

$$\tilde{\Delta}^{\gamma} = \{y | \psi_{\Delta^\sim}(y) \leqslant \gamma, y \in \Re\},\ \gamma \in [\delta, 1] \quad (24)$$

The γ–cut in closed interval is described as,

$$\tilde{\Delta}^{\gamma} = [L_{\Delta^\sim}(\gamma), R_{\Delta^\sim}(\gamma)] \quad (25)$$

where $L_{\tilde{\Delta}}(\gamma) = \frac{(1-\gamma)b+(\gamma-\delta)a}{1-\delta}$and $R_{\tilde{\Delta}}(\alpha) = \frac{(1-\gamma)b+(\gamma-\delta)c}{1-\delta}$.

Definition 9. *Let $\tilde{\Delta}_1, \tilde{\Delta}_2 \in Z_{NF}$, such that $\tilde{\Delta}_1 = \{(a_1, b_1, c_1); \lambda_1, \omega_1, \delta_1\}$ and $\tilde{\Delta}_2 = \{(a_2, b_2, c_2); \lambda_2, \omega_2, \delta_2\}$, then algebraic operations are carried out as*:

- *Addition:*

$$\tilde{\Delta}_1 + \tilde{\Delta}_2 = \{(a_1 + a_2, b_1 + b_2, c_1 + c_2); \lambda_1 \wedge \lambda_2, \omega_1 \vee \omega_2, \delta_1 \vee \delta_2\} \quad (26)$$

- *Multiplication:*

$$\tilde{\Delta}_1 * \tilde{\Delta}_2 = \begin{cases} \{(a_1a_2, b_1b_2, c_1c_2); \lambda_1 \wedge \lambda_2, \omega_1 \vee \omega_2, \delta_1 \vee \delta_2\}, & c_1 > 0, c_2 > 0 \\ \{(a_1c_2, b_1b_2, c_1a_2); \lambda_1 \wedge \lambda_2, \omega_1 \vee \omega_2, \delta_1 \vee \delta_2\}, & c_1 < 0, c_2 > 0 \\ \{(c_1c_2, b_1b_2, a_1a_2); \lambda_1 \wedge \lambda_2, \omega_1 \vee \omega_2, \delta_1 \vee \delta_2\}, & c_1 < 0, c_2 < 0 \end{cases} \quad (27)$$

- *Crisp Multiplication:*

$$\upsilon \tilde{\Delta}_1 = \begin{cases} \{(\upsilon a_1, \upsilon b_1, \upsilon c_1); \lambda_1, \omega_1, \delta_1\}, & \upsilon > 0, \\ \{(\upsilon c_1, \upsilon b_1, \upsilon a_1); \lambda_1, \omega_1, \delta_1\}, & \upsilon > 0 \end{cases} \quad (28)$$

Modelling and analysis

Economic growth of any country is studied by using the production function that depends upon the four input factors; physical capital, labour, land and entrepreneurship. In particular, the input factors are reduced to just labour and physical capital, as it is considered that production aspects do not require all of these factors. In [5], production function F is defined as a consequence of amount of labour $\Omega(s)$ and physical capital invested $\Pi(s)$ i.e.,

$$F(\Pi(s), \Lambda\, \Omega(s)) = \Lambda\, (\Omega(s))^{\chi} (\Pi(s))^{1-\chi},\ \chi \in [0, 1] \quad (29)$$

$\Lambda$is taken as to be the total factor productivity that deals with the change in output in the absence of the input factors. χ shows the output elasticity that occurs as a result from a change in the input factors. Here, we consider the labour and physical capital inputs as SVTNFF with the assumption that in reality the investment on capital cannot be defined using certain phrases or values and so the skills and population of labor, in producing output. Hence, to make the production function more realistic, Eq. (29) is rewritten as,

$$F\left(\tilde{\Pi}(s), \tilde{\Lambda}\ \tilde{\Omega}(s)\right) = \tilde{\Lambda}\ \left(\tilde{\Omega}(s)\right)^{\chi}\left(\tilde{\Pi}(s)\right)^{1-\chi} \tag{30}$$

where now each factor is in SVTNFF symbolised with a tilde "~". Assume that the production function $F\left(\tilde{\Pi}, \tilde{\Lambda}\ \tilde{\Omega}\right)$ meet with the following facts:

i) $F\left(\lambda\ \tilde{\Pi}, \lambda\ \tilde{\Lambda}\ \tilde{\Omega}\right) = \lambda\, F\left(\tilde{\Pi}, \tilde{\Lambda}\ \tilde{\Omega}\right), \forall \lambda \in \Re, \forall\ \tilde{\Pi}, \tilde{\Lambda},\ \tilde{\Omega} \in Z_{NF}$ (constant return to scale)

ii) $F(0,0) = F\left(\tilde{\Pi}, 0\right) = F\left(0, \tilde{\Lambda}\ \tilde{\Omega}\right) = 0,\ \forall\, \tilde{\Pi}, \tilde{\Lambda}, \tilde{\Omega} \in Z_{NF}$

iii) $\frac{\partial F}{\partial\ \tilde{\Pi}} > 0,\ \frac{\partial F}{\partial \tilde{\Omega}} > 0,\ \frac{\partial^2 F}{\partial\ \tilde{\Pi}^2} < 0,\ \frac{\partial^2 F}{\partial \tilde{\Omega}^2} < 0,$

iv) $\lim\limits_{\tilde{\Pi}\to 0} \frac{\partial F}{\partial\ \tilde{\Pi}} = \lim\limits_{\tilde{\Omega}\to 0} \frac{\partial F}{\partial\ \tilde{\Omega}} = +\infty;\ \lim\limits_{\tilde{\Pi}\to +\infty} \frac{\partial F}{\partial\ \tilde{\Pi}} = \lim\limits_{\tilde{\Omega}\to +\infty} \frac{\partial F}{\partial\ \tilde{\Omega}} = 0$ (Inada conditions).

Here, appending the concept of logistic growth and SVTNFN numbers, we consider the gross investment to depend upon the savings and production with a logistic growth. It will generate a carrying capacity parameter, which measures the maximum amount of investment that can be done. Thus, taking into account these assumptions along with the fractional order change, the dynamical model of capital is structured as:

$${}^{C}D_s^{\psi}\ \tilde{\Pi}(s) = \tilde{t}\, F\left(\tilde{\Pi}(s), \tilde{\Lambda}\ \tilde{\Omega}(s)\right)\left(1 - \frac{\tilde{\Pi}(s)}{\tilde{f}}\right) - \tilde{m}\ \tilde{\Pi}(s) \tag{31}$$

with initial condition $\tilde{\Pi}(0) = \tilde{\Pi}_0$. The parameters $\tilde{t}$ and $\tilde{m}$ represent the saving and depreciation of capital, respectively and $\tilde{f}$ is the maximum sustainable amount for investment. If $\tilde{\Pi}_0 > \tilde{f}$, then $\tilde{\Pi}(s)$ decays towards $\tilde{g}$and remains at steady-state. The novelty of taking gross investment as a logistic function is more realistic and competent for the reason that it postulates a maximum investment level that a country can attain under the existing state of affairs of the economy. Besides, SVTNFN will let the uncertain situations, which are only defined in linguistic explanations up till now, be involved in the equation while measuring the capital rate of change. Next, the fractional order rate of change of labour amount $\tilde{\Omega}(s)$ is defined as:

$${}^{C}D_s^{\psi}\,\tilde{\Omega}(s) = \tilde{\xi}\ \tilde{\Omega}(s)\left(1 - \frac{\tilde{\Omega}(s)}{\tilde{p}}\right)\left(\frac{\tilde{\Omega}(s)}{\tilde{q}} - 1\right) \tag{32}$$

with the initial conditions $\tilde{\Omega}(0) = \tilde{\Omega}_0$. Here $\tilde{\xi}$is the intrinsic growth rate of labours, $\tilde{p}$is the carrying capacity of the labours and $\tilde{q}$is the Allee threshold. The Eq. (32) verifies the following properties:

i) $\tilde{\Omega}_0 > 0, {}^{C}D_t^{\rho}\,\tilde{\Omega}(s) > 0,\ \ \forall s \geqslant 0.$

ii) If $\tilde{\Omega}_0 > \tilde{p}$, then $\tilde{\Omega}(s)$ decays towards $\tilde{p}$and remains at steady-state.

iii) If $\tilde{q} < \tilde{\Omega}_0 < \tilde{p}$, then $\tilde{\Omega}(s)$ gradually rises to $\tilde{p}$ in an asymptotic manner and remains at steady-state.

iv) If $0 < \tilde{\Omega}_0 < \tilde{q}$, then $\tilde{\Omega}(s)$ decreases toward zero.

Assuming the degrees of uncertainties of SVTNFN i.e., truth, indeterminacy and falsity, Eqs. (31) and (32) convert into (α, β, γ)-cuts as:

$$\begin{aligned}
&\left[{}^{C}D_s^{\psi}L_{\tilde{\Pi}(s)}(s;\alpha),\ {}^{C}D_s^{\psi}R_{\tilde{\Pi}(s)}(s;\alpha)\right] = \left[L_{\tilde{t}}(\alpha), R_{\tilde{t}}(\alpha)\right]\\
&\left[\left(L_{\tilde{\Pi}(s)}(s;\alpha)\right)^{\chi}, \left(R_{\tilde{\Pi}(s)}(s;\alpha)\right)^{\chi}\right]\left[\left(L_{\tilde{\Lambda}}(\alpha)\right)^{1-\chi}, \left(R_{\tilde{\Lambda}}(\alpha)\right)^{1-\chi}\right]\\
&\left[\left(L_{\tilde{\Omega}(s)}(s;\alpha)\right)^{1-\chi}, \left(R_{\tilde{\Omega}(s)}(s;\alpha)\right)^{1-\chi}\right]\left(1 - \frac{\left[L_{\tilde{\Pi}(s)}(s;\alpha), R_{\tilde{\Pi}(s)}(s;\alpha)\right]}{\left[L_{\tilde{f}}(\alpha), R_{\tilde{f}}(\alpha)\right]}\right)\\
&-\left[L_{\tilde{n}}(\alpha), R_{\tilde{n}}(\alpha)\right]\ \left[L_{\tilde{\Pi}(s)}(s;\alpha), R_{\tilde{\Pi}(s)}(s;\alpha)\right],
\end{aligned} \tag{33}$$

$$\begin{aligned}
&\left[{}^{C}D_s^{\psi}L_{\tilde{\Pi}(s)}(s;\beta),\ {}^{C}D_s^{\psi}R_{\tilde{\Pi}(s)}(s;\beta)\right] = \left[L_{\tilde{t}}(\beta), R_{\tilde{t}}(\beta)\right]\left[\left(L_{\tilde{\Pi}(s)}(s;\beta)\right)^{\chi}, \left(R_{\tilde{\Pi}(s)}(s;\beta)\right)^{\chi}\right]\\
&\left[\left(L_{\tilde{\Lambda}}(\beta)\right)^{1-\chi}, \left(R_{\tilde{\Lambda}}(\beta)\right)^{1-\chi}\right]\left[\left(L_{\tilde{\Omega}(s)}(s;\beta)\right)^{1-\chi}, \left(R_{\tilde{\Omega}(s)}(s;\beta)\right)^{1-\chi}\right]\\
&\left(1 - \frac{\left[L_{\tilde{\Pi}(s)}(s;\beta), R_{\tilde{\Pi}(s)}(s;\beta)\right]}{\left[L_{\tilde{f}}(\beta), R_{\tilde{f}}(\beta)\right]}\right) - \left[L_{\tilde{n}}(\beta), R_{\tilde{n}}(\beta)\right]\ \left[L_{\tilde{\Pi}(s)}(s;\beta), R_{\tilde{\Pi}(s)}(s;\beta)\right],
\end{aligned} \tag{34}$$

$$\begin{aligned}
&\left[{}^{C}D_s^{\psi}L_{\tilde{\Pi}(s)}(s;\gamma),\ {}^{C}D_s^{\psi}R_{\tilde{\Pi}(s)}(s;\gamma)\right] = \left[L_{\tilde{t}}(\gamma), R_{\tilde{t}}(\gamma)\right]\left[\left(L_{\tilde{\Pi}(s)}(s;\gamma)\right)^{\chi}, \left(R_{\tilde{\Pi}(s)}(s;\gamma)\right)^{\chi}\right]\\
&\left[\left(L_{\tilde{\Lambda}}(\gamma)\right)^{1-\chi}, \left(R_{\tilde{\Lambda}}(\gamma)\right)^{1-\chi}\right]\left[\left(L_{\tilde{\Omega}(s)}(s;\gamma)\right)^{1-\chi}, \left(R_{\tilde{\Omega}(s)}(s;\gamma)\right)^{1-\chi}\right]\\
&\left(1 - \frac{\left[L_{\tilde{\Pi}(s)}(s;\gamma), R_{\tilde{\Pi}(s)}(s;\gamma)\right]}{\left[L_{\tilde{f}}(\gamma), R_{\tilde{f}}(\gamma)\right]}\right) - \left[L_{\tilde{n}}(\gamma), R_{\tilde{n}}(\gamma)\right]\ \left[L_{\tilde{\Pi}(s)}(s;\gamma), R_{\tilde{\Pi}(s)}(s;\gamma)\right],
\end{aligned} \tag{35}$$

$$\begin{aligned}
&\left[{}^{C}D_s^{\psi}L_{\tilde{\Omega}(s)}(s;\alpha),\ {}^{C}D_s^{\psi}R_{\tilde{\Omega}(s)}(s;\alpha)\right] = \left[L_{\tilde{\xi}}(\alpha), R_{\tilde{\xi}}(\alpha)\right]\left[L_{\tilde{\Omega}(s)}(s;\alpha), R_{\tilde{\Omega}(s)}(s;\alpha)\right]\\
&\left(1 - \frac{\left[L_{\tilde{\Omega}(s)}(s;\alpha), R_{\tilde{\Omega}(s)}(s;\alpha)\right]}{\left[L_{\tilde{p}}(\alpha), R_{\tilde{p}}(\alpha)\right]}\right)\left(\frac{\left[L_{\tilde{\Omega}(s)}(s;\alpha), R_{\tilde{\Omega}(s)}(s;\alpha)\right]}{\left[L_{\tilde{q}}(\alpha), R_{\tilde{q}}(\alpha)\right]} - 1\right),
\end{aligned} \tag{36}$$

$$\begin{aligned}
&\left[{}^{C}D_s^{\psi}L_{\tilde{\Omega}(s)}(s;\beta),\ {}^{C}D_s^{\psi}R_{\tilde{\Omega}(s)}(s;\beta)\right] = \left[L_{\tilde{\xi}}(\beta), R_{\tilde{\xi}}(\beta)\right]\left[L_{\tilde{\Omega}(s)}(s;\beta), R_{\tilde{\Omega}(s)}(s;\beta)\right]\\
&\left(1 - \frac{\left[L_{\tilde{\Omega}(s)}(s;\beta), R_{\tilde{\Omega}(s)}(s;\beta)\right]}{\left[L_{\tilde{p}}(\beta), R_{\tilde{p}}(\beta)\right]}\right)\left(\frac{\left[L_{\tilde{\Omega}(s)}(s;\beta), R_{\tilde{\Omega}(s)}(s;\beta)\right]}{\left[L_{\tilde{q}}(\beta), R_{\tilde{q}}(\beta)\right]} - 1\right)
\end{aligned} \tag{37}$$

and

$$\begin{aligned}
&\left[{}^{C}D_s^{\psi}L_{\tilde{\Omega}(s)}(s;\gamma),\ {}^{C}D_s^{\psi}R_{\tilde{\Omega}(s)}(s;\gamma)\right] = \left[L_{\tilde{\xi}}(\gamma), R_{\tilde{\xi}}(\gamma)\right]\left[L_{\tilde{\Omega}(s)}(s;\gamma), R_{\tilde{\Omega}(s)}(s;\gamma)\right]\\
&\left(1 - \frac{\left[L_{\tilde{\Omega}(s)}(s;\gamma), R_{\tilde{\Omega}(s)}(s;\gamma)\right]}{\left[L_{\tilde{p}}(\gamma), R_{\tilde{p}}(\gamma)\right]}\right)\left(\frac{\left[L_{\tilde{\Omega}(s)}(s;\gamma), R_{\tilde{\Omega}(s)}(s;\gamma)\right]}{\left[L_{\tilde{q}}(\gamma), R_{\tilde{q}}(\gamma)\right]} - 1\right),
\end{aligned} \tag{38}$$

accordingly. Subjected to the initial conditions defined for Eqs. (33)-(35) as:

$$\begin{aligned}
&\left[L_{\tilde{\Pi}(0)}(0;\alpha), R_{\tilde{\Pi}(0)}(0;\alpha)\right] = \left[L_{\tilde{\Pi}_0}(\alpha), R_{\tilde{\Pi}_0}(\alpha)\right],\\
&\left[L_{\tilde{\Pi}(0)}(0;\beta), R_{\tilde{\Pi}(0)}(0;\beta)\right] = \left[L_{\tilde{\Pi}_0}(\beta), R_{\tilde{\Pi}_0}(\beta)\right],
\end{aligned} \tag{39}$$

and

$$\left[L_{\Pi^{\sim}(0)}(0;\gamma), R_{\Pi^{\sim}(0)}(0;\gamma)\right] = \left[L_{\Pi^{\sim}{}_0}(\gamma), R_{\Pi^{\sim}{}_0}(\gamma)\right],$$

Similarly, for Eqs. (36)-(38),

$$\begin{aligned}
&\left[L_{\tilde{\Omega}(0)}(0;\alpha), R_{\tilde{\Omega}(0)}(0;\alpha)\right] = \left[L_{\tilde{\Omega}_0}(\alpha), R_{\tilde{\Omega}_0}(\alpha)\right],\\
&\left[L_{\tilde{\Omega}(0)}(0;\beta), R_{\tilde{\Omega}(0)}(0;\beta)\right] = \left[L_{\tilde{\Omega}_0}(\beta), R_{\tilde{\Omega}_0}(\beta)\right]
\end{aligned}$$

and

$$\left[L_{\Omega^{\sim}(0)}(0;\gamma),R_{\Omega^{\sim}(0)}(0;\gamma)\right]=\left[L_{\Omega^{\sim}{}_0}(\gamma),R_{\Omega^{\sim}{}_0}(\gamma)\right]. \tag{40}$$

Further elaboration of the proposed assumptions and to clear the concepts, a detailed evaluation with a numerical example is given in next section.

Numerical findings and illustration

Here, numerical illustrations are carried out by using *Mathematica 11.0* software, which has built-in algorithms that go through several stages of generating solutions, depending on the type of equations. These programs can solve a wide range of differential equations such as linear and nonlinear partial, delay, differential–algebraic etc., with initial as well as boundary conditions. For the system of differential equations, it uses the time integration stages, which has the options of few methods namely, Runge-Kutta method, predictor–corrector Adams method, and implicit backward differentiation formulas, etc. We set out the manipulation by employing the definition 2 of Laplace transform on the Eqs. (33) - (38), which converts the equations into nonlinear SVTNF ordinary differential equations in the fractional context as:

$$\begin{gathered}\left[\dot{L}_{\tilde{\Pi}(s)}(s;\alpha),\dot{R}_{\tilde{\Pi}(s)}(s;\alpha)\right]=\\ \tfrac{1}{\psi}\left[L_{\tilde{t}}(\alpha),R_{\tilde{t}}(\alpha)\right]\left[\left(L_{\tilde{\Pi}(s)}(s;\alpha)\right)^{\chi},\left(R_{\tilde{\Pi}(s)}(s;\alpha)\right)^{\chi}\right]\left[\left(L_{\tilde{\Lambda}}(\alpha)\right)^{1-\chi},\left(R_{\tilde{\Lambda}}(\alpha)\right)^{1-\chi}\right]\\ \left[\left(L_{\tilde{\Omega}(s)}(s;\alpha)\right)^{1-\chi},\left(R_{\tilde{\Omega}(s)}(s;\alpha)\right)^{1-\chi}\right]\left(1-\frac{\left[L_{\tilde{\Pi}(s)}(s;\alpha),R_{\tilde{\Pi}(s)}(s;\alpha)\right]}{\left[L_{\tilde{f}}(\alpha),R_{\tilde{f}}(\alpha)\right]}\right)-\tfrac{1}{\psi}\left[L_{\tilde{n}}(\alpha),R_{\tilde{n}}(\alpha)\right]\\ \left[L_{\tilde{\Pi}(s)}(s;\alpha),R_{\tilde{\Pi}(s)}(s;\alpha)\right]-\left(\tfrac{1}{\psi}-1\right)\left(\left[L_{\tilde{\Pi}(s)}(s;\alpha),R_{\tilde{\Pi}(s)}(s;\alpha)\right]-\left[L_{\tilde{\Pi}_0}(\alpha),R_{\tilde{\Pi}_0}(\alpha)\right]\right),\end{gathered} \tag{41}$$

$$\begin{gathered}\left[\dot{L}_{\tilde{\Pi}(s)}(s;\beta),\dot{R}_{\tilde{\Pi}(s)}(s;\beta)\right]=\\ \tfrac{1}{\psi}\left[L_{\tilde{t}}(\beta),R_{\tilde{t}}(\beta)\right]\left[\left(L_{\tilde{\Pi}(s)}(s;\beta)\right)^{\chi},\left(R_{\tilde{\Pi}(s)}(s;\beta)\right)^{\chi}\right]\left[\left(L_{\tilde{\Lambda}}(\beta)\right)^{1-\chi},\left(R_{\tilde{\Lambda}}(\beta)\right)^{1-\chi}\right]\\ \left[\left(L_{\tilde{\Omega}(s)}(s;\beta)\right)^{1-\chi},\left(R_{\tilde{\Omega}(s)}(s;\beta)\right)^{1-\chi}\right]\left(1-\frac{\left[L_{\tilde{\Pi}(s)}(s;\beta),R_{\tilde{\Pi}(s)}(s;\beta)\right]}{\left[L_{\tilde{f}}(\beta),R_{\tilde{f}}(\beta)\right]}\right)-\tfrac{1}{\psi}\left[L_{\tilde{n}}(\beta),R_{\tilde{n}}(\beta)\right]\\ \left[L_{\tilde{\Pi}(s)}(s;\beta),R_{\tilde{\Pi}(s)}(s;\beta)\right]-\left(\tfrac{1}{\psi}-1\right)\left(\left[L_{\tilde{\Pi}(s)}(s;\beta),R_{\tilde{\Pi}(s)}(s;\beta)\right]-\left[L_{\tilde{\Pi}_0}(\beta),R_{\tilde{\Pi}_0}(\beta)\right]\right),\end{gathered} \tag{42}$$

$$\begin{gathered}\left[\dot{L}_{\tilde{\Pi}(s)}(s;\gamma),\dot{R}_{\tilde{\Pi}(s)}(s;\gamma)\right]=\\ \tfrac{1}{\psi}\left[L_{\tilde{t}}(\gamma),R_{\tilde{t}}(\gamma)\right]\left[\left(L_{\tilde{\Pi}(s)}(s;\gamma)\right)^{\chi},\left(R_{\tilde{\Pi}(s)}(s;\gamma)\right)^{\chi}\right]\left[\left(L_{\tilde{\Lambda}}(\gamma)\right)^{1-\chi},\left(R_{\tilde{\Lambda}}(\gamma)\right)^{1-\chi}\right]\\ \left[\left(L_{\tilde{\Omega}(s)}(s;\gamma)\right)^{1-\chi},\left(R_{\tilde{\Omega}(s)}(s;\gamma)\right)^{1-\chi}\right]\left(1-\frac{\left[L_{\tilde{\Pi}(s)}(s;\gamma),R_{\tilde{\Pi}(s)}(s;\gamma)\right]}{\left[L_{\tilde{f}}(\gamma),R_{\tilde{f}}(\gamma)\right]}\right)-\tfrac{1}{\psi}\left[L_{\tilde{n}}(\gamma),R_{\tilde{n}}(\gamma)\right]\\ \left[L_{\tilde{\Pi}(s)}(s;\gamma),R_{\tilde{\Pi}(s)}(s;\gamma)\right]-\left(\tfrac{1}{\psi}-1\right)\left(\left[L_{\tilde{\Pi}(s)}(s;\gamma),R_{\tilde{\Pi}(s)}(s;\gamma)\right]-\left[L_{\tilde{\Pi}_0}(\gamma),R_{\tilde{\Pi}_0}(\gamma)\right]\right),\end{gathered} \tag{43}$$

$$\begin{gathered}\left[\dot{L}_{\tilde{\Omega}(s)}(s;\alpha),\dot{R}_{\tilde{\Omega}(s)}(s;\alpha)\right]=\\ \tfrac{1}{\psi}\left[L_{\tilde{\xi}}(\alpha),R_{\tilde{\xi}}(\alpha)\right]\left[L_{\tilde{\Omega}(s)}(s;\alpha),R_{\tilde{\Omega}(s)}(s;\alpha)\right]\\ \left(1-\frac{\left[L_{\tilde{\Omega}(s)}(s;\alpha),R_{\tilde{\Omega}(s)}(s;\alpha)\right]}{\left[L_{\tilde{p}}(\alpha),R_{\tilde{p}}(\alpha)\right]}\right)\left(\frac{\left[L_{\tilde{\Omega}(s)}(s;\alpha),R_{\tilde{\Omega}(s)}(s;\alpha)\right]}{\left[L_{\tilde{q}}(\alpha),R_{\tilde{q}}(\alpha)\right]}-1\right)\\ -\left(\tfrac{1}{\psi}-1\right)\left(\left[L_{\tilde{\Omega}(s)}(s;\alpha),R_{\tilde{\Omega}(s)}(s;\alpha)\right]-\left[L_{\tilde{\Omega}_0}(\alpha),R_{\tilde{\Omega}_0}(\alpha)\right]\right),\end{gathered} \tag{44}$$

$$\begin{gathered}\left[\dot{L}_{\tilde{\Omega}(s)}(s;\beta),\dot{R}_{\tilde{\Omega}(s)}(s;\beta)\right]=\\ \tfrac{1}{\psi}\left[L_{\tilde{\xi}}(\beta),R_{\tilde{\xi}}(\beta)\right]\left[L_{\tilde{\Omega}(s)}(s;\beta),R_{\tilde{\Omega}(s)}(s;\beta)\right]\\ \left(1-\frac{\left[L_{\tilde{\Omega}(s)}(s;\beta),R_{\tilde{\Omega}(s)}(s;\beta)\right]}{\left[L_{\tilde{p}}(\beta),R_{\tilde{p}}(\beta)\right]}\right)\left(\frac{\left[L_{\tilde{\Omega}(s)}(s;\beta),R_{\tilde{\Omega}(s)}(s;\beta)\right]}{\left[L_{\tilde{q}}(\beta),R_{\tilde{q}}(\beta)\right]}-1\right)\\ -\left(\tfrac{1}{\psi}-1\right)\left(\left[L_{\tilde{\Omega}(s)}(s;\beta),R_{\tilde{\Omega}(s)}(s;\beta)\right]-\left[L_{\tilde{\Omega}_0}(\beta),R_{\tilde{\Omega}_0}(\beta)\right]\right)\end{gathered} \tag{45}$$

and

$$\begin{gathered}\left[\dot{L}_{\tilde{\Omega}(s)}(s;\gamma),\dot{R}_{\tilde{\Omega}(s)}(s;\gamma)\right]=\\ \tfrac{1}{\psi}\left[L_{\tilde{\xi}}(\gamma),R_{\tilde{\xi}}(\gamma)\right]\left[L_{\tilde{\Omega}(s)}(s;\gamma),R_{\tilde{\Omega}(s)}(s;\gamma)\right]\left(1-\frac{\left[L_{\tilde{\Omega}(s)}(s;\gamma),R_{\tilde{\Omega}(s)}(s;\gamma)\right]}{\left[L_{\tilde{p}}(\beta),R_{\tilde{p}}(\beta)\right]}\right)\\ \left(\frac{\left[L_{\tilde{\Omega}(s)}(s;\gamma),R_{\tilde{\Omega}(s)}(s;\gamma)\right]}{\left[L_{\tilde{q}}(\gamma),R_{\tilde{q}}(\gamma)\right]}-1\right)-\left(\tfrac{1}{\psi}-1\right)\left(\left[L_{\tilde{\Omega}(s)}(s;\gamma),R_{\tilde{\Omega}(s)}(s;\gamma)\right]-\left[L_{\tilde{\Omega}_0}(\gamma),R_{\tilde{\Omega}_0}(\gamma)\right]\right),\end{gathered} \tag{46}$$

respectively. Parametric expansion of Laplace transform is a very efficient technique for fractional order differential equations, due to its ability to convert the equations in integer order without completely eliminating the fractional index. It has been widely used nowadays in the areas where fractional differential equations get to be challenging to attain the analytical solutions directly. After the Laplace transform expansion, to make the illustration of the model easily fathomable, we assume some numerical values for the parameters in Eqs. (41)- (46), as shown in Table 2. Let $\tilde{\xi}$, $\tilde{p}$ and $\tilde{q}$ quantify the population in billions and $\tilde{t}$, $\tilde{n}$, $\tilde{f}$ and $\tilde{\Lambda}$ in billion dollars, whereas time s is measured in years. All the attained solutions are illustrated through different types of graphs, to visualize the effects more efficaciously. Figs. 1-3 represent the evolution of capital for $\alpha=\beta=\gamma=0.35$, $\psi=0.9,\ 0.95,\ 1$ and $s\in[0,1]$. It can be clearly seen that the left and right levels of capital growth function $\tilde{\Pi}(t)$ declines towards the maximum sustainable levels $L_{\tilde{f}}(\alpha)=L_{\tilde{f}}(\beta)=L_{\tilde{f}}(\gamma)=111.667$ and $R_{\tilde{f}}(\alpha)=R_{\tilde{f}}(\beta)=R_{\tilde{f}}(\gamma)=128.333$billions, respectively and remains in a steady state. On the flip side, since

$$\left[L_{q^{\sim}}(\alpha),R_{q^{\sim}}(\alpha)\right]<\left[L_{\Omega^{\sim}{}_0}(\alpha),R_{\Omega^{\sim}{}_0}(\alpha)\right]<\left[L_{p^{\sim}}(\alpha),R_{p^{\sim}}(\alpha)\right],$$

$$\left[L_{q^{\sim}}(\beta),R_{q^{\sim}}(\beta)\right]<\left[L_{\Omega^{\sim}{}_0}(\beta),R_{\Omega^{\sim}{}_0}(\beta)\right]<\left[L_{p^{\sim}}(\beta),R_{p^{\sim}}(\beta)\right],$$

and

$$\left[L_{q^{\sim}}(\gamma),R_{q^{\sim}}(\gamma)\right]<\left[L_{\Omega^{\sim}{}_0}(\gamma),R_{\Omega^{\sim}{}_0}(\gamma)\right]<\left[L_{p^{\sim}}(\gamma),R_{p^{\sim}}(\gamma)\right],$$

Table 2
Numerical values for the parameters.

Parameters	SVTNFNs
	Population (in billions)
$\tilde{p}$	(200, 300, 400; 0.6, 0.4, 0.54)
$\tilde{\xi}$	(298, 300, 302; 0.6,0.4, 0.54)
$\tilde{q}$	(0.0357, 0.0407, 0.0457; 0.6,0.4, 0.54)
$\tilde{\Omega}_0$	(5, 8, 11; 0.6,0.4, 0.54)
	Amount (in billions)
$\tilde{f}$	(\$100, \$120, \$140; 0.6,0.4, 0.54)
$\tilde{t}$	(\$20000, \$20200, \$20400; 0.6, 0.4, 0.54)
$\tilde{\Lambda}$	(\$500, \$510, \$520; 0.6,0.4, 0.54)
$\tilde{n}$	(\$0.0347, \$0.2347, \$0.4347; 0.6, 0.4, 0.54)
$\tilde{\Pi}_0$	(\$405, \$408, \$411; 0.6, 0.4, 0.54)

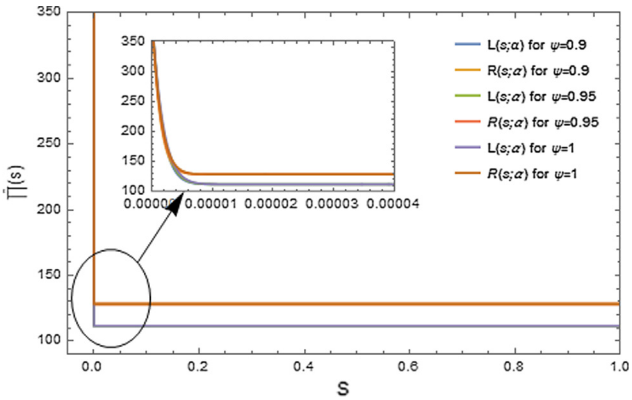

Fig. 1. The impact on the physical capital with truth membership degree $\alpha = 0.35$ of fractional order parameter ψ.

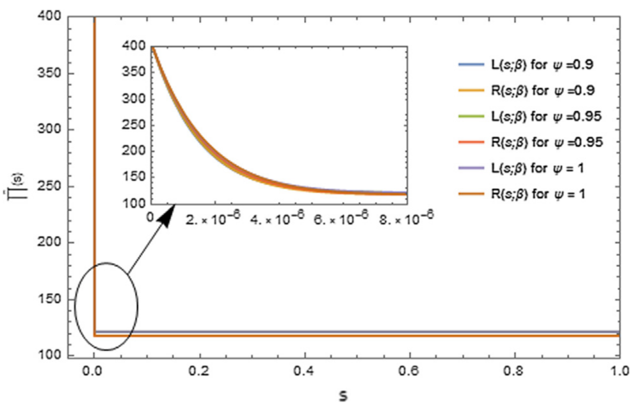

Fig. 2. The impact on the physical capital with indeterminacy membership degree $\beta = 0.35$ of fractional order parameter ψ.

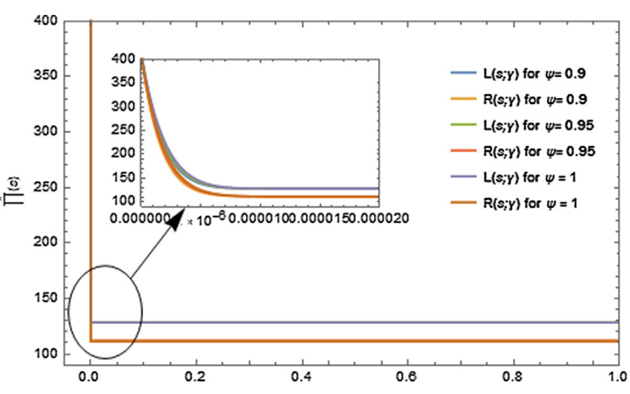

Fig. 3. The impact on the physical capital with falsity membership degree $\gamma = 0.35$of fractional order parameter ψ.

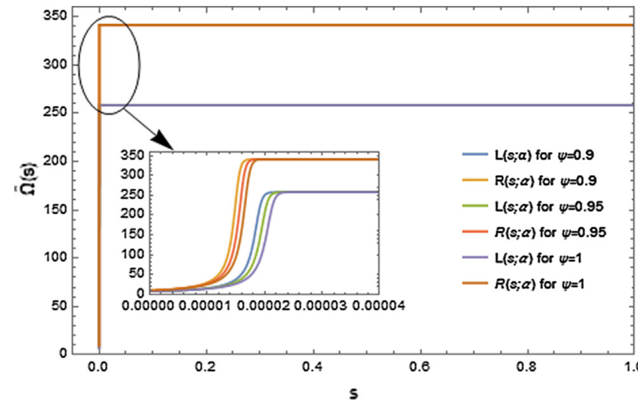

Fig. 4. The impact on the labour function with truth membership degree $\alpha = 0.35$of fractional order parameter ψ.

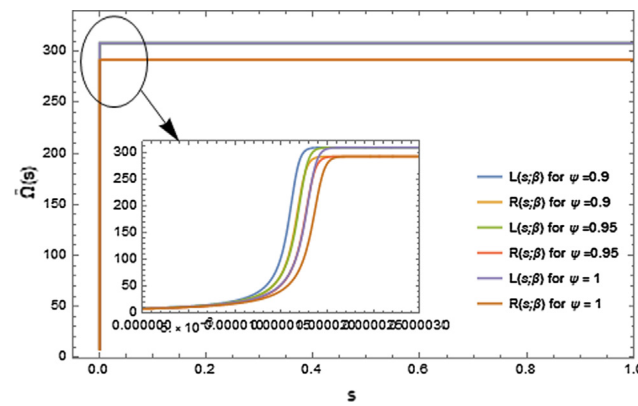

Fig. 5. The impact on the labour function with indeterminacy membership degree $\beta = 0.35$of fractional order parameter ψ.

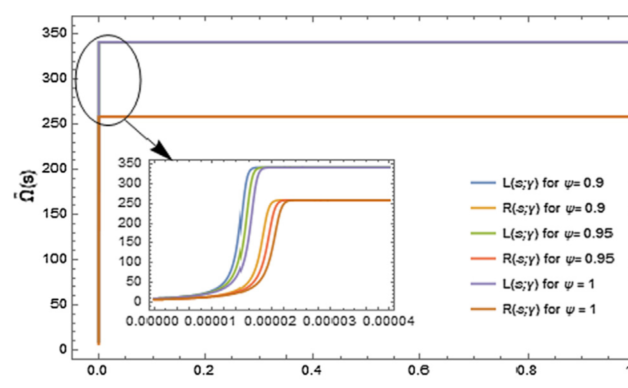

Fig. 6. The impact on the labour function with falsity membership degree $\gamma = 0.35$of fractional order parameter ψ.

therefore, Figs. 4-6 evidently display rising growth of labor population towards the maximum carrying capacity levels $L_{\tilde{p}}(\alpha) = L_{\tilde{p}}(\beta) = L_{\tilde{p}}(\gamma) = 258.333$ and $R_{\tilde{p}}(\alpha) = R_{\tilde{p}}(\beta) = R_{\tilde{p}}(\gamma) = 341.667$ billion populations, accordingly, for $\alpha = \beta = \gamma = 0.35$, $\psi = 0.9, 0.95, 1$ and $s \in [0,\ 1]$. Fig. 7(a-c) and 8(a-c) screen the contours of capital $\tilde{\Pi}(s)$ and labour $\tilde{\Omega}(s)$ functions for $s = 100$ years and $\psi \in [0.9,\ 1]$ with truth, indeterminacy and falsity degrees; $\alpha \in [0,\ 0.6]$, $\beta \in [0.4,\ 1]$ and $\gamma \in [0.54,\ 1]$. Specifically, these plots are attained by making slices of the three-dimensional figures of the proposed multivariate SVTNF capital and labour functions. Each curve generated from the segments, inspecting different values of capital and labour functions, are then assembled to form the contours. The space between the curves demonstrates the extent acquired to change the value of the function to move to the next level or curve. For instance, smaller the distance implies very small change in the parameters will trigger the value of the function and vice versa. Also, the areas shaded with light colours correspond to the smaller distance and darker regions for large distance. In Fig. 7(a-c), the framed value on each curve represents the amount

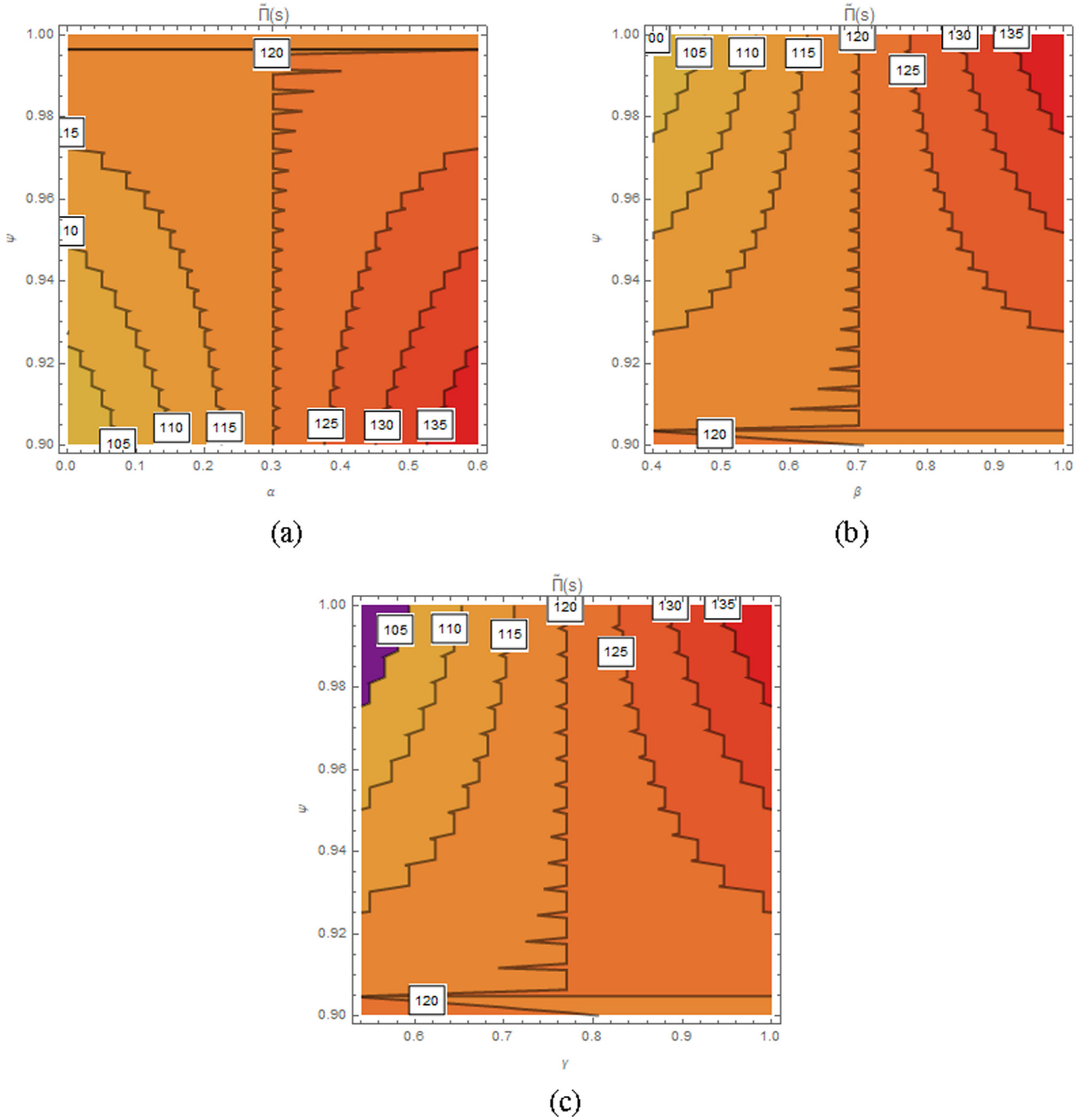

Fig. 7. (a-c). Contours of capital with respect to truth, indeterminacy and falsity membership degrees for $s = 100$.

of capital investment in billion dollars, which will remain same for each value of parameters on that particular curve. Analogously, Fig. 8(a-c) read the framed values of population of labours in billions.

If we postulate the aforementioned conjectures and graphical analysis on the study of economic growth of a country, one can easily scrutinize the maximum investment level of a country with the carrying capacity of the labours and vice versa. Also, if the maximum sustainable investment on a capital is predefined, then it can also be investigated that how production growth will gradually reach its maximum or minimum level. Correspondingly, the study about the performance of the labours to produce output with their maximum capability to survive in the milieu and utilize the available resources, can be done efficiently. For instance, from the above numerical results, with the maximum sustainable capital investment of [$111.667, $128.333] billions and maximum survival of [258.333, 341.667] billion labours along with other fix values of parameters, the production will gradually increase and attain a value [$0, $140000] billions till 2000 years.

Conclusions

We comprehended a system of nonlinear fractional order single-valued triangular neutrosophic fuzzy differential equations to interpret the dynamics of capital and labour in the economic growth. The theory of NF helped to deal with impreciseness of the parameters with truth, indeterminacy and falsity membership

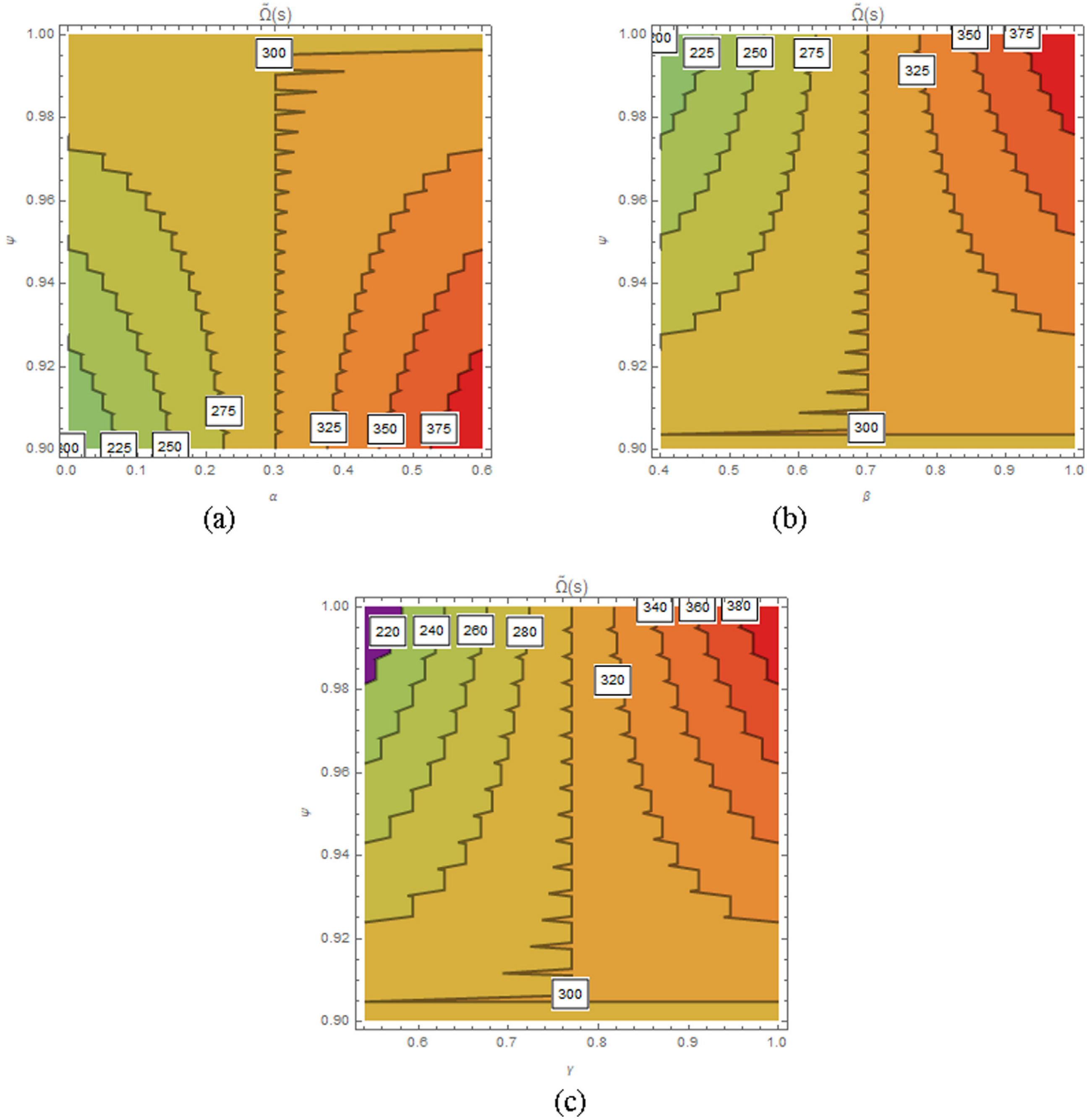

Fig. 8. (a-c). Contours of labours with respect to truth, indeterminacy and falsity membership degrees for $s = 100$.

degrees. In addition, growth of economy is analysed more effectively by using logistic equation with Allee effect in population growth of labour and capital. Graphical results were also added in discussion in the form of contours and cylindrical coordinate systems.

The attained facts and figures of the endeavour evidence that the amalgamation of NF, fractional order derivative and Allee effect can be used effectively in modelling different aspects. Moreover, the numerical experiments and illustrations may provide the policymakers or economic analysts a new school of thought in forecasting, planning and effectuating strategies for economic growth of the state.

In future, we will study the applications of Allee effect and neutrosophic fuzzy numbers on some other models of science and engineering. With new methodologies, we will also explore new graphical views of the results for effective visualization of the outcomes.

Compliance with Ethics Requirements

This article does not contain any studies with human or animal subjects

This is a research article for SI: Fractional Calculus Model.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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