Fractional and fractal processes applied to cryptocurrencies price series

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HIGHLIGHTS

- The dynamics of the cryptocurrencies is analyzed.
- Several tools for investigating the dynamic behavior of the cryptocurrencies are adopted.
- Characteristics of the cryptocurrencies price time series such as persistence, randomness, predictability and chaoticity are addressed.
- With exception of for the Bitcoin, the other five cryptocurrencies analyzed are mean reverting.

GRAPHICAL ABSTRACT
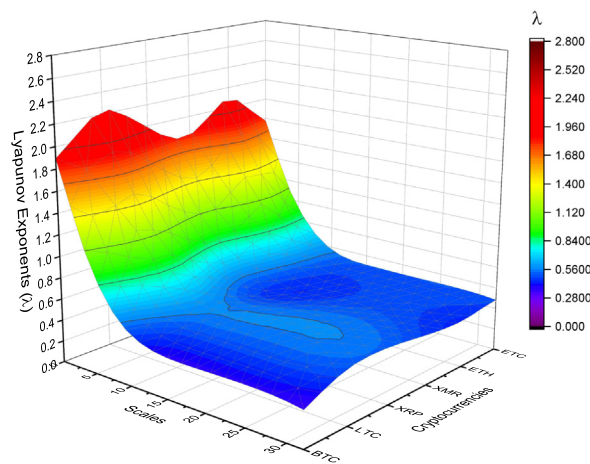
ABSTRACT

Introduction: Cryptocurrencies have been attracting the attention from media, investors, regulators and academia during the last years. In spite of some scepticism in the financial area, cryptocurrencies are a relevant subject of academic research.

Objectives: In this paper, several tools are adopted as an instrument that can help market agents and investors to more clearly assess the cryptocurrencies price dynamics and, thus, guide investment decisions more assertively while mitigating risks.

Methods: We consider three methods, namely the Auto-Regressive Integrated Moving Average (ARIMA), Auto-Regressive Fractionally Integrated Moving Average (ARFIMA) and Detrended Fluctuation Analysis, and three indices given by the Hurst and Lyapunov exponents or the Fractal Dimension. This information allows assessing the behaviour of the time series, such as their persistence, randomness, predictability and chaoticity.

Results: The results suggest that, except for the Bitcoin, the other cryptocurrencies exhibit the characteristic of mean reverting, showing a lower predictability when compared to the Bitcoin. The results for the Bitcoin also indicate a persistent behavior that is related to the long memory effect.

Conclusions: The ARFIMA reveals better predictive performance than the ARIMA for all cryptocurrencies. Indeed, the obtained residual values for the ARFIMA are smaller for the auto and partial auto correlations functions, as well as for confidence intervals.

1. Introduction

Automation and cognitive innovation continue to apace, creating opportunities to radically simplify the processes adopted by the human kind. As this transition picks up speed the capacity to add value tend also to be unleashed. In this context, blockchain may play a key role in the future, addressing concepts and tools, such as artificial intelligence, 5G communications, internet of things, and digital finance [1].

Important countries such as China [2,3] launched an ambitious effort to improve blockchain technology, for issuing digital money and to streamle government services [4]. This strategy evinces that governments understand that to become a high-tech power it is necessary to be positioned at the forefront of the new technologies and that the blockchain is an integral part of this process. Indeed, financial transactions are becoming touchless, as automation and blockchain mature accelerating, therefore, this trend. In this scenario, cryptocurrencies play a major role and become more popular as their use spreads around the world. Indeed, cryptocurrencies can change the procedures adopted for financial transactions [5–7] and attention must be paid to this novel paradigm.

The first and most important digital currency at the present date is the bitcoin (BTC). The BTC first records are dated in 2008 and was introduced with the initial objective of mitigating costs related to electronic transactions (e-commerce) [8]. Nonetheless, the BTC became more popular due to the anonymity that offers in transactions, as well as its independence from traditional financial providers, such as banks or brokers, among others.

Baeck and Elbeck [9] point that the BTC is more a speculative commodity rather than a currency. Nonetheless, investors have employed BTC not only as a currency, but also as an investment [10].

Over the year 2013 the BTC skyrocketed to an amazing 8000% of growth of its value, becoming, consequently, the major bull market of that period [8]. Nonetheless, problems appeared due to the increasing amount of users and the popularization of digital transactions, such as the so-called "double-spending", which is related to the possibility of a same digital currency being involved in two transactions at the same time. Due to this practice, the cryptocurrency became an easy target for hackers [11,12]. One idea that emerged as a possible solution for the problem was to create a public ledger, where all the information about negotiation would be recorded, including reports about the buyer and the seller of BTC. However, this idea did not completely solved the problem, since it carried to the centralization of the cryptocurrency, and also the lack of anonymity of the transactions [8,11].

A protection system denoted "blockchain" was created, consisting of a structure where all the cryptocurrencies are negotiated, serving as a public ledger and being transparent to all transactions. Blockchain is an exceptional anti-fraud system, since every modification in a given transaction (i.e., a block) results in a complete modification of all the precedent blocks posing, therefore, a significant computational load [13,14].

In recent years, the emergence of the altcoins, that is, cryptocurrencies derivated from the BTC open code, contributed to liquidity and competition in the digital currency market. Among the multitude of present day altcoins, we highlight those explored in this work: Litecoin (LTC), Ripple (XRP), Monero (XMR), Ethereum (ETH), and Ethereum Classic (ETC) released during 2011, 2012, 2014, 2015 and 2016, respectively. How the altcoins will influence the BTC price dynamics is a frequent and still open question.

Cryptocurrencies and especially the BTC, have shown to be an innovative and alternative asset that has evolved in the context of absolute returns, volatilities and correlations [15]. At the same time, investors in innovative assets demand for more information to make sure that they are making an appropriate investment while mitigating risks. It is well known that investments in traditional markets require access to data from companies, industries and even the global economy in order to obtain the aforementioned information. Usually, financial advisors and investors rely on analysts to process this data and to provide insight into those assets, that is, to provide an extended and solid investment analysis. Nonetheless, crypto-assets are not related to traditional companies and the processes of evaluating and monitoring their price dynamics is different from those adopted for company stock prices [16].

The main contribution of this work is the use of fractional and fractal mathematical tools, so that market agents and investors have a more refined instrument to assess clearly the cryptocurrencies price dynamics. In this context, recent advances [15] suggest that fractals and fractional calculus are interesting tools to deal with complex behavior with memory effect [17–21]. Non-integer orders of differentiation and integration describe phenomena such as long-term memory, non-locality, persistence, predictability, chaoticity, thus encouraging their application in oscillatory phenomena [22], control algorithms [23,24], biosystems [25–30], economy and finance [31–34], among others [35–37]. The prediction of the cryptocurrencies prices is of key importance since a small advantage obtained through accurate information can lead to significant profits [38–40,10]. Lahmiri and Bekiros [38] implemented a novel deep learning model to forecast the Bitcoin, Digital Cash, and Ripple digital currencies prices series. These researchers observed that a chaotic dynamics, led by long memory effects, can be better understand using fractional operators. Corbet et al. [39] investigated the effect of futures contracts for the Bitcoin market and concluded that the increasing volatility leads to a null effectivity of pricing protection tools, such as, the hedging technique. Thus, speculative approaches are necessary in order to perform in the Bitcoin futures trading. Balcilar et al. [40] examined the causal relation between Bitcoin returns, volatility, and trading volume. It was observed a significant evidence of causality between the returns and trading volume, but no evidence of any causal relation with the volatility was found. This pattern can help models using the return-volume relationship, except during bear and bull market regimes. Despite that, Urquhart [10] found evidence of inefficiency in the Bitcoin market indicating a less random market with memory effects in the prices. In fact, Bitcoin may be moving towards market efficiency. Therefore, for investors to succeed in the digital currency market, it is important to understand the price dynamics of such cryptocurrencies, especially for the fractal dynamics and long-memory effects.

Mensi et al. [41] observed that there is a double long-memory process between the BTC and the ETH prices. Aharon and Qadan [42], and Caporale and Plastum [43] investigated the relationships between the day of the week and the BTC price and, furthermore, the relationship between the prices of the BTC and other altcoins. The so-called "Monday effect" was found for the returns of the BTC. However, that behavior was not observed for the LTC, XRP and Dash (DASH) cryptocurrencies. Balcilar et al. [40] identified another BTC feature when exploring the relationship between the trading volume, BTC returns, and volatility. The study suggested that the trading volume does not helps in predicting the volatility of the BTC returns. Briere et al. [44] showed that the BTC allows the diversification of the investors benefits, and Cheung et al. [45] unveiled the presence of bubbles in the BTC market. Corbet and Katsiampa [46] noted the asymmetries in the BTC price return series' persistence and found evidences of an higher persistence of positive than negative returns. Corbet et al. [47] explored the effectiveness of technical trading rules in the cryptocurrency markets and obtained results supporting that a variable-length moving

average strategy in buy signals outperforms the sell signals in terms of returns. Catania et al. [48,49] verified that, besides the long-memory, the leverage effect also contributes to the volatility of the price time series (TS), because of the different foreign exchange currencies. Bouri et al. [50] analysed the long-memory process in the structural breaks along with the records of the BTC price. Caporale et al. [51] found the same result for other alt-coins, such as the LTC, XRP, and DASH. Since the Efficient-Market hypothesis (EMH) is related to a random-walk type of phenomenon [51,52], studies also suggest that the BTC may be in the process of moving towards an efficient market over the last years [10,53–55].

This paper explores the dynamics of the price TS for the BTC, LTC, XRP, XMR, ETH and ETC cryptocurrencies. Several mathematical tools are adopted, namely the Auto-Regressive Integrated Moving Average (ARIMA), Auto-Regressive Fractionally Integrated Moving Average (ARFIMA) and Detrended Fluctuation Analysis (DFA) algorithms. Additionally, three metrics corresponding to the Hurst (H) index, Lyapunov (λ) exponent and Fractal Dimension (d_A) are also used. The information provided by the distinct approaches captures different characteristics of the price TS, such as, persistence, randomness, predictability and chaoticity.

The paper has the following organization. Section 2 introduces the price TS and the proposed methods. Section 3 discusses the results obtained by the mathematical and algorithmic tools. Finally, Section 4 gives the main conclusions.

2. Data and methods

Cryptocurrencies have shown to be highly volatile, allowing significant profit opportunities for experts in this market [56]. Nonetheless, this market, as well as the financial and economic systems, requires accurate mathematical models to understand their complex behavior [57,58].

The ARIMA and ARFIMA techniques are usually employed in studies involving TS. David et al. [59,60] applied these tools in the description of energy and agricultural commodities and concluded that the ARFIMA shows a better performance than the ARIMA. Several papers adopt the Hurst exponent H to estimate the market efficiency [61,51,62–64]. The Lyapunov exponent λ is also an effective index to indicate the presence of chaos and provides relevant information about the TS predictability [65,66]. The concept of fractals and multifractals have been also extensively applied to evaluate the market efficiency of cryptocurrencies and other financial TS [67–71,53,72]. Bearing these facts in mind and to achieve an effective forecasting, we adopt the aforementioned mathematical tools to explore the dynamics of cryptocurrencies price TS.

The data was obtained in the website https://www.invest-ing.com/crypto/currencies. The TS prices available for the ETC starts in the year of 2016. For this reason, we use daily data (7 values per week and without holidays) of closure prices from 2016 onwards. Furthermore, we consider the same period for all cryptocurrencies investigated in this work, as depicted in Fig. 1. The time interval explored goes from July/2016 to March/2019 for the prices TS composition, and from April/2019 to Jun/2019 for the future prices prediction.

2.1. The ARIMA and ARFIMA models

The ARIMA(p,d,q) and ARFIMA(p,d,q) models are applied in this work to predict the prices of the cryptocurrencies. The parameters p and $q \in \mathbb{N}$ and $d \in \mathbb{R}$ stand for the order (number of time lags) of the autoregressive models, the degree of differencing and the order of the moving average, respectively. The ARIMA can be interpreted

as a combination of the autoregressive and moving-average models. The ARFIMA [73] generalizes the ARIMA, so that the parameter d can assume non-integer values.

The price TS, P_t, is integrated and leads to combined process of auto-regressive (AR), integrated (I) and moving average (MA). The ARIMA can be written by means of the discrepancy operator B that, by definition, can be approximated using the Wold decomposition [74],

$$\Psi(B) \approx \frac{\Theta(B)}{\phi(B)}, \tag{1}$$

where $\Psi(B)$ is an infinite lag polynomial, and $\phi(B) = 1 - \varphi_1 B - \ldots - \varphi_p B^p$ and $\Theta(B) = 1 + \theta_1 B + \ldots + \theta_q B^q$ represent the autoregressive and moving-average operators, respectively. Therefore, the ARIMA(p,d,q) can be written as:

$$\phi(B)(1-B)^d P_t = \Theta(B)\varepsilon_t, \tag{2}$$

where ε_t denotes a white noise process, and $(1-B)^d$ is the differencing operator in the autoregressive model parameters. Similarly, the ARFIMA(p,d,q) is defined by:

$$\Phi(B)P_t = \Theta(B)(1-B)^{-d}\varepsilon_t, \tag{3}$$

where d can assume non-integer values $-0.5 \leqslant d \leqslant 0.5$ [75], and $\Phi(B), \Theta(B)$ and ε_t have the same meaning as before. The polynomials $\Phi(B)$ and $\Theta(B)$ have no common roots, B is the backward shift operator and $(1-B)^{-d}$ is the fractional differencing operator given by [76]:

$$(1-B)^{-d} = \sum_{j=0}^{\infty} \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)} B^j = \sum_{j=0}^{\infty} n_j B^j. \tag{4}$$

An asymptotic approximation of n_j is given by:

$$n_j = \frac{\Gamma(j+d)}{\Gamma(j+d)\Gamma(d)}, \tag{5}$$

where Γ is the gamma function.

The ARFIMA(p,d,q) can grasp the dynamics of a long-range memory process [75,73,76].

The Autocorrelation Function (ACF) and the Partial Autocorrelation Function (PACF) correlograms are plotted for obtaining the orders of p and q of the prediction models (see A.7 and A.8). If necessary, the TS are differenced to become stationary. To compare the orders of the models, the Bayesian Information Criterion (BIC) criterion [77] is adopted in the follow-up. Also, the Seasonal and Trend decomposition using the Loess (STL) procedure is applied, where Loess represents an estimating technique for nonlinear relationships [77]. Table 1 lists the BIC values obtained by means of the ARIMA and ARFIMA for the six cryptocurrencies. The model with the smallest BIC value is selected for each technique. The smaller the BIC value, the better the fitting for the cryptocurrency TS [77].

2.2. The detrended fluctuation analysis method and Hurst exponent

It is known that financial TS may exhibit long-range dependence [78,79,77,59]. The same characteristic can be observed in cryptocurrencies price TS [77,41,48–51]. The long-range dependence is related to the fractional Brownian motion (fBm) [80] that generalizes the classical Brownian motion (Bm) describing a random walk. The properties of the Bm and fBm can be quantified by the Hurst exponent H [62]. Bearing in mind that $H \in (0, 1]$, the main characteristics of the Hurst exponent can be summarized as: a) $H = 1/2$ for a random walk (Bm) process, i.e., without long-memory behavior, b) $H > 1/2$ for a persistent process (fBm) and long-memory effect, and c) $H < 1/2$ for an anti-persistent process related to the short-term memory.
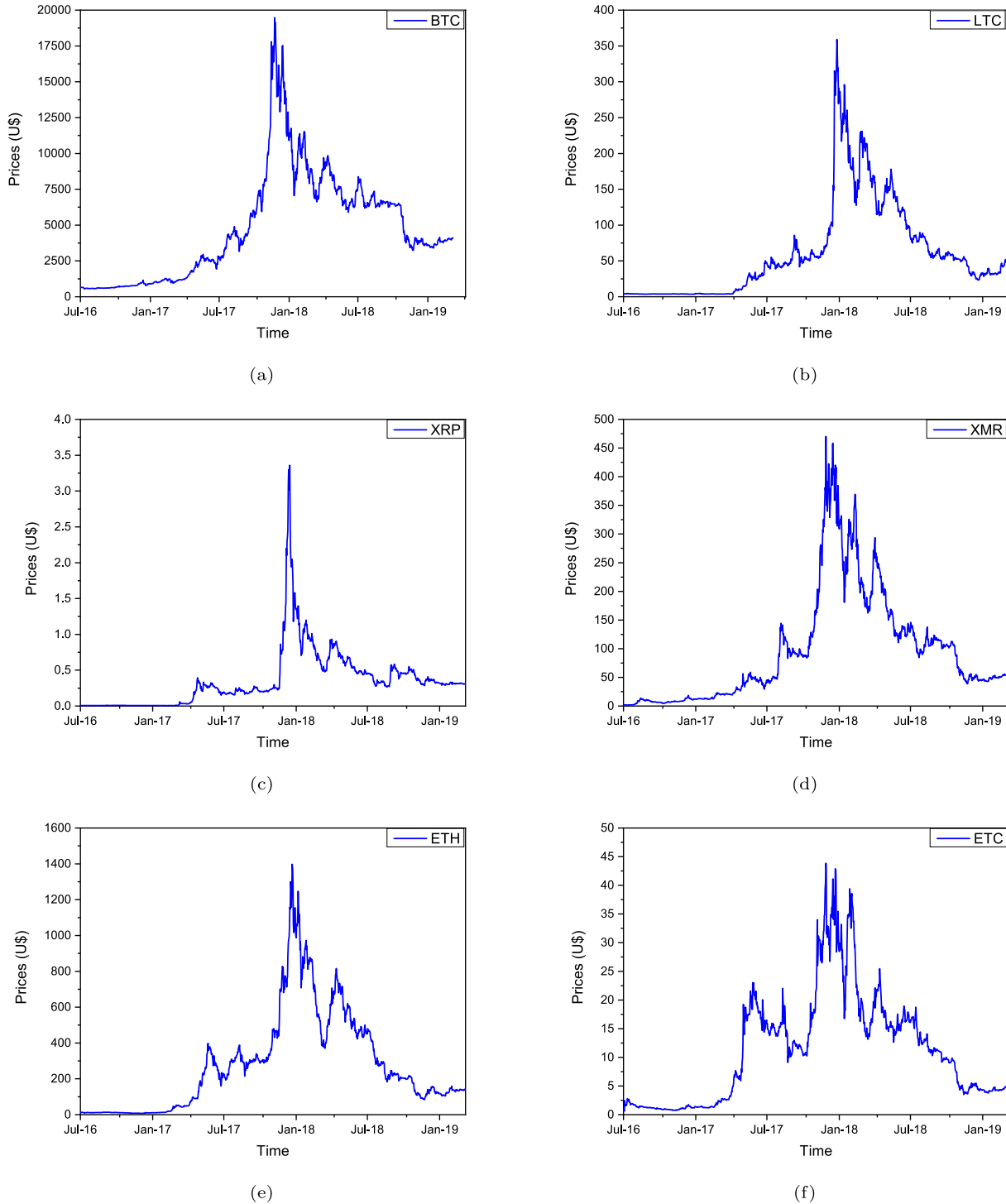
Fig. 1. The six cryptocurrencies (BTC, LTC, XRP, XMR, ETH, ETC from a to f, respectively) price TS from July/2016 to March/2019.

The fBm described by a persistent H is a centered Gaussian process $B^H = \left(B_t^H \right)_{t \geqslant 0}$ that has the covariance function [80] given by:

$$E\left[B^{H_t} B^{H_s} \right] = \frac{1}{2} \left(s^{2H} + t^{2H} - |t - s|^{2H} \right). \tag{6}$$

The closer the value of H to 1, the higher is the probability for the next step to be positive if the last one was also positive.

Several different techniques can be applied to estimate the H exponent of a TS. One of the first methodologies is the rescaled range analysis (R/S), [81,82]. An approach based on the Fourier

analysis, implemented by means of the FFT algorithm, was applied in [83,84]. The DFA method has also been employed in several works [85,86]. This technique avoids the false detection of correlation of self-similarities and can be used in the evaluation of non-stationary series, as well as on the detection of long-memory processes.

The DFA involves successive steps for calculating the H exponent. Let us consider a stochastic prices TS $p(j), j = 1, \ldots, N$, where $N \in \mathbb{N}$ represents the number of observations of the TS. The DFA algorithm involves the three steps: i) Compute the TS mean,

Table 1
The BIC values for the ARIMA and ARFIMA models.

Cryptocurrency	Model	BIC
BTC	STL + ARIMA (1,1,1)	14214.42
	STL + ARFIMA (0,0.5,5)	3047.85
LTC	STL + ARIMA (3,1,2)	6664.00
	STL + ARFIMA (0,0.5,5)	3109.76
XRP	STL + ARIMA (3,1,2)	−2692.24
	STL + ARFIMA (1,0.25,1)	3115.67
XMR	STL + ARIMA (1,1,0)	7500.60
	STL + ARFIMA (0,0.5,5)	3191.99
ETH	STL + ARIMA (0,1,0)	9084.16
	STL + ARFIMA (0,0.5,5)	3201.11
ETC	STL + ARIMA (4,1,3)	3272.17
	STL + ARFIMA (1,0,0)	3106.87

$\hat{p} = \frac{1}{N}\sum_{j=1}^{N} p(j)$, and write the integrated TS by the estimation of $P(j) = \sum_{j=1}^{N}(p(j) - \hat{p})$, ii) Calculate the fluctuation sequence (or quantity), $F(n) = \sqrt{\frac{1}{N}\sum_{j=1}^{N}[P(j) - P_n(j)]^2}$, where $P_n(j)$ is the ordinary least squares method for removing trend and is subtracted from $P(j)$, and iii) Repeat the process so that the slope of the straight line relating $\log(F(n))$ versus $\log(n)$ provides the scaling H exponent.

Assuming that the kth order auto-covariance given by:

$$\gamma(k) = Cov[P_t, P_{t+k}], \qquad (7)$$

is the kth order autocorrelation, the autocorrelation function ρ can be determined as:

$$\rho = \frac{\gamma(k)}{\sqrt{Var(P_t)}\sqrt{Var(P_{t+k})}} = \frac{\gamma(k)}{\gamma(0)}. \qquad (8)$$

As stated by Peters [87], there is a relation between H and ρ described by:

$$\rho = 2^{2H-1} - 1. \qquad (9)$$

Section 3 shows the H exponents determined for the six cryptocurrencies.

2.3. The Lyapunov exponent

The Hurst and Lyapunov exponents are important indices for characterizing non-linear and chaotic systems. The H index measures the irregularity of the TS, that is, captures the rate of chaos. On the other hand, the exponent λ indicates how the presence of chaos conditions influences the prediction of the future.

The value of the λ reflects the sensitive dependence on the initial conditions by measuring the exponential divergence of adjacent orbits. Therefore, the evaluation of how distinct trajectories, with nearby initial conditions, diverge, is related to the expansion or contraction of directions in the phase space [88].

Since the values of H and d_A of a given TS obey the formula $d_A = 2 - H$, a relation between the Hurst and Lyapunov exponents can be estimated from the global dimension d_G. This relationship is used to find the neighboring points in the TS and must be at least equal to or greater than $2d_A$, and one can write [89,90]:

$$d_G \geqslant 4 - 2H. \qquad (10)$$

For the calculation of λ, its local dimension D must be determined. The value of D is related to the dimension of Jacobian matrices as stated by Bryant et al. [89]. The values of D must be not much greater than d_A and, therefore, an option is to choose the next integer value. Also, if we have D equal to d_G then both conditions can be satisfied.

The system dynamics maps a D-sphere of states into a D-ellipsoid. Consequently, when chaotic motion emerges, some kind of complex dynamics is present. The instabilities and stabilities are associated with the directions where stretching and contraction occur [91].

If we consider $\varepsilon_i(t) = \varepsilon_0 b^{\lambda t}$, where $\varepsilon_i(t)$ represents the deformed hyper-volume at time t and b is a given basis, then the Lyapunov exponents can be obtined as [88,91],

$$\lambda_i = \lim_{t \to \infty}\left(\frac{1}{t}\log_b\left(\frac{\varepsilon_i(t)}{\varepsilon_0}\right)\right), i = 1, 2, \ldots, D. \qquad (11)$$

Expression (11) gives a common algorithm for obtaining the Lyapunov spectrum of a system with known equations of motion. For the case of a TS, the Lyapunov exponent can be estimated through the algorithm proposed by Wolf et al. [65]

$$\lambda_i(t) = \frac{1}{t_M - t_0}\sum_{k=1}^{M}\log_2\left(\frac{\varepsilon_i(t_k)}{\varepsilon_0(t_{k-1})}\right), \qquad (12)$$

where M and $t_k - t_{k-1} = \Delta$ represent the total number of replacement steps and the time step, respectively.

The albebraic signs of λ_i provide information about the system's dynamics and recognize chaotic motion since a positive value indicates that the system is chaotic. Furthermore, the Lyapunov exponent can indicate how far future forecasting can be tried in a TS [92]. Bearing this fact in mind, we applied such technique to calculate the λ for the cryptocurrencies price TS.

2.4. Rolling sample approach for the H and d_A indices

A rolling sample calculation is applied to obtain the values of H and d_A, by considering a movable window with fixed length of $n = 100$ samples, that is, by starting at the first one-hundred observations and rolling until the last group of hundred samples. Indeed, the same methodology is described in SubSection 2.2 to obtain H. With this approach, H is calculated along time and it is possible to analyze the cryptocurrencies behavior from July/2016 to March/2019.

Similarly to H, the properties of d_A [93,94] are also related to the memory processes and can be summarized as:

(a) $1 < d_A \leqslant 2$,
(b) $d_A = 3/2$, for a random walk (Bm) indicates that the TS does not have a long-memory process and local anti-correlations,
(c) $d_A < 3/2$, indicates a persistence process (long-memory or correlated), corresponding to fBm,
(d) $d_A > 3/2$, indicates an anti-persistent process (short-term memory, anti-correlated).

The index d_A is calculated through the Hall-Wood (HW) and Robust Genton (RG) estimators [95] described in the follow-up.
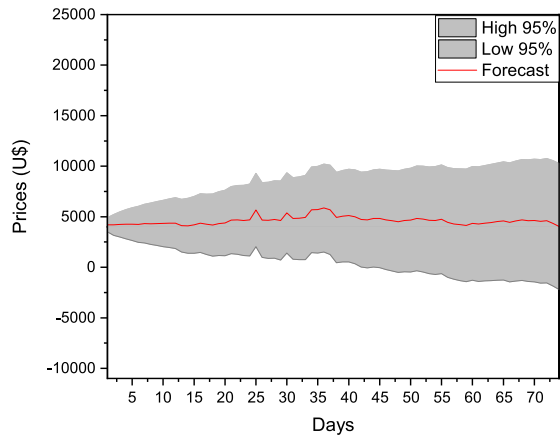
2.4.1. The Hall-Wood estimator

The HW estimator [96] is a box-counting algorithm. Let us have a scale $\epsilon_l = l/n$, where $l = 1, 2, 3, \ldots, n$. The area of the boxes covers the curve is
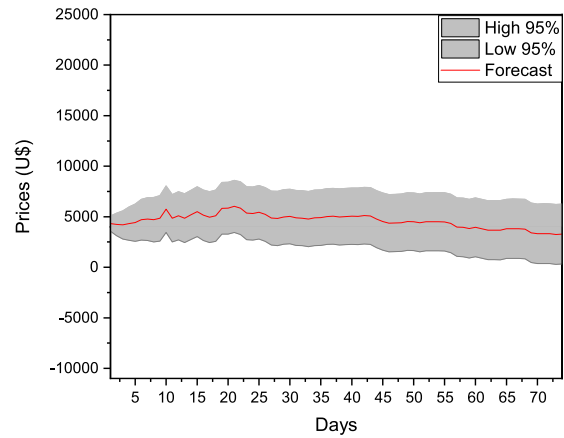
$$\widehat{A}(l/n) = (l/n)\sum_{i=1}^{[n/l]}\left(x_{il/n} - x_{(i-1)l/n}\right), \qquad (13)$$

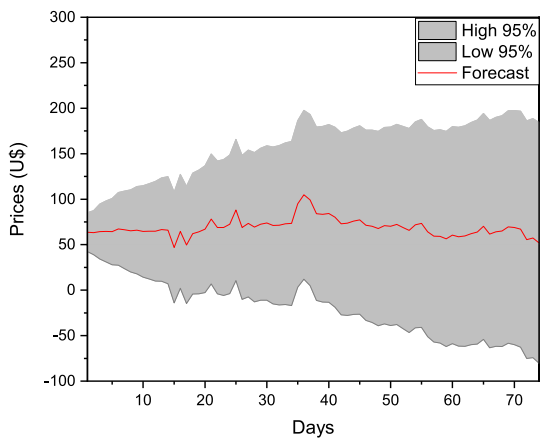where the operator $[n/l]$ calculates the integer part of the argument. The HW estimator is given by

$$d_{A_{HW}} = 2 - \left(\sum_{i=1}^{L}(s_i - \bar{s})\log\left(\widehat{A}(l/n)\right)\right)\left(\sum_{i=1}^{L}(s_i - \bar{s})^2\right)^{-1}, \qquad (14)$$
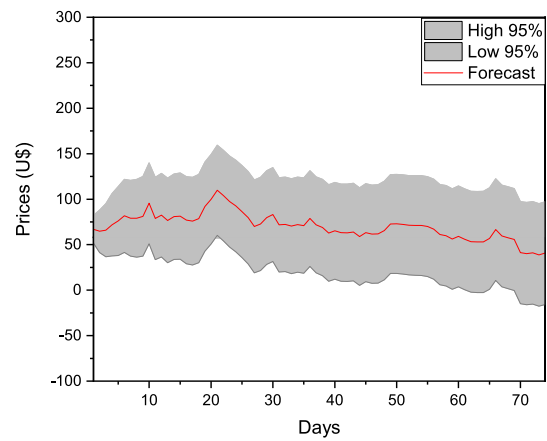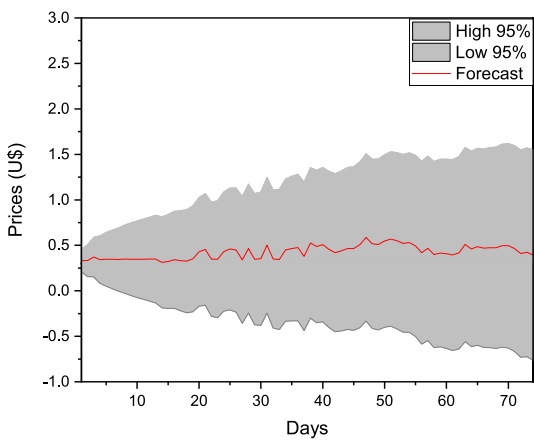
(a) BTC - ARIMA

(b) BTC - ARFIMA

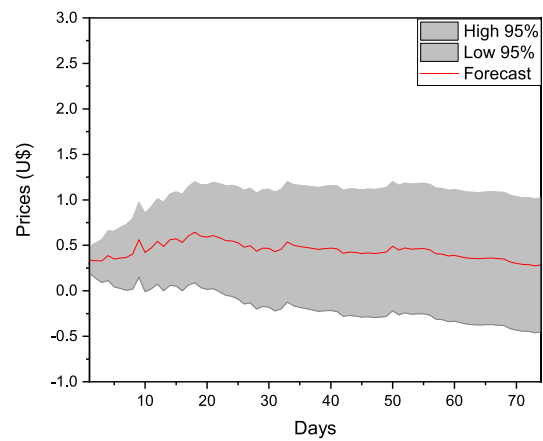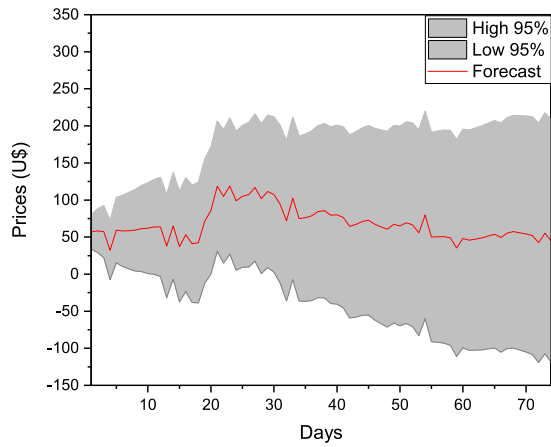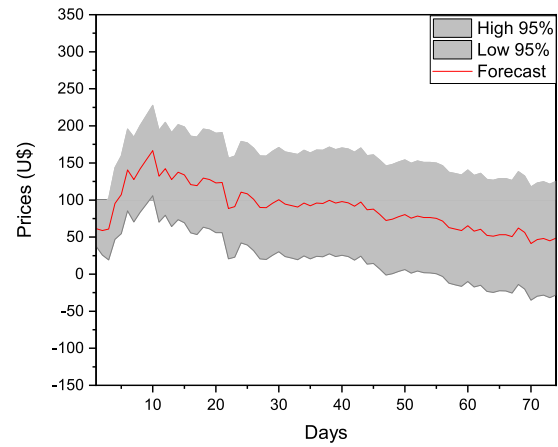(c) LTC - ARIMA

(d) LTC - ARFIMA
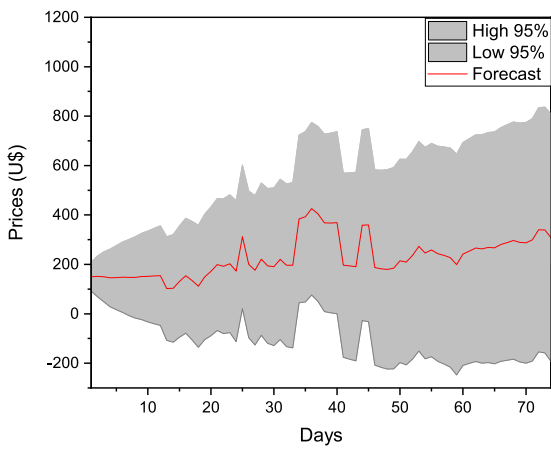
(e) XRP - ARIMA

(f) XRP - ARFIMA

Fig. 2. The ARIMA (left) and ARFIMA (right) predictions for the BTC, LTC and XRP cryptocurrencies.
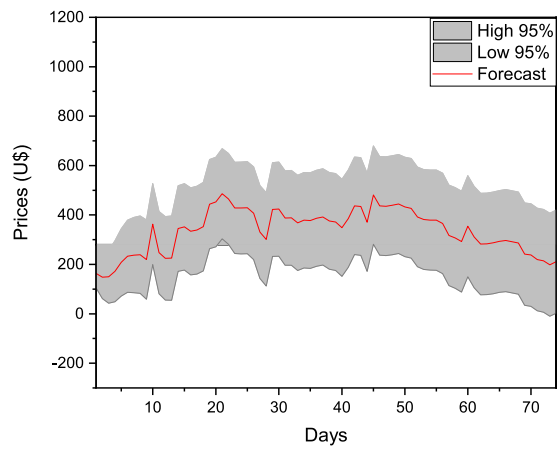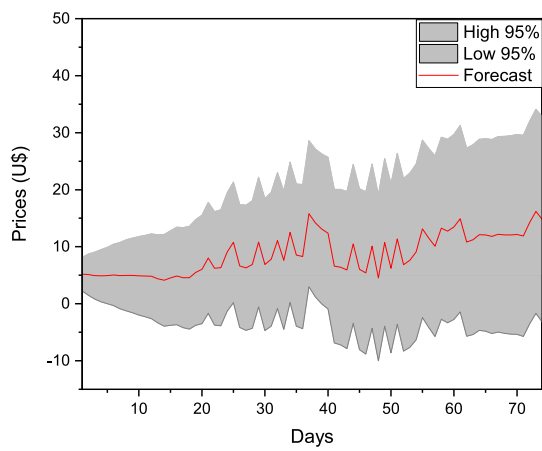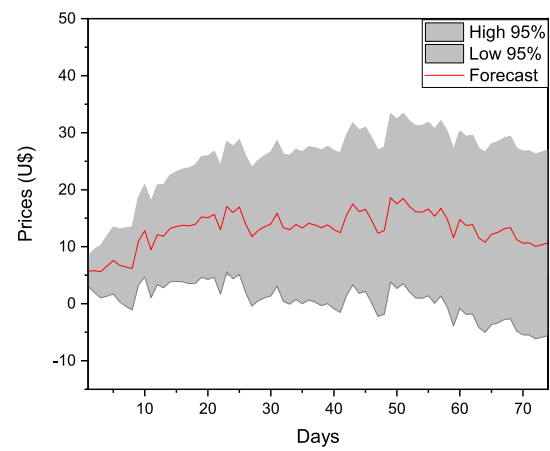
(a) XMR - ARIMA

(b) XMR - ARFIMA

(c) ETH - ARIMA

(d) ETH - ARFIMA

(e) ETC - ARIMA

(f) ETC - ARFIMA

Fig. 3. The ARIMA (left) and ARFIMA (right) predictions for the XMR, ETH and ETC cryptocurrencies.

Table 2

The Hurst, Fractal Dimension and Lyapunov exponents values for the six cryptocurrencies.

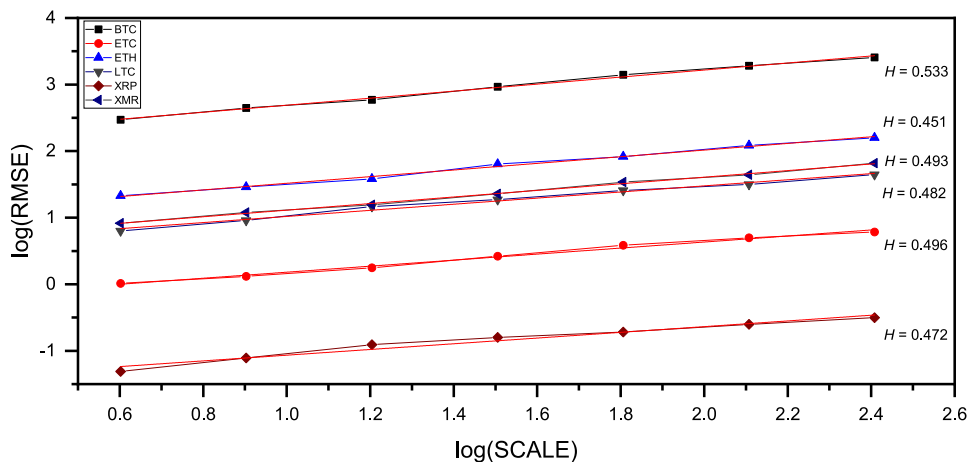| Cryptocurrency	Hurst (H)	Fractal Dimension (d_A)	Lyapunov (λ)	$1/\lambda$ (days)
BTC	0.533	1.467	0.331	3.020
LTC	0.482	1.518	0.565	1.769
XRP	0.472	1.528	0.507	1.972
XMR	0.493	1.507	0.461	2.169
ETH	0.496	1.504	0.389	2.570
ETC	0.451	1.549	0.545	1.835

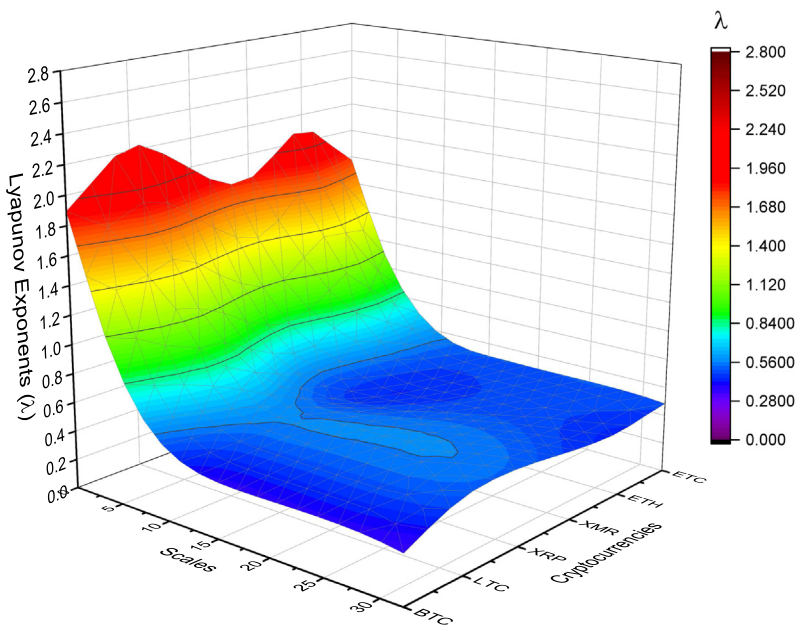**Fig. 4.** DFA-Hurst values for the six cryptocurrencies.

Fig. 5. Average Lyapunov local exponents (λ_1) for the six cryptocurrencies.

where $L \geqslant 2, s_l = \log(l/n)$ and $\bar{s} = (1/L)\sum_{i=1}^{L} s_i$. If we adopt $L = 2$, as pointed by Hall-Wood to avoid bias, then we obtain

$$d_{A_{HW}} = 2 - \frac{\log\left(\widehat{A}(2/n)\right) - \log\left(\widehat{A}(l/n)\right)}{\log(2)}. \tag{15}$$
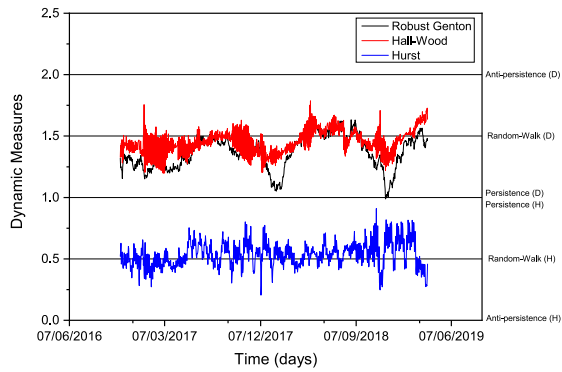
2.4.2. The robust Genton estimator

The RG [97] is based on the moments estimator of scale. However, the scheme is not robust and, therefore, the algorithm developed by Genton is adopted. The calculation yields

$$\hat{V}_2(l/n) = \frac{1}{2(l-n)}\sum_{i=l}^{n}(X_{i/n} - X_{(i-l)/n})^2, \tag{16}$$

and we obtain the RG estimator as

$$d_{A_{RG}} = 2 - \frac{1}{2}\left(\sum_{i=1}^{L}(s_i - \bar{s})\log\left(\hat{V}_2(l/n)\right)\right)\left(\sum_{i=1}^{L}(s_i - \bar{s})^2\right)^{-1}, \tag{17}$$

with $L \geqslant 2, s_l = \log(l/n)$ and $\bar{s} = (1/L)\sum_{i=1}^{L} s_i$. Using $L = 2$ to mitigate bias, one obtains
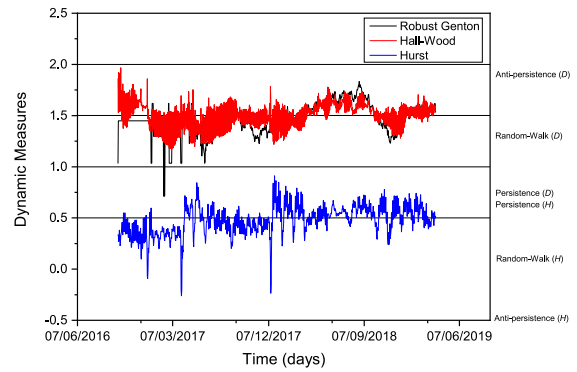
Fig. 6. Rolling window approach for the six cryptocurrencies (BTC, LTC, XRP, XMR, ETH and ETC from a to f, respectively) price TS.

$$d_{A_{RG}} = 2 - \frac{\log\left(\hat{V}_2(2/n)\right) - \log\left(\hat{V}_2(l/n)\right)}{2\log(2)}. \tag{18}$$

3. Results and discussion

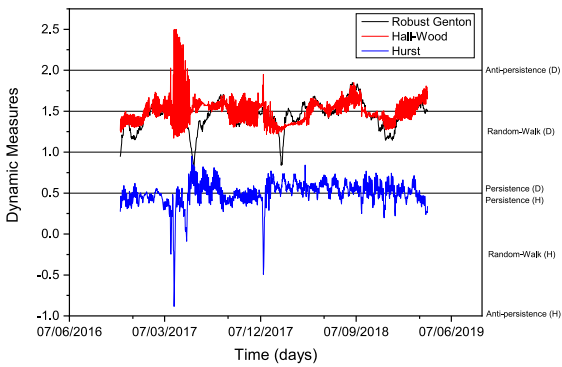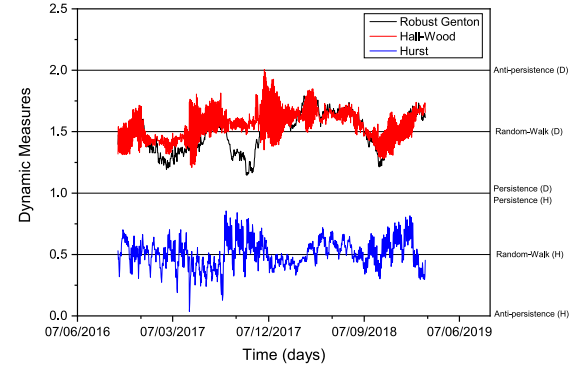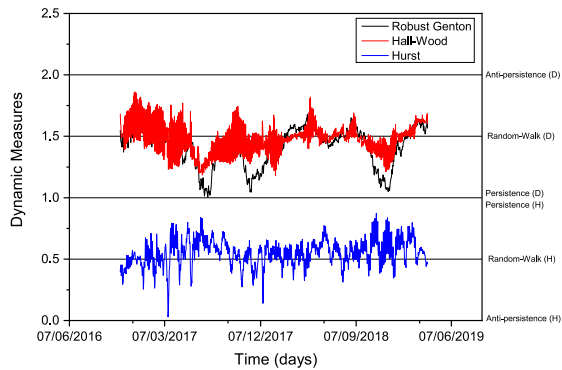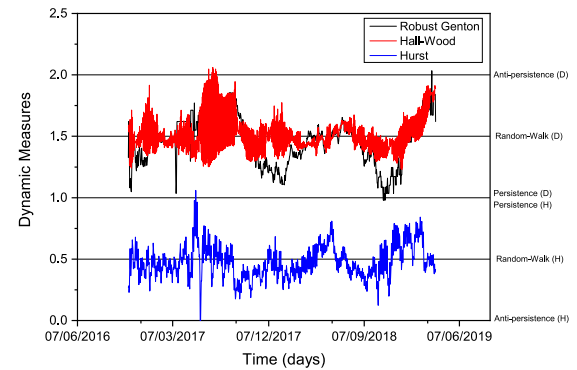The ARIMA and ARFIMA prediction results are depicted in Figs. 2 and 3. From Table 1 it is possible to note that an identical order ($d = 0.5$) is calculated by the ARFIMA for the BTC, ETH, LTC and XMR cryptocurrencies. The range of values higher than $d = 0$ indicate that these cryptocurrencies have a dynamic fBm. The XRP cryptocurrency also presents a long-memory effect. On the other hand, the ETC price series reveals a short-term memory process. However, other fractional tools, such as the Hurst (H), Fractal Dimension (d_A) and Lyapunov (λ) exponents are necessary to determine the dominant process for each price series. Also, Figs. A.7 and A.8 suggest that the ARFIMA has a better fitting for all the cryptocurrencies, since the presence of the residual reveals smaller correlations for the ACF and PACF. Additionally, Figs. 2 and 3 points out smaller confidence intervals values for the ARFIMA that can lead a more precise prediction measurement.

The Hurst, Fractal Dimension and Lyapunov exponents, as well as the prediction horizon ($1/\lambda$) for the cryptocurrencies, are listed in Table 2.

The Hurst exponent and fractal dimension shown in Table 2 and Fig. 4 suggest persistence (long-memory) solely for the BTC, since for the other virtual coins an anti-persistence phenomenon is observed, and are related to a short-memory process.

The local and global dimensions, D and d_G, with values equal to 3, were identified for all cryptocurrencies, as a way of satisfying the Hurst, Fractal Dimension and Lyapunov conditions.

One can note from Table 2 and Fig. 5, that λ is positive for the six cryptocurrencies pointing to a chaotic dynamics. The numerical value $\frac{1}{\lambda}$ can be viewed as information about the predictability of the future TS price based on its past. The smallest value of λ occurs for the BTC and, consequently, the predictability for the BTC is limited to $\frac{1}{\lambda} = 3.020$ day. Nonetheless, this value corresponds to the higher horizon prediction among the six cryptocurrencies. The lowest predictability occurs for the LTC with a value of $\frac{1}{\lambda} = 1.769$ day.

The dynamics in time of the H, $d_{A_{HW}}$ and $d_{A_{RG}}$ for the six cryptocurrencies are depicted in Fig. 6.

Fig. 6a shows that, for the most part of the period, the BTC behaves as persistent (long-memory effect) when the indices H, $d_{A_{HW}}$ and $d_{A_{RG}}$ are observed.

One can also note from Fig. 6 that the other five cryptocurrencies alternate between persistent and anti-persistent behavior. Nonetheless, a predominant anti-persistent (short-memory effect) phenomenon can be observed (Fig. 6b, d and e) for the ETH, XRP and LTC, respectively. The ETC and XMR (Figs. 6c and f) also fluctuate between persistent and anti-persistent behavior, but it seems clear that they remain preferably with anti-persistence.

The results achieved for the BTC are consistent with the one mentioned in the literature [41,46,48–50]. We highlighted that despite the classical fixed window technique showing evidence for an specific effect for a TS (for example, a long-memory effect), the dynamic behavior of an TS can significantly bias its analysis. For this reason, it is possible that the volatility of an certain asset can eventually impair the efficiency of the prediction, when only the aforesaid technique is used.

In our study, besides the adoption of a fixed window, we also consider the rolling window approach, that is, we take into account its important dynamic behavior during the analysis, mitigating possible bias phenomena. Fig. 6 shows the time evolution of the three indices H, $d_{A_{HW}}$ and $d_{A_{RG}}$.

Differently of our findings, previous studies [48,41,50,49] point to a persistent behavior for the cryptocurrencies. Nonetheless, such papers do not employ rolling windows and important information may have been overlooked.

Despite the price volatility revealed by the oscillation between persistence and anti-persistence along the whole period, it is possible to observe that the measures for H and d_A point to similar behaviors as listed in Table 2, evincing the consistency of the results.

4. Conclusions

This study employed a variety of mathematical tools for the analysis of six significant cryptocurrencies. Its main contribution is related to the use of fractional and fractal mathematical tools as an instrument that can help market agents and investors to more clearly assess the cryptocurrencies price dynamics and, thus, guide investment decisions more assertively while mitigating risks. Classical and fractional integration were explored by means of the ARIMA and ARFIMA processes. ARFIMA performed better than the ARIMA for all cryptocurrencies, since the residual values revealed smaller correlations for the ACF and PACF. Moreover, the smaller confidence interval values for the ARFIMA indicated a more precise prediction measurement. The DFA method and the Wolf algorithm were used for obtaining the Hurst index and the Lyapunov exponent, respectively. The fractal dimension was computed by means of HW and RG estimators. The Hurst exponent and fractal dimension versus time were calculated using sliding windows of constant width, that is, the so-called "rolling sample approach". The BTC was the only cryptocurrency that presented more consistent long-memory behavior and the smallest value of the Lyapunov exponent. The LTC exhibited the lowest predictable horizon compared to the other cryptocurrencies, pointing to a chaotic behavior and presenting the highest Lyapunov exponent. The ETH and the XMR presented values of H near to the random walk phenomenon. Nevertheless, they behaved mainly as an anti-persistent process showing a short memory effect.

With exception of the BTC, the other five cryptocurrencies TS are mean reverting, showing a lower predictability than the BTC, revealing a behavior that was verified to be persistent.

Following the results, a future research topic is the multivariate analysis about the influence or the price transmission between the altcoins and the BTC.

Compliance with Ethics Requirements

This article does not contain any studies with human or animal subjects.

Appendix A

To measure the accurary of the prediction models, several criteria are adopted for comparison. We consider the Entropy-Theil's

(E) measure, Mean Absolute Error (MAE), Auto Correlation Function at lag 1 (ACF1) and Mean Absolute Scaled Error (MASE),

$$E = \frac{\sqrt{Avg\left(P_t - \hat{P}_t\right)^2}}{\sqrt{Avg\left(P_t^2\right)} + \sqrt{Avg\left(\hat{P}_t^2\right)}}, \tag{A.1}$$

$$MAE = Avg(|e_t|), \tag{A.2}$$

$$MASE = Avg\left(\frac{|e_t|}{\frac{1}{n-1}\sum_{i=2}^{n}|P_i - P_{i-1}|}\right), \tag{A.3}$$

where $\hat{P}_t$ represents the forecast value and $e_t = P_t - \hat{P}_t$, which is the error value of t.

The results for the six cryptocurrencies are summarized in Table A.3. The ACF and PACF residuals are plotted in Figs. A.7 and A.8.

Table A.3
Results of the criteria for comparing the ARIMA(p, d, q) and ARFIMA(p, d, q).

| Cryptocurrency	Model	E	MAE	ACF1	MASE
BTC	ARIMA(1,1,1)	7.478	1925.365	0.942	12.002
	ARFIMA(0,0.5,5)	8.252	2027.310	0.956	12.638
LTC	ARIMA(3,1,2)	5.041	24.774	0.867	7.993
	ARFIMA(0,0.5,5)	5.415	25.326	0.911	8.171
XRP	ARIMA(3,1,2)	5.736	0.074	0.665	3.702
	ARFIMA(1,0.25,1)	8.806	0.122	0.917	6.055
XMR	ARIMA(1,1,0)	8.028	30.028	0.872	4.917
	ARFIMA(0,0.5,5)	11.627	40.399	0.941	6.691
ETH	ARIMA(0,1,0)	10.193	54.456	0.788	4.515
	ARFIMA(0,0.5,5)	20.340	139.231	0.877	11.544
ETC	ARIMA(4,1,3)	9.619	3.658	0.622	4.791
	ARFIMA(1,0,0)	19.372	6.320	0.813	10.454
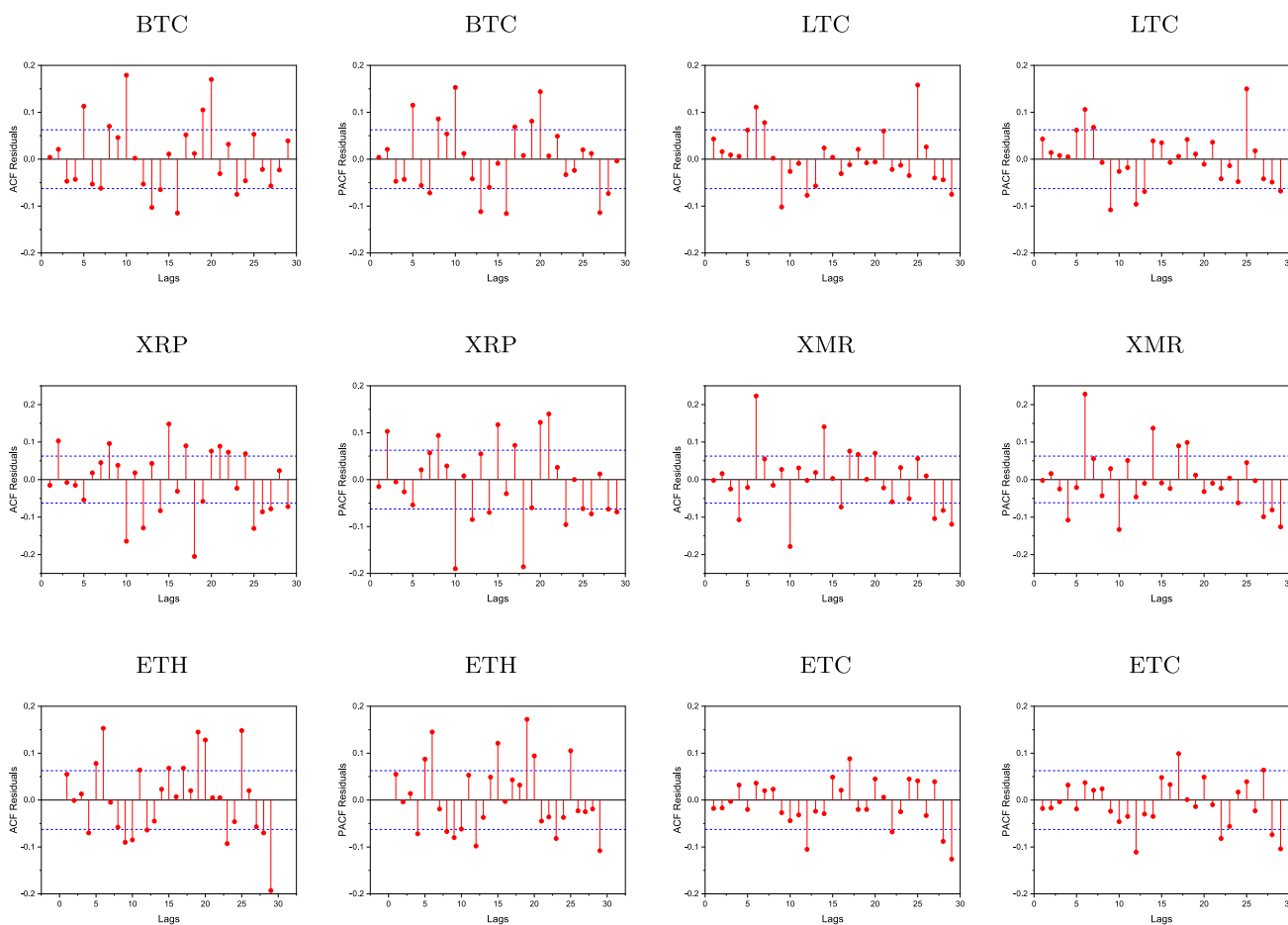
Fig. A.7. The ACF and PACF residuals for the ARIMA for the six cryptocurrencies.
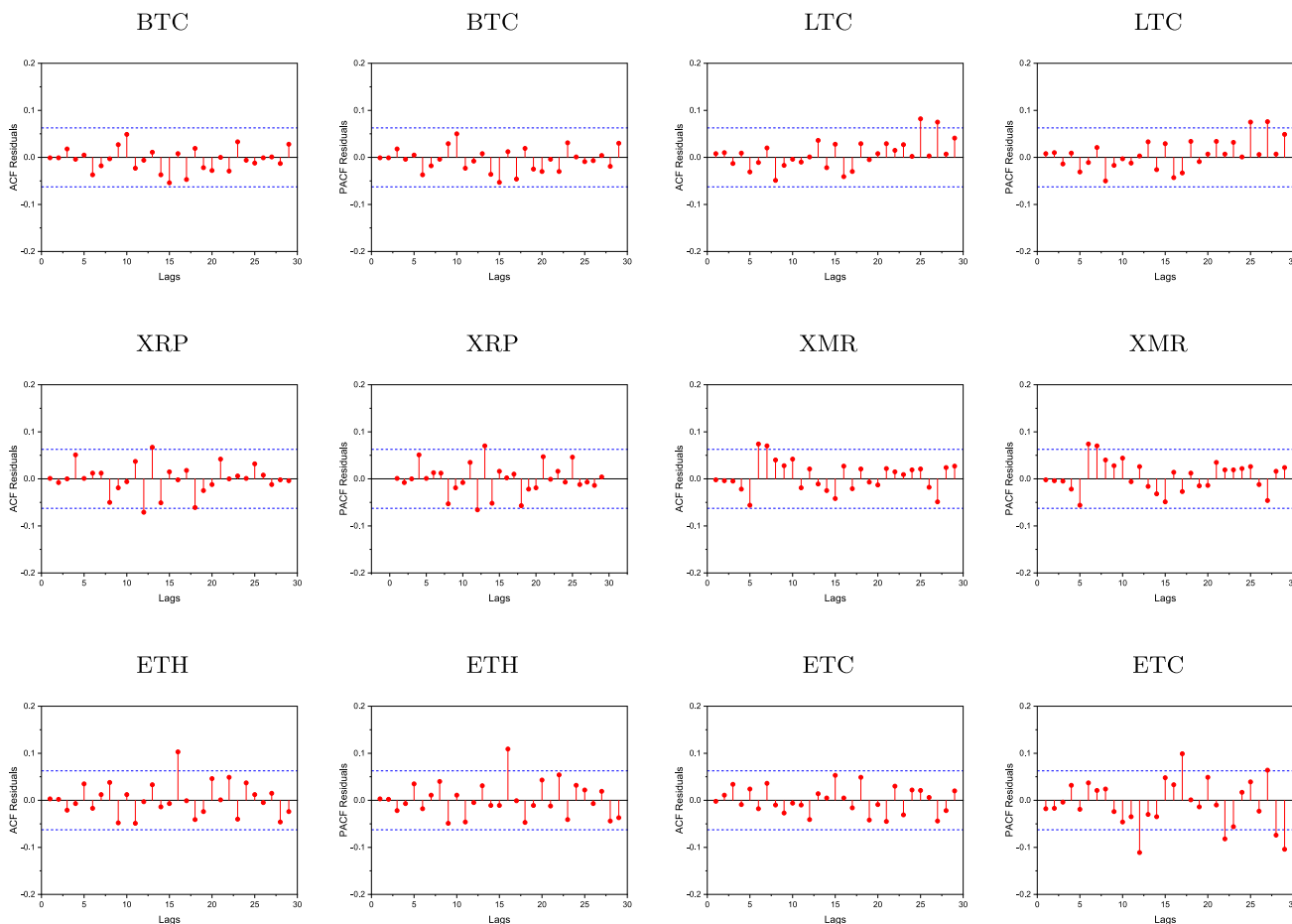
Fig. A.8. The ACF and PACF residuals for the ARFIMA for the six cryptocurrencies.

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