

RESEARCH ARTICLE

Hopf Bifurcation of an Epidemic Model with Delay

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Abstract

A spatiotemporal epidemic model with nonlinear incidence rate and Neumann boundary conditions is investigated. On the basis of the analysis of eigenvalues of the eigenpolynomial, we derive the conditions of the existence of Hopf bifurcation in one dimension space. By utilizing the normal form theory and the center manifold theorem of partial functional differential equations (PFDEs), the properties of bifurcating periodic solutions are analyzed. Moreover, according to numerical simulations, it is found that the periodic solutions can emerge in delayed epidemic model with spatial diffusion, which is consistent with our theoretical results. The obtained results may provide a new viewpoint for the recurrent outbreak of disease.



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Introduction

Currently, new infectious diseases continuously emerge, and existing diseases recurrently outbreak [1–9]. Ebola virus was firstly discovered in 1976, which began to outbreak in Guinea in February 2014, then spread to West Africa. It caused serious death and social panic. After the outbreak of 2014, Ebola once again emerged in Guinea in March 2016 [10–12]. These diseases have brought a great threat to the public health. In order to provide some suggestions for the prevention and control of the disease, it is necessary to establish rational mathematics model based on infectious mechanism of disease, the route of transmission, and the symptoms of the infected individuals. In particular, the incidence rate describes the number of new infections per unit time, which largely reflects the transmission mechanism of the disease [13–17]. For example, Capasso et al. proposed saturated incidence rate $\beta SI/(1 + kI)$ to model the cholera epidemics in Bari in 1973, which reflects the psychological effect or the inhibition effect [18]. By taking appropriate preventive measures, May and Anderson gave nonlinear incidence rate $\beta(SI/(1 + \alpha S))$ [19]. Therefore, some reasonable suggestions can be provided for the prevention and effective control of infectious diseases.

It takes an individual a period of time to show the corresponding symptoms based on the infectious mechanism of disease, after an individual is infected disease, such as, dengue, rabies, cholera and so on [20–27]. Therefore, time delay describing the incubation period of disease is a significant quantity. In fact, these potentially asymptomatic individuals (incubation individuals) may promote the wide spread of disease [28, 29]. Thus, it is necessary for us to introduce time delay in the epidemic models.

Because of all the species living in the space environment, and they could diffuse the surrounding area. The individual diffusion in space has an effect on the disease contagion. For example, Zhang et al. indicated that dog movement led to the traveling wave of dog and human rabies and had a large influence on the minimal wave speed [30]. However, previous works on epidemic models did not account for the spatial diffusion factors. McCluskey proved that the endemic equilibrium was globally asymptotically stable whenever it existed for an SIR epidemic model with delay and nonlinear incidence rate [31]. A delayed predator-prey system with disease in the prey was investigated by Han et al., they considered the existence of Hopf bifurcation with time delay in terms of degree 2 [32]. Hence, it is more suitable for us to consider time delay and spatial factor in epidemic model.

This paper is organized as below. In Section II, the eigenpolynomials of spatiotemporal epidemic model with nonlinear incidence rate are given, we further analyze the existence of Hopf bifurcation for two cases. In Section III, by using the normal form theory and the center manifold theorem, some properties of Hopf bifurcation are showed. In Section IV, on the basis of numerical simulations, we show that the epidemics will display recurrent behavior if time delay exceeds a critical point. Finally, some conclusions are obtained.

Materials and Methods

Existence of Hopf bifurcation

We consider a SI epidemic model with nonlinear incidence rate $\beta S^p I^q$ with $p > 0, q > 0$. This form of nonlinear incidence rate was firstly proposed by Liu et al., and exhibited qualitatively different dynamical behaviors [33, 34]. Therefore, it is helpful to interpret some complex epidemic phenomena. In this paper, let $p = 1$ and $q = 2$. Since the time delay describing incubation period of transmission process widely exists in most epidemiological models [35–37], thus we need to introduce the time delay into the infected population. Furthermore, we consider Neumann boundary conditions. Consequently, the following system with Neumann boundary conditions is given:

$$\begin{cases} \frac{\partial S(x, t)}{\partial t} = A - \beta S(x, t)(I(x, t - \tau))^2 - dS(x, t) + d_1 \nabla^2 S(x, t), \\ \frac{\partial I(x, t)}{\partial t} = \beta S(x, t)(I(x, t - \tau))^2 - (\mu + d)S(x, t) + d_2 \nabla^2 I(x, t), t \geq 0, x \in (0, \pi), \\ \frac{\partial S(x, t)}{\partial x} \Big|_{x=0, \pi} = 0, \frac{\partial I(x, t)}{\partial x} \Big|_{x=0, \pi} = 0, t \geq 0, \\ S(x, t) = \phi_1(x, t) \geq 0, I(x, t) = \phi_2(x, t) \geq 0, (x, t) \in [0, \pi] \times [-\tau, 0], \end{cases} \quad (1)$$

where $S(x, t)$ represents the number of the susceptible at location x and time t , $I(x, t)$ the number of the infectious at location x and time t , A represents the recruitment rate of the susceptible, d and μ are natural death rate and the disease-related death rate due to the infected, respectively. d_1 and d_2 are diffusion coefficients. x represents the one dimensional space, and $\nabla^2 = \frac{\partial^2}{\partial x^2}$ denotes the usual Laplacian operator.

Assuming $\phi = (\phi_1, \phi_2)^T \in \varphi = C([-\tau, 0], X)$, $\tau > 0$ and X is defined as

$$X = \left\{ (S(x, t), I(x, t))^T : S(x, t), I(x, t) \in W^{2,2}(0, \pi); \frac{\partial S(x, t)}{\partial x} \Big|_{x=0, \pi} = \frac{\partial I(x, t)}{\partial x} \Big|_{x=0, \pi} = 0 \right\}$$

with the inner product $\langle \cdot, \cdot \rangle$.

System (1) without diffusion and delay corresponds to the following system:

$$\begin{cases} \frac{dS(t)}{dt} = A - \beta S(t)(I(t))^2 - dS(t), \\ \frac{dI(t)}{dt} = \beta S(t)(I(t))^2 - (\mu + d)I(t). \end{cases} \tag{2}$$

The [system \(2\)](#) has three equilibria, $E^0(\frac{A}{d}, 0)$, the saddle

$$E^1 = \left(\frac{A\beta + \sqrt{A^2\beta^2 - 4d^3\beta - 8d^2\beta\mu - 4d\beta\mu^2}}{2d\beta}, \frac{2d(d + \mu)}{A\beta + \sqrt{A^2\beta^2 - 4d^3\beta - 8d^2\beta\mu - 4d\beta\mu^2}} \right),$$

and the stable node $E^*(S^*, I^*)$, where

$$S^* = \frac{A\beta - \sqrt{A^2\beta^2 - 4d^3\beta - 8d^2\beta\mu - 4d\beta\mu^2}}{2d\beta},$$

$$I^* = \frac{2d(d + \mu)}{A\beta - \sqrt{A^2\beta^2 - 4d^3\beta - 8d^2\beta\mu - 4d\beta\mu^2}}.$$

Based on the biological meaning, E^1 and E^* are satisfied the following conditions [\[38\]](#):

$$(A1) \quad A^2\beta > 4d(d + \mu)^2.$$

Let $\bar{S} = S - S^*, \bar{I} = I - I^*$, then [system \(1\)](#) can be transformed into:

$$\begin{cases} \frac{\partial \bar{S}}{\partial t} = A - \beta(\bar{S} + S^*)(\bar{I}(t - \tau) + I^*)^2 - d(\bar{S} + S^*) + d_1 \nabla^2 \bar{S}, \\ \frac{\partial \bar{I}}{\partial t} = \beta(\bar{S} + S^*)(\bar{I}(t - \tau) + I^*)^2 - (\mu + d)(\bar{I} + I^*) + d_2 \nabla^2 \bar{I}. \end{cases} \tag{3}$$

One can define

$$f^{(1)}(\bar{S}, \bar{I}, \bar{I}(t - \tau)) = A - \beta(\bar{S} + S^*)(\bar{I}(t - \tau) + I^*)^2 - d(\bar{S} + S^*),$$

$$f^{(2)}(\bar{S}, \bar{I}, \bar{I}(t - \tau)) = \beta(\bar{S} + S^*)(\bar{I}(t - \tau) + I^*)^2 - (\mu + d)(\bar{I} + I^*),$$

and for $i, j, l = 0, 1, 2, \dots$, let

$$f_{ijl}^{(1)} = \frac{\partial^{i+j+l} f^{(1)}}{\partial \bar{S}^i \partial \bar{I}^j \partial \bar{I}^l(t - \tau)}(0, 0, 0), \quad i + j + l \geq 1, \quad f_{ijl}^{(2)} = \frac{\partial^{i+j+l} f^{(2)}}{\partial \bar{S}^i \partial \bar{I}^j \partial \bar{I}^l(t - \tau)}(0, 0, 0), \quad i + j + l \geq 1.$$

In the phase space $\varphi = C([- \tau, 0]; X)$, the abstract differential equation of the [system \(3\)](#) is

$$\frac{dU(t)}{dt} = D\Delta U(t) + L(U_t) + F(U_t), \tag{4}$$

where $U = (u_1, u_2)^T, \bar{S}(x, t) = u_1(x, t), \bar{I}(x, t) = u_2(x, t), D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$. We set $U_t(\theta) =$

$U(t+\theta), \phi = (\phi_1, \phi_2)^T \in \varphi, \phi(\theta) = U_t(\theta)$, and $\theta \in [- \tau, 0]$.

Let $L: \varphi \rightarrow X$ and $F: \varphi \rightarrow X$ are given by

$$L(\phi) = \begin{pmatrix} a_{11}\phi_1(0) + a_{12}\phi_2(0) + a_{13}\phi_2(-\tau) \\ a_{21}\phi_1(0) + a_{22}\phi_2(0) + a_{23}\phi_2(-\tau) \end{pmatrix},$$

where $a_{11} = f_{100}^{(1)}(0, 0, 0) = -\beta(I^*)^2 - d$, $a_{12} = f_{010}^{(1)}(0, 0, 0) = 0$, $a_{13} = f_{001}^{(1)}(0, 0, 0) = -2\beta S^* I^*$, $a_{21} = f_{100}^{(2)}(0, 0, 0) = \beta(I^*)^2$, $a_{22} = f_{010}^{(2)}(0, 0, 0) = -(d + \mu)$, $a_{23} = f_{001}^{(2)}(0, 0, 0) = 2\beta S^* I^*$, and

$$F(\phi) = \left(\begin{array}{c} \sum_{i+j+l \geq 2} \frac{1}{i!j!l!} f_{ijl}^{(1)}(0, 0, 0) \phi_1^i(0) \phi_2^j(0) \phi_2^l(-\tau) \\ \sum_{i+j+l \geq 2} \frac{1}{i!j!l!} f_{ijl}^{(2)}(0, 0, 0) \phi_1^i(0) \phi_2^j(0) \phi_2^l(-\tau) \end{array} \right),$$

where $f_{101}^{(1)}(0, 0, 0) = -2\beta I^*$, $f_{002}^{(1)}(0, 0, 0) = -2\beta S^*$, $f_{101}^{(2)}(0, 0, 0) = 2\beta I^*$, $f_{002}^{(2)}(0, 0, 0) = 2\beta S^*$, $f_{102}^{(1)}(0, 0, 0) = -2\beta$, $f_{102}^{(2)}(0, 0, 0) = 2\beta$.

The linearized part of system (4) is given by

$$\frac{dU(t)}{dt} = D\Delta U(t) + L(U_t), \tag{5}$$

then we set $U(t) = ye^{\lambda t}$, and $y = (y_1, y_2)^T$, hence the characteristic equation is

$$\lambda y - D \frac{\partial^2 y}{\partial x^2} - L(e^{\lambda \cdot} y) = 0, \tag{6}$$

where $y \in \text{dom}(\frac{\partial^2}{\partial x^2})$ and $y \neq 0$, $\text{dom}(\frac{\partial^2}{\partial x^2}) \in X$.

On the basis of the Laplacian operator in the bound domain, $\frac{\partial^2}{\partial x^2}$ on X have eigenvalues $-k^2$ with the corresponding eigenfunctions $\beta_k^1 = \begin{pmatrix} \cos kx \\ 0 \end{pmatrix}$, $\beta_k^2 = \begin{pmatrix} 0 \\ \cos kx \end{pmatrix}$, $k \in N_0 = \{0, 1, 2, \dots\}$, namely, a basis of the phase space X is $\{\beta_k^1, \beta_k^2\}_{k=0}^\infty$. Thus, for $\forall y \in X$, y can be expanded as Fourier series in the following form:

$$y = \sum_{k=0}^\infty Y_k^T \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix}, \quad \text{and} \quad Y_k = \begin{pmatrix} \langle y, \beta_k^1 \rangle \\ \langle y, \beta_k^2 \rangle \end{pmatrix}. \tag{7}$$

Furthermore, through simple computations, we have

$$L \left(\phi^T \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix} \right) = L(\phi)^T \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix}, \quad k \in N_0. \tag{8}$$

From the above Eqs (7) and (8), Eq (6) can be written as

$$\sum_{k=0}^\infty Y_k^T \left[\lambda I_2 + Dk^2 - \begin{pmatrix} a_{11} & a_{13}e^{-\lambda\tau} \\ a_{21} & a_{22} + a_{23}e^{-\lambda\tau} \end{pmatrix} \right] \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix} = 0, \tag{9}$$

then the eigenpolynomial associated with λ of system (1) is given by:

$$\lambda^2 + [(d_1 + d_2)k^2 - a_{11} - a_{22} - a_{23}e^{-\lambda\tau}]\lambda + d_1 d_2 k^4 - (d_2 a_{11} + d_1 a_{22} + d_1 a_{23} e^{-\lambda\tau})k^2 + a_{11} a_{22} + (a_{11} a_{23} - a_{13} a_{21}) e^{-\lambda\tau} = 0, \tag{10}$$

where $a_{11} = -\beta(I^*)^2 - d$, $a_{13} = -2\beta S^* I^*$, $a_{21} = \beta(I^*)^2$, $a_{22} = -(d + \mu)$, $a_{23} = 2\beta S^* I^*$, then $a_{11} a_{23} - a_{13} a_{21} < 0$ can be derived.

Considering $k = 0$, the eigenpolynomial Eq (10) becomes

$$\lambda^2 - (a_{11} + a_{22} + a_{23}e^{-\lambda\tau})\lambda + a_{11}a_{22} + (a_{11}a_{23} - a_{13}a_{21})e^{-\lambda\tau} = 0. \tag{11}$$

By replacing λ with $iw (w > 0)$ in Eq (11), then

$$(iw)^2 - (a_{11} + a_{22} + a_{23}e^{-iw\tau})iw + a_{11}a_{22} + (a_{11}a_{23} - a_{13}a_{21})e^{-iw\tau} = 0, \\ -w^2 - [a_{11} + a_{22} + a_{23}(\cos w\tau - i\sin w\tau)]iw + a_{11}a_{22} + (a_{11}a_{23} - a_{13}a_{21})(\cos w\tau - i\sin w\tau) = 0,$$

Through separating the real and imaginary parts of above equations, the following equations are obtained:

$$\begin{cases} -w^2 + a_{11}a_{22} = a_{23}w\sin w\tau - (a_{11}a_{23} - a_{13}a_{21})\cos w\tau, \\ -(a_{11} + a_{22})w = a_{23}w\cos w\tau + (a_{11}a_{23} - a_{13}a_{21})\sin w\tau. \end{cases} \tag{12}$$

Further, by squaring and adding the two parts of Eq (12), we get

$$w^4 + Bw^2 + C = 0, \tag{13}$$

where $B = a_{11}^2 + a_{22}^2 - a_{23}^2$, $C = a_{11}^2a_{22}^2 - (a_{11}a_{23} - a_{13}a_{21})^2 > 0$. Thus we give

$$w^2 = \frac{-B + \sqrt{B^2 - 4C}}{2}, \quad \hat{w}^2 = \frac{-B - \sqrt{B^2 - 4C}}{2},$$

if the formula $B < 0$, and $B^2 - 4C > 0$ then $w^2 > 0$ and $\hat{w}^2 > 0$ can hold simultaneously. Moreover, the corresponding condition is

$$(A2) \quad \beta(I^*)^2 + d - \sqrt{3}(d + \mu) < 0,$$

$$(A3) \quad \beta^4(I^*)^8 + 4d\beta^3(I^*)^6 - 2\beta^2(2d^2 + 10d\mu + 5\mu^2)(I^*)^4 \\ - 4d\beta(4d^2 + 10d\mu + 5\mu^2)(I^*)^2 + (4d^2 + 6d\mu + 3\mu^2)^2 > 0,$$

therefore, Eq (11) has two groups of simple imaginary roots $\pm iw_0, \pm i\hat{w}_0$.

In the following part, we take into account the imaginary roots $\pm iw_0$, the other one is similar.

From Eq (12), we can obtain

$$\cos(w_0\tau) = -\frac{(a_{22}a_{23} + a_{13}a_{21})w_0^2 + a_{11}a_{22}(a_{11}a_{23} - a_{13}a_{21})}{a_{23}^2w_0^2 + (a_{11}a_{23} - a_{13}a_{21})^2} = P(w_0), \\ \sin(w_0\tau) = \frac{w_0[a_{11}a_{22}a_{23} - a_{23}w_0^2 - (a_{11} + a_{22})(a_{11}a_{23} - a_{13}a_{21})]}{a_{23}^2w_0^2 + (a_{11}a_{23} - a_{13}a_{21})^2} = Q(w_0).$$

Moreover, some simple derivations show that

$$\tau_0^j = \begin{cases} \frac{1}{w_0}(\arccos(P(w_0)) + 2j\pi), & \text{when } Q(w_0) \geq 0, \\ \frac{1}{w_0}(2\pi - \arccos(P(w_0)) + 2j\pi), & \text{when } Q(w_0) \leq 0, j = 0, 1, 2, \dots \end{cases}$$

$\lambda(\tau) = \alpha(\tau) + iw(\tau)$ is the root of Eq (11) near τ_0^j , which satisfies $\alpha(\tau_0^j) = 0$ and $w(\tau_0^j) = w_0$, where $j = 0, 1, 2, \dots$

Next, taking the derivative with respect to τ on two sides of Eq (11), then we derive

$$[2\lambda - (a_{11} + a_{22} + a_{23}e^{-\lambda\tau})] \frac{d\lambda}{d\tau} + [a_{23}\lambda - (a_{11}a_{23} - a_{13}a_{21})]\tau e^{-\lambda\tau} \frac{d\lambda}{d\tau} + [a_{23}\lambda - (a_{11}a_{23} - a_{13}a_{21})]\lambda e^{-\lambda\tau} = 0.$$

By the above expression, one can derive

$$\begin{aligned} \operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_0^j} &= \operatorname{Re}\left\{\frac{2\lambda - (a_{11} + a_{22} + a_{23}e^{-\lambda\tau})}{[-a_{23}\lambda + (a_{11}a_{23} - a_{13}a_{21})]\lambda e^{-\lambda\tau}} + \frac{[a_{23}\lambda - (a_{11}a_{23} - a_{13}a_{21})]\tau e^{-\lambda\tau}}{[-a_{23}\lambda + (a_{11}a_{23} - a_{13}a_{21})]\lambda e^{-\lambda\tau}}\right\}_{\tau=\tau_0^j} \\ &= \operatorname{Re}\left\{\frac{2\lambda e^{\lambda\tau} - (a_{11}e^{\lambda\tau} + a_{22}e^{\lambda\tau} + a_{23})}{[-a_{23}\lambda + (a_{11}a_{23} - a_{13}a_{21})]\lambda} - \frac{\tau}{\lambda}\right\}_{\tau=\tau_0^j} \\ &= \left\{\frac{2w_0^2[(a_{11}a_{23} - a_{13}a_{21})\cos w_0\tau - a_{23}w_0\sin w_0\tau] - a_{23}^2w_0^2}{[a_{23}^2w_0^2 + (a_{11}a_{23} - a_{13}a_{21})^2]w_0^2} + \frac{-(a_{11} + a_{22})w_0[a_{23}w_0\cos w_0\tau + (a_{11}a_{23} - a_{13}a_{21})\sin w_0\tau]}{[a_{23}^2w_0^2 + (a_{11}a_{23} - a_{13}a_{21})^2]w_0^2}\right\}_{\tau=\tau_0^j} \\ &= \frac{[2w_0^2 + a_{11}^2 + a_{22}^2 - a_{23}^2]w_0^2}{w_0^2[a_{23}^2w_0^2 + (a_{11}a_{23} - a_{13}a_{21})^2]}, \\ &= \frac{2w_0^2 + B}{a_{23}^2w_0^2 + (a_{11}a_{23} - a_{13}a_{21})^2} > 0, \\ &= \frac{\sqrt{B^2 - 4C}}{a_{23}^2w_0^2 + (a_{11}a_{23} - a_{13}a_{21})^2} > 0. \end{aligned}$$

So the transversality condition

$$\left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)\right\}_{\tau=\tau_0^j} > 0$$

is deduced.

Theorem 2.1 If (A1), (A2) and (A3) are all satisfied, system (1) without diffusion experiences a spatially homogeneous Hopf bifurcation at equilibrium $E^* = (S^*, I^*)$ when $\tau = \tau_0^j$, and period solution will appear.

Lemma 2.1 (S1) If there is a certain $k_0 \in N = \{1, 2, \dots\}$ such that

$$d_1d_2k_0^4 - (d_2a_{11} + d_1a_{22} + d_1a_{23})k_0^2 + a_{11}a_{22} + (a_{11}a_{23} - a_{13}a_{21}) < 0, \tag{14}$$

then Eq (10) has a pair purely imaginary roots $\pm iw_{k_0}$, and

$$w_{k_0} = \frac{\sqrt{2}}{2} \sqrt{-B_{k_0} + \sqrt{B_{k_0}^2 - 4C_{k_0}}}. \tag{15}$$

Proof: If we assume $k = k_0 \in N$, and $\lambda = iw(w > 0)$ be a root of Eq (10). By inserting iw ($w > 0$) into Eq (10) and using the same method as before, then Eq (10) can be translated into:

$$w^4 + Bw^2 + C = 0, \tag{16}$$

where $B = (d_1^2 + d_2^2)k^4 - 2d_1a_{11}k^2 - 2d_2a_{22}k^2 + a_{11}^2 + a_{22}^2 - a_{23}^2$, $C = [d_1d_2k^4 - d_2a_{11}k^2 - d_1a_{23}k^2 + a_{11}a_{23}]^2 - [d_1a_{22}k^2 - (a_{11}a_{22} - a_{12}a_{21})]^2$, for $\forall k \in N$. Besides, we set $C = C_1 \times C_2$, where

$$C_1 = d_1d_2k^4 - (d_2a_{11} + d_1a_{22} - d_1a_{23})k^2 + a_{11}a_{22} - (a_{11}a_{23} - a_{13}a_{21}),$$

$$C_2 = d_1d_2k^4 - (d_2a_{11} + d_1a_{22} + d_1a_{23})k^2 + a_{11}a_{22} + (a_{11}a_{23} - a_{13}a_{21}).$$

Further, we can deduce

$$w^2 = \frac{-B \pm \sqrt{B^2 - 4C}}{2}. \tag{17}$$

It is clear that $C_1(k) > 0$ for $\forall k \in N$, according to assumption (S1), $C(k_0) < 0$ is given, then we get $w_{k_0}^2 > 0$,

$$w_{k_0} = \frac{\sqrt{2}}{2} \sqrt{-B_{k_0} + \sqrt{B_{k_0}^2 - 4C_{k_0}}}.$$

Lemma 2.1 shows that the critical value of bifurcation parameter τ can be found. Similar to the method for the case of $k = 0$,

$$\begin{aligned} \cos(w_{k_0} \tau) &= \frac{d_1^2 d_2 a_{23} k_0^6 + [-d_1 d_2 (2a_{11} a_{23} - a_{13} a_{21}) - d_1^2 a_{22} a_{23}] k_0^4}{a_{23}^2 w_{k_0}^2 + [d_1 a_{23} k_0^2 - (a_{11} a_{23} - a_{13} a_{21})]^2} \\ &+ \frac{d_2 [a_{11} (a_{11} a_{23} - a_{13} a_{21}) + a_{23} w_{k_0}^2] k_0^2 + d_1 a_{22} (2a_{11} a_{23} - a_{13} a_{21}) k_0^2}{a_{23}^2 w_{k_0}^2 + [d_1 a_{23} k_0^2 - (a_{11} a_{23} - a_{13} a_{21})]^2} \\ &+ \frac{-a_{11} a_{22} (a_{11} a_{23} - a_{13} a_{21}) - a_{13} a_{21} w_{k_0}^2 - a_{22} a_{23} w_{k_0}^2}{a_{23}^2 w_{k_0}^2 + [d_1 a_{23} k_0^2 - (a_{11} a_{23} - a_{13} a_{21})]^2} = X(w_{k_0}), \\ \sin(w_{k_0} \tau) &= \frac{-d_1^2 a_{23} w_{k_0} k_0^4 + [d_1 (2a_{11} a_{23} - a_{13} a_{21}) - d_2 a_{13} a_{21}] w_{k_0} k_0^2}{a_{23}^2 w_{k_0}^2 + [d_1 a_{23} k_0^2 - (a_{11} a_{23} - a_{13} a_{21})]^2} \\ &+ \frac{-(a_{11} a_{23} - a_{13} a_{21}) a_{11} w_{k_0} + a_{13} a_{21} a_{22} w_{k_0} - a_{23} w_{k_0}^3}{a_{23}^2 w_{k_0}^2 + [d_1 a_{23} k_0^2 - (a_{11} a_{23} - a_{13} a_{21})]^2} = Y(w_{k_0}), \end{aligned}$$

then we get

$$\tau_{k_0}^j = \begin{cases} \frac{1}{w_{k_0}} (\arccos(X(w_{k_0})) + 2j\pi), & \text{when } Y(w_{k_0}) \geq 0, \\ \frac{1}{w_{k_0}} (2\pi - \arccos(X(w_{k_0})) + 2j\pi), & \text{when } Y(w_{k_0}) \leq 0, j = 0, 1, 2, \dots \end{cases}$$

Let $\lambda(\tau) = \alpha(\tau) + iw(\tau)$ be the root of (10) near $\tau_{k_0}^j$ which satisfies $\alpha(\tau_{k_0}^j) = 0$ and $w(\tau_{k_0}^j) = w_{k_0}$, where $j = 0, 1, 2, \dots$

Lemma 2.2 If condition (S1) is established, then the transversality condition

$$\left\{ \text{Re} \left(\frac{d\lambda}{d\tau} \right) \right\}_{\tau=\tau_{k_0}^j} > 0$$

is derived.

The proof can be found in [S1 File](#).

Theorem 2.2 In the presence of space, if the conditions (A1) and (S1) are satisfied, then [system \(1\)](#) undergoes a Hopf bifurcation at $E^* = (S^*, I^*)$ when $\tau = \tau_{k_0}^j$, and period solution will emerge.

Results

The properties of Hopf bifurcating period solutions

The above section gives the conditions of the existence of Hopf bifurcation for two cases. In this section, we investigate properties of these bifurcating periodic solutions from the positive constant steady state $E^*(S^*, I^*)$ of [system \(1\)](#) by employing the normal form theory and the center manifold theorem of partial functional differential equations (PFDEs) [39–42], these properties include the direction, stability and period. It's simple for mathematical calculation to mark $\tau_c = \tau_k^j (k = 0, k_0; j = 0, 1, 2, \dots)$.

Let $\hat{S}(t, x) = \bar{S}(\tau t, x)$, $\hat{I}(t, x) = \bar{I}(\tau t, x)$, then [system \(3\)](#) can be expressed as

$$\begin{cases} \frac{\partial \hat{S}(x, t)}{\partial t} = \tau d_1 \frac{\partial^2 \hat{S}(x, t)}{\partial x^2} + \tau [a_{11} \hat{S}(x, t) + a_{13} \hat{I}(x, t - 1)] \\ \quad + \tau \sum_{i+j+l \geq 2} \frac{1}{i!j!l!} f_{ijl}^{(1)}(0, 0, 0) \hat{S}^i(x, t) \hat{I}^j(x, t) \hat{I}^l(x, t - 1), \\ \frac{\partial \hat{I}(x, t)}{\partial t} = \tau d_2 \frac{\partial^2 \hat{I}(x, t)}{\partial x^2} + \tau [a_{21} \hat{S}(x, t) + a_{22} \hat{I}(x, t) + a_{23} \hat{I}(x, t - 1)] \\ \quad + \tau \sum_{i+j+l \geq 2} \frac{1}{i!j!l!} f_{ijl}^{(2)}(0, 0, 0) \hat{S}^i(x, t) \hat{I}^j(x, t) \hat{I}^l(x, t - 1). \end{cases} \tag{18}$$

In the space $\wp = C([-1, 0], X)$, let $\tau = \tau_c + \alpha (\alpha \in R)$,

$U = (u_1, u_2)^T$, $\hat{S}(x, t) = u_1(x, t)$, $\hat{I}(x, t) = u_2(x, t)$, then [system \(18\)](#) can be rewritten as:

$$\frac{dU(t)}{dt} = \tau_c D\Delta U(t) + L(\tau_c)U_t + F(U_t, \alpha). \tag{19}$$

Let $\phi = (\phi_1, \phi_2)^T \in \wp$, $U_t(\theta) = U(t+\theta)$, and $\phi(\theta) = U_t(\theta)$ for $\theta \in [-1, 0]$. Defining $L(b)(\cdot): R \times \wp \rightarrow X$ (b is τ_c or α) and $F: \wp \times R \rightarrow X$ as

$$L(b)(\phi) = b \begin{pmatrix} a_{11}\phi_1(0) + a_{13}\phi_2(-1) \\ a_{21}\phi_1(0) + a_{22}\phi_2(0) + a_{23}\phi_2(-1) \end{pmatrix},$$

and

$$F(\phi, \alpha) = \alpha D\Delta \phi(0) + L(\alpha)(\phi) + f(\phi, \alpha),$$

where

$$f(\phi, \alpha) = (\tau_c + \alpha) \begin{pmatrix} \sum_{i+j+l \geq 2} \frac{1}{i!j!l!} f_{ijl}^{(1)}(0, 0, 0) \phi_1^i(0) \phi_2^j(0) \phi_2^l(-1) \\ \sum_{i+j+l \geq 2} \frac{1}{i!j!l!} f_{ijl}^{(2)}(0, 0, 0) \phi_1^i(0) \phi_2^j(0) \phi_2^l(-1) \end{pmatrix}.$$

Next, the linear part of the [system \(19\)](#) is given by

$$\dot{U}(t) = \tau_c D\Delta U(t) + L(\tau_c)U_t. \tag{20}$$

From the conclusions of section II, an equilibrium of the [system \(20\)](#) is the origin, the corresponding characteristic equation of the [system \(20\)](#) at origin has two pairs of purely imaginary eigenvalues $\pm i\omega_0 \tau_c$, $\pm i\hat{\omega}_0 \tau_c$ for $k = 0$, and only a pair of purely imaginary eigenvalues $\pm i\omega_k \tau_c$

for $k \in N$. We account for purely imaginary eigenvalues $\pm iw_0 \tau_c$ for the case $k = 0$, and set $\Lambda_0 = \{iw_k \tau_c, -iw_k \tau_c\}$, ($k = 0, k_0$).

Considering the ordinary functional differential equation:

$$\dot{X}(t) = -\tau_c Dk^2 X(t) + L(\tau_c)(X_t). \tag{21}$$

For $\phi \in C([-1, 0], X)$, according to the Riesz representation theorem, there is a 2×2 matrix function $\eta(\theta, \tau_c)$ ($-1 \leq \theta \leq 0$), then we have [40]

$$-\tau_c Dk^2 \phi(0) + L(\tau_c)(\phi) = \int_{-1}^0 d[\eta(\theta, \tau_c)]\phi(\theta), \tag{22}$$

where

$$\eta(\theta, \tau_c) = \begin{cases} \tau_c \begin{pmatrix} -d_1 k^2 + a_{11} & 0 \\ a_{21} & -d_2 k^2 + a_{22} \end{pmatrix}, & \theta = 0, \\ 0, & \theta \in (-1, 0), \\ \tau_c \begin{pmatrix} 0 & -a_{13} \\ 0 & -a_{23} \end{pmatrix}, & \theta = -1. \end{cases} \tag{23}$$

For $\phi \in C([-1, 0], X)$, defining semigroup induced by the solution of the linear eq (20), and the infinitesimal generator $A(\tau_c)$ of the semigroup is

$$A(\tau_c)\phi(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^0 d_\theta \eta(\theta, \tau_c)\phi(\theta), & \theta = 0. \end{cases} \tag{24}$$

For $\psi \in C([0, 1], X)$, the formal adjoint operators of $A(\tau_c)$ is $A^*(\tau_c)$ which denotes [43]

$$A^*(\tau_c)\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^0 d_s \eta(s, \tau_c)\psi(-s), & s = 0. \end{cases} \tag{25}$$

Here, the bilinear pairing form associated $A(\tau_c)$ with $A^*(\tau_c)$ is

$$\begin{aligned} (\psi, \phi) &= \psi(0)\phi(0) - \int_{-1}^0 \int_{\xi=0}^0 \psi(\xi - \theta) d\eta(\theta, \tau_c)\phi(\xi) d\xi \\ &= \psi(0)\phi(0) - \tau_c \int_{-1}^0 \psi(\xi + 1) \begin{pmatrix} 0 & -a_{13} \\ 0 & -a_{23} \end{pmatrix} \phi(\xi) d\xi. \end{aligned} \tag{26}$$

On the basis of the discussion of section II, $A(\tau_c)$ has a pair purely imaginary eigenvalues $\pm iw_k \tau_c$, which are also eigenvalues of $A^*(\tau_c)$. Furthermore, the generalized eigenspaces of $A(\tau_c)$ and $A^*(\tau_c)$ associated with Λ_0 are the center subspaces P and P^* , respectively. P^* is the adjoint space of P and $\dim P = \dim P^* = 2$ [42].

By some computations, the following Lemma is directly given:

Lemma 2.3 A basis of P with Λ_0 is

$$q_1(\theta) = e^{iw_k\tau_c\theta}(1, \zeta)^T, \quad q_2(\theta) = \overline{q_1(\theta)}, \quad -1 \leq \theta \leq 0,$$

and a basis of P^* with Λ_0 is

$$q_1^*(s) = (1, \eta)e^{-iw_k\tau_c s}, \quad q_2^*(s) = \overline{q_1^*(s)}, \quad 0 \leq s \leq 1,$$

where

$$\zeta = \frac{(iw_k + d_1k^2 - a_{11})}{a_{13}e^{-iw_k\tau_c}}, \quad \eta = \frac{(iw_k + d_1k^2 - a_{11})}{a_{21}}. \tag{27}$$

$\Phi = (\Phi_1, \Phi_2)$ and $\Phi^* = (\Phi_1^*, \Phi_2^*)^T$ are obtained by separating the real and imaginary parts of $q_1(\theta)$ and $q_1^*(s)$, respectively. Obviously, Φ is the basis of P , Φ^* is the basis of P^* , and

$$\Phi_1(\theta) = \begin{pmatrix} \operatorname{Re}\{e^{iw_k\tau_c\theta}\} \\ \operatorname{Re}\{\zeta e^{iw_k\tau_c\theta}\} \end{pmatrix} = \begin{pmatrix} \cos w_k\tau_c\theta \\ \frac{(d_1k^2 - a_{11})\cos w_k\tau_c(\theta + 1) - w_k\sin w_k\tau_c(\theta + 1)}{a_{13}} \end{pmatrix},$$

$$\Phi_2(\theta) = \begin{pmatrix} \operatorname{Im}\{e^{iw_k\tau_c\theta}\} \\ \operatorname{Im}\{\zeta e^{iw_k\tau_c\theta}\} \end{pmatrix} = \begin{pmatrix} \sin w_k\tau_c\theta \\ \frac{(d_1k^2 - a_{11})\sin w_k\tau_c(\theta + 1) + w_k\cos w_k\tau_c(\theta + 1)}{a_{13}} \end{pmatrix}.$$

$$\Phi_1^*(s) = \begin{pmatrix} \operatorname{Re}\{e^{-iw_k\tau_c s}\} \\ \operatorname{Re}\{\eta e^{-iw_k\tau_c s}\} \end{pmatrix}^T = \begin{pmatrix} \cos w_k\tau_c s \\ \frac{w_k\sin w_k\tau_c s + (d_1k^2 - a_{11})\cos w_k\tau_c s}{a_{21}} \end{pmatrix}^T,$$

$$\Phi_2^*(s) = \begin{pmatrix} \operatorname{Im}\{e^{-iw_k\tau_c s}\} \\ \operatorname{Im}\{\eta e^{-iw_k\tau_c s}\} \end{pmatrix}^T = \begin{pmatrix} \sin w_k\tau_c s \\ \frac{w_k\cos w_k\tau_c s - (d_1k^2 - a_{11})\sin w_k\tau_c s}{a_{21}} \end{pmatrix}^T.$$

According to the bilinear pairing form Eq (26), we can compute:

$$\begin{aligned} (\Phi_1^*, \Phi_1) &= 1 + \frac{(d_1k^2 - a_{11})^2 \cos w_k\tau_c - w_k(d_1k^2 - a_{11})\sin w_k\tau_c}{a_{13}a_{21}} \\ &\quad + \frac{1}{2}\tau_c [a_{13}a_{21}(d_1k^2 - a_{11}) + a_{23}(d_1k^2 - a_{11})^2] \left(1 + \frac{\sin 2w_k\tau_c}{2w_k\tau_c}\right) \\ &\quad - \frac{1}{2}a_{13}a_{21}(\sin w_k\tau_c)^2 - \frac{1}{2}\tau_c a_{23}w_k^2 \left(1 - \frac{\sin 2w_k\tau_c}{2w_k\tau_c}\right), \end{aligned}$$

$$\begin{aligned} (\Phi_1^*, \Phi_2) &= \frac{(d_1k^2 - a_{11})^2 \sin w_k\tau_c + w_k(d_1k^2 - a_{11})\cos w_k\tau_c}{a_{13}a_{21}} \\ &\quad + \tau_c \frac{a_{23}w_k(d_1k^2 - a_{11})}{a_{13}a_{21}} + \frac{1}{2}\tau_c w_k \left(1 + \frac{\sin 2w_k\tau_c}{2w_k\tau_c}\right) \\ &\quad + \tau_c \frac{a_{13}a_{21}(d_1k^2 - a_{11}) + a_{23}[w_k^2 + (d_1k^2 - a_{11})^2]}{a_{13}a_{21}} \frac{(\sin w_k\tau_c)^2}{2w_k\tau_c}, \end{aligned}$$

$$\begin{aligned}
 (\Phi_2^*, \Phi_1) &= \frac{w_k(d_1 k^2 - a_{11}) \cos w_k \tau_c - w_k^2 \sin w_k \tau_c}{a_{13} a_{21}} \\
 &+ \tau_c \frac{a_{23} w_k (d_1 k^2 - a_{11})}{a_{13} a_{21}} - \frac{1}{2} \tau_c w_k \left(1 - \frac{\sin 2 w_k \tau_c}{2 w_k \tau_c} \right) \\
 &+ \tau_c \frac{a_{13} a_{21} (d_1 k^2 - a_{11}) - a_{23} [w_k^2 + (d_1 k^2 - a_{11})^2] (\sin w_k \tau_c)^2}{a_{13} a_{21} 2 w_k \tau_c}, \\
 (\Phi_2^*, \Phi_2) &= \frac{w_k (d_1 k^2 - a_{11}) \sin w_k \tau_c + (w_k)^2 \cos w_k \tau_c}{a_{13} a_{21}} \\
 &+ \frac{1}{2} \tau_c \left\{ \frac{1}{\tau_c} (\sin w_k \tau_c)^2 + \frac{a_{23} w_k^2}{a_{13} a_{21}} \left(1 + \frac{\sin 2 w_k \tau_c}{2 w_k \tau_c} \right) \right\} \\
 &+ \frac{1}{2} \tau_c \frac{a_{13} a_{21} (d_1 k^2 - a_{11}) - a_{23} (d_1 k^2 - a_{11})^2}{a_{13} a_{21}} \left(1 - \frac{\sin 2 w_k \tau_c}{2 w_k \tau_c} \right).
 \end{aligned}$$

Next, we construct a new basis Ψ for P^* , where $\Psi = (\Psi_1, \Psi_2)^T = (\Phi^*, \Phi)^{-1} \Phi^*$ and $(\Phi^*, \Phi) = (\Phi_t^*, \Phi_t)$, $(t, \ell = 1, 2)$. $(\Psi, \Phi) = I_2$ needs to be satisfied. In addition, $f_k = (\beta_k^1, \beta_k^2)$, where $\beta_k^1 = \begin{pmatrix} \cos kx \\ 0 \end{pmatrix}$, $\beta_k^2 = \begin{pmatrix} 0 \\ \cos kx \end{pmatrix}$. For $c = (c_1, c_2) \in C([-1, 0], X)$, we define $c \cdot f_k = c_1 \beta_k^1 + c_2 \beta_k^2$.

On the basis of the theory of decomposition of the phase space, we have $\wp = P_{CN\wp} + P_s\wp$, where $P_{CN\wp}$ is the center subspace of linear Eq (20),

$$P_{CN\wp}(\phi) = \Phi(\Psi, \langle \phi, f_k \rangle) \cdot f_k, \quad \phi \in \wp, \tag{28}$$

and $P_s\wp$ is the complement subspace of $P_{CN\wp}$.

Since the infinitesimal generator $A(\tau_c)$ is induced by the solution of Eq (20), then Eq (18) can be translated into:

$$\dot{U}_t = A(\tau_c) U_t + X_0 F(U_t, \alpha), \tag{29}$$

$$\text{where } X_0(\theta) = \begin{cases} 0, & \theta \in [-1, 0), \\ I, & \theta = 0. \end{cases}$$

According to the phase space decomposition $\wp = P_{CN\wp} + P_s\wp$ and Eq (28), the solution of Eq (19) is written as

$$U_t = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_k + h(x_1, x_2, \alpha), \tag{30}$$

where $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = (\Psi, \langle U_t, f_k \rangle)$, and $h(x_1, x_2, \alpha) \in P_s\wp$, $h(0,0,0) = 0$, $Dh(0,0,0) = 0$. Moreover, the solution of Eq (19) on center manifold is

$$U_t = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_k + h(x_1, x_2, 0). \tag{31}$$

Let $z = x_1 - ix_2$, $\Psi(0) = (\Psi_1(0), \Psi_2(0))^T$, and $q_1 = \Phi_1 + i\Phi_2$, thus

$$\Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_k = (\Phi_1, \Phi_2) \begin{pmatrix} \frac{z + \bar{z}}{2} \\ \frac{i(z - \bar{z})}{2} \end{pmatrix} \cdot f_k = \frac{1}{2} (q_1 z + \bar{q}_1 \bar{z}) \cdot f_k. \tag{32}$$

By using the previous variable substitution, Eq (31) can be transformed into:

$$U_t = \frac{1}{2} (q_1 z + \bar{q}_1 \bar{z}) \cdot f_k + W(z, \bar{z}), \tag{33}$$

where $W(z, \bar{z}) = h(\frac{z+\bar{z}}{2}, \frac{i(z-\bar{z})}{2}, 0)$, and setting

$$W(z, \bar{z}) = W_{20} \frac{z^2}{2} + W_{11} z\bar{z} + W_{02} \frac{\bar{z}^2}{2} + \dots \tag{34}$$

According to the conclusions of Ref. [42], z satisfies

$$\dot{z} = iw_k \tau_c z + g(z, \bar{z}), \tag{35}$$

where

$$g(z, \bar{z}) = (\Psi_1(0) - i\Psi_2(0)) \langle F(U_t, 0), f_k \rangle = (\Psi_1(0) - i\Psi_2(0)) \langle f(U_t, 0), f_k \rangle, \tag{36}$$

and setting

$$g(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11} z\bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{21} \frac{z^2 \bar{z}}{2} \dots \tag{37}$$

From $f(\phi, a)$ and Eq (31), it is easy to compute

$$\begin{aligned} \langle f(U_t, 0), f_k \rangle &= \frac{\tau_c}{4} \cdot \left(2f_{101}^{(1)} \zeta e^{-iw_k \tau_c} + f_{002}^{(1)} \zeta^2 e^{-2iw_k \tau_c} \right) \cdot \frac{1}{\pi} \int_0^\pi (\cos kx)^3 dx \cdot \frac{z^2}{2} \\ &+ \frac{\tau_c}{4} \cdot \left(f_{101}^{(1)} (\bar{\zeta} e^{iw_k \tau_c} + \zeta e^{-iw_k \tau_c}) + f_{002}^{(1)} \zeta \bar{\zeta} \right) \cdot \frac{1}{\pi} \int_0^\pi (\cos kx)^3 dx \cdot z\bar{z} \\ &+ \frac{\tau_c}{4} \cdot \left(2f_{101}^{(1)} \bar{\zeta} e^{iw_k \tau_c} + f_{002}^{(1)} \bar{\zeta}^2 e^{2iw_k \tau_c} \right) \cdot \frac{1}{\pi} \int_0^\pi (\cos kx)^3 dx \cdot \frac{\bar{z}^2}{2} \\ &+ \frac{\tau_c}{4} \cdot \left(2f_{101}^{(2)} \bar{\zeta} e^{iw_k \tau_c} + f_{002}^{(2)} \bar{\zeta}^2 e^{2iw_k \tau_c} \right) \cdot \frac{1}{\pi} \int_0^\pi (\cos kx)^3 dx \cdot \frac{\bar{z}^2}{2} \end{aligned}$$

$$\begin{aligned}
 & \left(\begin{aligned}
 & f_{101}^{(1)} \left\langle \left(W_{11}^{(2)}(-1) + \frac{1}{2} W_{20}^{(2)}(-1) \right) \cdot \cos kx, \cos kx \right\rangle \\
 & + f_{101}^{(1)} \left\langle \left(W_{11}^{(1)}(0) \xi e^{-i w_k \tau_c} + \frac{1}{2} W_{20}^{(1)}(0) \bar{\xi} e^{i w_k \tau_c} \right) \cdot \cos kx, \cos kx \right\rangle \\
 & + f_{002}^{(1)} \left\langle \left(W_{11}^{(2)}(-1) \xi e^{-i w_k \tau_c} + \frac{1}{2} W_{20}^{(2)}(-1) \bar{\xi} e^{i w_k \tau_c} \right) \cdot \cos kx, \cos kx \right\rangle \\
 & \quad + \frac{1}{8} f_{102}^{(1)} \left\langle (2 \xi \bar{\xi} + \xi^2 e^{-2i w_k \tau_c}) \cos^3 kx, \cos kx \right\rangle \\
 & f_{101}^{(2)} \left\langle \left(W_{11}^{(2)}(-1) + \frac{1}{2} W_{20}^{(2)}(-1) \right) \cdot \cos kx, \cos kx \right\rangle \\
 & + f_{101}^{(2)} \left\langle \left(W_{11}^{(1)}(0) \xi e^{-i w_k \tau_c} + \frac{1}{2} W_{20}^{(1)}(0) \bar{\xi} e^{i w_k \tau_c} \right) \cdot \cos kx, \cos kx \right\rangle \\
 & + f_{002}^{(2)} \left\langle \left(W_{11}^{(2)}(-1) \xi e^{-i w_k \tau_c} + \frac{1}{2} W_{20}^{(2)}(-1) \bar{\xi} e^{i w_k \tau_c} \right) \cdot \cos kx, \cos kx \right\rangle \\
 & \quad + \frac{1}{8} f_{102}^{(2)} \left\langle (2 \xi \bar{\xi} + \xi^2 e^{-2i w_k \tau_c}) \cos^3 kx, \cos kx \right\rangle
 \end{aligned} \right) \cdot \frac{z^2 \bar{z}}{2} + \dots,
 \end{aligned}$$

where

$$\left\langle W_{ij}^{(m)}(\theta), \cos kx \right\rangle = \frac{1}{\pi} \int_0^\pi W_{ij}^{(m)}(\theta)(x) \cos kx dx,$$

and $i, j = 0, 1, 2, \dots, m = 1, 2$.

Let $(\psi_1, \psi_2) = \Psi_1 - i\Psi_2$, then we can obtain

$$g_{20} = \begin{cases} 0, & k \in N, \\ \frac{\tau_c}{4} [(2f_{101}^{(1)} \xi e^{-i w_k \tau_c} + f_{002}^{(1)} \xi^2 e^{-2i w_k \tau_c}) \psi_1 + (2f_{101}^{(2)} \bar{\xi} e^{-i w_k \tau_c} + f_{002}^{(2)} \bar{\xi}^2 e^{-2i w_k \tau_c}) \psi_2], & k = 0, \end{cases}$$

$$g_{11} = \begin{cases} 0, & k \in N, \\ \frac{\tau_c}{4} \{ [f_{101}^{(1)} (\bar{\xi} e^{i w_k \tau_c} + \xi e^{-i w_k \tau_c}) + f_{002}^{(1)} \bar{\xi} \xi] \psi_1 + [f_{101}^{(2)} (\bar{\xi} e^{i w_k \tau_c} + \xi e^{-i w_k \tau_c}) + f_{002}^{(2)} \bar{\xi} \xi] \psi_2 \}, & k = 0, \end{cases}$$

$$g_{02} = \overline{g_{20}},$$

$$g_{21} = \tau_c \left(\begin{array}{l} f_{101}^{(1)} \left\langle \left(W_{11}^{(2)}(-1) + \frac{1}{2} W_{20}^{(2)}(-1) \right) \cdot \cos kx, \cos kx \right\rangle \\ + f_{101}^{(1)} \left\langle \left(W_{11}^{(1)}(0) \xi e^{-i w_k \tau_c} + \frac{1}{2} W_{20}^{(1)}(0) \bar{\xi} e^{i w_k \tau_c} \right) \cdot \cos kx, \cos kx \right\rangle \\ + f_{002}^{(1)} \left\langle \left(W_{11}^{(2)}(-1) \xi e^{-i w_k \tau_c} + \frac{1}{2} W_{20}^{(2)}(-1) \bar{\xi} e^{i w_k \tau_c} \right) \cdot \cos kx, \cos kx \right\rangle \\ + \frac{1}{8} f_{102}^{(1)} \left\langle (2 \xi \bar{\xi} + \xi^2 e^{-2 i w_k \tau_c}) \cos^3 kx, \cos kx \right\rangle \end{array} \right) \psi_1$$

$$+ \tau_c \left(\begin{array}{l} f_{101}^{(2)} \left\langle \left(W_{11}^{(2)}(-1) + \frac{1}{2} W_{20}^{(2)}(-1) \right) \cdot \cos kx, \cos kx \right\rangle \\ + f_{101}^{(2)} \left\langle \left(W_{11}^{(1)}(0) \xi e^{-i w_k \tau_c} + \frac{1}{2} W_{20}^{(1)}(0) \bar{\xi} e^{i w_k \tau_c} \right) \cdot \cos kx, \cos kx \right\rangle \\ + f_{002}^{(2)} \left\langle \left(W_{11}^{(2)}(-1) \xi e^{-i w_k \tau_c} + \frac{1}{2} W_{20}^{(2)}(-1) \bar{\xi} e^{i w_k \tau_c} \right) \cdot \cos kx, \cos kx \right\rangle \\ + \frac{1}{8} f_{102}^{(2)} \left\langle (2 \xi \bar{\xi} + \xi^2 e^{-2 i w_k \tau_c}) \cos^3 kx, \cos kx \right\rangle \end{array} \right) \psi_2, \quad k \in \{0, N\}.$$

Since the expression of g_{21} containing $W_{20}(\theta)$ and $W_{11}(\theta)$ for $\theta \in [-1, 0]$, it is necessary to compute them. From Eq (34), we can derive

$$\dot{W}(z, \bar{z}) = W_{20} z \dot{z} + W_{11} \dot{z} \bar{z} + W_{11} z \dot{\bar{z}} + W_{02} \bar{z} \dot{\bar{z}} + \dots, \tag{38}$$

$$A(\tau_c)W = A(\tau_c)W_{20} \frac{z^2}{2} + A(\tau_c)W_{11} z \bar{z} + A(\tau_c)W_{02} \frac{\bar{z}^2}{2}. \tag{39}$$

Meanwhile, from the conclusion of literature [42],

$$\dot{W} = A(\tau_c)W + H(z, \bar{z}), \tag{40}$$

where

$$H(z, \bar{z}) = H_{20} \frac{z^2}{2} + H_{11} z \bar{z} + H_{02} \frac{\bar{z}^2}{2} + \dots = X_0 f(U_t, 0) - \Phi(\Psi, \langle X_0 f(U_t, 0), f_k \rangle) \cdot f_k, \tag{41}$$

with $H_{ij} \in P^*$, $i, j = 0, 1, 2, \dots$

Therefore, from Eqs (35) and (37)–(41), the following form can be given by:

$$\begin{cases} (2i w_k \tau_c - A(\tau_c))W_{20} = H_{20}, \\ -A(\tau_c)W_{11} = H_{11}. \end{cases} \tag{42}$$

Because $A(\tau_c)$ has only two eigenvalues $\pm i w_k \tau_c$, Eq (42) has unique solution W_{ij} in the following form:

$$\begin{cases} W_{20} = (2i w_k \tau_c - A(\tau_c))^{-1} H_{20}, \\ W_{11} = -A(\tau_c)^{-1} H_{11}. \end{cases} \tag{43}$$

From Eq (41), for $-1 \leq \theta < 0$,

$$\begin{aligned} H(z, \bar{z}) &= -\Phi(\theta)\Psi(0)\langle f(U_t, 0), f_k \rangle \cdot f_k \\ &= -\left(\frac{q_1(\theta) + q_2(\theta)}{2}, \frac{q_1(\theta) - q_2(\theta)}{2i}\right) (\Psi_1(0), \Psi_2(0)) \cdot \langle f(U_t, 0), f_k \rangle \cdot f_k \\ &= -\frac{1}{2}(q_1(\theta)g_{20} + q_2(\theta)\overline{g_{02}}) \cdot f_k \frac{z^2}{2} - \frac{1}{2}(q_1(\theta)g_{11} + q_2(\theta)\overline{g_{11}}) \cdot f_k z\bar{z} + \dots \end{aligned}$$

Thus, for $-1 \leq \theta < 0$,

$$H_{20}(\theta) = \begin{cases} 0, & k \in N, \\ -\frac{1}{2}(q_1(\theta)g_{20} + q_2(\theta)\overline{g_{02}}) \cdot f_0, & k = 0, \end{cases} \tag{44}$$

$$H_{11}(\theta) = \begin{cases} 0, & k \in N, \\ -\frac{1}{2}(q_1(\theta)g_{11} + q_2(\theta)\overline{g_{11}}) \cdot f_0, & k = 0. \end{cases} \tag{45}$$

For $\theta = 0$, $H(z, \bar{z})(0) = f(U_t, 0) - \Phi(\Psi, \langle f(U_t, 0), f_k \rangle) \cdot f_k$, we have

$$H_{20}(0) = \begin{cases} \frac{\tau_c}{4} \begin{pmatrix} 2f_{101}^{(1)} \zeta e^{-i\omega_k \tau_c} + f_{002}^{(1)} \zeta^2 e^{-2i\omega_k \tau_c} \\ 2f_{101}^{(2)} \bar{\zeta} e^{-i\omega_k \tau_c} + f_{002}^{(2)} \bar{\zeta}^2 e^{-2i\omega_k \tau_c} \end{pmatrix} \cdot (\cos kx)^2, & k \in N, \\ \frac{\tau_c}{4} \begin{pmatrix} 2f_{101}^{(1)} \zeta e^{-i\omega_k \tau_c} + f_{002}^{(1)} \zeta^2 e^{-2i\omega_k \tau_c} \\ 2f_{101}^{(2)} \bar{\zeta} e^{-i\omega_k \tau_c} + f_{002}^{(2)} \bar{\zeta}^2 e^{-2i\omega_k \tau_c} \end{pmatrix} - \frac{1}{2}(q_1(0)g_{20} + q_2(0)\overline{g_{02}}) \cdot f_0, & k = 0, \end{cases} \tag{46}$$

$$H_{11}(0) = \begin{cases} \frac{\tau_c}{4} \begin{pmatrix} f_{101}^{(1)} (\bar{\zeta} e^{i\omega_k \tau_c} + \zeta e^{-i\omega_k \tau_c}) + f_{002}^{(1)} \zeta \bar{\zeta} \\ f_{101}^{(2)} (\bar{\zeta} e^{i\omega_k \tau_c} + \zeta e^{-i\omega_k \tau_c}) + f_{002}^{(2)} \zeta \bar{\zeta} \end{pmatrix} \cdot (\cos kx)^2, & k \in N, \\ \frac{\tau_c}{4} \begin{pmatrix} f_{101}^{(1)} (\bar{\zeta} e^{i\omega_k \tau_c} + \zeta e^{-i\omega_k \tau_c}) + f_{002}^{(1)} \zeta \bar{\zeta} \\ f_{101}^{(2)} (\bar{\zeta} e^{i\omega_k \tau_c} + \zeta e^{-i\omega_k \tau_c}) + f_{002}^{(2)} \zeta \bar{\zeta} \end{pmatrix} - \frac{1}{2}(q_1(0)g_{11} + q_2(0)\overline{g_{11}}) \cdot f_0, & k = 0. \end{cases} \tag{47}$$

Based on the definition of infinitesimal generator $A(\tau_c)$, then Eq (42) is transformed into

$$\dot{W}_{20}(\theta) = 2i\omega_k \tau_c W_{20}(\theta) + \frac{1}{2}(q_1(\theta)g_{20} + q_2(\theta)\overline{g_{02}}) \cdot f_0, \tag{48}$$

and $-1 \leq \theta < 0$.

From $q_1(\theta) = q_1(0)e^{i\omega_k \tau_c \theta}$, $-1 \leq \theta \leq 0$, we have

$$W_{20}(\theta) = \frac{1}{2} \left[\frac{i g_{20}}{\omega_k \tau_c} q_1(\theta) + \frac{i \overline{g_{02}}}{3\omega_k \tau_c} q_2(\theta) \right] \cdot f_k + C_1 e^{2i\omega_k \tau_c \theta}, \tag{49}$$

further we obtain

$$C_1 = \begin{cases} W_{20}(0), & k \in N, \\ W_{20}(0) - \frac{1}{2} \left[\frac{ig_{20}}{w_k \tau_c} q_1(0) + \frac{\overline{ig_{02}}}{3w_k \tau_c} q_2(0) \right] \cdot f_0, & k = 0. \end{cases} \quad (50)$$

For $k = 0, \theta = 0$, in the light of the definition of $A(\tau_c)$ and Eq (49), the first Eq of Eq (42) becomes

$$\begin{aligned} & 2iw_k \tau_c \left[\frac{1}{2} \left(\frac{ig_{20}}{w_k \tau_c} q_1(0) + \frac{\overline{ig_{02}}}{3w_k \tau_c} q_2(0) \right) \cdot f_0 + C_1 \right] \\ & - \tau_c D \frac{\partial^2}{\partial x^2} \left[\frac{1}{2} \left(\frac{ig_{20}}{w_k \tau_c} q_1(0) + \frac{\overline{ig_{02}}}{3w_k \tau_c} q_2(0) \right) \cdot f_0 + C_1 \right] \\ & - L(\tau_c) \left[\frac{1}{2} \left(\frac{ig_{20}}{w_k \tau_c} q_1(\theta) + \frac{\overline{ig_{02}}}{3w_k \tau_c} q_2(\theta) \right) \cdot f_0 + C_1 e^{2iw_k \tau_c \theta} \right] \\ & = \frac{\tau_c}{4} \left(2f_{101}^{(1)} \zeta e^{-iw_k \tau_c} + f_{002}^{(1)} \zeta^2 e^{-2iw_k \tau_c} \right) - \frac{1}{2} (q_1(0)g_{20} + q_2(0)\overline{g_{02}}) \cdot f_0. \end{aligned}$$

So we can derive

$$2iw_k \tau_c C_1 - \tau_c D \frac{\partial^2}{\partial x^2} C_1 - L(\tau_c)(C_1 e^{2iw_k \tau_c \theta}) = \frac{\tau_c}{4} \begin{pmatrix} 2f_{101}^{(1)} \zeta e^{-iw_k \tau_c} + f_{002}^{(1)} \zeta^2 e^{-2iw_k \tau_c} \\ 2f_{101}^{(2)} \zeta e^{-iw_k \tau_c} + f_{002}^{(2)} \zeta^2 e^{-2iw_k \tau_c} \end{pmatrix} (\cos kx)^2. \quad (51)$$

From Eq (51), the formula of C_1 can be derived

$$C_1 = \frac{1}{4} \begin{pmatrix} 2iw_k + d_1 k^2 - a_{11} & -a_{13} e^{-2iw_k \tau_c} \\ -a_{21} & 2iw_k + d_2 k^2 - a_{22} - a_{23} e^{-2iw_k \tau_c} \end{pmatrix}^{-1} J_1, \quad (52)$$

where

$$J_1 = \begin{pmatrix} 2f_{101}^{(1)} \zeta e^{-iw_k \tau_c} + f_{002}^{(1)} \zeta^2 e^{-2iw_k \tau_c} \\ 2f_{101}^{(2)} \zeta e^{-iw_k \tau_c} + f_{002}^{(2)} \zeta^2 e^{-2iw_k \tau_c} \end{pmatrix} (\cos kx)^2.$$

For $-1 \leq \theta < 0$, similar to the above case, $W_{11}(\theta)$ can be obtained

$$\dot{W}_{11}(\theta) = \frac{1}{2} (q_1(\theta)g_{11} + q_2(\theta)\overline{g_{11}}) \cdot f_0, \quad (53)$$

$$W_{11}(\theta) = \frac{1}{2} \left(\frac{-ig_{11}}{w_k \tau_c} q_1(\theta) + \frac{\overline{ig_{11}}}{w_k \tau_c} q_2(\theta) \right) \cdot f_k + C_2, \quad (54)$$

further we derive

$$C_2 = \frac{1}{4} \begin{pmatrix} d_1 k^2 - a_{11} & -a_{13} \\ -a_{21} & d_2 k^2 - a_{22} - a_{23} \end{pmatrix}^{-1} J_2, \quad (55)$$

where

$$J_2 = \left(\begin{matrix} f_{101}^{(1)}(\bar{\zeta} e^{i w_k \tau_c} + \zeta e^{-i w_k \tau_c}) + f_{002}^{(1)} \bar{\zeta} \zeta \\ f_{101}^{(2)}(\bar{\zeta} e^{i w_k \tau_c} + \zeta e^{-i w_k \tau_c}) + f_{002}^{(2)} \bar{\zeta} \zeta \end{matrix} \right) (\cos kx)^2.$$

Through the above calculations of $W_{20}(\theta)$ and $W_{11}(\theta)$, we obtain the expression of g_{21} . Consequently, in order to determine the properties of Hopf bifurcating period solutions at the critical value τ_c , we can compute the following values:

$$\begin{aligned} c_1(0) &= \frac{i}{2w_k \tau_c} \left(g_{20} g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{1}{2} g_{21}, \\ \mu_2 &= -\frac{\text{Re}(c_1(0))}{\text{Re}(\lambda'(\tau_k^j))}, \\ \beta_2 &= 2\text{Re}(c_1(0)), \\ T_2 &= -\frac{1}{w_k \tau_c} (\text{Im}(c_1(0)) + \mu_2 \text{Im}(\lambda'(\tau_k^j))). \end{aligned} \tag{56}$$

$\mu_2 > 0$ ($\mu_2 < 0$) determines the direction of the Hopf bifurcation is supercritical ($\tau > \tau_c$) (subcritical ($\tau < \tau_c$)); if $\beta_2 < 0$ ($\beta_2 > 0$) indicates that the bifurcating period solutions on center manifold are asymptotically stable (unstable); furthermore, T_2 can determine the period of the bifurcating period solutions, namely, $T_2 < 0$ ($T_2 > 0$) represents the decrease (increase) of the period.

Numerical results

Compared with the theoretical analyses, we perform a series of extensive numerical simulations of the spatiotemporal epidemic model with nonlinear incidence rate in one-dimensional space, and investigate the incubation period how to affect the spread of epidemics. We solve the numerical solutions of [system \(1\)](#) by using Matlab. The reaction-diffusion system is solved in a discrete domain with $N_x \times N_y$ lattice sites. The Laplacian describing diffusion is approximated by using finite differences, and we also discretize the time evolution.

In case $k = 0$, we set $d_1 = 6$, $d_2 = 1$, $A = 1$, $\beta = 32$, $\mu = 1.8$, $d = 1$, the equilibrium is $E^* = (S^*, I^*) = (0.43, 0.20)$. By some calculations, $\tau_0^0 = 1.33$, $c_1(0) = -9.81 + 22.15i$ are obtained. Through the formulae of properties of Hopf bifurcating period solutions in section III, we get $\mu_2 > 0$, $\beta_2 < 0$ and $T_2 > 0$. These parameter values shows E^* is asymptotically stable for $0 \leq \tau < \tau_c$. With the increase of τ , E^* loses its stability and Hopf bifurcation occurs at critical point τ_c , these bifurcating period solutions are stable, the direction of bifurcation is forward and the period increases, which are presented in [Fig 1](#).

In case $k = 1$, setting $d_1 = 6$, $d_2 = 1$, $A = 1$, $\beta = 32$, $\mu = 1.8$, $d = 1$, then the equilibrium is $E^* = (S^*, I^*) = (0.43, 0.20)$. Furthermore, by using the formulae derived in section III, we compute $\tau_1^0 = 0.45$, $c_1(0) = 2.30 \times 10^2 - 7.55 \times 10^1 i$. By computing the formulae (58), $\mu_2 > 0$, $\beta_2 < 0$ and $T_2 < 0$ are obtained, which indicates that these bifurcating period solutions are stable, the direction of Hopf bifurcation is forward, and the period decreases. These phenomena are showed in [Fig 2](#).

Discussion

In this study, the characteristic equation at the positive constant steady state $E^*(S^*, I^*)$ is derived. In order to study the influence of incubation period on epidemic transmission, we choose time delay τ as a bifurcation parameter. Moreover, we get the two classes conditions of

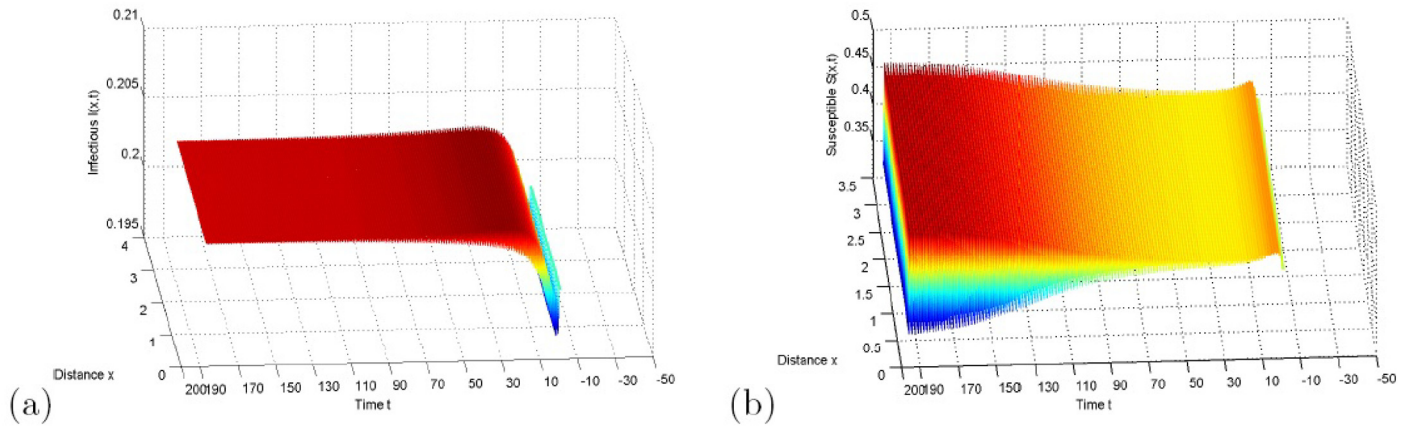


Fig 1. Hopf bifurcation with $k = 0$. (a) The constant steady state E^* is asymptotically stable for $\tau = 1.2$ with initial conditions $S(x, t) = 0.42, I(x, t) = 0.20, (x, t) \in [0, \pi] \times [-1.2, 0]$; (b) The bifurcating periodic solutions are asymptotically stable for $\tau = 1.6$ with initial conditions $S(x, t) = 0.42, I(x, t) = 0.20, (x, t) \in [0, \pi] \times [-1.6, 0]$.

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the existence of Hopf bifurcation: one is the absence of diffusion $k = 0$, the other is the presence of diffusion $k = k_0 \in N$. With increasing of parameter τ , the stability of positive constant steady state $E^*(S^*, I^*)$ will change, and Hopf bifurcation will concurrently occur in system (1) at the critical point $\tau_c(\tau_0^j \text{ or } \tau_{k_0}^j)$. In the following, we obtain the properties of bifurcating period solutions including direction, stability and period by utilizing the normal form theory and the center manifold theorem of partial functional differential equations (PFDs).

It should be noted that spatial pattern may be found in epidemic model (1). Based on pattern dynamics of model (1), one can obtain the pattern structures in different parameters space [44, 45]. In this case, we can reveal the distributions of disease with high density or low density and thus provide useful control measures to eliminate the disease.

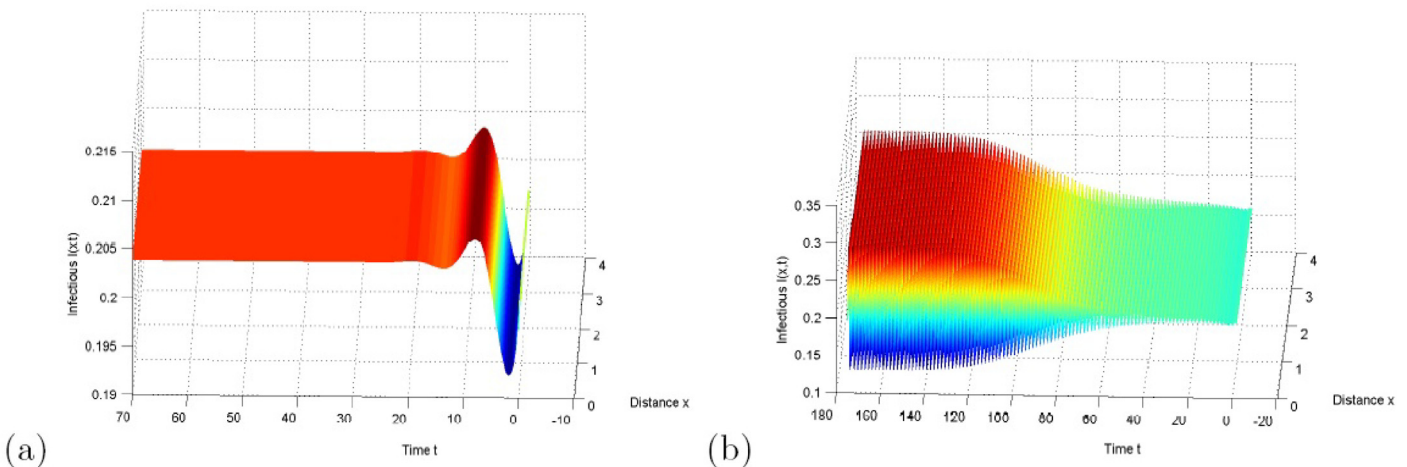


Fig 2. Hopf bifurcation with $k = 1$. (a) When $\tau = 0.3$, the constant steady state E^* is asymptotically stable with initial conditions $S(x, t) = 0.42, I(x, t) = 0.20, (x, t) \in [0, \pi] \times [-0.3, 0]$; (b) When $\tau = 1.5$, the bifurcating periodic solutions are asymptotically stable with initial conditions $S(x, t) = 0.42, I(x, t) = 0.20, (x, t) \in [0, \pi] \times [-1.5, 0]$.

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Conclusion

The numerical results validate our theoretical findings, which show that the length of the incubation period have significant impacts on epidemic transmission. The biennial outbreaks of measles is the signature of an endemic infectious disease, which becomes non-endemic if there were a minor increase in infectivity or a decrease in the length of the incubation period [15]. Based on this paper, we provide a possible mechanism to explain the recurrent outbreak of disease.

Supporting Information

S1 Fig. Hopf bifurcation with $k = 0$. (a) The constant steady state E^* is asymptotically stable for $\tau = 1.2$ with initial conditions $S(x, t) = 0.42, I(x, t) = 0.20, (x, t) \in [0, \pi] \times [-1.2, 0]$; (b) The bifurcating periodic solutions are asymptotically stable for $\tau = 1.6$ with initial conditions $S(x, t) = 0.42, I(x, t) = 0.20, (x, t) \in [0, \pi] \times [-1.6, 0]$.
(EPS)

S2 Fig. Hopf bifurcation with $k = 1$. (a) When $\tau = 0.3$, the constant steady state E^* is asymptotically stable with initial conditions $S(x, t) = 0.42, I(x, t) = 0.20, (x, t) \in [0, \pi] \times [-0.3, 0]$; (b) When $\tau = 1.5$, the bifurcating periodic solutions are asymptotically stable with initial conditions $S(x, t) = 0.42, I(x, t) = 0.20, (x, t) \in [0, \pi] \times [-1.5, 0]$.
(EPS)

S1 File. Transversality condition. The relationship between real part of eigenvalues and time delay.
(PDF)

Author Contributions

Conceived and designed the experiments: LPS XQD LPF QS. Performed the experiments: LPS XQD LPF QS. Analyzed the data: LPS XQD LPF QS. Contributed reagents/materials/analysis tools: LPS XQD LPF QS. Wrote the paper: LPS XQD LPF QS.

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