Heliyon 10 (2024) e30690

Contents lists available at ScienceDirect

Heliyon

journal homepage: www.cell.com/heliyon

Research article

Generalization of Odd Ramos-Louzada generated family of distributions: Properties, characterizations, and applications to diabetes and cancer survival datasets

John Kwadey Okutu [a,*], Nana K. Frempong [a], Simon K. Appiah [a,b], Atinuke O. Adebanji [a]

[a] Department of Statistics and Actuarial Science, College of Science, Kwame Nkrumah University of Science and Technology, Kumasi, Ghana
[b] Department of Mathematics, College of Arts and Sciences, Howard University, Washington, DC, 20059, USA

ARTICLE INFO

Keywords:
Kumaraswamy Odd Ramos-Louzada generator
Kumaraswamy Odd Ramos-Louzada Burr XII
Statistical properties
ML estimation
Simulations
and Characterizations

ABSTRACT

Probability distributions offer the best description of survival data and as a result, various lifetime models have been proposed. However, some of these survival datasets are not followed or sufficiently fitted by the existing proposed probability distributions. This paper presents a novel Kumaraswamy Odd Ramos-Louzada-G (KumORL-G) family of distributions together with its statistical features, including the quantile function, moments, probability-weighted moments, order statistics, and entropy measures. Some relevant characterizations were obtained using the hazard rate function and the ratio of two truncated moments. In light of the proposed KumORL-G family, a five-parameter sub-model, the Kumaraswamy Odd Ramos-Louzada Burr XII (KumORLBXII) distribution was introduced and its parameters were determined with the maximum likelihood estimation (MLE) technique. Monte Carlo simulation was performed and the numerical results were used to evaluate the MLE technique. The proposed probability distribution's significance and applicability were empirically demonstrated using various complete and censored datasets on the survival times of cancer and diabetes patients. The analytical results showed that the KumORLBXII distribution performed well in practice in comparison to its sub-models and several other competing distributions. The new KumORL-G for diabetes and cancer survival data is found extremely efficient and offers an enhanced and novel technique for modeling survival datasets.

1. Introduction

Diabetes and cancer are prevalent medical conditions that pose a serious threat to public health worldwide. Millions of people are diagnosed with these noninfectious diseases each year, and many of them die as a result of the complications of these illnesses. Cancer and diabetes are among the top 10 causes of death globally, accounting for approximately 10 million (one in every six) and 1.5 million deaths, respectively in 2020 [1,2]. Statistical models have contributed much to reducing the case burden of cancer and diabetes through prudent decisions from research. These models have frequently been used to model large datasets, including demographic,

* Corresponding author.
E-mail addresses: jkokutu@st.knust.edu.gh (J.K. Okutu), nkfrempong@knust.edu.gh (N.K. Frempong), skappiah@knust.edu.gh (S.K. Appiah), aoadebanji.cos@knust.edu.gh (A.O. Adebanji).

https://doi.org/10.1016/j.heliyon.2024.e30690
Received 8 November 2023; Received in revised form 15 April 2024; Accepted 2 May 2024
Available online 6 May 2024

clinical, genetic, and behavioral risk factors collected on patients' survival times. Several research studies have used statistical modeling approaches in healthcare management. Notable among these are De Villiers et al. [3], Lama et al. [4], Sandberg et al. [5], Altaf-Ul-Amin et al. [6], Ratnovsky et al. [7], Onchonga et al. [8], and Illescas et al. [9] which provide more details.

Numerous statistical models have been suggested and applied owing to the acknowledged significance of statistical models in healthcare management. For example, El-Morshedy et al. [10] employed an updated form of the Weibull model to simulate bladder cancer and leukemia patients. Kumar and Nair [11] offered a version of the log-inverse Weibull distribution to model cancer datasets, Liu et al. [12] suggested the arcsine-modified Weibull model to study and assess the COVID-19 patients' survival rates in China, while Mohammed et al. [13] developed a new lifetime Weibull density for the analysis of bladder cancer datasets, and Klakattawi [14] developed a novel, extended Weibull model to examine patient survival rates across different cancer subtypes. The power Lomax distribution for bladder cancer patients' remission periods was examined by Rady et al. [15]. At the same time, Shah et al. [16] introduced a novel generalized logarithmic Weibull density and applied it to head cancer data.

Although a wide range of distributions is available in the statistical literature for modeling healthcare and biomedical data, not all the salient statistical features, particularly the skewness patterns in the data are captured or modeled. Besides, most of these distributions are less adaptable and unfit in various scenarios for various survival datasets. Thus, it becomes necessary for new probability distributions to be developed to expand and generalize those that already exist. These newly proposed generated distributions typically are more flexible, and better suited to explain circumstances in healthcare survival data. In this regard, diverse studies have proposed and used various strategies to develop new families of distributions. Most of these generated families can be found in Mahmood et al. [17], Marshall and Olkin [18], Rastic and Balakrishnan [19], Cordeiro and de Castro [20], Alzaatreh and Ghosh [21], Bourguignon et al. [22], Zografos and Balakrishnan [23], Rahman et al. [24], Ahmad [25], and Al-Shomrani et al. [26], among others.

Okutu et al. [27] recently introduced the Odd Ramos-Louzada generator (ORL-G), from which numerous sub-models have been developed. Despite its desirable qualities, the ORL-G lacks a shape parameter, making it less versatile in modeling different skewness patterns that most survival data exhibit. To address this shortcoming of the ORL-G model, a novel extended version known as the Kumaraswamy Odd Ramos-Louzada generator (KumORL-G) with two extra shape parameters is presented by utilizing the Kumaraswamy-G (Kum-G) [20] approach. The additional parameters are intended to regulate the tail weights and skewness of the developed model. One appealing aspect of this family is that the two shape parameters allow for better control over the weights in both the tails and the center of the distribution. The Kum-G approach is used because of its mathematical simplicity and flexibility in modeling survival data. Thus, this study sought to obtain the more efficient KumORL-G model, derive and also investigate some of its statistical properties; develop some characterizations of the model employing the hazard rate function and the ratio of two truncated moments; utilize the MLE approach to estimate the model's parameters and evaluate the estimators' performance using a simulation analysis; and demonstrate how the new sub-model from the proposed generator offers a good fit to diabetes and cancer survival datasets.

The KumORL-G model can generate new, more flexible distributions that can explain various lifetime data with non-monotonic and monotonic hazard rates, making it useful for lifetime data analysis. The distributions from this family can be efficiently applied in modeling data that are heavy-tailed, extremely skewed, monotonic, and non-monotonic failure rates in various fields such as engineering, business, agriculture, and health. As a result, models from this family are anticipated to be more widely used in the domains mentioned earlier.

The introduction of this new family of distributions is inspired by four motivations. First, most survival data are highly skewed; therefore, the new model seeks to enhance the degree of flexibility in capturing different skewness patterns than the classical statistical distributions such as the Weibull, Raleigh, and exponential, which frequently have few parameters. The second choice of this new family with more parameters, offers greater flexibility in capturing a wider range of survival patterns. The third choice of this new family is the ability to capture additional features of the survival data that make it better equipped to estimate survival probabilities and make better predictions. Finally, we are inspired by the introduction of the KumORL-G family due to its ability to offer improved fits to the public health data as well as its ability to enhance flexibility for describing various forms of hazard-rate functions, such as "upside-down bathtub" and "bathtub" shapes, which are extensively encountered in real-world data.

The present study makes a substantial contribution to the field of probability theory (or mathematical statistics) by establishing a novel and innovative probability generator for modifying previously developed distributions to enhance flexibility in describing survival data. Within the KumORL-G class, we are especially interested in the new KumORLBXII distribution, which performs well in modeling diabetes and cancer survival datasets. Its improved adaptability and capacity to model the complicated structure of the data provide an effective tool for modeling survival data. A thorough examination of its statistical features such as entropy measures, moments, order statistics, and quantile functions among others, as well as the results of characterizations, distinguishes the KumORL-G class and its sub-model from other models. These results contribute to a better understanding of the new model's features and flexibility. The findings obtained herein provide both researchers and practitioners with strong statistical tools for effectively analyzing and modeling datasets with varied hazard rate functions. Also, such modeling methods improve knowledge regarding failure rates and provide better decision-making in a broad range of fields, including public health, engineering, finance, and many more.

The remaining portion of the paper is set out as follows: The PDF, CDF, and hazard rate function of the KumORL-G density and KumORLBXII model, a specific sub-model of the KumORL-G, are presented in Section 2. Section 3 contains a useful expansion along with certain essential statistical characteristics of the suggested probability model. Section 4 presents some derived characterization results for the KumORL-G density while the MLE of the model parameters and simulation analyses carried out are given in Section 5. Section 6 discusses the relevance of the KumORL-G density to the complete and censored real datasets, showing its adaptability and utility, while Section 7 concludes the study, stating the summary, limitations of the suggested model, the future direction of the study, and practical implications based on the findings of the study.

2. Model formulation

This section defines the PDF, CDF, survival, and hazard rate functions of the KumORL-G model. The section also includes graphs of the hazard rate function and the PDF of the novel KumORLBXII, a special member of the family.

2.1. The Kumaraswamy odd ramos-louzada generated family of distributions

Given that X is an ORL-G random variable with CDF represented by (1); where $G(x;\epsilon)$ is the baseline CDF and ϵ is a parameter vector connected to the CDF,

$$Z_{ORL-G}(x;\ \theta,\epsilon)=1-\left(1+\frac{G(x;\ \epsilon)}{\theta(\theta-1)(1-G(x;\ \epsilon))}\right)e^{-\frac{G(x;\ \epsilon)}{\theta(1-G(x;\ \epsilon))}},x>0,\theta\geq 2, \tag{1}$$

along with the corresponding PDF provided by (2);

$$z_{ORL-G}(x;\theta,\epsilon)=\frac{G(x;\ \epsilon)}{\theta^2(\theta-1)(1-G(x;\ \epsilon))^2}\left(\theta^2-2\theta+\frac{G(x;\ \epsilon)}{1-G(x;\ \epsilon)}\right)e^{-\frac{G(x;\ \epsilon)}{\theta(1-G(x;\ \epsilon))}}, \tag{2}$$

Equations (3,4) provide the PDF and CDF of the Kum-G model, respectively;

$$f(x;\alpha,\beta)=\alpha\beta z(x;\ \boldsymbol{\epsilon})[Z(x;\boldsymbol{\epsilon})]^{\alpha-1}[1-\{Z(x,\boldsymbol{\epsilon})\}^{\alpha}]^{\beta-1}, \tag{3}$$

$$F(x;\alpha,\beta)=1-[1-\{Z(x,\boldsymbol{\epsilon})\}^{\alpha}]^{\beta},x>0,\alpha>0,\beta>0, \tag{4}$$

Suppose the random variable X has a distribution within the KumORL-G family, then the CDF of the KumORL-G family in (5) is obtained by merging the CDFs of the ORL-G and Kum-G families, from which we obtain;

$$F_{KumORL-G}(x;\ \theta,\alpha,\beta,\boldsymbol{\epsilon})=1-\left(1-\left\{1-\left(1+\frac{G(x;\ \boldsymbol{\epsilon})}{\theta(\theta-1)(1-G(x;\ \boldsymbol{\epsilon}))}\right)e^{-\frac{G(x;\ \boldsymbol{\epsilon})}{\theta(1-G(x;\ \boldsymbol{\epsilon}))}}\right\}^{\alpha}\right)^{\beta},x>0 \tag{5}$$

The PDF in (6) corresponding to (5) is obtained by directly finding the first derivative of (5) or setting (1, 2) into (3);

$$f_{KumORL-G}(x;\ \theta,\alpha,\beta,\boldsymbol{\epsilon})=\frac{\alpha\beta g(x;\ \epsilon)}{\theta^2(\theta-1)(\overline{G}(x;\ \boldsymbol{\epsilon}\ ))^3}\left(\overline{G}(x;\boldsymbol{\epsilon})\left(\theta^2-2\theta\right)+G(x;\ \boldsymbol{\epsilon})\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}$$
$$\times\left(1-\left(1+\frac{G(x;\ \boldsymbol{\epsilon})}{\theta(\theta-1)\overline{G}(x;\ \boldsymbol{\epsilon})}\right)e^{-\frac{G(x;\ \boldsymbol{\epsilon})}{\theta\overline{G}(x;\ \boldsymbol{\epsilon})}}\right)^{\alpha-1}\left[1-\left(1-\left(1+\frac{G(x;\ \boldsymbol{\epsilon})}{\theta(\theta-1)\overline{G}(x;\ \boldsymbol{\epsilon}\ )}\right)e^{-\frac{G(x;\ \boldsymbol{\epsilon})}{\theta\overline{G}(x;\ \boldsymbol{\epsilon})}}\right)^{\alpha}\right]^{\beta-1},\quad x>0 \tag{6}$$

where $\alpha>0$ and $\beta>0$ are shape parameters, $\theta\geq 2,$ is a scale parameter and $\boldsymbol{\epsilon}$ is a $\boldsymbol{k}\times 1$ parameter vector of the baseline CDF.
The KumORL-G survival and hazard functions are respectively given by (7,8):

$$\mathrm{S}_{\mathrm{KumORL-G}}(\mathrm{x};\ \theta,\alpha,\beta,\boldsymbol{\epsilon})=\left(1-\left\{1-\left(1+\frac{\mathrm{G}(\mathrm{x};\ \boldsymbol{\epsilon})}{\theta(\theta-1)(1-\mathrm{G}(\mathrm{x};\ \boldsymbol{\epsilon}))}\right)\mathrm{e}^{-\frac{\mathrm{G}(\mathrm{x};\ \boldsymbol{\epsilon})}{\theta(1-\mathrm{G}(\mathrm{x};\ \boldsymbol{\epsilon}))}}\right\}^{\alpha}\right)^{\beta},x>0 \tag{7}$$

$$\mathrm{h}_{\mathrm{KumRL-G}}(\mathrm{x};\ \theta,\alpha,\beta,\boldsymbol{\epsilon})=\frac{\alpha\beta g(x,\boldsymbol{\epsilon})(1-G(x;\ \boldsymbol{\epsilon}))^{\frac{1}{\theta}-1}\left(\theta^2-2\theta-\log\left(1-G(x;\boldsymbol{\epsilon}\ )\right)\left\{1-\left(1-\frac{\log\left(1-G(x;\ \boldsymbol{\epsilon})\right)}{\theta(\theta-1)}\right)(1-G(x;\ \boldsymbol{\epsilon}))^{1/\theta}\right\}^{\alpha-1}\right.}{\theta^2(\theta-1)\left(1-\left\{1-\left(1-\frac{\log\left(1-G(x;\ \boldsymbol{\epsilon})\right)}{\theta(\theta-1)}\right)(1-G(x;\ \boldsymbol{\epsilon}))^{1/\theta}\right\}^{\alpha}\right)},x>0, \tag{8}$$

The KumORL-G family has the following important sub-families:

- The exponentiated ORL-G model is obtained when $\beta=1$;
- A power function ORL-G model is derived when $\alpha=1$; and
- the baseline model, that is the ORL-G family of distributions is obtained when $\alpha=\beta=1$.

2.2. The new Kumaraswamy Odd Ramos-Louzada Burr XII distribution

We introduce the Kumaraswamy Odd Ramos-Louzada Burr XII (KumORLBXII) model, a unique member of the KumORL-G family. Consider the Burr XII as the baseline model, its CDF and PDF are provided by the following, respectively;

$$G(x;\ \omega,\lambda) = 1 - \left(1+x^{\lambda}\right)^{-\omega} \text{and } g(x;\ \omega,\lambda) = \omega\lambda x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\omega-1}, x>0, \omega>0, \lambda>0,$$

then the CDF of KumORLBXII in (9) is expressed as;

$$F_{\text{KumORLBXII}}(\mathrm{x};\theta,\beta,\alpha,\omega,\lambda) = 1 - \left(1 - \left\{1 - \left(1 + \frac{\left(1+x^{\lambda}\right)^{\omega}-1}{\theta(\theta-1)}\right)e^{-\frac{\left(1+x^{\lambda}\right)^{\omega}-1}{\theta}}\right\}^{\alpha}\right)^{\beta}, x, \alpha, \beta, \omega, \lambda > 0, \theta \geq 2 \tag{9}$$

The PDF, survival function, and hazard function are respectively expressed in (10-12) as;

$$f_{\text{KumORLBXII}}(\mathrm{x};\theta,\beta,\alpha,\omega,\lambda) = \frac{\alpha\beta\omega\lambda x^{\lambda-1}\left(1+x^{\lambda}\right)^{\omega-1}}{\theta^{2}(\theta-1)}\left(\theta^{2}-2\theta+\left(1+x^{\lambda}\right)^{\omega}-1\right)e^{-\frac{\left(1+x^{\lambda}\right)^{\omega}-1}{\theta}}$$
$$\times\left(1-\left(1+\frac{\left(1+x^{\lambda}\right)^{\omega}-1}{\theta(\theta-1)}\right)e^{-\frac{\left(1+x^{\lambda}\right)^{\omega}-1}{\theta}}\right)^{\alpha-1}\left[1-\left(1-\left(1+\frac{\left(1+x^{\lambda}\right)^{\omega}-1}{\theta(\theta-1)}\right)e^{-\frac{\left(1+x^{\lambda}\right)^{\omega}-1}{\theta}}\right)^{\alpha}\right]^{\beta-1}, x>0 \tag{10}$$

$$S_{\text{KumORLBXII}}(\mathrm{x};\theta,\beta,\alpha,\omega,\lambda) = \left(1 - \left\{1 - \left(1 + \frac{\left(1+x^{\lambda}\right)^{\omega}-1}{\theta(\theta-1)}\right)e^{-\frac{\left(1+x^{\lambda}\right)^{\omega}-1}{\theta}}\right\}^{\alpha}\right)^{\beta}, x>0, \tag{11}$$

$$h_{\text{KumORLBXII}}(\mathrm{x};\ \theta,\beta,\alpha,\omega,\lambda) = \frac{\alpha\beta\omega\lambda x^{\lambda-1}\left(1+x^{\lambda}\right)^{\omega-1}\left(\theta^{2}-2\theta+\left(1+x^{\lambda}\right)^{\omega}-1\right)e^{-\frac{\left(1+x^{\lambda}\right)^{\omega}-1}{\theta}}\left(1-\left(1+\frac{\left(1+x^{\lambda}\right)^{\omega}-1}{\theta(\theta-1)}\right)e^{-\frac{\left(1+x^{\lambda}\right)^{\omega}-1}{\theta}}\right)^{\alpha-1}}{\theta^{2}(\theta-1)\left[1-\left(1-\left(1+\frac{\left(1+x^{\lambda}\right)^{\omega}-1}{\theta(\theta-1)}\right)e^{-\frac{\left(1+x^{\lambda}\right)^{\omega}-1}{\theta}}\right)^{\alpha}\right]}, x > 0, \tag{12}$$

The graphs in Fig. 1 show the KumORLBXII distribution, taking various "decreasing" and "unimodal" shapes including "left skewed" (green), "almost symmetric" (black), "right skewed" (red), and "reversed -J" (purple/blue) shapes. The hazard rate function graphs are shown in Fig. 2, with some well-behaved "decreasing" (purple), "reversed-J" (black), "bathtub" (red), and "bathtub" followed by "inverted bathtub" (green/blue) shapes. The shapes of these graphs of KumORLBXII's PDF and hazard function demonstrate that the model is very flexible and possesses the ability to model data of any shape with monotonic and non-monotonic failure rates.

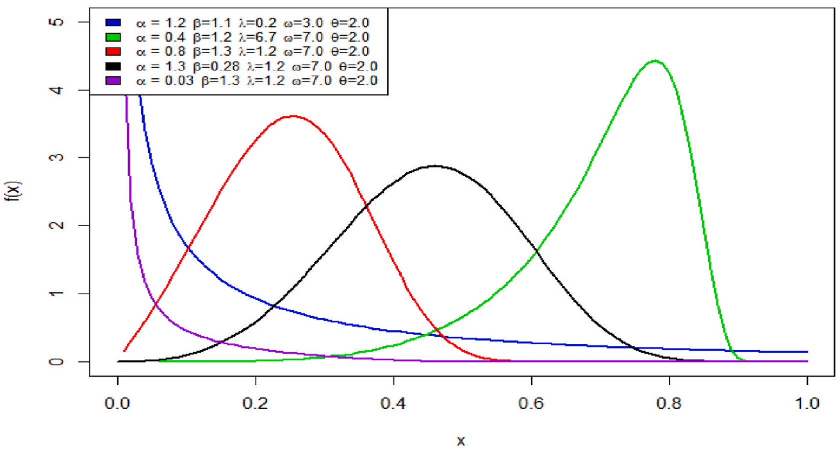

Fig. 1. KumORLBXII PDF graphs with various values of the parameters.

Table 1 shows some important sub-models of the KumORLBXII which are relevant for modeling data from various life phenomena. Each parameter assumes a value of 1, where for example, the following family members are obtained: ORLBXII, for $\alpha, \beta = 1$ [27]; GRL, for $\alpha, \beta, \omega = 1$ [28]; RL, for $\alpha, \beta, \omega, \lambda = 1$ [28]; KumGRL, for $\omega = 1$; ExORLLx, for $\beta, \lambda = 1$; and KumRL, for $\omega, \lambda = 1$.

3. Useful expansion and relevant statistical characteristics

In this section, a useful expansion of the new probability model is obtained to aid in deriving certain essential statistical features of the KumORL-G and its sub-model.

3.1. Expansion of the KumORL-G density

Consider the following series expansion, for $|z| < 1$ and $\beta > 0$,

$$(1-z)^{\beta-1} = \sum_{q=0}^{\infty} \frac{(-1)^q \Gamma(\beta)}{\Gamma(\beta-q)q!} z^q \tag{13}$$

$$e^{-\beta z} = \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k}{k!} z^k \tag{14}$$

Applying (13) thrice followed by (6-14), we obtain the expression in (15) after some simplifications

$$\begin{aligned} f_{\text{KumORL}-G}(x;\ \theta, \alpha, \beta, \epsilon) = {} & \frac{\alpha\beta g(x;\ \epsilon)}{\theta^2(\theta-1)\left(\overline{G}(x;\ \epsilon)\right)^3} \left(\overline{G}(x;\epsilon)\left(\theta^2-2\theta\right) + G(x;\ \epsilon)\right) \times \sum_{q=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty} \\ & \times \sum_{k=0}^{\infty} \frac{(-1)^{k+m+q}\Gamma(\beta)\Gamma(\alpha(q+1))\Gamma(1+m)(m+1)^k}{\Gamma(\beta-q)q!\Gamma(\alpha(q+1)-m)m!\Gamma(1+m-l)l!\theta^k k!\theta^l(\theta-1)^l} \frac{(G(x;\ \epsilon))^{k+l}}{\left(\overline{G}(x;\ \epsilon)\right)^{k+l}} \end{aligned} \tag{15}$$

Removing brackets and applying (13) twice, followed by substituting $G(x;\ \epsilon) = 1 - (1 - G(x;\ \epsilon))$, and applying (13) again with some simplifications becomes;

$$\begin{aligned} & f_{\text{KumORL}-G}(x; \theta, \alpha, \beta, \epsilon) \\ & = \left(\frac{\alpha\beta(\theta^2-2\theta)}{\theta^2(\theta-1)} \sum_{j=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{v=0}^{\infty}\sum_{u=0}^{\infty} \binom{l+k+j}{u}\binom{u}{v} \right. \\ & \quad \times \frac{(-1)^{m+k+q+u+v}\Gamma(\beta)\Gamma(\alpha(1+q))\Gamma(m+1)(m+1)^k\Gamma(j+k+l+2)}{\Gamma(\beta-q)q!\Gamma(\alpha(1+q)-m)m!\Gamma(m+1-l)l!\theta^k k!\theta^l(\theta-1)^l\Gamma(l+k+2)j!} g(x;\ \epsilon)(G(x;\ \epsilon))^v \\ & \quad + \frac{\alpha\beta}{\theta^2(\theta-1)} \sum_{j=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{v=0}^{\infty}\sum_{u=0}^{\infty} \binom{l+k+j+1}{u}\binom{u}{v} \\ & \quad \left. \times \frac{(-1)^{m+k+q+u+v}\Gamma(\beta)\Gamma(\alpha(1+q))\Gamma(m+1)(m+1)^k\Gamma(j+k+l+3)}{\Gamma(\beta-q)q!\Gamma(\alpha(1+q)-m)m!\Gamma(m+1-l)l!\theta^k k!\theta^l(\theta-1)^l\Gamma(l+k+3)j!} g(x;\ \epsilon)(G(x;\ \epsilon))^v \right) \end{aligned} \tag{16}$$

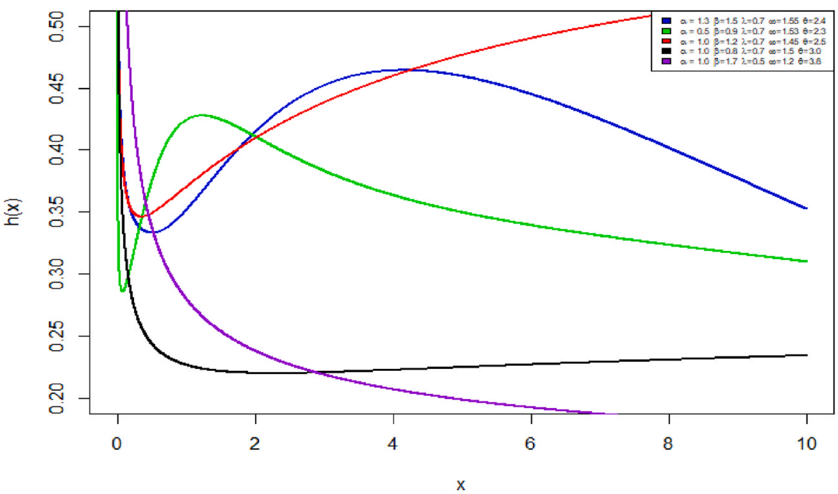

Fig. 2. Hazard rate function plots of KumORLBXII with various values of the parameters and their corresponding monotonic and non-monotonic graphs.

Table 1
An overview of the KumORLBXII distribution's sub-models.

S/N	α	β	θ	ω	λ	Name of Distribution
1	1	1	–	–	–	Odd Ramos-Louzada Burr XII (ORLBXII) [27]
2	–	1	–	–	–	Exponentiated Odd Ramos-Louzada Burr XII (ExORLBXII)
3	–	–	–	1	–	Kumaraswamy Generalized Ramos-Louzada (KumGRL)
4	–	–	–	–	1	Kumaraswamy Odd Ramos-Louzada Lomax (KumORLLx)
5	–	1	–	–	1	Exponentiated Odd Ramos-Louzada Lomax (ExORLLx)
6	–	1	–	1	–	Exponentiated Generalized Ramos-Louzada (ExGRL)
7	–	–	–	1	1	Kumaraswamy Ramos-Louzada (KumRL)
8	–	1	–	1	1	Exponentiated Ramos-Louzada (ERL)
9	1	1	–	1	–	Generalized Ramos-Louzada (GRL) [29]
10	1	1	–	1	1	Ramos-Louzada (RL) [28]

The useful expansion of the KumORL-G density is obtained in (17) after simplifying (16):

$$\mathrm{f}_{\mathrm{KumORL-G}}(\mathrm{x};\ \theta,\alpha,\beta,\boldsymbol{\epsilon}) = \sum_{j=0}^{\infty}\sum_{q=o}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{v=0}^{\infty}\sum_{u=0}^{\infty}\varpi_{jklmquv}g(x;\ \boldsymbol{\epsilon})(G(x;\ \boldsymbol{\epsilon}))^{v}, \tag{17}$$

where .

$$\varpi_{jklmquv} = \alpha\beta\left(\frac{(\theta^2-2\theta)}{\theta^2(\theta-1)}\binom{j+k+l}{u}\frac{\Gamma(j+k+l+2)}{\Gamma(k+l+2)}+\frac{1}{\theta^2(\theta-1)}\binom{l+k+j+1}{u}\frac{\Gamma(j+l+k+3)}{\Gamma(l+k+3)}\right)$$
$$\times\binom{u}{v}\frac{(-1)^{m+k+q+u+v}\Gamma(\beta)\Gamma(\alpha(1+q))\Gamma(1+m)(m+1)^k}{\Gamma(\beta-q)q!\Gamma(\alpha(q+1)-m)m!\Gamma(1+m-l)l!\theta^k k!\theta^l(\theta-1)^l j!}$$

Expressing (17) in the form of exponentiated-G (exp-G) density function, we have (18):

$$\mathrm{f}_{\mathrm{KumORL-G}}(\mathrm{x};\ \theta,\alpha,\beta,\boldsymbol{\epsilon}) = \sum_{j=0}^{\infty}\sum_{q=o}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{v=0}^{\infty}\sum_{u=0}^{\infty}\varpi'_{jklmquv}h_{v+1}(x;\ \boldsymbol{\epsilon}), \tag{18}$$

where $\varpi'_{jklmquv} = \frac{\varpi_{jklmquv}}{v+1}$ and $h_{v+1}(x;\ \boldsymbol{\epsilon}) = (1+v)g(x;\ \boldsymbol{\epsilon})(G(x;\ \boldsymbol{\epsilon}))^{v}$, is the exp-G density with the power parameter $v+1$. The expansion of the KumORLBXII density is subsequently obtained in (19) by substituting the BXII density into (17) which gives the following result:

$$f_{\mathrm{KumORLBXII}}(\mathrm{x};\theta,\alpha,\beta,\omega,\lambda) = \omega\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{q=o}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{v=0}^{\infty}\sum_{u=0}^{\infty}\varphi_{ijklmquv}x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\omega(1+i)-1}, \tag{19}$$

where $\varphi_{ijklmquv} = \binom{v}{i}(-1)^i\varpi_{jklmquv}$.

3.2. Relevant statistical characteristics

We now discuss the relevant statistical characteristics of the KumORL-G along with its sub-model. Among them are the quantile function, moments, Renyi entropy, order statistics, and probability-weighted moments of the new model.

- **Quantile function**

The inverse of a distribution function is the quantile function. The KumORL-G family's quantile function is determined as follows: Let $\mathrm{F}_{\mathrm{KumORL-G}}(Q(p),\theta,\alpha,\beta,\boldsymbol{\epsilon}) = p, 0 \le p \le 1$, solving for $Q(p)$ gives the following expression in (20) for the quantile function of the KumORL-G family,

$$Q(p) = \mathrm{G}^{-1}\left\{\frac{-\theta(\theta-1)-\theta W_{-}\left(-\left[(\theta-1)-(\theta-1)\left(1-(1-p)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right]e^{-(\theta-1)}\right)}{1-\theta(\theta-1)-\theta W_{-}\left(-\left[(\theta-1)-(\theta-1)\left(1-(1-p)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right]e^{-(\theta-1)}\right)}\right\}, \tag{20}$$

where $W_{-}(.)$ signifies the Lambert function's negative branch [30].

Various features of the KumORL-G random variable, such as simulation analysis, skewness, kurtosis, and median, can be determined using (20). From the CDF of the Burr XII distribution, $G^{-1}(x) = \left[(1-x)^{-\frac{1}{\omega}}-1\right]^{\frac{1}{\lambda}}$, and using (20), the KumORLBXII model's quantile function is obtained in (21).

$$Q(p) = \left[\left(1 - \theta^2 + \theta - \theta W_{-}\left(-(\theta-1)\left[1-\left(1-(1-p)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right]\right)e^{-(\theta-1)}\right)^{\omega} - 1\right]^{\frac{1}{\lambda}} \tag{21}$$

- **The r^{th} non-central moments**

By definition, the KumORL-G family's r^{th} non-central moment is provided by (22) as;

$$\mu_r' = E(X^r) = \int_0^\infty x^r f_{\text{KumORL-G}}(x;\ \theta,\alpha,\beta,\epsilon)dx, r = 1,2,3,\ldots \tag{22}$$

Using (17) and (22), we obtain;

$$\mu_r' = \sum_{j=0}^\infty \sum_{q=o}^\infty \sum_{m=0}^\infty \sum_{l=0}^\infty \sum_{k=0}^\infty \sum_{v=0}^\infty \sum_{u=0}^\infty \varpi_{jklmquv} \int_0^\infty x^r g(x;\ \epsilon)(G(x;\ \epsilon))^v dx,$$

from which we derive the KumORLBXII r^{th} non-central moments as;

$$\mu_r' = \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{q=o}^\infty \sum_{m=0}^\infty \sum_{l=0}^\infty \sum_{k=0}^\infty \sum_{u=0}^\infty \sum_{v=0}^\infty \varphi_{ijklmquv} \int_0^\infty \omega\lambda x^r x^{\lambda-1}(1+x^\lambda)^{-\omega(1+i)-1}dx. \tag{23}$$

To evaluate the integral in (23), we let:

$$z = (1+x^\lambda)^{-1} \Rightarrow x = (1-z)^{\frac{1}{\lambda}} z^{-\frac{1}{\lambda}}, as\ x \to 0, z \to 1, as\ x \to \infty, z \to 0$$

from which we have $dx = \frac{-dz}{\lambda x^{\lambda-1}(1+x^\lambda)^{-2}} = \frac{-dz}{\lambda x^{\lambda-1} z^2}$ and obtain:

$$\int_0^\infty \omega\lambda x^{r+\lambda-1}(1+x^\lambda)^{-\omega(i+1)-1}dx = -\int_1^0 \omega\lambda x^{\lambda-1}(1-z)^{\frac{r}{\lambda}} z^{-\frac{r}{\lambda}} z^{\omega(i+1)+1} \frac{dz}{\lambda x^{\lambda-1} z^2}$$

$$= \omega \int_0^1 (1-z)^{\frac{r}{\lambda}} z^{\omega(i+1)-\frac{r}{\lambda}} - 1 dz = \omega \boldsymbol{B}\left(\omega(i+1) - \frac{r}{\lambda}, \frac{r}{\lambda}+1\right)$$

and hence:

$$\mu_r' = \omega \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{q=o}^\infty \sum_{m=0}^\infty \sum_{l=0}^\infty \sum_{k=0}^\infty \sum_{v=0}^\infty \sum_{u=0}^\infty \varphi_{ijklmquv} \boldsymbol{B}\left(\omega(i+1) - \frac{r}{\lambda}, \frac{r}{\lambda}+1\right), \tag{24}$$

where $\boldsymbol{B}(b,a) = \int_0^1 (1-z)^{a-1} z^{b-1} dz$ is the complete beta function. By setting $r = 1,2,3,4$ into (24), we obtain the KumORLBXII first four moments. The following relations are utilized to compute the mean (μ), variance (σ^2), coefficient of skewness (CS), and coefficient of kurtosis (CK);

- $\mu = \mu_1'$;
- $\sigma^2 = \mu_2' - \mu^2$;
- CS $= \frac{\mu_3' - 3\mu_2' + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}}$;
- CK $= \frac{\mu_4' - 4\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2' - \mu^2)^2}$.

Now we note an approximately symmetric model occurs if $|\text{CS}| < \frac{1}{2}$, for extremely skewed model $|\text{CS}| \geq 1$, and if $\frac{1}{2} \leq |CS| < 1$, a slightly skewed model is obtained. About kurtosis, a mesokurtic model occurs if CK $= 3$, the light tails (platykurtic) model is obtained if CK < 3, and the greater tails (leptokurtic) if CK > 3. These measures of dispersion, skewness, and kurtosis were estimated numerically for various values of the parameters using R software and are presented in Table 2. Generally, the findings displayed in Table 2 indicate that as each parameter increases and the others are fixed, the following observations are made:

1. The mean decreases, with the parameter α having more effects on the mean, while the variance decreases.
2. The KumORLBXII density can be right- or left-skewed. Increasing α and θ, respectively, increases the skewness to the right or left.
3. The kurtosis can be less than or more than 3. Increasing α and θ increases the kurtosis whereas increasing ω decreases the amount of kurtosis in the KumORLBXII distribution.

These results show that the new KumORLBXII model can represent very skewed, approximately skewed, and essentially symmetric data with greater and lighter tails, depending on the values of the parameters used.

- **Incomplete moments**

The r^{th} incomplete moment $m_r(t)$ of the KumORL-G family is defined in Eq. (25) by;

$$m_r(t) = \int_0^t x^r f_{\mathrm{KumORL-G}}(x;\ \theta, \alpha, \beta, \boldsymbol{\epsilon}) dx, r = 1, 2, \ldots \tag{25}$$

from (17, 25), we obtain the expression for the r^{th} incomplete moment of the KumORL-G family in (26):

$$m_r(t) = \sum_{j=0}^{\infty}\sum_{q=o}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{v=0}^{\infty}\sum_{u=0}^{\infty} \varpi_{jklmquv} \int_0^t x^r g(x;\ \boldsymbol{\epsilon})(G(x;\ \boldsymbol{\epsilon}))^v dx, \tag{26}$$

from which we obtain the r^{th} incomplete moment of KumORLBXII as;

$$m_r(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{q=o}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{v=0}^{\infty}\sum_{u=0}^{\infty} \varphi_{ijklmquv} \int_0^t \omega\lambda x^r x^{\lambda-1}\left(1 + x^\lambda\right)^{-\omega(1+i)-1} dx$$

Equation (27) is obtained as the r^{th} incomplete moment of KumORLBXII after simplifying the integral in the last expression;

$$m_r(t) = \omega \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{q=o}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{v=0}^{\infty}\sum_{u=0}^{\infty} \varphi_{ijklmquv} \boldsymbol{B}\left(t^\lambda; \omega(1+i) - \frac{r}{\lambda}, \frac{r}{\lambda} + 1\right), \tag{27}$$

where $\boldsymbol{B}(t; a, b) = \int_0^t (1-y)^{a-1} y^{b-1} dy$ is the incomplete Beta function of the second type. The first incomplete moment is found by setting $r = 1$;

- **Probability-weighted moments**

If a random variable X has $KumORL - G$ family of distributions, the probability-weighted moments (PWM), denoted by $\Delta_{r,p}$ can be calculated through the following relation in (28);

$$\Delta_{r,p} = E(X^r(F_{\mathrm{KumORL-G}}(x; \beta, \theta, \alpha, \boldsymbol{\epsilon}))^p) = \int_0^\infty x^r f_{\mathrm{KumORL-G}}(x; \beta, \theta, \alpha, \boldsymbol{\epsilon})(F_{\mathrm{KumORL-G}}(x; \beta, \theta, \alpha, \boldsymbol{\epsilon}))^p, \tag{28}$$

We simplify the expression $f_{\mathrm{KumORL-G}}(x; \beta, \theta, \alpha, \boldsymbol{\epsilon})(F_{\mathrm{KumORL-G}}(x; \beta, \theta, \alpha, \boldsymbol{\epsilon}))^p$ in a similar way (17) is obtained. Equation (29) is the final result of $\Delta_{r,p}$.

Table 2
Numerical results of the kurtosis, skewness, variance, and the mean of the KumORLBXII model for selected parametric values.

α	β	λ	ω	θ	μ_1'	σ^2	CS	CK
0.2	1.05	0.7	0.7	2.1	8.899184	733.68762	7.48931	105.14595
0.01					0.48650	43.40047	31.77536	1807.16462
0.005					0.24383	21.80009	44.87711	3599.59025
0.0001					4.88775×10^{-3}	0.43797	316.91518	1.793×10^5
0.5	1.3	0.5	1.8	2.0	1.56482	4.03860	2.42828	12.04041
		1.5			0.88941	0.33249	−0.13511	2.24043
		3.5			0.82157	0.19339	−1.07141	2.67789
		7.5			0.80646	0.16893	−1.37989	3.06954
0.9	1.1	12	0.7	2.1	1.11990	0.05641	−3.71152	17.90736
	1.3				1.11767	0.048099	−3.96412	20.58964
	5				0.53279	0.304298	0.083997	1.03454
	30				1.23736×10^{-4}	1.33496×10^{-4}	93.45876	8741.93441
0.01	2.0	0.5	0.9	2.5	8.11039	2368.56057	14.80418	399.37181
			2.0		0.28003	1.11807	6.28136	56.72254
			5.0		0.02242	5.67774×10^{-3}	4.58918	28.43350
			10.0		4.55137×10^{-3}	2.16813×10^{-4}	4.20367	23.53127
0.5	1.2	8.7	6.0	2.0	0.63817	0.135670	−1.12657	2.33118
				2.2	0.71627	0.092840	−1.84581	4.60051
				2.3	0.76291	0.06213	−2.56701	8.07004
				2.4	0.81706	0.02134	−4.48633	25.06834

$$\Delta_{r,p}=\sum_{j=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{v=0}^{\infty}\sum_{u=0}^{\infty}\sum_{t=0}^{p}\varpi^{*}_{jklmquvt}\int_{0}^{\infty}x^{r}g(x;\ \epsilon)(G(x;\ \epsilon))^{v}dx, \tag{29}$$

where

$$\varpi^{*}_{jklmquvt}=\alpha\beta\left(\frac{(\theta^{2}-2\theta)}{\theta^{2}(\theta-1)}\binom{j+k+l}{u}\frac{\Gamma(l+k+j+2)}{\Gamma(l+k+2)}+\frac{1}{\theta^{2}(\theta-1)}\binom{l+k+j+1}{u}\frac{\Gamma(l+k+j+3)}{\Gamma(l+k+3)}\right)$$

$$\times\binom{u}{v}\binom{p}{t}\frac{(-1)^{m+k+q+u+v+t}\Gamma(\beta t+\beta)\Gamma(\alpha(1+q))\Gamma(m+1)(m+1)^{k}}{\Gamma(\beta-q)q!\Gamma(\alpha(q+1)-m)m!\Gamma(m+1-l)l!\theta^{k}k!\theta^{l}(\theta-1)^{l}j!}$$

By setting the BXII densities in (29) we obtain the PWM of KumORLBXII in (30);

$$\Delta_{r,p}=\omega\lambda\sum_{j=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{v=0}^{\infty}\sum_{u=0}^{\infty}\sum_{i=0}^{\infty}\sum_{t=0}^{p}\varpi^{**}_{jklmquvti}\int_{0}^{\infty}x^{r+\lambda-1}\left(1+x^{\lambda}\right)^{-\omega i-\omega-1}dx, \tag{30}$$

$$\varpi^{**}_{jklmquvti}=\varpi^{*}_{jklmquvt}(-1)^{i}\binom{v}{i}.$$

Using the result of the integration in (23), we obtain (31) as the simplified form of (30);

$$\Delta_{r,p}=\omega\sum_{j=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{v=0}^{\infty}\sum_{u=0}^{\infty}\sum_{i=0}^{\infty}\sum_{t=0}^{p}\varpi^{**}_{jklmquvti}\boldsymbol{B}\left(\omega(i+1)-\frac{r}{\lambda},\frac{r}{\lambda}+1\right), \tag{31}$$

- **Order statistics**

The PDF of $KumORL-G$ family h^{th} order statistics of a random sample of size n is defined in (32) by;

$$f_{h:n}(x)=\frac{n!}{(h-1)!(n-h)!}f(\mathrm{x},\beta,\theta,\alpha,\boldsymbol{\epsilon})(\mathrm{F}_{\mathrm{KumORL-G}}(\mathrm{x},\beta,\theta,\alpha,\boldsymbol{\epsilon}))^{h-1}(1-\mathrm{F}_{\mathrm{KumORL-G}}(\mathrm{x},\beta,\theta,\alpha,\boldsymbol{\epsilon}))^{n-h}, \tag{32}$$

Applying the generalized binomial series to the expression in (32) gives;

$$f_{h:n}(x)=\frac{n!}{(h-1)!(n-h)!}\sum_{w=0}^{n-h}\binom{n-h}{w}(-1)^{w}\mathrm{f}_{\mathrm{KumORL-G}}(\mathrm{x};\ \beta,\theta,\alpha,\boldsymbol{\epsilon})(\mathrm{F}_{\mathrm{KumORL-G}}(\mathrm{x},\beta,\theta,\alpha,\boldsymbol{\epsilon}))^{w+h-1}, \tag{33}$$

By setting $w+h-1=p$ in (29), substituting the results into [31], and simplifying becomes;

$$f_{h:n}(x)=\sum_{j=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{v=0}^{\infty}\sum_{u=0}^{\infty}\sum_{w=0}^{n-h}\sum_{t=0}^{w+h-1}\varpi^{***}_{jklmquvwt}g(x;\ \epsilon)(G(x;\ \epsilon))^{v}, \tag{34}$$

$$\varpi^{***}_{jklmquvwt}=\alpha\beta\left(\frac{(\theta^{2}-2\theta)}{\theta^{2}(\theta-1)}\binom{l+k+j}{u}\frac{\Gamma(l+k+j+2)}{\Gamma(l+k+2)}+\frac{1}{\theta^{2}(\theta-1)}\binom{l+k+j+1}{u}\frac{\Gamma(l+k+j+3)}{\Gamma(l+k+3)}\right)$$

$$\times\frac{n!}{(h-1)!(n-h)!}\binom{u}{v}\binom{w+h-1}{t}\binom{n-h}{w}\frac{(-1)^{m+k+q+u+v+t+w}\Gamma(\beta t+\beta)\Gamma(\alpha(1+q))\Gamma(m+1)(m+1)^{k}}{\Gamma(\beta-q)q!\Gamma(\alpha(1+q)-m)m!\Gamma(m+1-l)l!\theta^{k}k!\theta^{l}(\theta-1)^{l}j!}$$

From (32), the h^{th} order statistics PDF of a random sample of size n from the KumORLBXII model can be represented by (35):

$$f_{h:n}(x)=\omega\lambda\sum_{j=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{u=0}^{\infty}\sum_{i,v=0}^{\infty}\sum_{w=0}^{n-h}\sum_{t=0}^{w+h-1}\varpi^{****}_{ijklmquvwt}x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\omega(1+i)-1}, \tag{35}$$

where $\varpi^{****}_{ijklmquvwt}=(-1)^{i}\binom{v}{i}\varpi^{***}_{jklmquvwt}$.

- **Entropy measure**

The amount of unpredictability in the KumORL-G random variable is quantified by using an entropy. The Renyi entropy of KumORL-G denoted by $I_{R}(\delta)$ is defined as;

$$I_{R}(\delta)=\frac{1}{1-\delta}log\int_{0}^{\infty}f^{\delta}_{KumORL-G}(\mathrm{x};\ \theta,\alpha,\beta,\boldsymbol{\epsilon})dx,\delta>0\ \text{and}\ \delta\neq1, \tag{36}$$

We first obtain an expansion $f^{\delta}_{KumORL-G}(x;\ \theta,\alpha,\beta,\epsilon)$, which is an important function in most formulae for entropy. Setting the density function of KumORL-G into (36) and applying the binomial series twice, we obtain;

$$f^{\delta}_{KumORL-G}(x;\ \theta,\alpha,\beta,\boldsymbol{\epsilon}) = \left[\frac{(\alpha\beta)^{\delta}(g(x,\boldsymbol{\epsilon}))^{\delta}}{\theta^{2}\delta(\theta-1)^{\delta}(1-G(x;\boldsymbol{\epsilon}))^{2\delta}}\left(\theta^{2}-2\theta+\frac{G(x;\boldsymbol{\epsilon})}{1-G(x;\boldsymbol{\epsilon})}\right)^{\delta}\sum_{l=0}^{\infty}\right.$$

$$\left.\times\sum_{m=0}^{\infty}\binom{\delta(\beta-1)}{m}\binom{\alpha m+\delta(\alpha-1)}{l}(-1)^{m+l}\left(1+\frac{G(x;\ \boldsymbol{\epsilon})}{\theta(\theta-1)\overline{G}(x;\ \boldsymbol{\epsilon})}\right)^{l}e^{-(l+\delta)\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}\right]$$

Again applying (14) and binomial series expansion followed by some simplifications, we obtain the expression in (37) for $f^{\delta}_{KumORL-G}(x;\ \theta,\alpha,\beta,\boldsymbol{\epsilon})$ as;

$$f^{\delta}_{KumORL-G}(x;\ \theta,\alpha,\beta,\boldsymbol{\epsilon}) = \frac{(\alpha\beta)^{\delta}}{\theta^{2\delta}(\theta-1)^{\delta}}\sum_{h=0}^{\infty}\sum_{i,j,k,l,m=0}^{\infty}\binom{\delta(\beta-1)}{m}\binom{\alpha m+\delta(\alpha-1)}{l}\binom{l}{k}\binom{\delta}{i}\binom{i+j+k+2\delta+h-1}{h}$$
$$\times\frac{(-1)^{m+l+j}(\delta+l)^{j}(\theta^{2}-2\theta)^{\delta-i}}{\theta^{k+j}(\theta-1)^{k}j!}(g(x,\boldsymbol{\epsilon}))^{\delta}(G(x;\ \boldsymbol{\epsilon}))^{i+j+k+h} \tag{37}$$

Finally, $I_R(\delta)$ can be expressed as;

$$I_R(\delta)=\frac{1}{1-\delta}log\left(\frac{(\alpha\beta)^{\delta}}{\theta^{2\delta}(\theta-1)^{\gamma}}\varphi_{i,j,k,l,m,h}\int_{0}^{\infty}(g(x,\boldsymbol{\epsilon}))^{\delta}(G(x;\ \boldsymbol{\epsilon}))^{i+j+k+h}dx\right), \tag{38}$$

where;

$$\varphi_{i,j,k,l,m,h}=\sum_{h=0}^{\infty}\sum_{i,j,k,l,m=0}^{\infty}\binom{\delta(\beta-1)}{m}\binom{\alpha m+\delta(\alpha-1)}{l}\binom{l}{k}\binom{\delta}{i}\binom{i+j+k+2\delta+h-1}{h}\frac{(-1)^{m+l+j}(\delta+l)^{j}(\theta^{2}-2\theta)^{\delta-i}}{\theta^{k+j}(\theta-1)^{k}j!}$$

We now obtain the Renyi entropy formula for KumORLBXII by substituting the Burr XII densities into (38) and simplifying as follows:

$$\int_{0}^{\infty}(g(x,\boldsymbol{\epsilon}))^{\delta}(G(x;\ \boldsymbol{\epsilon}))^{i+j+k+h}dx=(\omega\lambda)^{\delta}\sum_{p=0}^{\infty}(-1)^{p}\binom{i+j+k+h}{p}\int_{0}^{\infty}x^{\delta(\lambda-1)}(1+x^{\lambda})^{-\omega(p+\delta)-\delta}dx,$$

Let $y=(1+x^{\lambda})^{-1}, x=(1-y)^{\frac{1}{\lambda}}y^{-\frac{1}{\lambda}}, as\ x\to0, y\to1, as\ x\to\infty, y\to0, dx=-\frac{dy}{\lambda x^{\lambda-1}y^{2}}$.

$$\int_{0}^{\infty}(g(x,\boldsymbol{\epsilon}))^{\delta}(G(x;\ \boldsymbol{\epsilon}))^{i+j+k+h}dx=\frac{(\omega\lambda)^{\delta}}{\lambda}\sum_{p=0}^{\infty}(-1)^{p}\binom{i+j+k+h}{p}\int_{0}^{1}(1-y)^{\delta-\frac{\delta}{\lambda}+\frac{1}{\lambda}-1}y^{\omega(p+\delta)+\frac{\delta}{\lambda}-\frac{1}{\lambda}-1}dy$$

$$=\frac{(\omega\lambda)^{\delta}}{\lambda}\sum_{p=0}^{\infty}(-1)^{p}\binom{i+j+k+h}{h}\boldsymbol{B}\left(\delta-\frac{1}{\lambda}(\delta-1),\omega(p+\delta)+\frac{1}{\lambda}(\delta-1)\right)$$

and hence for our new KumORLBXII model, we have the Renyi entropy expression in (39);

$$I_R(\delta)=\frac{1}{1-\delta}log\left(\frac{(\alpha\beta\omega\lambda)^{\delta}}{\theta^{2\delta}\lambda(\theta-1)^{\delta}}\varphi^{*}_{i,j,k,l,m,h,p}\boldsymbol{B}\left(\delta-\frac{1}{\lambda}(\delta-1),\omega(p+\delta)+\frac{1}{\lambda}(\delta-1)\right)\right), \tag{39}$$

where $\varphi^{*}_{i,j,k,l,m,h,p}=\sum_{p=0}^{\infty}(-1)^{p}\binom{i+j+k+h}{p}\varphi_{i,j,k,l,m,h}$.

Table 3 displays the numerical solutions of (39) that were obtained using the R software based on some selected parameter values. The positive and negative values indicate the flexibility in the randomness of KumORLBXII distribution.

4. Characterizations of KumORL-G

The two primary characterizations of the KumORL-G model that are covered in this section hinge on the hazard function and the ratio of two truncated moments.

4.1. Characterization based on the ratio of two truncated moments

The KumORL-G model is characterized using a simple relationship between two truncated moments based on the Glanzel theorem (see Appendix A1, Theorem A).

Proposition 1. *Given a continuous random variable* $X:\Omega\rightarrow(0,\infty)$, *and let*

$$v_1(x)=\left[1-\left(1-\left(1+\frac{G(x;\ \epsilon)}{\theta(\theta-1)\overline{G}(x;\ \epsilon)}\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}\right)^{\alpha}\right] \text{ and } v_2(x)=v_1(x)\left[1-\left(1-\left(1+\frac{G(x;\ \epsilon)}{\theta(\theta-1)\overline{G}(x;\ \epsilon)}\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}\right)^{\alpha}\right]^{\beta}, x>0.$$

The random variable X has PDF in (6) provided the function $\Lambda(x)$ specified in Theorem A has the expression in (40);

$$\Lambda(x)=\frac{\beta+1}{2\beta+1}\left[1-\left(1-\left(1+\frac{G(x;\ \epsilon)}{\theta(\theta-1)\overline{G}(x;\ \epsilon)}\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}\right)^{\alpha}\right]^{\beta}, x>0, \tag{40}$$

Proof.
Suppose X has PDF in equation (6), then;

$$(1-F_{KumORL-G}(x,\theta,\epsilon))E(v_1(x)|\ X\geq x)=\frac{\beta}{\beta+1}\left[1-\left(1-\left(1+\frac{G(x;\ \epsilon)}{\theta(\theta-1)\overline{G}(x;\ \epsilon)}\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}\right)^{\alpha}\right]^{\beta+1}$$

$$(1-F_{KumORL-G}(x,\theta,\epsilon))E(v_2(x)|\ X\geq x)=\frac{\beta}{2\beta+1}\left[1-\left(1-\left(1+\frac{G(x;\ \epsilon)}{\theta(\theta-1)\overline{G}(x;\ \epsilon)}\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}\right)^{\alpha}\right]^{2\beta+1}$$

and so,

$$\Lambda(x)=\frac{(1-F_{KumORL-G}(x,\theta,\epsilon))E(v_2(x)|\ X\geq x)}{(1-F_{KumORL-G}(x,\theta,\epsilon))E(v_1(x)|\ X\geq x)}=\frac{\beta+1}{2\beta+1}\left[1-\left(1-\left(1+\frac{G(x;\ \epsilon)}{\theta(\theta-1)\overline{G}(x;\ \epsilon)}\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}\right)^{\alpha}\right]^{\beta}$$

as

$$\Lambda'(x)=-\frac{\beta+1}{2\beta+1}\frac{\alpha\beta g(x;\ \epsilon)}{\theta^2(\theta-1)(\overline{G}(x;\ \epsilon))^3}\left(\overline{G}(x;\epsilon)(\theta^2-2\theta)+G(x;\ \epsilon)\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}$$

Table 3
Numerical results of Renyi entropy for KumORLBXII distribution for some chosen parameter values.

Parameter					Entropy		
α	β	λ	ω	θ	$\gamma=1.5$	$\gamma=6.0$	$\gamma=12.5$
0.05	2.8	3.0	2.8	2.0	0.0211	0.4434	1.1222
0.5					0.0895	0.7458	1.7653
1.5					−0.1951	−0.4018	−0.6289
5.0					−0.9460	−3.3919	−6.8848
0.5	0.5	0.3	0.8	2.0	−0.4802	−1.4980	−2.5060
	1.5				−0.3415	−1.1688	−1.8229
	5.0				−0.2514	−1.7683	−3.1003
	20.0				−0.2359	−2.6025	−5.8418
0.7	3.8	0.5	2.8	2.0	−0.0105	0.1824	0.5345
		3.5			0.0301	0.5835	1.4558
		6.5			0.0598	0.8861	2.1538
		10.5			0.0843	1.1346	2.7263
1.7	1.2	0.5	2.8	2.0	−0.0403	−0.0921	−0.0850
			5.0		0.0413	0.6837	1.6660
			10.0		0.1293	1.5467	3.6436
			15.0		0.1781	2.0296	4.7533
10.0	2.0	1.5	5.0	2.0	−1.5023	−5.3673	−10.8823
				2.1	−1.4439	−5.1121	−10.3472
				2.2	−1.4537	−5.1050	−10.3179
				2.5	−1.5794	−5.5356	−11.1759

$$\times\left(1-\left(1+\frac{G(x;\ \epsilon)}{\theta(\theta-1)\overline{G}(x;\ \epsilon)}\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}\right)^{\alpha-1}\left[1-\left(1-\left(1+\frac{G(x;\ \epsilon)}{\theta(\theta-1)\overline{G}(x;\ \epsilon)}\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}\right)^{\alpha}\right]^{\beta-1},$$

Conversely, if $\Lambda(x)$ is defined as above, then;

$$\Lambda(x)v_1(x)-v_2(x)=-\frac{\beta}{2\beta+1}v_1(x)\left[1-\left(1-\left(1+\frac{G(x;\ \epsilon)}{\theta(\theta-1)\overline{G}(x;\ \epsilon)}\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}\right)^{\alpha}\right]^{\beta}$$

In contrast, if $\Lambda(x)$ is defined as previously, then;

$$s'(x)=\frac{\Lambda'(x)v_1(x)}{\Lambda(x)v_1(x)-v_2(x)}=$$

$$\frac{\beta+1}{\beta}\frac{\alpha\beta g(x;\ \epsilon)}{\theta^2(\theta-1)(\overline{G}(x;\ \epsilon))^3}\left(\overline{G}(x;\epsilon)\left(\theta^2-2\theta\right)+G(x;\ \epsilon)\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}$$
$$\times\left(1-\left(1+\frac{G(x;\ \epsilon)}{\theta(\theta-1)\overline{G}(x;\ \epsilon)}\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}\right)^{\alpha-1}\left[1-\left(1-\left(1+\frac{G(x;\ \epsilon)}{\theta(\theta-1)\overline{G}(x;\ \epsilon)}\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}\right)^{\alpha}\right]^{-1},$$

and hence;

$$s(x)=-(\beta+1)ln\left[1-\left(1-\left(1+\frac{G(x;\ \epsilon)}{\theta(\theta-1)\overline{G}(x;\ \epsilon)}\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}\right)^{\alpha}\right]$$

$$e^{-s(x)}=\left[1-\left(1-\left(1+\frac{G(x;\ \epsilon)}{\theta(\theta-1)\overline{G}(x;\ \epsilon)}\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}\right)^{\alpha}\right]^{\beta+1},$$

As a result of Theorem A, X has a PDF in (6).

Corollary 1. *Given a continuous random variable $X:\Omega\rightarrow(0,\infty)$ and a function $v_1(x)$ as specified in Proposition 1, the PDF of X is (6) provided the functions $v_2(x)$ and $\Lambda(x)$ given in Theorem A satisfy the differential equation;*

$$\frac{\Lambda'(x)v_1(x)}{\Lambda(x)v_1(x)-v_2(x)}=\frac{\beta+1}{\beta}\frac{\alpha\beta g(x;\ \epsilon)}{\theta^2(\theta-1)(\overline{G}(x;\ \epsilon))^3}\left(\overline{G}(x;\epsilon)\left(\theta^2-2\theta\right)+G(x;\ \epsilon)\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}$$
$$\times\left(1-\left(1+\frac{G(x;\ \epsilon)}{\theta(\theta-1)\overline{G}(x;\ \epsilon)}\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}\right)^{\alpha-1}\left[1-\left(1-\left(1+\frac{G(x;\ \epsilon)}{\theta(\theta-1)\overline{G}(x;\ \epsilon)}\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}\right)^{\alpha}\right]^{-1},$$

The differential equation in *corollary 1* has a general solution represented as;

$$\Lambda(x)=\left[1-\left(1-\left(1+\frac{G(x;\ \epsilon)}{\theta(\theta-1)\overline{G}(x;\ \epsilon)}\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}\right)^{\alpha}\right]^{-\beta-1}$$

$$\times\int\left[\begin{array}{c}-\frac{(\beta+1)\alpha g(x;\ \epsilon)}{\theta^2(\theta-1)(\overline{G}(x;\ \epsilon))^3}\left(\overline{G}(x;\epsilon)\left(\theta^2-2\theta\right)+G(x;\ \epsilon)\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}\left(1-\left(1+\frac{G(x;\ \epsilon)}{\theta(\theta-1)\overline{G}(x;\ \epsilon)}\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}\right)^{\alpha-1}\\ \times\left[1-\left(1-\left(1+\frac{G(x;\ \epsilon)}{\theta(\theta-1)\overline{G}(x;\ \epsilon)}\right)e^{-\frac{G(x;\ \epsilon)}{\theta\overline{G}(x;\ \epsilon)}}\right)^{\alpha}\right]^{\beta}(v_1(x))^{-1}v_2(x)\end{array}\right]dx+C,$$

Where C is a constant. We would like to emphasize that Proposition 1 with C=0 has one set of functions that fulfill the aforementioned differential equation. There are alternative triplets of $(v_1(x),v_2(x),\Lambda(x))$ that satisfy the criteria of Theorem A in Appendix A1.

Considering the KumORLBXII distribution, suppose $v_1(x) = 1 - \left(1 - \left(1 + \frac{(1+x^{\lambda})^{\omega}-1}{\theta(\theta-1)}\right)e^{-\frac{(1+x^{\lambda})^{\omega}-1}{\theta}}\right)^{\alpha}, x > 0$ and $v_2(x) = \left(1 - \left\{1 - \left(1 + \frac{(1+x^{\lambda})^{\omega}-1}{\theta(\theta-1)}\right)e^{-\frac{(1+x^{\lambda})^{\omega}-1}{\theta}}\right\}^{\alpha}\right)^{\beta}, x > 0$, then X follows the KumORLBXII distribution provided $\Lambda(x)$ in Proposition 1 is

$$\Lambda(x) = \frac{\beta+1}{2\beta+1}\left(1 - \left\{1 - \left(1 + \frac{(1+x^{\lambda})^{\omega}-1}{\theta(\theta-1)}\right)e^{-\frac{(1+x^{\lambda})^{\omega}-1}{\theta}}\right\}^{\alpha}\right)^{\beta}, x > 0,$$

4.2. Characterization of KumORL-G under hazard rate function

A twice differentiable distribution function, G(x) with hazard function rate h(x) fulfills the first-order differential equation in (41);

$$\frac{g'(x)}{g(x)} = \frac{h'(x)}{h(x)} - h(x), \tag{41}$$

It needs to be emphasized that the hazard rate function in (41) is the sole differential equation available for many univariate continuous distributions.

Proposition 2. *Suppose $X : \Omega \to (0, \infty)$ is a random variable that is continuous from the KumORL-G family, then (6) is the PDF of X if and only if its hazard rate function $h_{KumORL-G}(x;\theta,\alpha,\beta,\epsilon)$ fulfills the differential equation that follows.*

$$h'_{\text{KumORL-G}}(x;\theta,\alpha,\beta,\epsilon) + \frac{g(x;\epsilon)}{\theta(\overline{G}(x;\epsilon))^2}h_{\text{KumORL-G}}(x;\theta,\alpha,\beta,\epsilon) = \frac{\alpha\beta}{\theta^2(\theta-1)}e^{-\frac{G(x;\epsilon)}{\theta\overline{G}(x;\epsilon)}}\frac{d}{dx}\left\{\frac{g(x;\epsilon)\left(\theta^2 - 2\theta + \frac{G(x;\epsilon)}{\overline{G}(x;\epsilon)}\right)\left(1 - \left(1 + \frac{G(x;\epsilon)}{\theta(\theta-1)\overline{G}(x;\epsilon)}\right)e^{-\frac{G(x;\epsilon)}{\theta\overline{G}(x;\epsilon)}}\right)^{\alpha-1}}{(\overline{G}(x;\epsilon))^2\left[1 - \left(1 - \left(1 + \frac{G(x;\epsilon)}{\theta(\theta-1)\overline{G}(x;\epsilon)}\right)e^{-\frac{G(x;\epsilon)}{\theta\overline{G}(x;\epsilon)}}\right)^{\alpha}\right]}\right\},$$

for $x > 0$, and with the initial condition : $h_{\text{KumORL-G}}(0) = 0$ for $\theta \geq 2$ (42)

Proof: The differential equation in (42) exists if [6] represents the PDF of KumORL-G random variable. Now, considering the validity of the differential equation, then:

$$\frac{d}{dx}\left(e^{\frac{G(x;\epsilon)}{\theta\overline{G}(x;\epsilon)}}h_{\text{KumORL-G}}(x;\theta,\epsilon)\right) = \frac{\alpha\beta}{\theta^2(\theta-1)}\frac{d}{dx}\left\{\frac{g(x;\epsilon)\left(\theta^2 - 2\theta + \frac{G(x;\epsilon)}{\overline{G}(x;\epsilon)}\right)\left(1 - \left(1 + \frac{G(x;\epsilon)}{\theta(\theta-1)\overline{G}(x;\epsilon)}\right)e^{-\frac{G(x;\epsilon)}{\theta\overline{G}(x;\epsilon)}}\right)^{\alpha-1}}{(\overline{G}(x;\epsilon))^2\left[1 - \left(1 - \left(1 + \frac{G(x;\epsilon)}{\theta(\theta-1)\overline{G}(x;\epsilon)}\right)e^{-\frac{G(x;\epsilon)}{\theta\overline{G}(x;\epsilon)}}\right)^{\alpha}\right]}\right\},$$

from which we obtain the hazard function of KumORL-G in (8).

The proof ends.

5. Parameter estimation and simulation analysis of KumORL-G

We now examine and implement the ML estimators of the KumORL-G distribution for both the complete and right-censored data in this section. A simulation analysis is also performed to evaluate the performances of the ML estimators.

5.1. Estimation of parameters for complete and censored datasets

5.1.1. Parameter estimation for the complete datasets

Let $x_1, x_2, \ldots, x_n$ represent a size n of a random sample drawn from the KumORL-G family described in (6). The log-likelihood function for the vector of parameter $\Omega = (x, \beta, \theta, \alpha, \epsilon)'$ is given in Eq. [43];

$$\ell(\Omega) = n\log\alpha + n\log\beta + \sum_{i=1}^{n}\log g(x_i;\ \epsilon) - 3\sum_{i=1}^{n}\log\overline{G}(x_i;\ \epsilon) - 2n\log\theta - n\log(\theta-1)$$

$$+\sum_{i=1}^{n}\log\left((\theta^2-2\theta)(\overline{G}(x_i;\ \epsilon)) + G(x_i,\epsilon)\right) - \frac{1}{\theta}\sum_{i=1}^{n}\frac{G(x_i,\epsilon)}{\overline{G}(x_i;\ \epsilon)}$$

$$+(\alpha-1)\sum_{i=1}^{n}\log\left(1-\left(1+\frac{G(x_i;\ \epsilon)}{\theta(\theta-1)\overline{G}(x_i;\ \epsilon)}\right)e^{-\frac{G(x_i;\ \epsilon)}{\theta\overline{G}(x_i;\ \epsilon)}}\right) + (\beta-1)\sum_{i=1}^{n}\log\left[1-\left(1-\left(1+\frac{G(x_i;\ \epsilon)}{\theta(\theta-1)\overline{G}(x_i;\ \epsilon)}\right)e^{-\frac{G(x_i;\ \epsilon)}{\theta\overline{G}(x_i;\ \epsilon)}}\right)^{\alpha}\right] \tag{43}$$

Calculating the partial derivatives of (43) relative to $\theta, \alpha, \beta,$ and ϵ leads to equations (44) to (47);

$$\frac{\partial\ell(\Omega)}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log A(x_i;\theta,\epsilon) - (\beta-1)\sum_{i=1}^{n}\frac{A^{\alpha}(x_i;\theta,\epsilon)\log A(x_i;\theta,\epsilon)}{1-A^{\alpha}(x_i;\theta,\epsilon)} \tag{44}$$

$$\frac{\partial\ell(\Omega)}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log(1-A^{\alpha}(x_i;\theta,\epsilon)), \tag{45}$$

$$\frac{\partial\ell(\Omega)}{\partial\theta} = \frac{1}{\theta^2}\sum_{i=1}^{n}\frac{G(x_i,\epsilon)}{\overline{G}(x;\ \epsilon)} - \frac{2n}{\theta} - \frac{n}{\theta-1} + \sum_{i=1}^{n}\frac{(2\theta-2)\overline{G}(x;\ \epsilon)}{\left((\theta^2-2\theta)\overline{G}(x;\ \epsilon) + G(x_i,\epsilon)\right)} + (\alpha-1)\sum_{i=1}^{n}\frac{A_{\theta}(x_i;\theta,\epsilon)}{A(x_i;\theta,\epsilon)}$$
$$-\alpha(\beta-1)\sum_{i=1}^{n}\frac{A^{\alpha-1}(x_i;\theta,\epsilon)A_{\theta}(x_i;\theta,\epsilon)}{1-A^{\alpha}(x_i;\theta,\epsilon)}, \tag{46}$$

$$\frac{\partial\ell(\Omega)}{\partial\epsilon} = \sum_{i=1}^{n}\frac{1}{g(x_i;\ \epsilon)}\ \frac{\partial g(x_i;\ \epsilon)}{\partial\epsilon} + 3\sum_{i=1}^{n}\frac{1}{\overline{G}(x;\ \epsilon)}\ \frac{\partial G(x_i;\ \epsilon)}{\partial\epsilon} + (1-(\theta^2-2\theta))\sum_{i=1}^{n}\frac{1}{(\theta^2-2\theta)(\overline{G}(x;\ \epsilon)) + G(x_i,\epsilon)}\ \frac{\partial G(x_i;\ \epsilon)}{\partial\epsilon}$$
$$-\frac{1}{\theta}\sum_{i=1}^{n}\frac{1}{(\overline{G}(x;\ \epsilon))^2}\ \frac{\partial G(x_i;\ \epsilon)}{\partial\epsilon} + (\alpha-1)\sum_{i=1}^{n}\frac{A_{\epsilon}(x_i;\theta,\epsilon)}{A(x_i;\theta,\epsilon)} - \alpha(\beta-1)\sum_{i=1}^{n}\frac{A^{\alpha-1}(x_i;\theta,\epsilon)A_{\epsilon}(x_i;\theta,\epsilon)}{1-A^{\alpha}(x_i;\theta,\epsilon)}, \tag{47}$$

where;

$$A(x_i;\theta,\epsilon) = \left(1-\left(1+\frac{G(x_i;\ \epsilon)}{\theta(\theta-1)\overline{G}(x_i;\ \epsilon)}\right)e^{-\frac{G(x_i;\ \epsilon)}{\theta\overline{G}(x_i;\ \epsilon)}}\right), A_{\theta}(x_i;\theta,\epsilon) = \frac{\partial A(x_i;\theta,\epsilon)}{\partial\theta} = A_{\theta}(x_i;\theta,\epsilon)$$

$$= -\left\{\frac{G(x_i;\ \epsilon)}{\theta^2\overline{G}(x_i;\ \epsilon)}e^{-\frac{G(x_i;\ \epsilon)}{\theta\overline{G}(x_i;\ \epsilon)}}\left(1-\frac{2\theta-1}{(\theta-1)^2} + \frac{G(x_i;\ \epsilon)}{\theta(\theta-1)\overline{G}(x_i;\ \epsilon)}\right)\right\}, A_{\epsilon}(x_i;\theta,\epsilon) = \frac{\partial A(x_i;\theta,\epsilon)}{\partial\epsilon}$$

$$A_{\epsilon}(x_i;\theta,\epsilon) = -\left\{\frac{1}{(\overline{G}(x_i;\ \epsilon))^2}\ \frac{\partial G(x_i;\ \epsilon)}{\partial\epsilon}e^{-\frac{G(x_i;\ \epsilon)}{\theta\overline{G}(x_i;\ \epsilon)}}\left[\frac{1}{\theta(\theta-1)} - \frac{1}{\theta}\left(1+\frac{G(x_i;\ \epsilon)}{\theta(\theta-1)\overline{G}(x_i;\ \epsilon)}\right)\right]\right\}$$

The ML estimates are calculated by equating each of (44) to (47) to zero and solving them simultaneously with numerical computations.

5.1.2. Parameter estimation for the censored datasets

Let $X = (X_1, X_2, \ldots, X_n)$ be a random sample from KumORL-G family of distributions with the parameter vector $\Omega = (x, \theta, \alpha, \beta, \epsilon)$ that can have right censored data over a fixed censoring time τ. Then, each X_i can be represented as (x_i, ϱ_i), where x_i is the failure time with censored observation and ϱ_i is the censoring index. If a failure occurs, $\varrho_i = 1$ and $\varrho_i = 0$, if censoring is observed. The likelihood function in (48) is obtained to suppose that the right censoring lacks information.

$$L(\Omega) = \prod_{i=1}^{n}\left(f_{KumORL-G}(x_i,\theta,\alpha,\beta,\epsilon)\right)^{\varrho_i}\left(S_{KumORL-G}(x_i,\theta,\alpha,\beta,\epsilon)\right)^{1-\varrho_i}, \tag{48}$$

The function for total log-likelihood can be obtained by;

Table 4
Results of the *AB, RMSE*, and *Av* simulations for KumORLBXII.

		I			II		
Parameter	*n*	*AB*	*RMSE*	*Av*	*AB*	*RMSE*	*Av*
α	10	7.9812	8.2145	15.2376	1.6424	1.2315	10.4521
	50	1.1151	2.7132	9.4821	1.4025	0.4982	2.5119
	150	0.2854	1.2975	1.4626	0.6217	0.2591	1.4530
	300	0.0232	0.3435	1.2987	0.3625	0.2171	0.8943
	500	0.0291	0.1023	1.2117	0.1530	0.0908	0.8175
	800	0.0092	0.0167	1.2081	0.0221	0.0152	0.8026
β	10	5.2132	12.2135	20.2317	3.2134	2.5673	25.2748
	50	2.2146	7.8704	17.4622	1.5623	1.5209	17.4207
	150	1.2574	3.1128	3.1146	0.5441	0.3681	3.4236
	300	0.7230	1.7113	1.3152	0.2522	0.1803	1.1237
	500	0.0228	0.8720	1.2318	0.0801	0.1017	0.8721
	800	0.0025	0.0215	1.2274	0.0126	0.0256	0.8243
θ	10	21.2312	10.2141	41.2201	20.2136	22.1133	55.1267
	50	18.2188	4.8712	30.1243	16.1772	12.3669	41.1243
	150	4.0125	1.3217	15.1019	2.8214	1.6432	20.5122
	300	1.5231	0.3723	9.2142	1.4521	0.5115	5.7957
	500	0.2210	0.0925	3.3421	0.2244	0.1232	3.7422
	800	0.0701	0.0054	2.7216	0.0196	0.0040	3.7218
λ	10	−0.8342	2.5600	18.0136	1.3466	3.1207	25.3216
	50	−0.5223	1.3743	14.4812	0.9647	1.3612	13.1923
	150	−0.1921	0.2388	3.4344	−0.4322	0.2389	6.2170
	300	−0.0117	0.1718	1.1231	−0.2171	0.1013	3.2435
	500	−0.0080	0.0921	0.3728	−0.0620	0.0250	2.6721
	800	−0.0016	0.0043	0.3019	−0.0026	0.0152	2.6212
ω	10	4.6712	4.2197	32.7789	2.6107	1.4783	20.2017
	50	3.2390	1.9851	24.6580	1.7902	0.5924	11.2487
	150	0.3487	0.7219	8.5499	0.7843	0.4031	5.7150
	300	0.1023	0.2116	2.5468	0.4295	0.3062	2.5218
	500	0.0287	0.0921	0.7901	0.1203	0.1088	1.5219
	800	0.0221	0.0108	0.7105	0.0620	0.0225	1.5138

		III			IV		
Parameter	*N*	*AB*	*RMSE*	*Av*	*AB*	*RMSE*	*AV*
α	10	20.0017	15.0362	21.4376	30.4276	21.1182	33.5522
	50	17.2072	10.3487	12.4511	20.8710	18.9105	10.7660
	150	8.1190	4.0333	9.8704	6.5490	11.4015	4.8744
	300	4.2053	1.4801	3.9329	1.0721	6.0017	3.4975
	500	1.0277	0.5691	2.5457	0.2463	0.9355	3.1341
	800	0.0041	0.00187	2.5629	0.00918	0.0281	3.0876
β	10	13.0023	17.4603	12.0943	19.3206	34.6577	17.3754
	50	9.3203	10.9405	7.4688	10.2051	28.2188	14.2051
	150	6.1045	3.4461	4.0015	4.3220	20.4421	8.2613
	300	3.6597	2.0439	2.8290	2.1076	6.2330	2.8587
	500	1.2722	0.4532	1.9061	0.8017	2.1958	2.8195
	800	0.0451	0.0007	1.9763	0.0014	0.8723	2.8523
θ	10	14.7125	16.1346	27.6107	22.7622	10.2172	13.4185
	50	9.5502	8.1472	18.4701	8.2301	7.3150	9.3244
	150	5.9214	4.6692	9.4771	3.2178	4.6360	3.7210
	300	2.6021	2.1793	2.5143	1.9659	2.8804	3.2421
	500	0.7698	0.9255	2.0425	0.0716	0.9356	3.2106
	800	0.0871	0.01843	2.0362	0.0087	0.0521	3.0502
λ	10	2.7119	4.2136	7.9205	−5.4302	10.6212	6.9263
	50	−1.7521	3.8192	3.5422	−3.7832	7.9750	5.1837
	150	−0.8320	0.8216	2.7704	−2.4339	3.4105	4.2156
	300	−0.1376	0.0321	1.2793	−0.9817	1.8547	3.1504
	500	−0.0214	0.0092	1.2143	−0.4427	0.6993	2.2845
	800	−0.0051	0.0004	1.2012	−0.0069	0.0072	2.1341
ω	10	11.0378	13.6521	8.3214	9.3266	12.0144	7.3243
	50	7.9954	9.2719	3.6534	5.6512	4.7557	4.8755
	150	3.4007	7.2265	2.8731	3.2783	2.5708	3.0962
	300	0.925	1.0743	2.1731	1.7630	1.6506	2.6034
	500	0.023	0.4217	2.0146	0.0692	0.0219	2.5098
	800	0.0072	0.0185	2.0095	0.0051	0.0036	2.5133

$$\ell(\Omega)=\sum_{i=1}^{n}\log\left[\left(f_{KumORL-G}(x_i,\theta,\alpha,\beta,\boldsymbol{\epsilon})\right)^{\varrho_i}\left(S_{KumORL-G}(x_i,\theta,\alpha,\beta,\boldsymbol{\epsilon})\right)^{1-\varrho_i}\right]$$

$$=\sum_{i=1}^{n}\left[\varrho_i\log\left(f_{KumORL-G}(x_i,\theta,\alpha,\beta,\boldsymbol{\epsilon})\right)+(1-\varrho_i)\log\left(S_{KumORL-G}(x_i,\theta,\alpha,\beta,\boldsymbol{\epsilon})\right)\right], \tag{49}$$

Now, we let $\varrho=\sum_{i=0}^{n}\varrho_i$ and setting (6,7) of the KumORL-G model into (49) produce;

$$\ell(\Omega)=\varrho\log\alpha+\varrho\log\beta+\sum_{i=1}^{n}\varrho_i\log g(x_i;\ \boldsymbol{\epsilon})-3\sum_{i=1}^{n}\varrho_i\log\overline{G}(x_i;\ \boldsymbol{\epsilon})-2\,\varrho\log\theta-\varrho\log(\theta-1)$$

$$+\sum_{i=1}^{n}\varrho_i\log\left((\theta^2-2\theta)\left(\overline{G}(x_i;\ \boldsymbol{\epsilon})\right)+G(x_i,\boldsymbol{\epsilon})\right)-\frac{1}{\theta}\sum_{i=1}^{n}\varrho_i\frac{G(x_i,\boldsymbol{\epsilon})}{\overline{G}(x_i;\ \boldsymbol{\epsilon})}$$

$$+(\alpha-1)\sum_{i=1}^{n}\varrho_i\log\left(1-\left(1+\frac{G(x_i;\ \boldsymbol{\epsilon})}{\theta(\theta-1)\overline{G}(x_i;\ \boldsymbol{\epsilon})}\right)e^{-\frac{G(x_i;\ \boldsymbol{\epsilon})}{\theta\overline{G}(x_i;\ \boldsymbol{\epsilon})}}\right)+(\beta-1)\sum_{i=1}^{n}\varrho_i\log\left[1\right.$$

$$\left.-\left(1-\left(1+\frac{G(x_i;\ \boldsymbol{\epsilon})}{\theta(\theta-1)\overline{G}(x_i;\ \boldsymbol{\epsilon})}\right)e^{-\frac{G(x_i;\ \boldsymbol{\epsilon})}{\theta\overline{G}(x_i;\ \boldsymbol{\epsilon})}}\right)^{\alpha}\right]$$

$$+\beta\sum_{i=1}^{n}(1-\varrho_i)\log\left[1-\left(1-\left(1+\frac{G(x_i;\ \boldsymbol{\epsilon})}{\theta(\theta-1)\overline{G}(x_i;\ \boldsymbol{\epsilon})}\right)e^{-\frac{G(x_i;\ \boldsymbol{\epsilon})}{\theta\overline{G}(x_i;\ \boldsymbol{\epsilon})}}\right)^{\alpha}\right], \tag{50}$$

The ML estimators are obtained by maximizing the log-likelihood in (50) using numerical methods.

5.2. Monte Carlo simulation analysis of KumORL-G

Using Monte Carlo simulation, the MLEs' performance of the five KumORLBXII parameters is evaluated. For different sample sizes n, the simulation is considered over several iterations with $N=5000$, with the following parameter values grouped into four different sets:

- I: $\alpha=1.2,\beta=1.2,\theta=2.6,\lambda=0.3,\omega=0.7$.
- II: $\alpha=0.8,\beta=0,8,\theta=3.7,\lambda=2.6,\omega=1.5$.
- III: $=2.5,\ \beta=1.9,\theta=2.0,\lambda=1.2,\omega=2.0$.
- IV: $\alpha=3.1,\beta=2.8,\theta=3.0,\lambda=2.1,\omega=2.5$.

The initial parameters selected for the simulation analysis were chosen without any strict guidelines since there were no issues. Any value of the parameters within the range could be selected. Three evaluation criteria are used to evaluate the estimators' performance. These evaluation criteria are the average bias (AB), root mean square error (RMSE), and average estimations (Av), defined as follows:

- $RMSE=\sqrt{\frac{1}{N}\sum_{i=1}^{N}(\widehat{\Omega}_i-\Omega)^2}$.
- $AB=\frac{1}{N}\sum_{i=1}^{N}(\widehat{\Omega}_i-\Omega)$.
- $Av=\frac{1}{N}\sum_{i=1}^{N}\widehat{\Omega}_i$.

Where $\Omega=(\alpha,\beta,\theta,\lambda,\omega)$.

The results of the simulation analysis obtained via the R software are displayed in Table 4. As the sample size n increases, we observe that the MLEs of $\alpha,\beta,\theta,\lambda,$ and ω tend to be stable, the biases decrease, and RMSE reduces to zero, proving that the MLE technique employed accurately estimates the KumORLBXII model's parameters.

6. Applications to real-life datasets

The fitting potential of the proposed probability model is evaluated using the four real-life complete and censored datasets on diabetes and cancer. In each case, the new model is compared with its sub-models (ExORLBXII, KumGRL, KumRL, ORLBXII, and RL) and other existing models, which include Burr XII (BXII) [32], Nadarajah, and Haghighi (NH) [31], New Generalized Inverse Weibull (NGIW) [33], Odd Kumaraswamy Inverse Weibull (OKIW) [34], and Exponentiated Kumaraswamy Inverse Weibull (EKIW) [35]. The

CDFs of the models used in the application are displayed in Appendix A2.

Different analytical tools and measures are employed to assess the model's adequacy. These include the consistent Akaike information criterion (CAIC), the Bayesian information criterion (BIC), the Akaike information criterion (AIC), Anderson–Darling (AD), Kolmogorov–Smirnov (KS) with its p-value, and likelihood ratio ($-2l$). The model that stands best is based on the lowest values of these statistics and the highest KS *p-value*. The mathematical expressions for these statistics are shown below:

$$\mathrm{CVM} = \sum_{j=1}^{k}\left[F(x_j) - \frac{2j-1}{2k}\right]^2 + \frac{1}{12k}, \mathrm{KS} = \sup_x[F_k(x) - \widehat{F}(x)],$$

$$\mathrm{AD} = -k - \frac{1}{k}\sum_{j=1}^{k}(2j-1)\left[\log\left(1 - F\left(x_{j-k+1}\right)\right) + \log F\left(x_j\right)\right], \mathrm{AIC} = 2p - 2\ell, \mathrm{BIC} = p\log(k) - 2\ell, \mathrm{CAIC} = \mathrm{AIC} + \frac{2p(p+1)}{k-p-1}$$

where k indicates the sample size, x_j defines the j^{th} observation when the data points are presented in increasing order, ℓ is the log ML function evaluated on the ML estimates, p denotes the number of parameters in the model, $\widehat{F}(x)$ represents the estimated density, whereas $F_k(x)$ denotes the empirical CDF, and sup_x is the maximum set of distances. We have also compared the fitness of the KumORLBXII with its sub-models via the likelihood ratio (LR) test. The LR statistic is $LR = 2[l(\widehat{\Omega}) - l(\widehat{\Delta})]$, where $l(\widehat{\omega})$ and $l(\widehat{\Delta})$ are the log-likelihood functions of the ML estimates of the full and reduced models respectively. Under the null hypothesis, LR is rejected if $LR > \chi^2_U$, where χ^2_U is the upper 100 % point of the χ^2 distribution, with degrees of freedom being the number of parameters of the full model minus that of the reduced model. All analyses in this section were completed employing the R program.

- **Complete Datasets:**

The first set of data provides the survival time in years before the start of diabetes of a randomly selected group of 105 individuals from the Upper East Region of Ghana's Bolgatanga Regional Hospital obtained from Zamanah et al. [36]. The dataset contains the following:

52,18, 69, 19, 28, 74, 25, 29, 56, 39, 76, 26, 81, 33, 34, 38, 38, 34, 35, 43, 45, 45, 63, 47, 46, 42, 42, 42, 41, 46, 45, 45, 43, 41, 40, 49, 49, 48,53, 53, 35, 54, 61, 54, 55, 55, 25, 73, 51, 74, 37, 56, 58, 58, 58, 57, 50,18,62,62, 81, 63, 47, 64, 64, 65, 67, 60, 36, 68, 19, 69, 61, 61, 61, 70,75,83, 52, 62, 33, 80, 26, 76, 75, 37, 29, 39, 51, 35, 59, 50, 82, 52, 52, 71, 51, 73, 24, 51, 48, 48, 40, 54, 36.

The second dataset is from Ijaz et al. [37] and it involves the time to remission of 128 patients with bladder cancer, expressed in months. The dataset values are as follows:

0.080, 0.200, 0.400, 0.500, 0.510, 0.810, 0.900, 1.050, 1.190, 1.260, 1.350, 1.400, 1.460, 1.760, 2.020, 2.020, 2.070, 2.090, 2.230, 2.260, 2.460, 2.540, 2.620, 2.640, 2.690, 2.690, 2.750, 2.830, 2.870, 3.020, 3.250, 3.310, 3.360, 3.360, 3.480, 3.520, 3.570, 3.640, 3.700, 3.820, 3.880, 4.180, 4.230, 4.260, 4.330, 4.340, 4.400, 4.500, 4.510, 4.870, 4.980, 5.060, 5.090, 5.170, 5.320, 5.320, 5.340, 5.410, 5.410, 5.490, 5.620, 5.710, 5.850, 6.250, 6.540, 6.760, 6.930, 6.940, 6.970, 7.090, 7.260, 7.280, 7.320, 7.390, 7.590, 7.620, 7.630, 7.660, 7.870, 7.930, 8.260, 8.370, 8.530, 8.650, 8.660, 9.020, 9.220, 9.470, 9.740, 10.06, 10.34,10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05.

Table 5 provides the statistical summaries including measures of skewness and peakness of the two complete data sets. The datasets, in particular, show both positive skewness traits, with the cancer dataset demonstrating slight skewness (nearly symmetric).

Fig. 3 depicts Aarset [38] scaled total test on time (TTT)-transform plots for both complete datasets. The plot in the left panel shows that the complete diabetes dataset has a concave shape, indicating that the empirical hazard rate is monotonic. However, the plot in the right panel produces a non-monotonic failure rate for the complete cancer dataset.

Tables 6 and 7 show the ML estimates and their standard errors (in parentheses) for the complete diabetes and cancer datasets respectively while Tables 8 and 9 display the relevant selection criteria measures for best-fitting models for the complete diabetes and cancer datasets, respectively. The KumORLBXII model is seen to be the best-competing model based on its highest *p-value* of KS and lowest values of these statistics, $-2l$, AIC, CAIC, CVM, KS, and AD.

Tables 10 and 11 display the results of the LR tests, which clearly show that at a 5 % level of significance, there exist notable distinctions between KumORLBXII and its sub-models used in the application, and hence the KumORLBXII proves superior over its sub-models.

Figs. 4 and 5 display the histograms and empirical distributions, with the fitted PDFs on the left panels and fitted CDFs on the right panels of the competing probability models, respectively for the observed complete two datasets. It is observed that KumORLBXII model closely reflects the histograms and empirical CDFs for the complete diabetes and cancer datasets.

Table 5
Summary statistics for the complete datasets on diabetes and cancer patients.

	N	median	Mean	Standard deviation	skewness	Kurtosis
Dataset 1 (Diabetes)	105	51.00	50.71	16.0004	0.01261	2.3751
Dataset 2 (Cancer)	128	6.3950	9.3660	10.5083	3.2866	18.4831

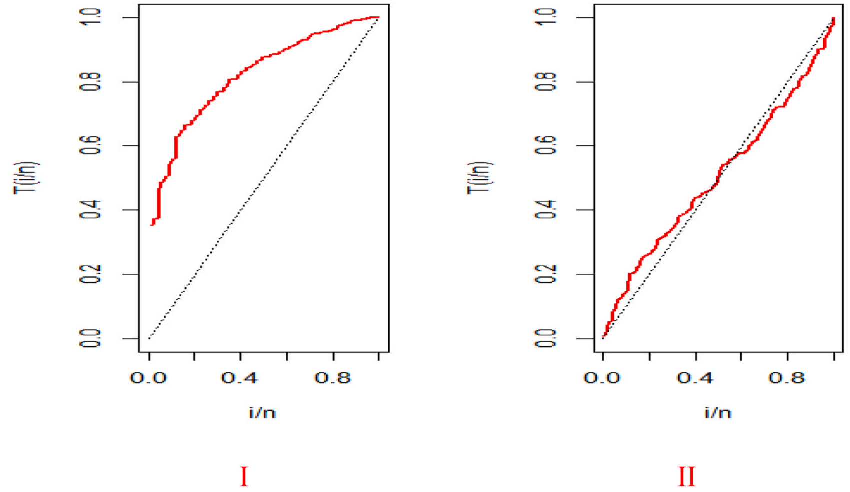

Fig. 3. Scaled TTT-transform plots for the diabetes dataset (left panel-I) and cancer dataset (right panel-II).

Table 6
Estimates of parameters and standard errors for the complete diabetes dataset.

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\omega}$
KumORLBXII	4.6761 (0.5838)	27.5718(30.4327)	117.9996(0.4266)	0.7544(0.2682)	1.4257(0.5063)
OKIW	19.1324 (26.7297)	0.9557 (0.4736)	12.0218(14.5167)	28.6061(39.1715)	0.4961(0.5139)
EKIW	27.5091(92.1535)	1.2487 (0.2851)	0.3029(0.1623)	12.8883(53.7527)	57.8262(36.2488)
NGIW	36.3553 (6.9941)	69.0844(39.1998)	51.1777(14.70372)	0.6657 (0.1030)	
ExORLBXII	5.9735 (0.9662)		120.9457(0.0157)	1.1877(0.6693)	1.2087(0.6798)
KumGRL	4.3605 (0.34523)	92.335 (2.02e-04)	125.81 (3.29e-03)	0.99972(2.33e-02)	
KumRL	5.2416 (1.4131)	20.6506(41.3857)	66.7199 (55.2123)		
ORLBXII	2.7823 (3.9620)	1.3153 (1.8730)	2599000 (6.8e-08)		
NH	0.0024 (1.64e-04)	6.3741 (6.59e-08)			
BXII	2.9406 (5.8255)	0.087884(0.1743)			
RL			49.6548(4.9512)		

Table 7
Estimates of parameters and standard errors for the complete cancer dataset.

distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\omega}$
KumORLBXII	1.9322 (0.82218)	17.1879 (57.40462)	4.56215 (7.04595)	0.9295 (0.39953)	0.44299 (0.3315)
OKIW	3.0208 (4.6896)	0.2453 (0.1165)	1.7781(0.9860)	3.6023(3.0203)	2.7333 (1.4432)
EKIW	4.3447 (0.8004)	0.2331 (0.0755)	0.8748(0.4720)	5.5467(0.8285)	100.3275(94.856)
NGIW	−0.1878(0.1279)	64.1552(64.9569)	7.3222(0.9176)	0.2527 (0.0596)	
ExORLBXII	4.9590 (2.2878)		14.3840 (0.0118)	0.2193(0.0579)	3.7800(0.5033)
KumGRL	2.2840 (0.9499)	5.5028 (10.404)	4.27329 (4.00528)	0.57158(0.2447)	
KumRL	1.1594 (0.1335)	1.51148 (1.4588)	11.4394 (11.71530)		
ORLBXII			86.6658 (0.0038)	0.4020(0.0409)	3.5611(0.2420)
NH	0.1217 (0.0344)	0.9226 (0.1516)			
BXII				2.3348(0.3541)	0.2338(0.0399)
RL			8.2691(0.8319)		

- **Censored Datasets:**

The third dataset available in Amadu [39] pertains to the weekly relapse rates of thirty individuals with leukemia undergoing similar therapies. The dataset values are:

1,1,2,4, 4, 6, 6, 6, 7,8, 9,9,10, 12,13,14, 18,19, 24,26, 29, 31,42*, 45, 50*, 57*, 60, 71*, 85*, 91, where the asterisks (*) indicate the censored observations.

The fourth dataset involves the remission periods in number of months experienced by a random sample of 137 bladder patients. The dataset values are as produced below:

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 24.80*, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32,

Table 8
Model selection criteria measures for the complete diabetes dataset.

Model	$-2l$	CVM	AD	KS	p-value	AIC	CAIC	BIC
KumORLBXII	872.5796	0.0190	0.2247	0.0340	0.9997	882.5797	883.1858	895.8494
EKIW	880.8270	0.0872	0.6004	0.0675	0.7247	890.8270	891.4331	904.0968
OKIW	875.2810	0.0274	0.2359	0.0386	0.9977	885.2810	885.8871	898.5508
NGIW	884.8680	0.1023	0.7300	0.0707	0.6701	892.8680	893.2680	903.4838
ExORLBXII	877.9742	0.0663	0.6526	0.0590	0.8576	885.9742	886.3742	896.5900
KumGRL	878.1938	0.0368	0.28109	0.0467	0.9763	886.1938	886.5938	896.8096
KumRL	879.0812	0.0305	0.3033	0.0455	0.9814	885.0812	885.3188	893.0431
ORLBXII	879.5864	0.0359	0.2951	0.0446	0.9850	885.5864	885.8240	893.5483
NH	984.1812	4.4055	21.064	0.3580	4.084e-12	988.1812	988.2988	993.4892
BXII	1306.7820	8.6773	40.087	0.3580	5.714e-06	1310.782	1310.899	1316.09
RL	1034.4630	4.5041	22.2780	0.3642	2.2e-16	1036.463	1036.5018	1039.117

Table 9
Model selection criteria measures for the complete cancer dataset.

Model	-2ℓ	CVM	AD	KS	p-value	AIC	CAIC	BIC
KumORLBXII	821.8511	0.0486	0.3191	0.0486	0.9225	831.8511	832.3429	846.1112
EKIW	823.2022	0.0550	0.3751	0.0517	0.8832	833.2022	833.6940	847.4624
OKIW	822.3197	0.0641	0.4293	0.0544	0.8433	832.3197	832.8115	846.5799
NGIW	822.3116	0.0576	0.3785	0.0515	0.8858	830.3116	830.6368	841.7197
ExORLBXII	827.0604	0.0645	0.3702	0.0550	0.8333	835.0604	835.3856	846.4685
KumGRL	827.1705	0.0829	0.5001	0.0589	0.7666	835.1705	835.4957	846.5786
KumRL	828.4712	0.1441	0.8080	0.0753	0.4626	834.4712	834.6647	843.0273
ORLBXII	828.2742	0.2875	1.8516	0.0875	0.2805	834.2742	834.4677	842.8303
NH	829.4513	0.1991	1.3038	0.0919	0.2296	833.4513	833.5473	839.1554
BXII	907.0332	2.7219	13.364	0.2507	2.1e-07	911.0332	911.1292	916.7373
RL	832.0356	0.1786	1.1556	0.0810	0.3699	834.0356	834.0673	836.8876

Table 10
Comparison of KumORLBXII and its sub-models for the complete diabetes dataset.

model	hypotheses	*LR*	critical value
ExORLBXII	$H_0 : \widehat{\beta} = 1$ vrs $H_1 : H_0$ is false	5.3946	3.841
KumGRL	$H_0 : \widehat{\omega} = 1$ vrs $H_1 : H_0$ is false	5.6142	3.841
KumRL	$H_0 : \widehat{\omega} = \widehat{\lambda} = 1$ vrs $H_1 : H_0$ is false	6.5016	5.991
ORLBXII	$H_0 : \widehat{\alpha} = \widehat{\beta} = 1$ vrs $H_1 : H_0$ is false	7.0068	5.991
RL	$H_0 : \widehat{\alpha} = \widehat{\beta} = \widehat{\lambda} = \widehat{\omega} = 1$ vrs $H_1 : H_0$ is false	161.8834	9.488

Table 11
Comparison of KumORLBXII and its sub-models for the complete cancer dataset.

model	hypotheses	*LR*	critical value
ExORLBXII	$H_0 : \widehat{\beta} = 1$ vrs $H_1 : H_0$ is false	5.2093	3.841
KumGRL	$H_0 : \widehat{\omega} = 1$ vrs $H_1 : H_0$ is false	5.3194	3.841
KumRL	$H_0 : \widehat{\omega} = \widehat{\lambda} = 1$ vrs $H_1 : H_0$ is false	6.6201	5.991
ORLBXII	$H_0 : \widehat{\alpha} = \widehat{\beta} = 1$ vrs $H_1 : H_0$ is false	6.4231	5.991
RL	$H_0 : \widehat{\alpha} = \widehat{\beta} = \widehat{\lambda} = \widehat{\omega} = 1$ vrs $H_1 : H_0$ is false	10.1845	9.488

7.32, 10.06, 14.77, 32.15, 0.87*, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 10.86*, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 4.33*, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 3.02*, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 19.36*, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 4.65*, 6.76, 8.60*, 12.07, 21.73, 2.07, 3.36, 4.70*, 6.93, 8.65, 12.63, 22.69, where the asterisks (*) indicate the observations with censorship.

Table 12 displays the summary statistics of the two censored datasets. The two datasets show positive skewness traits, with the bladder cancer dataset demonstrating high skewness and kurtosis.

Fig. 6 provides the scaled TTT-transform plots for both leukemia and bladder cancer datasets. The leukemia cancer dataset plot shows concave shape after convex, indicating that the empirical hazard rate is a bathtub (left panel-I), whereas the bladder cancer dataset plot appears to be the reverse, concave, and then convex (right panel-II). As a result, the empirical hazard rates for both

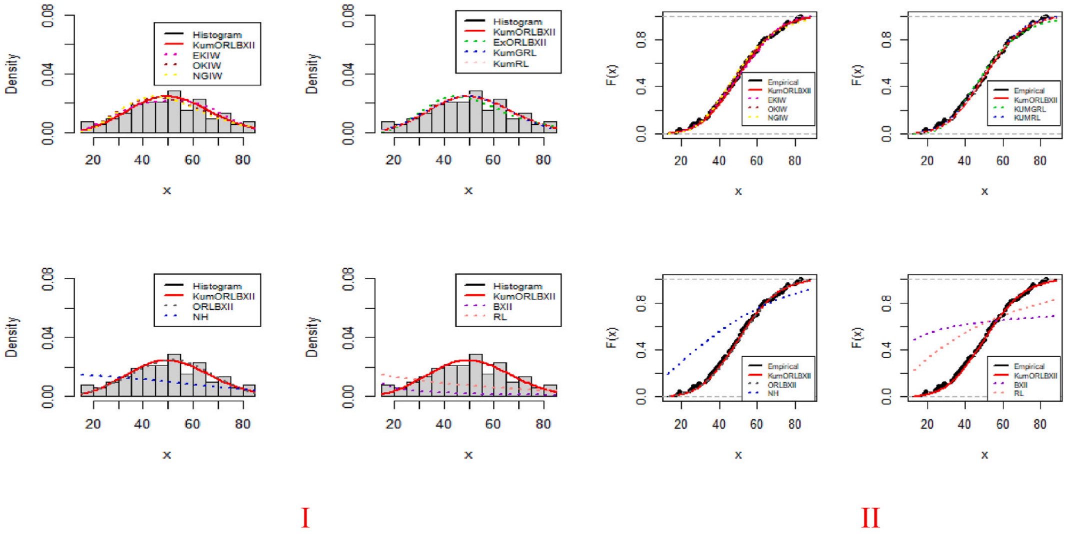

Fig. 4. Fitted PDFs (left panel-I) and fitted CDFs (right panel-II) vs. empirical for the complete diabetes dataset.

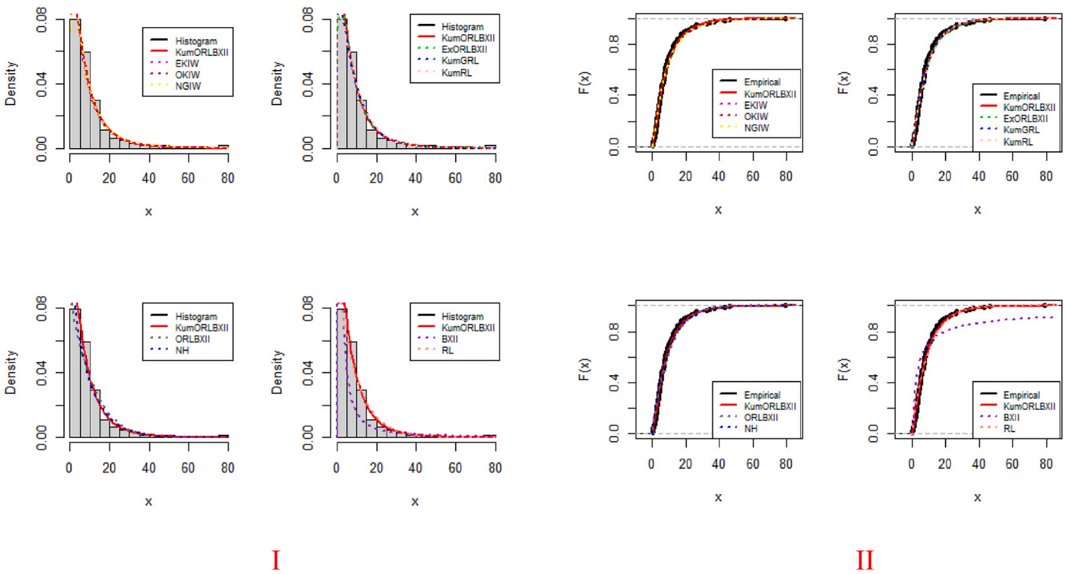

Fig. 5. Fitted PDFs (left panel-I) and fitted CDFs (right pane-Il) vs. empirical for complete cancer dataset.

datasets reveal that the failure rate is non-monotonic.

Tables 13 and 14 present the ML estimates and their standard errors (in parentheses) for the leukemia and bladder cancer datasets, respectively, Tables 15 and 16, provide the relevant selection criteria measures for the best-fitting models for the censored cancer datasets. It is observed from the results in Tables 15 and 16, that although the selection criteria (AIC, BIC, and CAIC) do not favor the KumORLBXII model, it is the best-fit model based on the greatest *p-value* of KS and least values of these statistics: $-2l$, CVM, AD, and KS. From the likelihood ratio test results in Tables 17 and 18, it is noted that there is no statistical difference between KumORLBXII and its sub-models, even though KumORLBXII has shown to be a good fit.

The plots of fitted PDFs with histograms on the left panels, and on the right is the empirical against fitted CDFs of the observed

Table 12
Summary statistics for censored leukemia and bladder cancer datasets.

	N	Median	Mean	Standard deviation	Skewness	kurtosis
Dataset 3 (Leukemia)	30	13.50	25.33	25.73115	1.192072	3.33419
Dataset 4 (Bladder)	137	6.250	9.343	10.34217	3.248328	18.4638

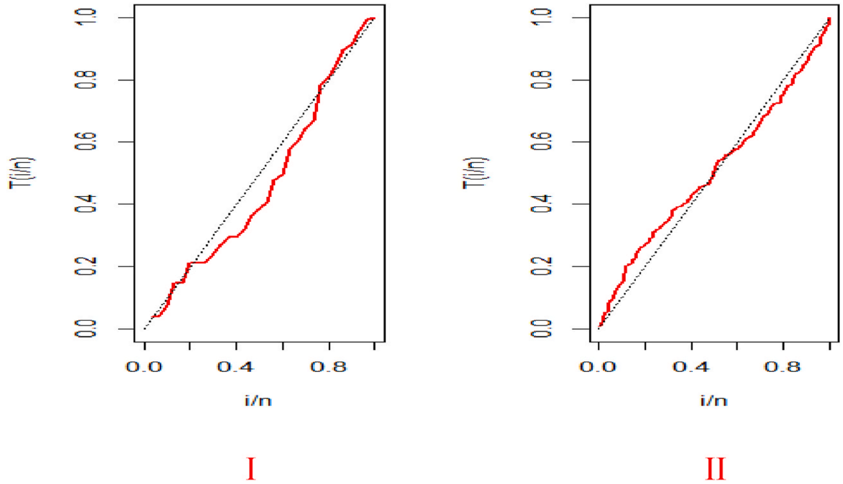

Fig. 6. Scaled TTT-Transform plots for the leukemia cancer dataset (left panel-I) and bladder cancer dataset (right panel-II).

Table 13
Estimates of parameters and standard errors (in parenthesis) for the leukemia cancer dataset.

distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\omega}$
KumORLBXII	13.9955 (30.796)	0.2116 (0.513)	18.0781 (2.526)	0.1415 (0.109)	5.2399 (1.338)
OKIW	1.2648 (32.31)	0.63272 (0.662)	2.24867 (36.862)	0.6311 (1.6794)	1.3019 (4.325)
EKIW	4.5003 (43.329)	0.4409 (0.502)	0.40521 (1.974)	5.8534 (26.675)	5.7573 (11.809)
NGIW	0.6788 (2.250)	11.6083 (41.794)	5.73037 (2.250)	0.2606 (0.282)	
ExORLBXII	36.0468 (0.002)		19.9021 (0.008)	0.0657 (0.010)	5.6230 (0.120)
KumGRL	1.1355 (0.814)	20.2567 (88.11)	150.0251 (4.251)	0.7745 (0.480)	
KumRL	0.8259 (0.160)	4.1160 (10.495)	173.0109 (501.833)		
ORLBXII			4.8185 (8.650)	4.3394 (8.634)	0.1379 (0.296)
BXII				12.6773 (30.467)	0.0248 (0.0597)
NH	0.1123(0.077)	0.4947 (0.170)			
RL			29.3677 (6.076)		

Table 14
Estimates of parameters and standard errors (in parenthesis) for the bladder cancer dataset.

distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\omega}$
KumORLBXII	5.9437 (5.380)	2.3666 (4.300)	188.356 (0.0024)	0.10097 (0.099)	7.2186 (0.761)
OKIW	9.1792 (0.485)	0.1119 (0.086)	2.8028 (1.879)	18.9989 (41.416)	4.0499 (2.783)
EKIW	4.3873 (0.188)	0.2106 (0.045)	0.9104 (0.505)	5.9662(0.671)	150.589 (0.0024)
NGIW	−0.1458 (0.105)	108.5478 (0.002)	7.74058 (0.303)	0.2231 (0.015)	
ExORLBXII	4.2907 (1.906)		188.4(1.9716e-03)	0.1357(3.23e-02)	7.0878 (0.572)
KumGRL	2.9264 (1.075)	20.0411 (10.248)	4.1262 (0.2823)	0.4383(3.929)	
KumRL	1.1665 (0.134)	1.3819 (1.322)	11.0951(11.251)		
ORLBXII			79.6635 (2.98e-03)	0.4089(3.81e-02)	3.5112 (0.244)
BXII				2.3353 (0.354)	0.2182 (0.037)
NH	0.1115(0.032)	0.9335 (0.161)			
RL			8.89736 (0.889)		

censored datasets have been presented in Figs. 7 and 8. The adequacy of KumORLBXII is demonstrated in Figs. 7 and 8 due to the closeness of its fits to the histograms and empirical CDFs of the censored data.

7. Conclusions, limitations, further research, and practical implications

The study proposed and examined the KumORL-G, an extension of the ORL-G family of distributions [27] with two more shape parameters to increase its versatility in modeling real-world datasets. Its statistical properties including quantile function expansions, ordinary, and incomplete moments, inequality measures, probability-weighted moments, order statistics, and Renyi entropy were thoroughly discussed and presented. The study also developed characterizations based on the hazard function and the ratio of two truncated moments. These results give an in-depth assessment of the proposed method. The KumORLBXII was introduced, along with

Table 15
Model selection criteria measures for the leukemia cancer dataset.

Distribution	$-2l$	AIC	BIC	CAIC	CVM	AD	KS	p-value
KumORLBXII	216.4471	226.4471	233.4531	228.9471	0.0355	0.3451	0.0966	0.9423
OKIW	216.4811	226.4811	233.4871	228.9811	0.0470	0.4418	0.1092	0.8671
EKIW	216.5025	226.5025	233.5085	229.0025	0.0460	0.4390	0.1125	0.8421
NGIW	220.4575	228.4575	234.0623	230.0575	0.0477	0.4600	0.1161	0.8134
ExORLBXII	216.5849	224.5849	230.1897	226.1849	0.0435	0.4157	0.1108	0.8548
KumGRL	219.2652	227.2652	232.8700	228.8652	0.0588	0.4451	0.1117	0.8485
KumRL	219.6124	225.6124	229.8160	226.5355	0.0882	0.5235	0.1222	0.7612
ORLBXII	216.6276	222.6276	226.8312	223.5507	0.0549	0.4728	0.0987	0.9321
NH	217.9807	221.9807	224.7831	222.4251	0.0582	0.4712	0.0996	0.9272
BXII	229.0087	233.0087	235.8111	233.4531	0.5659	3.0828	0.2645	0.0300
RL	220.735	222.735	224.1362	222.8779	0.1723	0.8504	0.1646	0.3907

Table 16
Model selection criteria measures for the bladder cancer dataset.

Distribution	$-2l$	AIC	BIC	CAIC	CVM	AD	KS	p-value
KumORLBXII	838.0985	848.0985	862.6984	848.5565	0.0429	0.2717	0.0463	0.9305
OKIW	839.3426	849.3426	863.9425	849.8006	0.0604	0.3820	0.0512	0.8657
EKIW	839.8167	849.8167	864.4166	850.2747	0.0929	0.5561	0.0589	0.7278
NGIW	839.1353	847.1353	858.8152	847.4383	0.0871	0.5025	0.0593	0.7216
ExORLBXII	838.1819	846.1819	857.8618	846.4849	0.1106	0.5791	0.0659	0.5916
KumGRL	840.2672	848.2672	859.9471	848.5702	0.0924	0.6940	0.0679	0.5535
KumRL	843.5731	849.5731	858.3331	849.7536	0.3142	1.4986	0.0996	0.1321
ORLBXII	854.9202	860.9202	869.6802	861.1007	0.2432	1.7669	0.0962	0.1582
BXII	924.6137	928.6137	934.4536	928.7033	2.4662	12.685	0.2332	6.7e-07
NH	845.3051	849.3051	855.1451	849.3947	0.2610	1.5678	0.0792	0.3568
RL	845.7131	847.7131	850.6331	847.7427	0.2829	1.5958	0.0886	0.2321

Table 17
Comparison of KumORLBXII and its sub-models for the leukemia cancer dataset.

model	hypotheses	*LR*	critical value
ExORLBXII	$H_0: \widehat{\beta} = 1$ vrs $H_1: H_0$ is false	0.1378	3.841
KumGRL	$H_0: \widehat{\omega} = 1$ vrs $H_1: H_0$ is false	2.8181	3.841
KumRL	$H_0: \widehat{\omega} = \widehat{\lambda} = 1$ vrs $H_1: H_0$ is false	3.1653	5.991
ORLBXII	$H_0: \widehat{\alpha} = \widehat{\beta} = 1$ vrs $H_1: H_0$ is false	0.1805	5.991
RL	$H_0: \widehat{\alpha} = \widehat{\beta} = \widehat{\lambda} = \widehat{\omega} = 1$ vrs $H_1: H_0$ is false	4.2879	9.488

Table 18
Comparison of KumORLBXII and its sub-models for the bladder cancer dataset.

model	hypotheses	*LR*	critical value
ExORLBXII	$H_0: \widehat{\beta} = 1$ vrs $H_1: H_0$ is false	0.0834	3.841
KumGRL	$H_0: \widehat{\omega} = 1$ vrs $H_1: H_0$ is false	2.1687	3.841
KumRL	$H_0: \widehat{\omega} = \widehat{\lambda} = 1$ vrs $H_1: H_0$ is false	5.4746	5.991
ORLBXII	$H_0: \widehat{\alpha} = \widehat{\beta} = 1$ vrs $H_1: H_0$ is false	16.9217	5.991
RL	$H_0: \widehat{\alpha} = \widehat{\beta} = \widehat{\lambda} = \widehat{\omega} = 1$ vrs $H_1: H_0$ is false	7.6146	9.488

its statistical features. It is a unique sub-model of the KumORL-G that employed the Burr XII as the baseline model. The KumORLBXII is a five-parameter extension of the RL distribution with many sub-models suited for modeling lifetime data. The MLE strategy was utilized to estimate the model parameters and evaluated using Monte Carlo simulation. Our findings show that as the sample size increases, the AB and RMSE continually drop, validating the consistency and robustness of the ML estimation approach. To assess the new model's performance and flexibility, a comprehensive application was carried out using complete and censored survival datasets. Different analytical tools and measures were employed to assess the model's adequacy. Among its competitors, the KumORLBXII distribution stands out as a potential and novel approach for modeling diabetes and cancer datasets, particularly because it can accurately represent the complex structure of survival dataset.

The proposed probability distribution will serve as an invaluable tool in survival analysis, with prospective uses in healthcare and biomedical data modeling. In addition, this probability model provides alternatives to existing distributions established in relevant

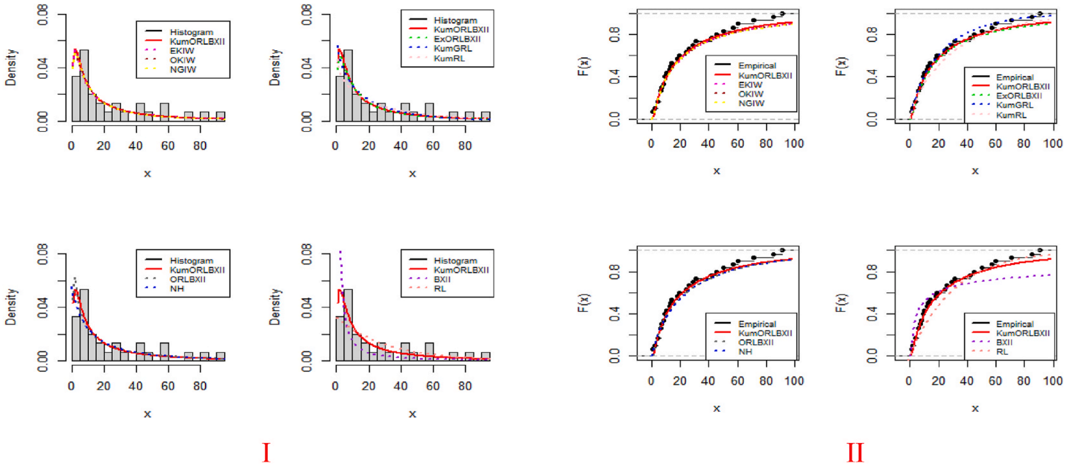

Fig. 7. Fitted PDFs (left panel-I) and fitted CDFs (right panel-II) vs. empirical for leukemia cancer dataset.

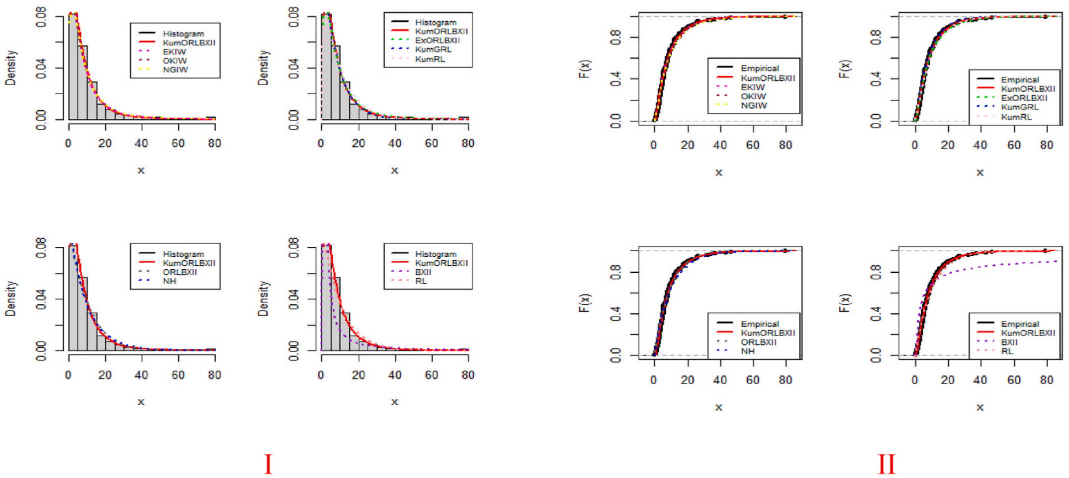

Fig. 8. Fitted PDFs (left panel-I) and fitted CDFs vs. empirical (right panel-II) for bladder cancer dataset.

studies and expands alternatives for modeling cancer and diabetes survival datasets. The new model's versatility in different data settings, and its reliable results with larger sets of data, encourage more research into its potential uses in healthcare, biomedical, and related sectors. Despite its numerous advantages, the KumORL-G model cannot be used to model discrete datasets since it is a continuous type distribution, and the formulations of the ML estimates and statistical attributes are difficult to convert to closed-form equations.

In future research, we hope to tackle some extensions such as other sub-models of the KumORL-G, a discrete version of KumORLBXII, and statistical properties that are not addressed in this paper. The estimation of the KumORL-G using other estimation methods, characterization results based on order statistics and record values, and applications of the new probability model to censored data with covariates will also be addressed in our future study. Researchers can increase our comprehension of the suggested model and its impact in a variety of domains by following these lines of inquiry.

Additionally, the use of this innovative model has the potential to provide essential information on the survival rates of diabetes and cancer patients, as well as aid in enhancing treatment and caregiving. The novel probability proposed in this paper has the potential to be an effective instrument for identifying populations at higher risk and projecting patient outcomes, which may assist healthcare practitioners build more efficient programs for diabetes and cancer screening and prevention. Healthcare professionals could gain from the findings of the study to invest in research and development of innovative treatments and technologies to reduce the impact of the diseases on individuals.

Statement of funding

No entity directly contributed a grant for this research.

Availability of data

The study's data is accessible in the text and is properly cited.

CRediT authorship contribution statement

John Kwadey Okutu: Writing – original draft, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Nana K. Frempong:** Writing – review & editing, Validation, Supervision. **Simon K. Appiah:** Writing – review & editing, Validation, Supervision. **Atinuke O. Adebanji:** Writing – review & editing, Validation, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A1

Theorem A. *Let* $(\Omega, \mathscr{F}, \mathrm{P})$ *represent a given probability space and let* $\mathscr{M} = [c, d]$ *be an interval for some* $c < d$ ($c = -\infty$, $d = \infty$ could as well be permitted). *Let* $X : \Omega \rightarrow \mathscr{M}$ *denotes a continuous random variable with CDF* $G(x)$ *and let* v_1 *and* v_2 *represent two real functions that are defined on the interval* $\mathscr{M}$ *such that;*

$$E(v_2(x)|\ X \geq x) = E(v_1(x)|\ X \geq x)\Lambda(x), x \in \mathscr{M},$$

is defined with some real function Λ. *Suppose that* v_1, $v_2 \in C^1(\mathscr{M})$, $\Lambda \in C^2(\mathscr{M})$ *and* $G(x)$ *is twice continuously differentiable and completely monotonic function on the set* $\mathscr{M}$. *Finally, assume that the equation* $\Lambda v_1 = v_2$ *has no real solution in the interior of* $\mathscr{M}$, *then the functions* Λ, v_1, v_2 *specifically define* $G(x)$ *by;*

$$G(x) = \int_a^x C\left|\frac{\Lambda'(u)}{\Lambda(u)v_1(u) - v_2(u)}\right| e^{-s(u)}du$$

where the function s is a solution of the differential equation $s'(x) = \frac{\Lambda' v_1}{\Lambda v_1 - v_2}$, and C is a constant, chosen to make $\int dG = 1$.

Appendix A2

$$F_{\mathrm{OKIW}}(x) = \left(1 - e^{1-\left(1-e^{\left(-\theta\left(\frac{\alpha}{x}\right)^{\beta}\right)}\right)^{-\lambda}}\right)^{\omega}, x, \alpha, \beta, \theta, \lambda, \omega > 0$$

$$F_{\mathrm{EKIW}}(x) = \left(1 - \left(1 - e^{\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)}\right)^{\omega}\right)^{\theta}, x, \alpha, \beta, \theta, \lambda, \omega > 0$$

$$F_{\mathrm{NGIW}}(x) = 1 - \left(1 - e^{\left(-\frac{\alpha}{x} - \theta\left(\frac{1}{x}\right)^{\lambda}\right)}\right)^{\beta}, x > 0, \alpha, \beta, \theta, \lambda > 0$$

$$F_{\mathrm{ExORLBXII}}(x) = \left(1 - \left(1 + \frac{(1+x^{\lambda})^{\omega} - 1}{\theta(\theta - 1)}\right)e^{-\frac{(1+x^{\lambda})^{\omega}-1}{\theta}}\right)^{\alpha}, x, \omega, \lambda, \alpha > 0, \theta \geq 2$$

$$F_{\mathrm{KumGRL}}(x) = 1 - \left(1 - \left\{1 - \left(1 + \frac{x^{\lambda}}{\theta(\theta - 1)}\right)e^{-\frac{x^{\lambda}}{\theta}}\right\}^{\alpha}\right)^{\beta}, x, \alpha, \beta, \lambda > 0, \theta \geq 2$$

$$F_{\mathrm{KumRL}}(x) = 1 - \left(1 - \left\{1 - \left(1 + \frac{x}{\theta(\theta - 1)}\right)e^{-\frac{x}{\theta}}\right\}^{\alpha}\right)^{\beta}, x, \alpha, \beta > 0, \theta \geq 2$$

$$F_{ORLBXII}(x)=1-\left(1+\frac{(1+x^{\lambda})^{\omega}-1}{\theta(\theta-1)}\right)e^{-\frac{(1+x^{\lambda})^{\omega}-1}{\theta}},x,\omega,\lambda>0,\theta\geq 2$$

$$F_{RL}(x)=1-\frac{(\theta^2+x-\theta)}{\theta(\theta-1)}e^{-\frac{x}{\theta}},x,\theta\geq 2$$

$$F_{BXII}(x)=1-(1+x^{\lambda})^{-\omega},x>0,\omega>0,\lambda>0$$

$$F_{NH}(x)=1-e^{\left(1-(1+\alpha x)^{\beta}\right)},x,\alpha,\beta>0$$

References

[1] Global burden of disease collaborative network, Global burden of disease Study 2019. Results, Institute for Health Metrics and Evaluation, 2020. https://vizhub.healthdata.org/gbd-results/.

[2] World Health Organization, Global action plan for the prevention and control of non- communicable diseases. https://www.who.int/news-room/fact-sheets/detail/cancer, 2020.

[3] C.B. de Villiers, M. Kroese, S. Moorthie, Understanding polygenic models, their development and the potential application of polygenic scores in healthcare, J. Med. Genet. 57 (2020) 725–732, https://doi.org/10.1136/jmedgenet-2019-106763.

[4] L. Lama, O. Wilhelmsson, E. Norlander, L. Gustafsson, A. Lager, et al., Machine learning for prediction of diabetes risk in middle-aged Swedish people, Heliyon 7 (2021) e07419, https://doi.org/10.1016/j.heliyon.2021.e07419.

[5] L. Sandberg, H. Taavola, Y. Aoki, R. Chandler, G.N. Noren, Risk factor considerations in statistical signal detection: using subgroup disproportionality to uncover risk groups for adverse drug reactions in VigiBase, Drug Safety 43 (2020) 999–1009, https://doi.org/10.1007/s40264-020-00957-w.

[6] M. Altaf-Ul-Amin, S. Kanaya, N. Ono, M. Huang, Recent trends in computational Biomedical research, Life 12 (2021) 27, https://doi.org/10.3390/life12010027.

[7] O.A. Ratnovsky, S. Rozenes, E. Bloch, P. Halpern, Statistical learning methodologies and admission prediction in an emergency department, Australas, Emergency Care 24 (2021) 241–247, https://doi.org/10.1016/j.auec.2020.11.004.

[8] D. Onchonga, E. Ngetich, W. Makunda, P. Wainaina, D. Wangeshi, Anxiety and depression due to 2019 SARS-CoV-2 among frontier healthcare workers in Kenya, Heliyon 7 (2021) e06351, https://doi.org/10.1016/j.heliyon.2021.e06351.

[9] A. Illescas, H. Zhong, C. Cozowicz, A. Gonzalez Della Valle, J. Liu, S.G. Memtsoudis, et al., Health services research in anesthesia: a brief overview of common methodologies, Anesth. Analg. 134 (2021) 540–547, https://doi.org/10.1213/ANE.0000000000005884.

[10] M. El-Morshedy, Z. Ahmad, E. tag-Eldin, Z. Almaspoor, M.S. Eliwa, Z. Iqbal, A new Statistical approach for modeling the bladder cancer and leukemia patients' data sets: Case studies in the medical sector, Mathematical Biosciences and Engineering 19 (10) (2022) 10474–10492.

[11] C.S. Kumar, S.R. Nair, A generalized Log-Weibull distribution with bio-medical applications, Int. J. Stat. Med. Res. 10 (2021) 10–21.

[12] X. Liu, Z. Ahmad, A.M. Gemeay, A.T. Abdulrahman, E.H. Hafez, N. Khalil, Modeling The survival times of the COVID-19 patients with a new statistical model: A case study from China, Plos One 16 (2021) e0254999.

[13] H.S. Mohammed, Z. Ahmad, A.T. Abdulrahman, S.K. Khosa, E.H. Hafez, M.M. Abd El-Raouf, M.M. Mohie El-Din, Statistical modeling for Bladder cancer disease using the NLTW distribution, AIMS Math 6 (2021) 9262–9276.

[14] H.S. Klakattawi, Survival analysis of cancer patients using a new extended Weibull distribution, PLoS ONE 17 (2) (2022) e0264229.

[15] El-Houssainy A. Rady, W.A. Hassanein, T.A. Elhaddad, The power Lomax distribution with an application to bladder cancer data, 5, Springer Plus, 2016, p. 1838.

[16] Z. Shah, D.M. Khan, Z. Khan, N. Faiz, S. Hussain, A. Anwar, T. Ahmad, K.I. Kim, A New Generalized Logarithmic-X Family of Distributions with Biomedical Data Analysis, App. Sci. 13 (2023) 3668, 103390/app13063668.

[17] Z. Mahmood, T.M. Jawa, N. Sayed-Ahmed, E.M. Khalil, A.H. Muse, A.H. Tolba, An Extended Cosine Generalized Family of Distributions for Reliability Modeling: Characteristics and Applications with Simulation Study, Mathematical Problems in Engineering 2022 (2022) 20, 3634698.

[18] A. Marshall, I. Olkin, New method for adding a parameter to a family of distributions with applications to the exponential and Weibull families, Biometrika 84 (1997) 641–652.

[19] M.M. Rastic, N. Balakrishnan, The gamma-exponentiated exponential distribution, Journal of Statistical Computation Simulation 82 (2012) 1191–1206.

[20] G.M. Cordeiro, Castro M. de, A new family of generalized distributions, J. Stat. Comput. Simul. 81 (2011) 883–893.

[21] A. Alzaatreh, I. Ghosh, On the Weibull-X family of distributions, Journal of Statistical Theory and Applications 14 (2) (2015) 169–183.

[22] M. Bourguignon, R.B. Silva, G.M. Cordeiro, The Weibull-G Family of Probability Distributions, Journal of Data Science 12 (2014) 53–68.

[23] K. Zografos, N. Balakrishnan, On families of beta- and generalized gamma- generated distributions and associated inference, Statistical Methodology 6 (2009) 344–362.

[24] M.M. Rahman, B. Al-Zahrani, M.Q. Shahbaz, A General Transmuted Family of Distributions, Pakistan Journal of Statistics and Operations Research 14 (2) (2018).

[25] Z. Ahmad, The Zubair-G Family of Distributions: Properties and applications, Ann. Data. Sci. 7 (2018) 195–208, https://doi.org/10.1007/s40745-018-0169-9.

[26] A. Al-Shomrani, O. Arif, A. Shawky, S. Hanif, M.Q. Shahbaz, Topp-Leone family of distributions: Some properties and application, Pak. J. Stat. Oper. Res. 12 (2016) 443–451.

[27] J.K. Okutu, N.K. Frempong, S.K. Appiah, A.O. Adebanji, The Odd Ramos-Louzada Generator of Distributions with Applications to Failure and Waiting Times, Journal of Scientific African 22 (2023) 1–21.

[28] P.L. Ramos, F. Louzada, A distribution for instantaneous failures, Stats 2 (2019) 247–256.

[29] H. Al-Mofleh, A.Z. Afify, N.A. Ibrahim, A new Extended Two-Parameter Distributions: Properties, Estimation Methods, and Applications in Medicine and Geology. mathematics, MDPI Journal 8 (1578) (2020).

[30] R.M. Corless, G.H. Gonnet, D.E. Hare, D.J. Jeffrey, G.E. Knuth, On the Lambert W function, Advances in Computational Mathematics 5 (1) (1996) 329–359.

[31] S. Nadarajah, F. Haghighi, An extension of the exponential distribution, Statistics 45 (2011) 543–558.

[32] I.W. Burr, Cumulative frequency functions, Annals of Mathematical Statistics 13 (2) (1942) 215–232.

[33] A. Dostmoradi, M. Zadkarami, V. Bahrami, The new generalized inverse Weibull distribution, Journal of Mathematical Research 1 (2) (2016) 39–50.

[34] M. Artem, G.O. Orwa, L. Mbugua, The Odd Kumaraswamy Inverse Weibull with applications to survival data, Advances and applications in statistics 51 (5) (2017) 309–335.

[35] J.A. Rodrigues, A.P.C.M. Silva, G.G. Hamedani, The Exponentiated Kumaraswamy Inverse Weibull Distribution with Application in Survival Analysis, Journal of Statistical Theory and Applications 15 (1) (2016) 8–24.

[36] Zamanah, E.; Nasiru, S.; Luguterah, A. Harmonic mixture Weibull-G Family of Distribution: properties, regression, and applications to medical data, Computational and Mathematical Methods (Wiley-Hindawi, vol. 2022..
[37] M. Ijaz, S.M. Asim, Farooq M. Alamgir, S.A. Khan, S. Manzoor, A Gull Alpha Power Weibull distribution with applications to real and simulated data, PLoS ONE 15 (6) (2020) e0233080.
[38] M.V. Aarset, How to identify bathtub hazard rate, IEEE Transactions on Reliability R-36 (1987) 106–108.
[39] Y. Amadu, Some contributions to odd family of distributions with applications to cancer datasets, University of Development Studies, Tamale, 2021. Phd Thesis, http://www.udsspace.uds.edu.gh.