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Research article

Improved exponential type variance estimators for population utilizing supplementary information

Mujeeb Hussain [a,*], Qamruz Zaman [a], Hijaz Ahmad [b,c], Olayan Albalawi [d], Soofia Iftikhar [e]

[a] Department of Statistics, University of Peshawar, Pakistan
[b] Near East University, Operational Research Center in Healthcare, TRNC Mersin 10, Nicosia, 99138, Turkey
[c] Department of Computer Science and Mathematics, Lebanese American University, Beirut, Lebanon
[d] Department of Statistics, Faculty of Science, University of Tabuk, Tabuk, Saudi Arabia
[e] Department of Statistics, Shaheed Benazir Bhutto Women University, Peshawar, KP, Pakistan

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ABSTRACT

This paper contributes to the existing literature on variance estimators by utilizing supplementary information. The variance estimation problem of a finite population is a significant matter as sometimes, it is tough to control the variation. For this purpose, an optimum family of exponential variance estimators is suggested under simple random sampling. Moreover, different specific members of the proposed estimators are identified by incorporating various known characteristics of the supplementary variable in the suggested generalized class of estimators. The derivations for the expressions of bias as well as mean square error (MSE) of the proposed estimators are conducted. The suggested family of estimators is studied in different situations by using sets of real data and simulation studies for their performance. To evaluate the efficiency of the suggested estimators, R software is used for the analysis. The study compares the performance of the proposed estimators against the traditional estimators. The theoretical and numerical comparisons show that the estimators suggested in the study are superior in efficiency as compared to the existing estimators.

1. Introduction

Practically, it is not easy to obtain complete information about a population. Therefore, conclusions are based on sample data for the whole population. A sampling survey is carried out to gain maximum information for the population characteristics to improve the efficiency of the estimators in a study in minimum time, cost and less human effort [1]. It is very often that in numerous populations, variation or extreme observations are present and by excluding such important information, the estimates of the population parameters lead to misleading. As a result, the estimates will be overestimated or underestimated. Variance estimation plays a significant role as naturally two objects in any category are not the same. In daily activities, variation exists everywhere. Several examples are present in our daily life i.e. blood pressure, temperature, customers' reaction regarding products and fuel prices, etc. For the estimation of the variance of a finite population, many researchers used the sample variance in survey sampling. Ref. [2] proposed variance estimators

* Corresponding author.
E-mail addresses: mujeebhussain.stat@gmail.com (M. Hussain), cricsportsresearchgroup@gmail.com (Q. Zaman), ahmad.hijaz@uninettuno.it (H. Ahmad), oalbalwi@ut.edu.sa (O. Albalawi), soofia.iftikhar@sbbwu.edu.pk (S. Iftikhar).

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under different sampling techniques of the finite population.

It frequently happens that using an auxiliary variable in a sampling survey is always found to be helpful. Such variable has some kind of strong negative or positive correlation with the study variable, which increase the efficiency of the estimates. Utilizing such information, many methods in sampling surveys are presently used for the variance estimation of the population. To increase the estimators' efficiency for the variance estimation of a population, the use of supplementary variable was initiated [3]. When estimating the parameters of the population such as variance, mean or total, different estimators are used by several authors. These estimators include regression, ratio, exponential and product, etc. and are utilized under different sampling designs using supplementary information [4]. The methods of auxiliary information describe certain estimation techniques that are very suitable in minimizing the variance of the estimators. Subsequently, using supplementary variables, many researchers have developed a variety of variance estimators for the population. Ref. [5] developed ratio and regression estimators for population variance utilizing supplementary information.

Using known parameters of the supplementary variable, several estimators are developed. These parameters values comprise variance, coefficient of variation and kurtosis etc. Incorporating the known characteristics of the supplementary variable, different ratio-type exponential-type estimators were proposed by Refs. [6,7], and several ratio estimators were proposed by Ref. [8]. Using different situations and sampling techniques, a number of authors put their efforts into using auxiliary information, so that their estimation methods become more fruitful; for details see Refs. [9–14].

In the present study, the researchers suggest an improved family of estimators for the variance of a population using the known parameters of the supplementary variable such as coefficient of variation, quartile deviation, quartiles and median. The bias and MSE of the recommended optimum estimators are derived up to the first order of approximation. Based on theoretical and numerical comparisons, the applications of the proposed family estimator are elaborated. Therefore, the suggested novel estimators have the potential to be helpful in a variety of applications and it offer a fresh and significant contribution regarding the estimation of population variance.

Notations

Let a finite population $\omega = \{\omega_1, \omega_2, \omega_3, ..., \omega_N\}$ of N identifiable elements. Suppose Y and X denotes the study and auxiliary variables respectively. From the population of ω elements, a sample of size n elements are selected without replacement. Let $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ shows the means of the sample observations corresponding to the true means $(\overline{Y}, \overline{X})$ where y_i and x_i shows the values of the study and auxiliary variables. The unbiased estimators of $S_Y = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left(y_i - \overline{Y}\right)^2}$ and $S_X = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \overline{X}\right)^2}$ are $s_Y = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(y_i - \overline{y}\right)^2}$ and $s_X = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2}$ respectively. The coefficient of variations is denoted by $C_y = \frac{S_y}{\overline{Y}}$ and $C_x = \frac{S_x}{\overline{X}}$ for Y and X respectively. Where S_Y and S_X denotes the population standard deviations for study and auxiliary variable respectively while s_Y and s_X are the sample standard deviations for sample for study and auxiliary variable respectively.

Let ρ_{yx} be the population correlation coefficient between the study variable and the auxiliary variable.

To obtain the bias and MSE of the proposed estimators, the following error terms are used:

$$s_y^2 = S_y^2(1+e_0), s_x^2 = S_x^2(1+e_1), \text{Where } E(e_0) = E(e_1) = 0,$$

$$E\left(e_0^2\right) = \gamma(\lambda_{40} - 1), E\left(e_1^2\right) = \gamma(\lambda_{04} - 1), E(e_0 e_1) = \gamma(\lambda_{22} - 1), \gamma = \frac{1}{n}$$

Also $\lambda_{rs} = \frac{\mu_{rs}}{\mu_{20}^{r/2}\mu_{02}^{s/2}}, \mu_{rs} = \frac{\sum_{i=1}^{N}(Y_i - \overline{Y})^r(X_i - \overline{X})^s}{N-1}$ $\lambda_{22}^* = (\lambda_{22} - 1), \lambda_{04}^* = (\lambda_{04} - 1), \lambda_{40}^* = (\lambda_{40} - 1)$ $\lambda_{40} = \beta_{2(y)}$, $\lambda_{04} = \beta_{2(x)}$ shows the coefficient of kurtosis of the population [15–17].

2. Literature review

(1) The traditional variance estimator is given in equation (1) as:

$$v_0 = s_y^2 \tag{1}$$

The variance of v_0 is given in equation (2) as:

$$Var(v_0) = \gamma S_y^4 \lambda_{40}^* \tag{2}$$

(2) The classical regression estimator v_{Isa1} proposed by Isaki [5] is provided in equation (3) as:

$$v_{Isa1} = s_y^2 + b_{\left(s_y^2, s_x^2\right)}\left(S_x^2 - s_x^2\right) \tag{3}$$

where $b_{\left(s_y^2, s_x^2\right)} = \frac{s_y^2 \lambda_{22}^*}{s_x^2 \lambda_{04}^*}$ is the sample regression coefficient.

In equation (4), the MSE of v_{Isa1} is given as:

$$MSE(v_{Isa1}) \cong \gamma S_y^4 \lambda_{40}^* \left(1 - \rho^{*2}\right) \quad (4)$$

Where $\rho^* = \frac{\lambda_{22}^*}{\sqrt{\lambda_{40}^*}\sqrt{\lambda_{04}^*}}$:

(3) The [18] suggested a ratio-type estimator v_{US}, defined in equation (5):

$$v_{US} = s_y^2 \left(\frac{S_x^2 + \lambda_{04}}{s_x^2 + \lambda_{04}}\right) \quad (5)$$

The bias of the estimator v_{US} is defined in equation (6) as:

$$Bias(v_{US}) \cong \gamma S_y^2 g_0 \left(g_0 \lambda_{04}^* - \lambda_{22}^*\right) \quad (6)$$

The MSE of the estimator v_{US} is given in equation (7):

$$MSE(v_{US}) \cong \gamma S_y^4 \left(\lambda_{40}^* + g_0^2 \lambda_{04}^* - 2 g_0 \lambda_{22}^*\right) \quad (7)$$

where $g_0 = \frac{S_x^2}{S_x^2 + \lambda_{04}}$:

(4) The [6] proposed the following ratio estimators $v_{KCj} (j = 1, 2, 3)$ given in equations (8)–(10):

$$v_{KC1} = s_y^2 \left(\frac{S_x^2 + C_x}{s_x^2 + C_x}\right) \quad (8)$$

$$v_{KC2} = s_y^2 \left(\frac{\lambda_{04} S_x^2 + C_x}{\lambda_{04} s_x^2 + C_x}\right) \quad (9)$$

$$v_{KC3} = s_y^2 \left(\frac{C_x S_x^2 + \lambda_{04}}{C_x s_x^2 + \lambda_{04}}\right) \quad (10)$$

The bias of v_{KCj} is given in equation (11):

$$Bias\left(v_{KCj}\right) \cong \gamma S_y^2 g_j \left(g_j \lambda_{04}^* - \lambda_{22}^*\right) \quad (11)$$

The MSE of v_{KCj} is given in equation (12):

$$MSE\left(v_{KCj}\right) \cong \gamma S_y^4 \left(\lambda_{40}^* + g_j^2 \lambda_{04}^* - 2 g_j \lambda_{22}^*\right) \quad (12)$$

where

$$g_1 = \frac{S_x^2}{S_x^2 + C_x}, g_2 = \frac{\lambda_{04} S_x^2}{\lambda_{04} S_x^2 + C_x}, g_3 = \frac{C_x S_x^2}{C_x S_x^2 + \lambda_{04}}$$

(5) The [19] also suggested the following modified ratio-type estimator v_{Mr} for variance and is given in equation (13):

$$v_{M_r} = s_y^2 \left(\frac{S_x^2 \lambda_{04} + M_x^2}{s_x^2 \lambda_{04} + M_x^2}\right) \quad (13)$$

The expression for bias v_{Mr} is defined in equation (14):

$$Bias(v_{M_r}) = \frac{1-f}{n} S_y^4 \lambda_{04}^* \varphi \left(\varphi - \frac{\lambda_{22}^*}{\lambda_{04}^*}\right) \quad (14)$$

The MSE of the estimator v_{Mr} is given in equation (15):

$$MSE(v_{M_r}) = \frac{1-f}{n} S_y^4 \left[\lambda_{40}^* + \varphi \lambda_{04}^* \left(\varphi - 2\frac{\lambda_{22}^*}{\lambda_{04}^*}\right)\right] \quad (15)$$

Where $f = \frac{n}{N}$, shows the sampling fraction and $\varphi = \frac{\lambda_{04} S_x^2}{\left(\lambda_{04} S_x^2 + M_x^2\right)}$:

(6) The [20] defined a ratio-type variance estimator v_{Isa2}, and is given in equation (16):

$$v_{Isa2} = s_y^2\left(\frac{S_x^2}{s_x^2}\right) \tag{16}$$

The bias of the estimator v_{Isa2} is given in equation (17):

$$Bias(v_{Isa2}) \cong \gamma S_y^2\left(\lambda_{04}^* - \lambda_{22}^*\right) \tag{17}$$

The expressions for MSE of v_{Isa2} is defined in equation (18):

$$MSE(v_{Isa2}) \cong \gamma S_y^4\left(\lambda_{40}^* + \lambda_{04}^* - 2\lambda_{22}^*\right) \tag{18}$$

The product estimator denoted by v_{Prd} and is defined in equation (19):

$$v_{Prd} = s_y^2\frac{s_x^2}{S_x^2} \tag{19}$$

The MSE of the estimator v_{Prd} is defined in equation (20):

$$MSE(v_{Prd}) \cong \gamma S_y^4\left(\lambda_{40}^* + \lambda_{04}^* + 2\lambda_{22}^*\right) \tag{20}$$

(7) The [21] suggested the ratio and product exponential-type variance estimators v_{Sr} and v_{Sp} respectively given in equations (21) and (22):

$$v_{Sr} = s_y^2 \exp\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right) \tag{21}$$

$$\text{and } v_{Sp} = s_y^2 \exp\left(\frac{s_x^2 - S_x^2}{s_x^2 + S_x^2}\right) \tag{22}$$

The MSEs of the estimator v_{Sr} is given in equation (23):

$$MSE(v_{Sr}) \cong \gamma S_y^4\left(\lambda_{40}^* + \frac{\lambda_{04}^*}{4} - \lambda_{22}^*\right) \tag{23}$$

Similarly, the MSEs of v_{Sp} is provided in equation (24):

$$MSE(v_{Sp}) \cong \gamma S_y^4\left(\lambda_{40}^* + \frac{\lambda_{04}^*}{4} + \lambda_{22}^*\right) \tag{24}$$

(8) The [22] suggested a hybrid type estimator v_{Sh}, given in equation (25):

$$v_{Sh} = s_y^2\left\{\alpha \exp\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right) + (1-\alpha)\exp\left(\frac{s_x^2 - S_x^2}{s_x^2 + S_x^2}\right)\right\} \tag{25}$$

The MSE of v_{Sh} at the minimum value of $\alpha = \frac{1}{2} + \frac{\lambda_{22}^*}{\lambda_{04}^*}$ is given in equation (26):

$$MSE(v_{Sh})_{\min} = \gamma S_y^4\left[\lambda_{40}^* - \frac{\lambda_{22}^{*2}}{\lambda_{40}^*}\right] \tag{26}$$

3. Optimum estimators

Motivated by Refs. [19,21,22], exponential family of optimum estimators v_{Pk} is proposed for estimating the variance of a finite population under simple random sampling, given in equation (27) as:

$$v_{Pk} = s_y^2\left[\tau_1\left(\frac{S_x^2}{s_x^2}\right) + \tau_2\left(\frac{s_x^2}{S_x^2}\right)\right]\exp\left[\frac{a(G-T)}{a(G+T)+2b}\right], \text{k} = 1, 2, 3, 4, 5, 6 \tag{27}$$

where τ_1 and τ_2 are constants which are unknown and their values are to be obtained, while a and b take the values (−1,0,1) in order to generate different types of estimators.

Also $G = (C_1 + C_2)S_x^2 + C_3 s_x^2$ and $T = (C_3 + C_2)S_x^2 + C_1 s_x^2$:

Where $C_1, C_2,$ and C_3 denotes different functions of the parameter of the supplementary variable such as quartiles, coefficient of variation, median, etc.

Rewriting the above equation in error terms and putting the values of G and T, we obtained equation (28):

$$v_{Pk} = S_y^2(1+e_0)\left[\tau_1\left(\frac{S_x^2}{S_x^2(1+e_1)}\right)+\tau_2\left(\frac{S_x^2(1+e_1)}{S_x^2}\right)\right] \exp\left[\frac{a\{(C_1+C_2)S_x^2+C_3s_x^2-(C_3+C_2)S_x^2-s_x^2C_1\}}{a\{(C_1+C_2)S_x^2+C_3s_x^2+(C_3+C_2)S_x^2+s_x^2C_2\}+2b}\right] \tag{28}$$

Using Taylor's Expansion and ignoring terms of higher order, we get equation (29):

$$v_{Pk} \cong S_y^2(1+e_0)\left[\tau_1\left(1-e_1+e_1^2\right)+\tau_2(1+e_1)\right]\exp\left[\frac{\theta_1 e_1}{2\left(1+\frac{\theta_2}{2}e_1\right)}\right] \tag{29}$$

The expressions for θ_1 and θ_2 are given in equations (30) and (31) respectively as:

$$\theta_1 = \frac{a(C_3-C_1)S_x^2}{\{a(C_3+C_2+C_1)S_x^2+b\}}, \tag{30}$$

and

$$\theta_2 = \frac{a(C_3+C_1)S_x^2}{\{a(C_3+C_2+C_1)S_x^2+b\}} \tag{31}$$

Simplifying equation (29), we obtained equation (32):

$$v_{Pk} \cong S_y^2(1+e_0)\left[\tau_1\left(1-e_1+e_1^2\right)+\tau_2(1+e_1)\right]\exp\left[\frac{\theta_1 e_1}{2}-\frac{\theta_1\theta_2}{4}e_1^2\right] \tag{32}$$

Using exponential series in the above equation, we obtained equation (33):

$$v_{Pk} \cong S_y^2(1+e_0)\left[\tau_1\left(1-e_1+e_1^2\right)+\tau_2(1+e_1)\right]\left[1+\frac{\theta_1 e_1}{2}-\frac{\theta_1\theta_2}{4}e_1^2+\frac{\theta_1^2 e_1^2}{8}\right] \tag{33}$$

Subtracting S_y^2 from both sides of equation (33), we get equation (34):

$$\left(v_{Pk}-S_y^2\right) \cong S_y^2\left[\left\{\begin{array}{l}\tau_1\left(1+e_0+\left(\frac{\theta_1}{2}-1\right)(e_1+e_0e_1)+\left(1-\frac{\theta_1}{2}-\frac{\theta_1\theta_2}{4}+\frac{\theta_1^2}{8}\right)e_1^2\right)\\ +\tau_2\left(1+e_0+\left(\frac{\theta_1}{2}+1\right)(e_1+e_0e_1)+\frac{\theta_1}{2}\left(1-\frac{\theta_2}{2}+\frac{\theta_1}{4}\right)e_1^2\right)\end{array}\right\}-1\right] \tag{34}$$

Taking equation (34), applying expectation both sides, we obtained equation (35):

$$Bias(v_{Pk}) \cong \frac{S_y^2}{n}\left[\left\{\begin{array}{l}\tau_1\left(n+\left(\frac{\theta_1}{2}-1\right)\lambda_{22}^*+\left(1-\frac{\theta_1}{2}-\frac{\theta_1\theta_2}{4}+\frac{\theta_1^2}{8}\right)\lambda_{04}^*\right)\\ +\tau_2\left(n+\left(\frac{\theta_1}{2}+1\right)\lambda_{22}^*+\frac{\theta_1}{2}\left(1-\frac{\theta_2}{2}+\frac{\theta_1}{4}\right)\lambda_{04}^*\right)\end{array}\right\}-n\right] \tag{35}$$

To find MSE, taking equation (34) and squaring on both sides, we get equation (36):

$$\left(v_{Pk}-S_y^2\right)^2 \cong S_y^2\left[\begin{array}{l}\left\{\tau_1\left(1+e_0+\left(\frac{\theta_1}{2}-1\right)(e_1+e_0e_1)+\left(1-\frac{\theta_1}{2}-\frac{\theta_1\theta_2}{4}+\frac{\theta_1^2}{8}\right)e_1^2\right)\right\}^2\\ +\left\{\tau_2\left(1+e_0+\left(\frac{\theta_1}{2}+1\right)(e_1+e_0e_1)+\frac{\theta_1}{2}\left(1-\frac{\theta_2}{2}+\frac{\theta_1}{4}\right)e_1^2\right)\right\}^2+\{-1\}^2\\ +2\tau_1\tau_2\left\{1+e_0+\left(\frac{\theta_1}{2}-1\right)(e_1+e_0e_1)+\left(1-\frac{\theta_1}{2}-\frac{\theta_1\theta_2}{4}+\frac{\theta_1^2}{8}\right)e_1^2\right\}\\ \left\{1+e_0+\left(\frac{\theta_1}{2}+1\right)(e_1+e_0e_1)+\frac{\theta_1}{2}\left(1-\frac{\theta_2}{2}+\frac{\theta_1}{4}\right)e_1^2\right\}\\ -2\left\{\tau_1\left(1+e_0+\left(\frac{\theta_1}{2}-1\right)(e_1+e_0e_1)+\left(1-\frac{\theta_1}{2}-\frac{\theta_1\theta_2}{4}+\frac{\theta_1^2}{8}\right)e_1^2\right)\right\}\\ -2\left\{\tau_2\left(1+e_0+\left(\frac{\theta_1}{2}+1\right)(e_1+e_0e_1)+\frac{\theta_1}{2}\left(1-\frac{\theta_2}{2}+\frac{\theta_1}{4}\right)e_1^2\right)\right\}\end{array}\right] \tag{36}$$

Applying expectation on both sides of the above equation, we get the MSE of v_{Pk} in equation (37):

$$MSE(v_{Pk}) \cong \frac{S_y^4}{n}\begin{bmatrix} n+\tau_1^2\left(n+\lambda_{40}^*+\left\{\left(\frac{\theta_1}{2}-1\right)^2+2\left(1-\frac{\theta_1}{2}-\frac{\theta_1\theta_2}{4}+\frac{\theta_1^2}{8}\right)\right\}\lambda_{04}^*+4\left(\frac{\theta_1}{2}-1\right)\lambda_{22}^*\right) \\ +\tau_2^2\left(n+\lambda_{40}^*+\left\{\left(\frac{\theta_1}{2}+1\right)^2+\theta_1\left(1-\frac{\theta_2}{2}+\frac{\theta_1}{4}\right)\right\}\lambda_{04}^*+4\left(\frac{\theta_1}{2}+1\right)\lambda_{22}^*\right) \\ +2\tau_1\tau_2\begin{Bmatrix} n+\lambda_{40}^*+\left\{2\left(\frac{\theta_1}{2}-1\right)+2\left(\frac{\theta_1}{2}+1\right)\right\}\lambda_{22}^*+ \\ \left\{\left(1-\frac{\theta_1}{2}-\frac{\theta_1\theta_2}{4}+\frac{\theta_1^2}{8}\right)+\left(\frac{\theta_1^2}{4}-1\right)+\frac{\theta_1}{2}\left(1-\frac{\theta_2}{2}+\frac{\theta_1}{4}\right)\right\}\lambda_{04}^* \end{Bmatrix} \\ -2\tau_1\left(n+\left(\frac{\theta_1}{2}-1\right)\lambda_{22}^*+\left(1-\frac{\theta_1}{2}-\frac{\theta_1\theta_2}{4}+\frac{\theta_1^2}{8}\right)\lambda_{04}^*\right) \\ -2\tau_2\left(n+\left(\frac{\theta_1}{2}+1\right)\lambda_{22}^*+\frac{\theta_1}{2}\left(1-\frac{\theta_2}{2}+\frac{\theta_1}{4}\right)\lambda_{04}^*\right) \end{bmatrix} \tag{37}$$

The MSE in the above equation, can be written as given in equation (38):

$$MSE(v_{Pk}) \cong \frac{S_y^4}{n}\left(n+\tau_1^2A_1+\tau_2^2A_2+2\tau_1\tau_2A_3-2\tau_1A_4-2\tau_2A_5\right) \tag{38}$$

where

$$A_1=\left(n+\lambda_{40}^*+\left\{\left(\frac{\theta_1}{2}-1\right)^2+2\left(1-\frac{\theta_1}{2}-\frac{\theta_1\theta_2}{4}+\frac{\theta_1^2}{8}\right)\lambda_{04}^*+4\left(\frac{\theta_1}{2}-1\right)\lambda_{22}^*\right\}\right)$$

$$A_2=\left(n+\lambda_{40}^*+\left\{\left\{\left(\frac{\theta_1}{2}+1\right)^2+\theta_1\left(1-\frac{\theta_2}{2}+\frac{1}{4}\theta_1\right)\right\}\lambda_{04}^*+4\left(\frac{\theta_1}{2}+1\right)\lambda_{22}^*\right\}\right)$$

$$A_3=\begin{Bmatrix} n+\lambda_{40}^*+\left(2\left(\frac{\theta_1}{2}-1\right)+2\left(\frac{\theta_1}{2}+1\right)\right)\lambda_{22}^*+\left(\left(1-\frac{\theta_1}{2}-\frac{\theta_1\theta_2}{4}+\frac{\theta_1^2}{8}\right)\right. \\ \left.+\left(\frac{\theta_1^2}{4}-1\right)+\frac{1}{2}\theta_1\left(1-\frac{\theta_2}{2}+\frac{\theta_1}{4}\right)\right)\lambda_{04}^* \end{Bmatrix}$$

$$A_4=\left(n+\left(\frac{\theta_1}{2}-1\right)\lambda_{22}^*+\left(1-\frac{\theta_1}{2}-\frac{\theta_1\theta_2}{4}+\frac{\theta_1^2}{8}\right)\lambda_{04}^*\right), A_5=\left(n+\left(\frac{\theta_1}{2}+1\right)\lambda_{22}^*+\frac{\theta_1}{2}\left(1-\frac{\theta_2}{2}+\frac{\theta_1}{4}\right)\lambda_{04}^*\right)$$

For optimum values of τ_1 and τ_2, differentiating equation (38) with respect to τ_1 and τ_2,

We have $\frac{\partial}{\partial\tau_1}MSE(v_{Pk}) = 0$.

Putting MSE of v_{Pk} in the above equation, we have equation (39):

$$\frac{\partial}{\partial\tau_1}\frac{S_y^4}{n}\left(n+\tau_1^2A_1+\tau_2^2A_2+2\tau_1\tau_2A_3-2\tau_1A_4-2\tau_2A_5\right)=0 \tag{39}$$

$$\text{and } \frac{\partial}{\partial\tau_2}MSE(v_{Pk})=0$$

Putting MSE of v_{Pk} in the above equation, we have equation (40):

$$\frac{\partial}{\partial\tau_2}\frac{S_y^4}{n}\left(n+\tau_1^2A_1+\tau_2^2A_2+2\tau_1\tau_2A_3-2\tau_1A_4-2\tau_2A_5\right)=0 \tag{40}$$

The optimum expressions of τ_1 and τ_2 are obtained as:

$$\tau_{1(opt)}=\frac{(A_2A_4-A_5A_3)}{(A_1A_2-A_3^2)}, \tau_{2(opt)}=\frac{(A_1A_5-A_3A_4)}{(A_1A_2-A_3^2)}$$

Putting $\tau_{1(opt)}$ and $\tau_{2(opt)}$ in equation (38), we obtained equation (41):

$$MSE(v_{Pk})_{\min} \cong \frac{S_y^4}{n}\left(n+\tau_{1(opt)}^2A_1+\tau_{2(opt)}^2A_2+2\tau_{1(opt)}\tau_{2(opt)}A_3-2\tau_{1(opt)}A_4-2\tau_{2(opt)}A_5\right) \tag{41}$$

From equations (30) and (31), we see that each $a, b, C_1, C_2,$ and C_3 are different. For this reason, θ_1 *and* θ_2 are not the same. Therefore, the minimum MSE values for each estimator $v_{P1}, v_{P2}, v_{P3}, v_{P4}, v_{P5},$ and v_{p6} are different as $\tau_1, \tau_2, A_1, A_2, A_3, A_4,$ and A_5 in equation (41) are computed by using θ_1 *and* θ_2.

From equation (27), some family members of the estimators are.

Case I. Putting $a=-1, b=0, C_1=Med, C_2=Q_1, C_3=QD$ in equations (30) and (31), we have:

$$\theta_{(I)1}=\frac{(QD-Med)S_x^2}{(QD+Q_1+Med)S_x^2},\theta_{(I)2}=\frac{(QD+Med)S_x^2}{(QD+Q_1+Med)S_x^2}$$

The bias for the estimator ν_{P1} is given in equation (42):

$$Bias(\nu_{P1})\cong\frac{S_y^2}{n}\left[\left\{\begin{array}{l}\tau_{(I)1(opt)}\left(n+\left(\frac{\theta_{(I)1}}{2}-1\right)\lambda_{22}^*+\left(1-\frac{\theta_{(I)1}}{2}-\frac{\theta_{(I)1}\theta_{(I)2}}{4}+\frac{\theta_{(I)1}^2}{8}\right)\lambda_{04}^*\right)\\+\tau_{(I)2(opt)}\left(n+\left(\frac{\theta_{(I)1}}{2}+1\right)\lambda_{22}^*+\frac{\theta_{(I)1}}{2}\left(1-\frac{\theta_{(I)2}}{2}+\frac{\theta_{(I)1}}{4}\right)\lambda_{04}^*\right)\end{array}\right\}-n\right] \tag{42}$$

The MSE for the estimator ν_{P1} is given in equation (43):

$$MSE(\nu_{P1})\cong\frac{S_y^4}{n}\left(n+\tau_{(I)1(opt)}^2A_{(I)1}+\tau_{(I)2(opt)}^2A_{(I)2}+2\tau_{(I)1(opt)}\tau_{(I)2(opt)}A_{(I)3}-2\tau_{(I)1(opt)}A_{(I)4}-2\tau_{(I)2(opt)}A_{(I)5}\right) \tag{43}$$

Where $\tau_{(I)1(opt)}^2,\tau_{(I)2(opt)}^2,A_{(I)1},A_{(I)2},A_{(I)3}A_{(I)4}$ and $A_{(I)5}$ are obtained by putting $\theta_{(I)1}$ and $\theta_{(I)2}$ in A_1,A_2,A_3,A_4 and A_5:

Case II. Putting $a=1, b=-1, C_1=Med, C_2=C_x, C_3=Q_3$ in equations (30) and (31), we have:

$$\theta_{(II)1}=\frac{(Q_3-Med)S_x^2}{\left\{(Q_3+C_x+Med)S_x^2-1\right\}},\theta_{(II)2}=\frac{(Q_3+Med)S_x^2}{\left\{(Q_3+C_x+Med)S_x^2-1\right\}}$$

The bias for the estimator ν_{P2} is given in equation (44):

$$Bias(\nu_{P2})\cong\frac{S_y^2}{n}\left[\left\{\begin{array}{l}\tau_{(II)1(opt)}\left(n+\left(\frac{\theta_{(II)1}}{2}-1\right)\lambda_{22}^*+\left(1-\frac{\theta_{(II)1}}{2}-\frac{\theta_{(II)1}\theta_{(II)2}}{4}+\frac{\theta_{(II)1}^2}{8}\right)\lambda_{04}^*\right)\\+\tau_{(II)2(opt)}\left(n+\left(\frac{\theta_{(II)1}}{2}+1\right)\lambda_{22}^*+\frac{\theta_{(II)1}}{2}\left(1-\frac{\theta_{(II)2}}{2}+\frac{\theta_{(II)1}}{4}\right)\lambda_{04}^*\right)\end{array}\right\}-n\right] \tag{44}$$

The MSE for the estimator ν_{P2} is given in equation (45):

$$MSE(\nu_{P2})\cong\frac{S_y^4}{n}\left(n+\tau_{(II)1(opt)}^2A_{(II)1}+\tau_{(II)2(opt)}^2A_{(II)2}+2\tau_{(II)1(opt)}\tau_{(II)2(opt)}A_{(II)3}-2\tau_{(II)1(opt)}A_{(II)4}-2\tau_{(II)2(opt)}A_{(II)5}\right) \tag{45}$$

Where $\tau_{(II)1(opt)}^2,\tau_{(II)2(opt)}^2,A_{(II)1},A_{(II)2},A_{(II)3}A_{(II)4}$ and $A_{(II)5}$ are obtained by putting $\theta_{(II)1}$ and $\theta_{(II)2}$ in A_1,A_2,A_3,A_4 and A_5:

Case III. Putting $a=1, b=-1, C_1=C_x, C_2=Q_1, C_3=Med$ in equations (30) and (31), we have:

$$\theta_{(III)1}=\frac{(Med-C_x)S_x^2}{\left\{(Med+Q_1+C_x)S_x^2-1\right\}},\theta_{(III)2}=\frac{(Med+C_x)S_x^2}{\left\{(Med+Q_1+C_x)S_x^2-1\right\}}$$

The bias for the estimator ν_{P3} is given in equation (46):

$$Bias(\nu_{P3})\cong\frac{S_y^2}{n}\left[\left\{\begin{array}{l}\tau_{(III)1(opt)}\left(n+\left(\frac{\theta_{(III)1}}{2}-1\right)\lambda_{22}^*+\left(1-\frac{\theta_{(III)1}}{2}-\frac{\theta_{(III)1}\theta_{(III)2}}{4}+\frac{\theta_{(III)1}^2}{8}\right)\lambda_{04}^*\right)\\+\tau_{(III)2(opt)}\left(n+\left(\frac{\theta_{(III)1}}{2}+1\right)\lambda_{22}^*+\frac{\theta_{(III)1}}{2}\left(1-\frac{\theta_{(III)2}}{2}+\frac{\theta_{(III)1}}{4}\right)\lambda_{04}^*\right)\end{array}\right\}-n\right] \tag{46}$$

The MSE for the estimator ν_{P3} is given in equation (47):

$$MSE(\nu_{P3})\cong\frac{S_y^4}{n}\left(n+\tau_{(III)1(opt)}^2A_{(III)1}+\tau_{(III)2(opt)}^2A_{(III)2}+2\tau_{(III)1(opt)}\tau_{(III)2(opt)}A_{(III)3}-2\tau_{(III)1(opt)}A_{(III)4}-2\tau_{(III)2(opt)}A_{(III)5}\right) \tag{47}$$

Where $\tau_{(III)1(opt)}^2,\tau_{(III)2(opt)}^2,A_{(III)1},A_{(III)2},A_{(III)3}A_{(III)4}$ and $A_{(III)5}$ are obtained by putting $\theta_{(III)1}$ and $\theta_{(III)2}$ in A_1,A_2,A_3,A_4 and A_5:

Case IV. Putting $a=-1, b=0, C_1=QD, C_2=C_x, C_3=Med$ in equations (30) and (31), we have:

$$\theta_{(IV)1}=\frac{(Med-QD)S_x^2}{(Med+C_x+QD)S_x^2},\theta_{(IV)2}=\frac{(Med+QD)S_x^2}{(Med+C_x+QD)S_x^2}$$

The bias for the estimator ν_{P4} is given in equation (48):

$$Bias(\nu_{P4}) \cong \frac{S_y^2}{n}\left[\left\{\begin{array}{l}\tau_{(IV)1(opt)}\left(n+\left(\frac{\theta_{(IV)1}}{2}-1\right)\lambda_{22}^{*}+\left(1-\frac{\theta_{(IV)1}}{2}-\frac{\theta_{(IV)1}\theta_{(IV)2}}{4}+\frac{\theta_{(IV)1}^2}{8}\right)\lambda_{04}^{*}\right)\\ +\tau_{(IV)2(opt)}\left(n+\left(\frac{\theta_{(IV)1}}{2}+1\right)\lambda_{22}^{*}+\frac{\theta_{(IV)1}}{2}\left(1-\frac{\theta_{(IV)2}}{2}+\frac{\theta_{(IV)1}}{4}\right)\lambda_{04}^{*}\right)\end{array}\right\}-n\right] \tag{48}$$

The MSE for the estimator ν_{P4} is given in equation (49):

$$MSE(\nu_{P4}) \cong \frac{S_y^4}{n}\left(n+\tau_{(IV)1(opt)}^2A_{(IV)1}+\tau_{(IV)2(opt)}^2A_{(IV)2}+2\tau_{(IV)1(opt)}\tau_{(IV)2(opt)}A_{(IV)3}-2\tau_{(IV)1(opt)}A_{(IV)4}-2\tau_{(IV)2(opt)}A_{(IV)5}\right) \tag{49}$$

Where $\tau_{(IV)1(opt)}^2, \tau_{(IV)2(opt)}^2, A_{(IV)1}, A_{(IV)2}, A_{(IV)3}A_{(IV)4}$ and $A_{(IV)5}$ are obtained by putting $\theta_{(IV)1}$ and $\theta_{(IV)2}$ in A_1, A_2, A_3, A_4 and A_5:

Case V. Putting $a=1, b=-1, C_1=C_x, C_2=QD, C_3=Med$ in equations (30) and (31), we have:

$$\theta_{(V)1}=\frac{(Med-C_x)S_x^2}{\{(Med+QD+C_x)S_x^2-1\}}, \theta_{(V)2}=\frac{(Med+C_x)S_x^2}{\{(Med+QD+C_x)S_x^2-1\}}$$

The bias for the estimator ν_{P5} is given in equation (50):

$$Bias(\nu_{P5}) \cong \frac{S_y^2}{n}\left[\left\{\begin{array}{l}\tau_{(V)1(opt)}\left(n+\left(\frac{\theta_{(V)1}}{2}-1\right)\lambda_{22}^{*}+\left(1-\frac{\theta_{(V)1}}{2}-\frac{\theta_{(V)1}\theta_{(V)2}}{4}+\frac{\theta_{(V)1}^2}{8}\right)\lambda_{04}^{*}\right)\\ +\tau_{(V)2(opt)}\left(n+\left(\frac{\theta_{(V)1}}{2}+1\right)\lambda_{22}^{*}+\frac{\theta_{(V)1}}{2}\left(1-\frac{\theta_{(V)2}}{2}+\frac{\theta_{(V)1}}{4}\right)\lambda_{04}^{*}\right)\end{array}\right\}-n\right] \tag{50}$$

The MSE for the estimator ν_{P5} is given in equation (51):

$$MSE(\nu_{P5}) \cong \frac{S_y^4}{n}\left(n+\tau_{(V)1(opt)}^2A_{(V)1}+\tau_{(V)2(opt)}^2A_{(V)2}+2\tau_{(V)1(opt)}\tau_{(V)2(opt)}A_{(V)3}-2\tau_{(V)1(opt)}A_{(V)4}-2\tau_{(V)2(opt)}A_{(V)5}\right) \tag{51}$$

Where $\tau_{(V)1(opt)}^2, \tau_{(V)2(opt)}^2, A_{(V)1}, A_{(V)2}, A_{(V)3}A_{(V)4}$ and $A_{(V)5}$ are obtained by putting $\theta_{(V)1}$ and $\theta_{(V)2}$ in A_1, A_2, A_3, A_4 and A_5:

Case VI. Putting $a=1, b=-1, C_1=Q_1, C_2=Med, C_3=Q_3$ in equations (30) and (31), we have:

$$\theta_{(VI)1}=\frac{(Med-QD)S_x^2}{\{(Med+C_x+QD)S_x^2-1\}}, \theta_{(VI)2}=\frac{(Med+QD)S_x^2}{\{(Med+C_x+QD)S_x^2-1\}}$$

The bias for the estimator ν_{P1} is given in equation (52):

$$Bias(\nu_{P6}) \cong \frac{S_y^2}{n}\left[\left\{\begin{array}{l}\tau_{(VI)1(opt)}\left(n+\left(\frac{\theta_{(VI)1}}{2}-1\right)\lambda_{22}^{*}+\left(1-\frac{\theta_{(VI)1}}{2}-\frac{\theta_{(VI)1}\theta_{(VI)2}}{4}+\frac{\theta_{(VI)1}^2}{8}\right)\lambda_{04}^{*}\right)\\ +\tau_{(VI)2(opt)}\left(n+\left(\frac{\theta_{(VI)1}}{2}+1\right)\lambda_{22}^{*}+\frac{\theta_{(VI)1}}{2}\left(1-\frac{\theta_{(VI)2}}{2}+\frac{\theta_{(VI)1}}{4}\right)\lambda_{04}^{*}\right)\end{array}\right\}-n\right] \tag{52}$$

The MSE for the estimator ν_{P6} is given in equation (53):

$$MSE(\nu_{P6}) \cong \frac{S_y^4}{n}\left(n+\tau_{(VI)1(opt)}^2A_{(VI)1}+\tau_{(VI)2(opt)}^2A_{(VI)2}+2\tau_{(VI)1(opt)}\tau_{(VI)2(opt)}A_{(VI)3}-2\tau_{(VI)1(opt)}A_{(VI)4}-2\tau_{(VI)2(opt)}A_{(VI)5}\right) \tag{53}$$

Where $\tau_{(VI)1(opt)}^2, \tau_{(VI)2(opt)}^2, A_{(VI)1}, A_{(VI)2}, A_{(VI)3}A_{(VI)4}$ and $A_{(VI)5}$are obtained by putting$\theta_{(VI)1}$and $\theta_{(VI)2}$ in A_1, A_2, A_3, A_4 and A_5:

4. Theoretical comparisons

Theoretical comparisons are made by generating different efficiency conditions in this section. We compare the MSE of the proposed and competitor estimators with the help of different conditions given as:

Condition (i) $MSE(\nu_{Pk})_{\min} < Var(\nu_0)$, if

$$\frac{1}{n}S_y^4\lambda_{40}^{*}-\frac{1}{n}S_y^4\left(n+\tau_{1(opt)}^2A_1+\tau_{2(opt)}^2A_2+2\tau_{1(opt)}\tau_{2(opt)}A_3-2\tau_{1(opt)}A_4-2\tau_{2(opt)}A_5\right)>0$$

or $\lambda_{40}^{*}-\left(n+\tau_{1(opt)}^2A_1+\tau_{2(opt)}^2A_2+2\tau_{1(opt)}\tau_{2(opt)}A_3-2\tau_{1(opt)}A_4-2\tau_{2(opt)}A_5\right)>0$:

Condition (ii) $MSE(\nu_{Pk})_{\min} < Var(\nu_{Isa1})$, if

$$\frac{1}{n}S_y^4\lambda_{40}^{*}(1-\rho^{*2})-\frac{1}{n}S_y^4\left(n+\tau_{1(opt)}^2A_1+\tau_{2(opt)}^2A_2+2\tau_{1(opt)}\tau_{2(opt)}A_3-2\tau_{1(opt)}A_4-2\tau_{2(opt)}A_5\right)>0$$

or $\lambda_{40}^{*}\left(1-\rho^{*2}\right)-\left(n+\tau_{1(opt)}^{2}A_{1}+\tau_{2(opt)}^{2}A_{2}+2\tau_{1(opt)}\tau_{2(opt)}A_{3}-2\tau_{1(opt)}A_{4}-2\tau_{2(opt)}A_{5}\right)>0.$

Condition(iii) $MSE(v_{Pk})_{\min}<Var(v_{US})$, if

$$\frac{1}{n}S_{y}^{4}\left(\lambda_{40}^{*}+g_{0}^{2}\lambda_{04}^{*}-2g_{0}\lambda_{22}^{*}\right)-\frac{1}{n}S_{y}^{4}\left(n+\tau_{1(opt)}^{2}A_{1}+\tau_{2(opt)}^{2}A_{2}+2\tau_{1(opt)}\tau_{2(opt)}A_{3}-2\tau_{1(opt)}A_{4}-2\tau_{2(opt)}A_{5}\right)>0$$

$$or\ \left(\lambda_{40}^{*}+g_{0}^{2}\lambda_{04}^{*}-2g_{0}\lambda_{22}^{*}\right)-\left(n+\tau_{1(opt)}^{2}A_{1}+\tau_{2(opt)}^{2}A_{2}+2\tau_{1(opt)}\tau_{2(opt)}A_{3}-2\tau_{1(opt)}A_{4}-2\tau_{2(opt)}A_{5}\right)>0$$

Condition(iv) $MSE(v_{Pk})_{\min}<Var(v_{KC1})$, if

$$\frac{1}{n}S_{y}^{4}\left(\lambda_{40}^{*}+g_{1}^{2}\lambda_{04}^{*}-2g_{1}\lambda_{22}^{*}\right)-\frac{1}{n}S_{y}^{4}\left(n+\tau_{1(opt)}^{2}A_{1}+\tau_{2(opt)}^{2}A_{2}+2\tau_{1(opt)}\tau_{2(opt)}A_{3}-2\tau_{1(opt)}A_{4}-2\tau_{2(opt)}A_{5}\right)>0$$

$$or\ \left(\lambda_{40}^{*}+g_{1}^{2}\lambda_{04}^{*}-2g_{1}\lambda_{22}^{*}\right)-\left(n+\tau_{1(opt)}^{2}A_{1}+\tau_{2(opt)}^{2}A_{2}+2\tau_{1(opt)}\tau_{2(opt)}A_{3}-2\tau_{1(opt)}A_{4}-2\tau_{2(opt)}A_{5}\right)>0$$

Condition(v) $MSE(v_{Pk})_{\min}<Var(v_{KC2})$, if

$$\frac{1}{n}S_{y}^{4}\left(\lambda_{40}^{*}+g_{2}^{2}\lambda_{04}^{*}-2g_{2}\lambda_{22}^{*}\right)-\frac{1}{n}S_{y}^{4}\left(n+\tau_{1(opt)}^{2}A_{1}+\tau_{2(opt)}^{2}A_{2}+2\tau_{1(opt)}\tau_{2(opt)}A_{3}-2\tau_{1(opt)}A_{4}-2\tau_{2(opt)}A_{5}\right)>0$$

$$or\ \left(\lambda_{40}^{*}+g_{2}^{2}\lambda_{04}^{*}-2g_{2}\lambda_{22}^{*}\right)-\left(n+\tau_{1(opt)}^{2}A_{1}+\tau_{2(opt)}^{2}A_{2}+2\tau_{1(opt)}\tau_{2(opt)}A_{3}-2\tau_{1(opt)}A_{4}-2\tau_{2(opt)}A_{5}\right)>0$$

Condition(vi) $MSE(v_{Pk})_{\min}<Var(v_{KC3})$, if

$$\frac{1}{n}S_{y}^{4}\left(\lambda_{40}^{*}+g_{3}^{2}\lambda_{04}^{*}-2g_{3}\lambda_{22}^{*}\right)-\frac{1}{n}S_{y}^{4}\left(n+\tau_{1(opt)}^{2}A_{1}+\tau_{2(opt)}^{2}A_{2}+2\tau_{1(opt)}\tau_{2(opt)}A_{3}-2\tau_{1(opt)}A_{4}-2\tau_{2(opt)}A_{5}\right)>0$$

$$or\ \left(\lambda_{40}^{*}+g_{3}^{2}\lambda_{04}^{*}-2g_{3}\lambda_{22}^{*}\right)-\left(n+\tau_{1(opt)}^{2}A_{1}+\tau_{2(opt)}^{2}A_{2}+2\tau_{1(opt)}\tau_{2(opt)}A_{3}-2\tau_{1(opt)}A_{4}-2\tau_{2(opt)}A_{5}\right)>0$$

Condition (vii) $MSE(v_{Pk})_{\min}<Var(v_{Mr})$, if

$$\frac{1-f}{n}S_{y}^{4}\left[\lambda_{40}^{*}+\varphi\lambda_{04}^{*}\left(\varphi-2\frac{\lambda_{22}^{*}}{\lambda_{04}^{*}}\right)\right]-\frac{1}{n}S_{y}^{4}\left(n+\tau_{1(opt)}^{2}A_{1}+\tau_{2(opt)}^{2}A_{2}+2\tau_{1(opt)}\tau_{2(opt)}A_{3}-2\tau_{1(opt)}A_{4}-2\tau_{2(opt)}A_{5}\right)>0$$

or $\left[\lambda_{40}^{*}+\varphi\lambda_{04}^{*}\left(\varphi-2\frac{\lambda_{22}^{*}}{\lambda_{04}^{*}}\right)\right]-\left(n+\tau_{1(opt)}^{2}A_{1}+\tau_{2(opt)}^{2}A_{2}+2\tau_{1(opt)}\tau_{2(opt)}A_{3}-2\tau_{1(opt)}A_{4}-2\tau_{2(opt)}A_{5}\right)>0.$

Condition (viii) $MSE(v_{Pk})_{\min}<Var(v_{Isa2})$, if

$$\frac{1}{n}S_{y}^{4}\left(\lambda_{40}^{*}+\lambda_{04}^{*}-2\lambda_{22}^{*}\right)-\frac{1}{n}S_{y}^{4}\left(n+\tau_{1(opt)}^{2}A_{1}+\tau_{2(opt)}^{2}A_{2}+2\tau_{1(opt)}\tau_{2(opt)}A_{3}-2\tau_{1(opt)}A_{4}-2\tau_{2(opt)}A_{5}\right)>0$$

or $\left(\lambda_{40}^{*}+\lambda_{04}^{*}-2\lambda_{22}^{*}\right)-\left(n+\tau_{1(opt)}^{2}A_{1}+\tau_{2(opt)}^{2}A_{2}+2\tau_{1(opt)}\tau_{2(opt)}A_{3}-2\tau_{1(opt)}A_{4}-2\tau_{2(opt)}A_{5}\right)>0.$

Condition(ix) $MSE(v_{Pk})_{\min}<Var(v_{\Pr d})$, if

$$\frac{1}{n}S_{y}^{4}\left(\lambda_{40}^{*}+\lambda_{04}^{*}+2\lambda_{22}^{*}\right)-\frac{1}{n}S_{y}^{4}\left(n+\tau_{1(opt)}^{2}A_{1}+\tau_{2(opt)}^{2}A_{2}+2\tau_{1(opt)}\tau_{2(opt)}A_{3}-2\tau_{1(opt)}A_{4}-2\tau_{2(opt)}A_{5}\right)>0$$

or $\left(\lambda_{40}^{*}+\lambda_{04}^{*}+2\lambda_{22}^{*}\right)-\left(n+\tau_{1(opt)}^{2}A_{1}+\tau_{2(opt)}^{2}A_{2}+2\tau_{1(opt)}\tau_{2(opt)}A_{3}-2\tau_{1(opt)}A_{4}-2\tau_{2(opt)}A_{5}\right)>0.$

Condition(x) $MSE(v_{Pk})_{\min}<Var(v_{Sr})$, if

$$\frac{1}{n}S_{y}^{4}\left(\lambda_{40}^{*}+\frac{\lambda_{04}^{*}}{4}-\lambda_{22}^{*}\right)-\frac{1}{n}S_{y}^{4}\left(n+\tau_{1(opt)}^{2}A_{1}+\tau_{2(opt)}^{2}A_{2}+2\tau_{1(opt)}\tau_{2(opt)}A_{3}-2\tau_{1(opt)}A_{4}-2\tau_{2(opt)}A_{5}\right)>0$$

$$or\ \left(\lambda_{40}^{*}+\frac{\lambda_{04}^{*}}{4}-\lambda_{22}^{*}\right)-\left(n+\tau_{1(opt)}^{2}A_{1}+\tau_{2(opt)}^{2}A_{2}+2\tau_{1(opt)}\tau_{2(opt)}A_{3}-2\tau_{1(opt)}A_{4}-2\tau_{2(opt)}A_{5}\right)>0$$

Condition(xi) $MSE(v_{Pk})_{\min}<Var\left(v_{Sp}\right)$, if

$$\frac{1}{n}S_{y}^{4}\left(\lambda_{40}^{*}+\frac{\lambda_{04}^{*}}{4}-\lambda_{22}^{*}\right)-\frac{1}{n}S_{y}^{4}\left(n+\tau_{1(opt)}^{2}A_{1}+\tau_{2(opt)}^{2}A_{2}+2\tau_{1(opt)}\tau_{2(opt)}A_{3}-2\tau_{1(opt)}A_{4}-2\tau_{2(opt)}A_{5}\right)>0$$

$$or\ \left(\lambda_{40}^{*}+\frac{\lambda_{04}^{*}}{4}+\lambda_{22}^{*}\right)-\left(n+\tau_{1(opt)}^{2}A_{1}+\tau_{2(opt)}^{2}A_{2}+2\tau_{1(opt)}\tau_{2(opt)}A_{3}-2\tau_{1(opt)}A_{4}-2\tau_{2(opt)}A_{5}\right)>0$$

Condition(xii) $MSE(v_{Pk})_{\min}<Var(v_{Sh})_{\min}$, if

Table 1
MSE and PRE values for suggested and existing estimators using different data sets.

S.No	Estimators	MSE/PRE	Data1	Data2	Data3	Data4
1	v_0	MSE	1.01E+18	4.87E+17	7193.72	59199136
		PRE	100	100	100	100
2	v_{Isa1}	MSE	1.72E+16	1.81E+17	2652.44	25644380
		PRE	5889.493	268.7481	271.2114	230.8464
3	v_{US}	MSE	1.68E+16	1.92E+17	3656.81	35382611
		PRE	6040.316	253.9958	186.93	142.0326
4	v_{KC1}	MSE	1.68E+16	1.92E+17	3848.35	40822921
		PRE	6040.316	253.9958	184.6091	142.0233
5	v_{KC2}	MSE	1.68E+16	1.92E+17	3896.73	41559289
		PRE	6040.316	253.9958	200.9739	142.1407
6	v_{KC3}	MSE	1.68E+16	1.92E+17	3579.43	34219845
		PRE	6040.316	253.9958	183.3575	142.0217
7	v_{Mr}	MSE	2.00E+18	1.03E+18	17863.85	84892264
		PRE	50.84236	47.39437	40.26971	38.15222
8	v_{Isa2}	MSE	1.68E+16	1.92E+17	3923.33	41683145
		PRE	6040.316	253.9958	183.3575	142.0217
9	v_{Prd}	MSE	3.95E+18	1.84E+18	31653.5	268666705
		PRE	25.66152	26.46205	22.72646	22.03441
10	v_{Sr}	MSE	2.73E+17	2.07E+17	2909.85	26447194
		PRE	371.612	235.147	247.2196	223.839
11	v_{Sp}	MSE	2.24E+18	1.03E+18	16774.93	139938973
		PRE	45.26397	47.21636	42.88376	42.30354
12	$v_{Sh(\min)}$	MSE	1.66E+16	1.66E+17	2657.47	25648008
		PRE	6109.189	293.5368	270.6981	230.8138
13	v_{P1}	MSE	1.63E+16	9.50E+16	**2220.61**	**16351707**
		PRE	6207.784	513.0336	**323.9521**	**362.0265**
14	v_{p2}	MSE	**3.09E+15**	7.96E+16	2296.27	16501935
		PRE	**32797.82**	612.1504	313.2785	358.7406
15	v_{p3}	MSE	1.473584 + 16	8.40E+16	2301.17	18073635
		PRE	6884.022	579.6681	312.6115	327.5442
16	v_{p4}	MSE	1.52E+16	**7.27E+16**	2369.075	16351739
		PRE	6681.792	**669.6936**	303.651	362.0357
17	v_{p5}	MSE	1.50E+16	9.66E+16	2307.754	17900364
		PRE	6741.731	504.1318	311.7194	330.7147
18	v_{P6}	MSE	1.27E+16	8.42E+16	2331.93	17367897
		PRE	7995.022	578.8302	308.4879	340.8538

$$\frac{1}{n}S_y^4\left(\lambda_{40}^* - \frac{\lambda_{22}^{*2}}{\lambda_{04}^*}\right) - \frac{1}{n}S_y^4\left(n + \tau_{1(opt)}^2 A_1 + \tau_{2(opt)}^2 A_2 + 2\tau_{1(opt)}\tau_{2(opt)}A_3 - 2\tau_{1(opt)}A_4 - 2\tau_{2(opt)}A_5\right) > 0$$

$$or \left(\lambda_{40}^* - \frac{\lambda_{22}^{*2}}{\lambda_{04}^*}\right) - \left(n + \tau_{1(opt)}^2 A_1 + \tau_{2(opt)}^2 A_2 + 2\tau_{1(opt)}\tau_{2(opt)}A_3 - 2\tau_{1(opt)}A_4 - 2\tau_{2(opt)}A_5\right) > 0$$

The aforementioned conditions (i)-(xii) are always held correctly. In consequence, the suggested estimators v_{Pk} will give better results than the existing estimators. The above conditions are also verified numerically in Section 5, and the numerical values are provided in Table 2.

5. Numerical Investigation

For comparison of the efficiencies of traditional and suggested estimators, four real data sets are used. Data 1 and data 2 taken from Ref. [6]. This data is concern to the production of apple level Y (1 unit = 1000 tons) as the variable of interest and number of apple trees X (1 unit = 100 trees) shows the supplementary variable in 171 and 94 villages of Mediterranean and Central Anatolia respectively. The rest of the two data sets are taken from Ref. [19].

Data 1

$$\mathrm{N} = 171, \mathrm{n} = 67, \overline{\mathrm{Y}} = 5588.01, \overline{\mathrm{X}} = 74364.68, \rho = 0.98 \quad S_y = 28643.42,$$
$$C_y = 5.12, S_x = 285603.13, C_x = 3.84, \widetilde{\mathrm{X}} = 17850, \mathrm{Q}_1 = 4700, \mathrm{Q}_3 = 37900,$$
$$\mathrm{QD} = 16600, \lambda_{22} = 98.94, \lambda_{40} = 100.97, \lambda_{04} = 96.6$$

Data 2

$$\mathrm{N} = 94, \mathrm{n} = 38, \overline{\mathrm{Y}} = 9384.31, \overline{\mathrm{X}} = 72409.95, \rho = 0.90 \quad S_y = 29907.48,$$
$$C_y = 3.19, S_x = 160757.31, C_x = 2.22, \widetilde{\mathrm{X}} = 18305, \mathrm{Q}_1 = 5758.5, \mathrm{Q}_3 = 58250,$$

$$\mathrm{QD} = 26245.5, \lambda_{22} = 20.58, \lambda_{40} = 23.14, \lambda_{04} = 25.14$$

Table 2
Numerical verification of conditions (i)-(xii) derived in efficiency comparison.

Conditions'	Existing Estimators	Data	Proposed Estimators					
			v_{P1}	v_{P2}	v_{P3}	v_{P4}	v_{P5}	v_{P6}
i	v_0	1	9.98E+17	1.01E+18	1.01E+18	9.99E+17	9.99E+17	1.00E+18
		2	3.92E+17	4.08E+17	4.03E+17	4.14E+17	3.91E+17	4.03E+17
		3	4973.11	4897.45	4892.55	4824.645	4885.966	4861.79
		4	4284742	42697201	41125501	42847397	41298772	41831239
ii	v_{Isa1}	1	8.83E+14	1.41E+16	1.72E+16	2.04E+15	2.18E+15	4.54E+15
		2	8.63E+16	1.02E+17	9.72E+16	1.09E+17	8.46E+16	9.71E+16
		3	431.83	356.17	351.27	283.365	344.686	320.51
		4	9292673	9142445	7570745	9292641	7744016	8276483
iii	v_{US}	1	4.53E+14	1.37E+16	1.68E+16	1.61E+15	1.75E+15	4.11E+15
		2	9.68E+16	1.12E+17	1.08E+17	1.19E+17	9.52E+16	1.08E+17
		3	1436.2	1360.54	1355.64	1287.735	1349.056	1324.88
		4	19030904	18880676	17308976	19030872	17482247	18014714
iv	v_{KC1}	1	4.53E+14	1.37E+16	1.68E+16	1.61E+15	1.75E+15	4.11E+15
		2	9.68E+16	1.12E+17	1.08E+17	1.19E+17	9.52E+16	1.08E+17
		3	1627.74	1552.08	1547.18	1479.275	1540.596	1516.42
		4	24471214	24320986	22749286	24471182	22922557	23455024
v	v_{KC2}	1	4.53E+14	1.37E+16	1.68E+16	1.61E+15	1.75E+15	4.11E+15
		2	9.68E+16	1.12E+17	1.08E+17	1.19E+17	9.52E+16	1.08E+17
		3	1676.12	1600.46	1595.56	1527.655	1588.976	1564.8
		4	25207582	25057354	23485654	25207550	23658925	24191392
vi	v_{KC3}	1	4.53E+14	1.37E+16	1.68E+16	1.61E+15	1.75E+15	4.11E+15
		2	9.68E+16	1.12E+17	1.08E+17	1.19E+17	9.52E+16	1.08E+17
		3	1358.82	1283.16	1278.26	1210.355	1271.676	1247.5
		4	17868138	17717910	16146210	17868106	16319481	16851948
vii	v_{Mr}	1	1.98E+18	1.99E+18	2.00E+18	1.98E+18	1.98E+18	1.98E+18
		2	9.33E+17	9.48E+17	9.44E+17	9.55E+17	9.31E+17	9.44E+17
		3	15643.24	15567.58	15562.68	15494.78	15556.1	15531.92
		4	68540557	68390329	66818629	68540525	66991900	67524367
viii	v_{Isa2}	1	4.53E+14	1.37E+16	1.68E+16	1.61E+15	1.75E+15	4.11E+15
		2	9.68E+16	1.12E+17	1.08E+17	1.19E+17	9.52E+16	1.08E+17
		3	1702.72	1627.06	1622.16	1554.255	1615.576	1591.4
		4	25331438	25181210	23609510	25331406	23782781	24315248
ix	v_{Prd}	1	3.94E+18	3.95E+18	3.95E+18	3.94E+18	3.94E+18	3.94E+18
		2	1.75E+18	1.76E+18	1.76E+18	1.77E+18	1.74E+18	1.76E+18
		3	29432.89	29357.23	29352.33	29284.43	29345.75	29321.57
		4	252314998	252164770	250593070	2.52E+08	2.51E+08	2.51E+08
x	v_{Sr}	1	2.57E+17	2.70E+17	2.73E+17	2.58E+17	2.58E+17	2.60E+17
		2	1.12E+17	1.28E+17	1.23E+17	1.34E+17	1.11E+17	1.23E+17
		3	689.24	613.58	608.68	540.775	602.096	577.92
		4	10095487	9945259	8373559	10095455	8546830	9079297
xi	v_{Sp}	1	2.22E+18	2.24E+18	2.24E+18	2.23E+18	2.23E+18	2.23E+18
		2	9.37E+17	9.52E+17	9.48E+17	9.59E+17	9.35E+17	9.48E+17
		3	14554.32	14478.66	14473.76	14405.86	14467.18	14443
		4	123587266	123437038	121865338	1.24E+08	1.22E+08	1.23E+08
xii	$v_{Sh(\min)}$	1	2.64E+14	1.35E+16	1.66E+16	1.42E+15	1.56E+15	3.92E+15
		2	7.10E+16	8.64E+16	8.19E+16	9.32E+16	6.93E+16	8.18E+16
		3	436.86	361.2	356.3	288.395	349.716	325.54
		4	9296301	9146073	7574373	9296269	7747644	8280111

Data 3

$$\mathrm{N}=80, \mathrm{n}=20, \overline{\mathrm{Y}}=51.826, \overline{\mathrm{X}}=11.265, \rho=0.941 \quad S_y=18.357,$$
$$C_y=0.354, S_x=8.456, C_x=0.751, \widetilde{\mathrm{X}}=10.300, \mathrm{Q}_1=5.150, \mathrm{Q}_3=16.975,$$
$$\mathrm{QD}=5.910, \lambda_{22}=2.221, \lambda_{40}=2.267, \lambda_{04}=2.866$$

Data 4

$$\mathrm{N}=70, \mathrm{n}=25, \overline{\mathrm{Y}}=96.700, \overline{\mathrm{X}}=175.267, \rho=0.729, S_y=60.714,$$
$$C_y=0.625, S_x=140.857, C_x=0.804, \widetilde{\mathrm{X}}=72.437, \mathrm{Q}_1=80.150, \mathrm{Q}_3=225.025,$$
$$\mathrm{QD}=72.437, \lambda_{22}4.604, \lambda_{40}=4.759, \lambda_{04}=7.095$$

The PRE of the suggested estimators v_{Pk} with respect to the estimators in literature are obtained by (see for formula [22]):

$$PRE=\frac{Var(v_0)}{\mathrm{MSE}(v_l)\ or\ MSE(v_l)_{\min}}\times 100 \tag{54}$$

Table 3
MSEs and PREs of different estimators using a simulation study.

Estimators	n = 60		n = 150		n = 200	
	MSE	PRE	MSE	PRE	MSE	PRE
ν_0	0.028961	100.00	0.010251	100.00	0.007025	100.00
ν_{Isa1}	0.007759	373.2526	0.002198	466.2551	0.001461	480.5977
ν_{US}	0.005946	487.058	0.002022	506.9277	0.001369	513.1908
ν_{KC1}	0.007236	400.2337	0.002576	397.888	0.001715	409.4578
ν_{KC2}	0.028961	100.00	0.010251	100.00	0.007025	100.00
ν_{KC3}	0.005946	487.058	0.002022	506.9277	0.001369	513.1908
ν_{Mr}	0.012988	222.967	0.004650	220.4471	0.00317	221.6083
ν_{Isa2}	0.005946	487.058	0.002022	506.9277	0.001369	513.1908
ν_{Prd}	0.116136	24.93701	0.038993	26.29167	0.026924	26.09483
ν_{Sr}	0.010272	281.9375	0.003659	280.1249	0.002457	285.8815
ν_{Sp}	0.062998	45.98258	0.021889	46.8359	0.015112	46.48933
$\nu_{Sh(\min)}$	0.013798	209.87	0.004948	207.1827	0.003351	209.6502
ν_{P1}	0.005883	492.2375	0.001979	518.012	0.001323	530.6555
ν_{P2}	0.005673	510.5023	0.001955	524.2359	0.001309	536.6171
ν_{P3}	**0.005590**	**517.9955**	**0.001946**	**526.662**	0.001302	539.4979
ν_{P4}	0.005605	516.6609	0.001948	526.2644	0.001303	538.8679
ν_{P5}	0.005609	516.304	0.001950	525.6454	**0.001301**	**539.7565**
ν_{P6}	0.005653	512.2667	0.001953	524.852	0.001307	537.2338

Where l = Isa1, US, KCj, Mr, Isa2, Prd, Sr, Sp, Sh and Pk.

Percentage relative efficiency (PRE) is used to measure and compare the efficiency of one estimator with other. The values of MSE and PRE for the proposed and the estimators mentioned in the literature computed from each population are given in Table 1.

The above figures represent the PREs of the suggested estimators and estimators in the literature using four types of data sets. The figures show that the height of a line graph is directly proportional to the efficiency of proposed estimators for all the data sets. More specifically, as the length of the line increases, the PREs of the estimators also increases.

The above table shows the numerical illustration of the comparison of efficiency conditions of the suggested estimators with the estimators in the literature. The results according to conditions (i) to (xii), show that all the values are greater than zero, therefore, the suggested class of estimators is efficient than the estimators in the literature.

6. Simulation study

In this section, MSEs and PREs of the recommended and existing estimators are compared using simulation study. We performed 10,000 simulations using the distribution of bivariate normal with the identical theoretical means of $[Y, X]$ as $\mu = [2,2]$. The correlation coefficient between the study variable and the concomitant variables ρ_{yx} is 0.95. The covariance matrix is

$$\sum = \begin{bmatrix} 9 & 5.7 \\ 5.7 & 4 \end{bmatrix}$$

Samples of size n = 60, 150 and 200 are selected by simple random sampling. MSE and PRE are calculated using equations (27) and (54) respectively, and results are provided in Table 3 as.

7. Results and discussions

In this paper, an exponential family of optimum estimators for population variance using supplementary information under simple random sampling is proposed. To evaluate the performance of the suggested family of estimators, we used four real data sets and simulation studies. The criteria of MSE and PRE are used for comparison of different estimators. The MSEs and PREs of the suggested and recommended estimators are provided in Table 1 for the real data sets while in Table 3, the values of the simulation study are given. Moreover, MSEs and PREs of the proposed and existing estimators are also presented through different diagrams for both real data and simulation study. The results of the proposed family of estimators vary, based on different choices of a, b, C_1, C_2 and C_3.

The following are some general findings given as.

i. Table 1 reveals that all the suggested estimators ν_{Pk} have the optimum MSE over the competitor estimators in the literature for all the data sets. This verifies that the suggested estimators ν_{Pk} perform the best than the existing estimators.
ii. The combinations of different choices of a, b, C_1, C_2 and C_3 impacts the outcome of the suggested estimators.
iii. The proposed estimator ν_{P1} has the minimum MSE 2220.61, when $a = -1, b = 0, C_1 = Med,\ C_2 = Q_1$ and $C_3 = QD$ using real data set 3.
iv. The first two data sets related to the study variable “production of apple level” also provided efficient results particularly for the estimators ν_{P2} and ν_{P4}.

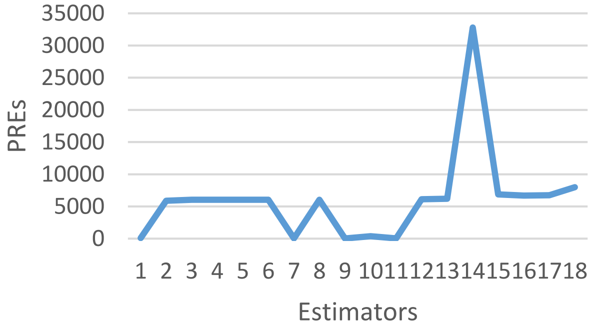

Fig. 1. PREs using Data 1.

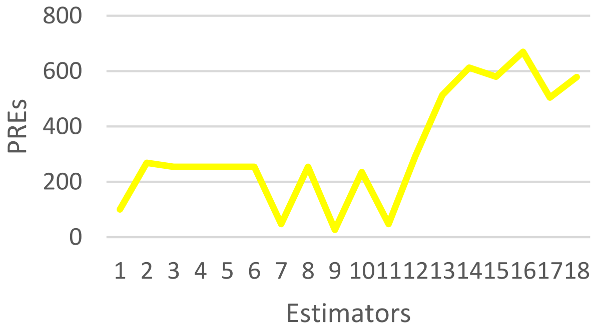

Fig. 2. PREs using Data 2.

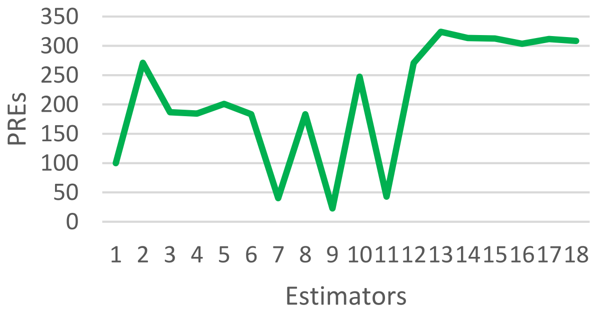

Fig. 3. PREs using Data 3.

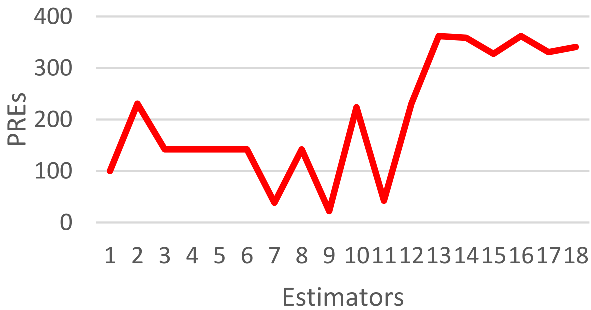

Fig. 4. PREs using Data 4.

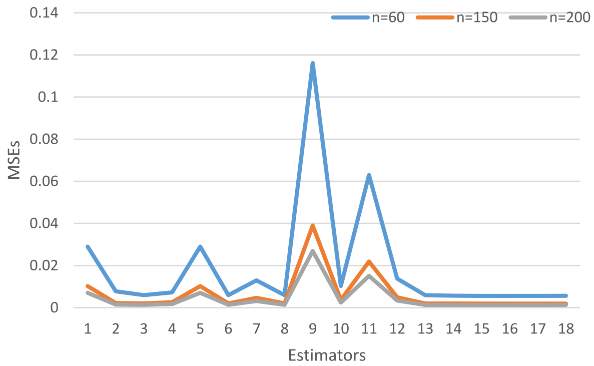

Fig. 5. Comparison by MSEs of proposed and existing estimators using simulation study.

v. Table 2 reveals that all the numerical differences are greater than zero, which verified the efficiency conditions theoretically described in Section 4.
vi. From Figs. 1–4 of real data sets, it is observed that there is maximum PRE for all the suggested estimators as compared to existing estimators as the line in the graph is going in the upward direction.
vii. The application of the suggested estimators is also demonstrated in Table 3, using simulation studies. The results reveal that all six suggested estimators have minimum MSE as compared to the competitor estimators.
viii. In simulation study, when $n = 60$, the variance of v_0 is 0.028961 and when $n = 150$, the value decreased to 0.010251. Similarly, comparing the value of the estimate $Var(v_0)$ when $n = 200$ its value further reduces to 0.007025. A Similar finding is obtained for the remaining existing and proposed estimators. Therefore, there is an inverse relationship between the value of n and MSE for both the suggested and existing estimators.
ix. Fig. 5 of the simulation study also verifies that all of the proposed estimators have smaller MSE than the existing estimators.

8. Conclusion

An improved family of estimators has been proposed for the variance of a finite population using supplementary information. Theoretical bias and MSE have been derived up-to the first order of approximation. The recommended and the existing estimators have been compared both theoretically, numerically and graphically by using PRE and MSE criteria. Four real-world data sets and simulation studies are used for numerical comparison. The results showed that the proposed estimator v_{P1} has minimum MSE than all the other proposed and existing estimators, utilizing $Med = 10.30,\ Q_1 = 5.15$ and $QD = 5.91$. The results of the real data as well as simulation study provided in different tables showed clearly that with the increase of sample size n, the MSE of all the proposed estimators v_{Pk} decreases. Overall, our proposed family of estimators are efficient performers and ensures that these exponential type estimators will be useful for practitioners. Therefore, we recommend our proposed estimators for the new survey in preference to the traditional estimators examined in this research for estimating the variance of the finite population using simple random sampling. This study can be extended by developing new estimators with the help of more selections of a, b, C_1, C_2 and C_3.

Data availability statement

The necessary data used for findings are provided in this paper.

CRediT authorship contribution statement

Mujeeb Hussain: Formal analysis, Data curation, Conceptualization. **Qamruz Zaman:** Supervision, Software, Investigation. **Hijaz Ahmad:** Writing – original draft, Resources, Methodology. **Olayan Albalawi:** Visualization, Project administration, Formal analysis. **Soofia Iftikhar:** Writing – review & editing, Writing – original draft, Visualization, Validation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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