Global Sensitivity Analysis of Various Numerical Schemes for the Heston Model

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Abstract. The pricing of financial options is usually based on statistical sampling of the evolution of the underlying under a chosen model, using a suitable numerical scheme. It is widely accepted that using low-discrepancy sequences instead of pseudorandom numbers in most cases increases the accuracy. It is important to understand and quantify the reasons for this effect. In this work, we use Global Sensitivity Analysis in order to study one widely used model for pricing of options, namely the Heston model. The Heston model is an important member of the family of the stochastic volatility models, which have been found to better describe the observed behaviour of option prices in the financial markets. By using a suitable numerical scheme, like those of Euler, Milstein, Kahl-Jäckel, Andersen, one has the flexibility needed to compute European, Asian or exotic options. In any case the problem of evaluating an option price can be considered as a numerical integration problem. For the purposes of modelling and complexity reduction, one should make the distinction between the model nominal dimension and its effective dimension. Another notion of "average dimension" has been found to be more practical from the computational point of view. The definitions and methods of evaluation of effective dimensions are based on computing Sobol' sensitivity indices. A classification of functions based on their effective dimensions is also known. In the context of quantitative finance, Global Sensitivity Analysis (GSA) can be used to assess the efficiency of a particular numerical scheme. In this work we apply GSA based on Sobol sensitivity indices in order to assess the interactions of the various dimensions in using the above mentioned schemes. We observe that the GSA offers useful insight on how to maximize the advantages of using QMC in these schemes.

1 Introduction to Option Pricing Under the Heston Stochastic Volatility Model

Financial options are instruments which allow their holder to obtain certain payout, which depends on the price of the underlying security. While the payout

We acknowledge the provided access to the e-infrastructure of the NCHDC – part of the Bulgarian National Roadmap on RIs, with the financial support by the Grant No DO1-271/16.12.2019.

V. V. Krzhizhanovskaya et al. (Eds.): ICCS 2020, LNCS 12143, pp. 520–528, 2020.
https://doi.org/10.1007/978-3-030-50436-6_38

of European options only depends on the price at the time of expiration, there are options that depend on the evolution of the price throughout certain time period in a complex way. A Monte Carlo method for determining the prices of options would be obtained by sampling paths for the price of the underlying following a chosen stochastic model. In such case the price of the financial option, whose payout is defined as a function of the evolution of the price: $F(\{S_t\}_{t=0}^{T})$ would be evaluated as the average of the (discounted) value of a function F' over the sampled paths, where F' uses only values of the price at certain points, determined by the discretisation used. It is obvious that such a method will have both stochastic and deterministic components of the error, i.e., the estimate would be biased.

The more sophisticated models involve also the evolution of the volatility, which is not directly observable. This evolution can be deterministic or following a stochastic process. It is widely accepted that stochastic volatility models can better explain the observed behaviours in financial markets, but pose numerical difficulties. One of the most popular stochastic volatility models is the model of Heston. The evolution of the Heston model is described by the following two equations:

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_s$$

$$dv_t = \kappa \left(\theta - v_t\right) dt + \sigma_v \sqrt{v_t} dW_v,$$

where S_t is the price and v_t is the volatility, while dW_s and dW_v are two Brownian motions that are correlated with a coefficient ρ. The parameters of the model are $\kappa, \theta, \sigma, S_0, V_0, \rho, r$. The determination of these parameters is out of scope of this paper. One can read more about the Heston model in [9]. The Monte Carlo numerical schemes are based on choosing a time step for discretization and then sampling the path of the underlying and the volatility. Brody and Kaya [7] demonstrated how it is possible to sample from the exact distributions of the price and volatility at the expense of more computational power requirements. In practice other numerical schemes require less computations and achieve sufficient accuracy when the time step is small enough. In this work we considered the schemes of Euler-Maruyama (see, e.g., [12]), Kahl-Jäckel [10], Milstein ([8]) and Andersen [5]. We denote them by the letters A, B, C, D respectively. For the scheme of Euler-Maruyama we apply the Lord's truncation method [11].

Under these schemes each time step requires the sampling of two random variables. Usually the inverse function method is used and thus we can assume that only random number uniformly distributed in $(0, 1)$ are used. Thus the constructive dimensionality of the algorithm in the sense, defined by Sobol', is $2n$, where n is the number of time steps. The practitioners in Mathematical Finance also need to compute various derivatives of the option price, which are generally known as Greeks. For example, the Delta of an option is the derivative of the price with respect to the (initial) price of the underlying, while the Theta is the derivative with respect to the remaining time to expiration. Such quantities can be estimated by introducing a small number ϵ and computing the option price also for values of the corresponding parameter with added $\pm\epsilon$ and using a formula for approximate computation of the derivative.

The Global Sensitivity Analysis methodology allows to assess the importance of each variable and quantify the various interactions between variables. In the next section we shall describe how this can be applied in our problem.

2 Computation of Global Sensitivity Indices in the Context of the Numerical Schemes for the Heston Model

The Global Sensitivity Analysis is based on the computation of the Sobol' sensitivity indices, which quantify the contribution of the various terms of the ANOVA decomposition of a function f:

$$f(x) = f_0 + \sum_{i=1}^{d} f_i(x_i) + \sum_{i=1}^{d} \sum_{j=i+1}^{d} f_{ij}(x_i, x_j) \dots$$

towards it's overall variance D. Sobol' defined the coefficients

$$S_{i_1,\dots,i_k} = D_{i_1,\dots,i_k}/D,$$

so that one can evaluate the sensitivity of the function to subsets of variables, where $D_{i_1,\dots,i_k}$ denotes the variance of the corresponding term in the ANOVA decomposition.

These coefficients sum to 1. There are substantially different algorithms for computing them. In our work we followed the approach proposed in [1], where formula (15) leads to efficient Monte Carlo and consequently quasi-Monte Carlo method for computing the indices. The total sensitivity indices are also important to consider, since they can be computed efficiently with a similar formula, while providing a numerical estimation for the total contribution of a variable to the variance of the function, summing all the coefficients where it is part of the subset. Various works establish the way to use the Sobol' sensitivity indices in order to compute other useful measures, for example, the mean dimensionality in [3] or the effective dimensions in [1]. In [4] one can see how GSA can be used to improve option pricing. Since the Heston model has stochastic volatility, one has to deal with a more complex numerical schemes and consequently more heterogenious distribution of the uncertainty.

All the numerical schemes for the Heston model that we consider in this work are built upon sampling two random variables for each time step. In most cases these variables are normally distributed, while the Andersen scheme is more complex. In practice one would use a pseudorandom number generator or a generator for a low-discrepancy sequence and then use the inverse function method in order to obtain the appropriate normally distributed number. Initially we used the natural ordering of the variables, so that each time step uses one odd and one even coordinate. In this way the sampling of a path with n steps requires

$2n$ pseudorandom numbers or $2n$ coordinates of a low-discrepancy sequence. Later on we shall discuss how, based on the results obtained from the GSA, one may reorder the variables in order to improve the accuracy of the computation.

Because of the constraints on the available compute power, in our analysis of these numerical schemes we limited ourselves to compute only coefficients and total coefficients of orders one and two, which already give idea about the behaviour of the schemes. The method we use requires us to effectively double the number of coordinates used. It is a well known fact that there is substantial advantage in using low-discrepancy sequence for such kinds of problems, but using Global Sensitivity Analysis we quantify the contribution of the different dimensions and obtain suggestions about possible re-orderings of the variables in order to improve the accuracy of the computation, when using low-discrepancy sequences.

3 Numerical Results and Discussion

First of all we compared pure Monte Carlo method for computing the price of an option using the above schemes with a quasi-Monte Carlo method which utilizes either modified Halton (see, e.g., [6]) or Sobol' sequences. The results cover the calculation of the price of an Asian option. The payout of an Asian option is defined as

$$\max\left(\frac{1}{N}\sum_{i=1}^{N} X_i - K,\, 0\right).$$

The parameters of the Heston model are $r = 3.19\%, \kappa = 6.21, \theta = 0.019, \sigma_v = 0.61, \rho = -0.7, S_0 = 100, V_0 = 0.010201$. One can see that with the increase in number of time steps the accuracy of the quasi-Monte Carlo method improves significantly more than that of the Monte Carlo method. In order to obtain the results in Table 1 the number of steps is fixed at 12 (corresponding to the number of months in the year). The scheme used is the Euler - Murayama scheme with the Lord full-truncation.

Table 1. Error from computation with 12 time steps, scheme A

N	MC	Sobol	Halton
256	0.60	0.23	0.13
512	0.23	0.18	0.08
1024	0.13	0.09	0.04
2048	0.08	0.01	0.02

For the computations in Table 2 the number of steps is fixed at 32, where the results for the price are on the left and the results for the delta are on the right. It is obvious that the accuracy for the delta is smaller. This time the more complex Andersen scheme (scheme D) is used. The parameters are

$$r = 0.\%, \kappa = 1.0606, \theta = 0.0733, \sigma_v = 0.3918, \rho = -0.3456,$$
$$S_0 = 100, V_0 = 0.0222, K = 100.$$

Table 2. Error from computation with 32 steps (nominal dimension 64), price (left) and delta (right), scheme D

N	MC	Sob.	Hal.	MC	Sob.	Hal.
256	0.61	0.12	0.40	1.02	.66	0.85
512	0.53	0.06	0.20	.75	0.44	0.59
1024	0.31	0.07	0.12	0.45	0.50	0.44

These results show the substantial advantage of using low-discrepancy sequences with these schemes. However, in such case we have the option to reorder of dimensions of the low-discrepancy sequence in order to position the first coordinates of the generated points to sample the most important variables. The importance of the variables is quantified by the Sobol sensitivity indices. Our first goal was to compute Sobol' one-dimensional sensitivity indices for the price under the different schemes. All the examples consider the same Heston model, with the following parameters: time steps - 52, time to expiration - 1, $\kappa = 1.0606$, interest rate $r = 0$, starting volatility $\theta_0 = 0.0222$, long-term mean volatility $\theta_v = 0.0733$, $\rho = 0.3456$, volatility of volatility $\varepsilon = 0.3918$. The starting price is $S_0 = 1.$ and we consider an Asian option with strike $K = 1$. We also considered a knock out-option with a knock out level $1.2\,K$, so that the owner receives payout

$$\max\left(X_N - K, 0\right)$$

only if all $X_i < 1.2\,K$, otherwise the payout is zero.

On Fig. 1 one can see the one-dimensional Sobol' sensitivity indices for the Asian option under the different schemes.

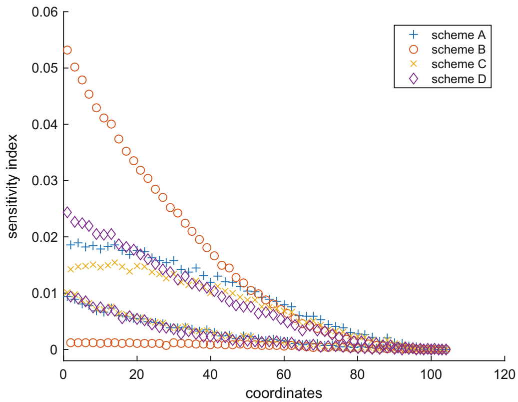

Fig. 1. One-dimensional Sobol' sensitivity indices for the price of an Asian option under the Heston model

We observe how three of the schemes lead to similar indices, while the Milstein scheme has substantially different behaviour. In all cases the indices decrease with the number of timesteps, which suggests that the leading dimensions of the low-discrepancy sequences should be used to sample the first timesteps. It is also noticeable how the even dimensions have much larger coefficients than the odd dimensions. Since this is the case for all schemes under considerations, the logical suggestion is to reorder the coordinates of the low-discrepancy sequence so that the first half of the coordinates are used to sample the odd coordinates and the second half of the coordinates are used to sample the even dimensions. We remind that two numbers are used for each timestep. These computations were carried out using the scrambled Sobol' sequence, with direction numbers provided by [2]. We used $4 \times 52 = 208$ dimensions and 2^{18} points. For the price of the knock-out option we make the same observation with regards to the odd and even dimensions (see Fig. 2). However, the last few coordinates have slightly increased importance, while also the accuracy of the computation of the indices is not as good as in the case of the Asian option.

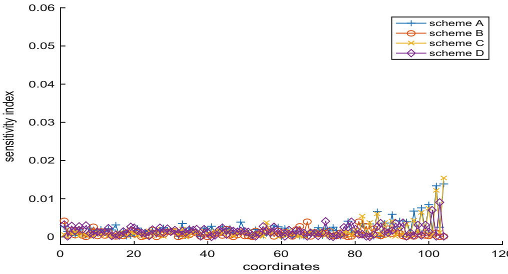

Fig. 2. One-dimensional Sobol' sensitivity indices for the price of a knock-out option under the Heston model

The accuracy improves when using the total Sobol' sensitivity indices. As we can see in Fig. 3, the Milstein scheme (scheme B) is again substantially different from the other 3 schemes in the observed behaviour of the indices. It seems that there is decreasing importance of the dimensions and the difference between odd and even dimensions is not so prounounced, except for the Milstein scheme. It would be logical to reorder the variables according to their importance as measured with the total Sobol' sensitivity indices, since they take into account the non-linear interactions.

The computation of sensitivity indices for the various "Greeks" is achieved by introducing a small parameter ϵ and using numerical differentiation formulae. Although we can expect decreased accuracy because of the differentiation, we can see in the next figure, showing the one-dimensional Sobol' sensitivity indices for the Delta of an Asian option, that sufficient accuracy is achieved and the conclusions with regards to the optimal use of the coordinates of the low-discrepancy sequence are similar to the case for the price (see Fig. 4).

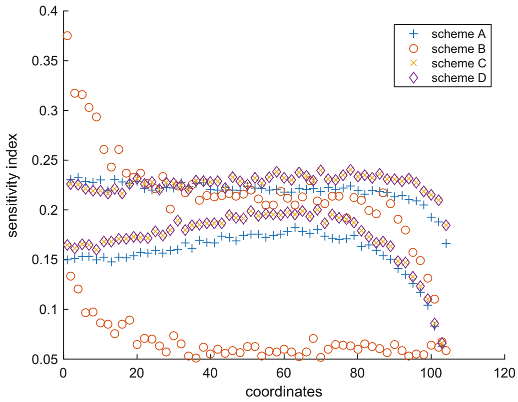

Fig. 3. One-dimensional Sobol' total sensitivity indices for the price of a knock-out option under the Heston model

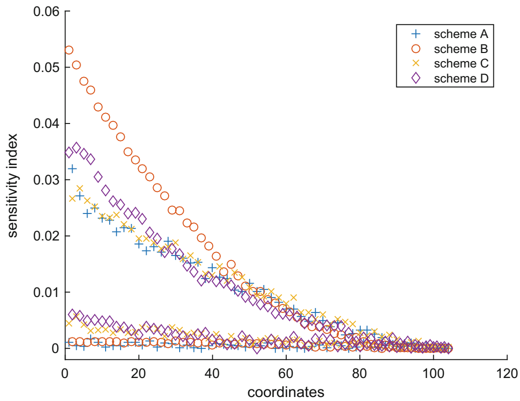

Fig. 4. One-dimensional Sobol' sensitivity indices for the Delta of an Asian option under the Heston model.

It is also possible to compute Sobol' sensitivity indices for larger subsets of variables. Unfortunately, this increases the computational complexity substantially. That is why we only show results obtained for pairs of variables. In this case, only the results for the Euler-Murayama scheme with the Lord's truncation (scheme A) are shown (see Fig. 5).

We show the interactions between indices for odd and even dimensions in 3 surfaces. As it can be expected, the highest importance is for pairs of indices that both correspond to even dimension. There is also visible decrease of the importance when increasing the index of the coordinate.

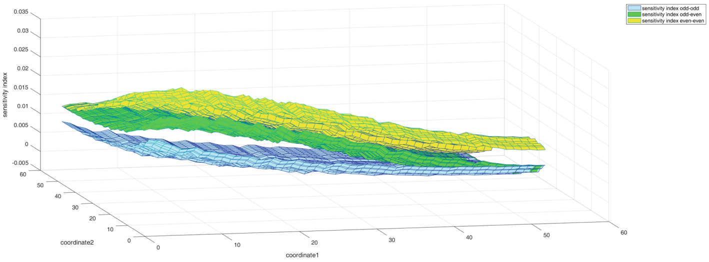

Fig. 5. Sobol' sensitivity indices for pairs of variables, when computing the price of an Asian option under the Heston model.

4 Conclusions

The various numerical schemes used for option pricing via the Heston model can benefit from use of low-discrepancy sequences. The Sobol' sequence with Owen scrambling and the modified Halton sequences proved to be effective. By studying the Sobol' sensitivity indices we can quantify the contribution of the various dimensions and their interactions to the overall variance, which gives us ideas on how to reorder the coordinates in order to increase the accuracy. Once it has been determined how much each variable contributes to the final result of the simulation, it becomes possible to optimise the parameters of the Sobol' or Halton sequences jointly with respect to a measure of quality of distribution that is weighted according to the compu. In a future work we plan to demonstrate how this approach improves the accuracy.

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