Dynamical Behavior of an SVIR Epidemiological Model with Two Stage Characteristics of Vaccine Effectiveness and Numerical Simulation

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Abstract. An SVIR epidemiological model with two stage characteristics of vaccine effectiveness is formulated. By constructing the appropriate Lyapunov functionals, it is proved that the disease free equilibrium of the system is globally stable when the basic reproduction number is less than or equal to one, and that the unique endemic equilibrium of the system is globally stable when the basic reproduction number is greater than one.

Keywords: Globally stability · Vaccine effectiveness

1 Introduction

In human history, infectious diseases have repeatedly brought great disaster to human survival. In recent years, the outbreak of some new infectious diseases (SARS, influenza A (H1N1), influenza A (H7N9), etc.) has caused a great impact on people's lives. Vaccines are biological agents made from bacteria, viruses, tumor cells and so on, which enable antibodies to produce specific immunity. Vaccination can provide immunity to those who are vaccinated, can eliminate the spread of some diseases (such as smallpox) [1].

In recent years, more and more authors study the epidemiological models with vaccination [2–5]. Some authors assume that vaccine recipients will not be infected [2, 3]; some other authors assume that vaccine recipients may still be infected [4, 5], but the probability of being infected is smaller than before vaccination. In fact, for some infectious diseases, the vaccinated individuals would not be infected for some time after vaccination. However, bacteria or viruses mutate as time goes by, and the efficacy of the vaccine is correspondingly affected, which makes it is possible for the vaccinated individuals to be infected. For example, the new H7N9 influenza virus mutates more quickly, and the effectiveness of the vaccine depends largely on the extent of the virus mutation [6]. Based on the above facts, we assume that vaccine effectiveness has two stage characteristics: in the first stage, the vaccinated individuals will not be infected; in the second stage, the vaccinated individuals will be infected, but the probability of infection will be smaller than before vaccination. Therefore, this

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paper studies the epidemiological model with two stage characteristics of vaccine effectiveness, On the basis of getting the basic reproductive number, by using appropriate functionals, the stability of the model is proved by the algebraic approach provided by the reference [8].

In this work, we study the following epidemiological model:

$$\begin{cases} S' = \mu(1-\mathrm{q})A - \beta SI - (\mu + \mathrm{p})\mathrm{S} + \mathrm{kV_2} \\ V_1' = \mu\,\mathrm{q}A + pS - \mu V_1 - \varepsilon V_1 \\ V_2' = \varepsilon V_1 - \beta\sigma V_2 I - \mu V_2 - kV_2 \\ I' = \beta SI + \beta\sigma V_2 I - (\mu + \alpha + \beta)I \\ R' = \gamma I - \mu R \end{cases} \tag{1}$$

The model (1) has the same dynamic behavior with the following system

$$\begin{cases} I' = \beta SI + \beta\sigma V_2 I - c_4 I \\ V_1' = \mu\,\mathrm{q}A + pS - c_2 V_1 \\ V_2' = \varepsilon V_1 - \beta\sigma V_2 I - c_3 V_2 \\ S' = \mu(1-\mathrm{q})A - \beta SI - c_1\mathrm{S} + \mathrm{kV_2} \end{cases} \tag{2}$$

2 Existence of Equilibria

Obviously, system (2) has a disease free equilibrium $P_0(\mathrm{S_0}, \mathrm{V_{10}}, \mathrm{V_{20}}, 0)$, where

$$S_0 = \frac{[\mathrm{c_2c_3} - \mu\,\mathrm{q}(\mathrm{c_3} + \varepsilon)]\mathrm{A}}{(\mathrm{c_3} + \varepsilon)p + \mathrm{c_2c_3}}, \quad V_{10} = \frac{(\mathrm{p} + \mu\,\mathrm{q})\mathrm{c_3A}}{(\mathrm{c_3} + \varepsilon)p + \mathrm{c_2c_3}}, \quad V_{20} = \frac{(\mathrm{p} + \mu\,\mathrm{q})\varepsilon\,\mathrm{A}}{(\mathrm{c_3} + \varepsilon)p + \mathrm{c_2c_3}}$$

Using [9], we have

$$R_0 = \frac{\beta A[\mathrm{c_2c_3} - \mu q(\mathrm{c_3} + \varepsilon)] + \beta\sigma\varepsilon(\mathrm{p} + \mu\,\mathrm{q})\mathrm{A}}{\mathrm{c_4}[(\mathrm{c_2} + p)\mathrm{c_3} + p\varepsilon]}$$

It can be found the unique endemic equilibrium $P^*(\mathrm{S^*}, \mathrm{V_1^*}, \mathrm{V_2^*}, \mathrm{I^*})$ from the following equations,

$$\begin{cases} \mu(1-\mathrm{q})A - \beta SI - c_1\mathrm{S} + \mathrm{kV_2} = 0 \\ \mu\,\mathrm{q}A + pS - c_2 V_1 = 0 \\ \varepsilon V_1 - \beta\sigma V_2 I - c_3 V_2 = 0 \\ \beta SI + \beta\sigma V_2 I - c_4 I = 0 \end{cases}$$

where

$$S^* = \frac{\mu A c_2(1-\mathrm{q})(\mathrm{c_3} + \beta\sigma\,\mathrm{I^*}) + \mathrm{k}\,\varepsilon\mu qA}{c_2(\mathrm{c_3} + \beta\sigma\,\mathrm{I^*})(\mathrm{c_1} + \beta\,\mathrm{I^*}) - \mathrm{k}\,\varepsilon\,\mathrm{p}}, \; V_2^* = \frac{[(\mathrm{c_1} + \beta\,\mathrm{I^*})\mathrm{q} + \mathrm{p}(1-\mathrm{q})]\varepsilon\mu A}{c_2(\mathrm{c_3} + \beta\sigma\,\mathrm{I^*})(\mathrm{c_1} + \beta\,\mathrm{I^*}) - \mathrm{k}\,\varepsilon\,\mathrm{p}},$$

$$V_1^* = \frac{[(c_1+\beta I^*)q + p(1-q)]\mu A(\beta\sigma I^* + c_3)}{c_2(c_3+\beta\sigma I^*)(c_1+\beta I^*) - k\varepsilon p},$$

and I^* satisfies the following equation

$$\begin{aligned} J(I) &= \frac{\beta\mu A c_2(1-q)(c_3+\beta\sigma I) + k\beta\varepsilon\mu q A}{-k\varepsilon p + c_2(c_3+\beta\sigma I)(c_1+\beta I)} - c_4 + \frac{[(c_1+\beta I)q + p(1-q)]\beta\sigma\varepsilon\mu A}{-k\varepsilon p + (c_3+\beta\sigma I)(c_1+\beta I)c_2} \\ &= 0 \end{aligned}$$

3 Stability of Equilibria

Theorem. When $R_0 \le 1$ the $P_0(S_0, V_{10}, V_{20}, 0)$ is global stable. And P^* is global stable when $R_0 > 1$.

Proof. The global stability of P_0 is firstly proved.
Consider the following Lyapunov functional

$$L_1 = (S - S_0 - S_0\ln\frac{S}{S_0}) + (V_2 - V_{20} - V_{20}\ln\frac{V_2}{V_{20}}) + \frac{\varepsilon}{c_2}(V_1 - V_{10} - V_{10}\ln\frac{V_1}{V_{10}}) + I.$$

so

$$\begin{aligned} L_1' &= \left(1-\frac{S_0}{S}\right)[c_1(S_0 - S) - \beta SI + k(V_2 - V_{20})] + \left(1-\frac{V_{20}}{V_2}\right)(\varepsilon v_1 - \beta\sigma V_2 I - c_3 V_2) \\ &\quad + \frac{\varepsilon}{c_2}\left(1-\frac{V_{10}}{V_1}\right)(\mu qA + pS - c_2V_1) + \beta SI + \beta\sigma V_2 I - c_4 I \\ &= H(V_2, S, V_1) + c_4(R_0 - 1)I \end{aligned}$$

where

$$\begin{aligned} H(V_2, S, V_1) = {} & c_1(S_0 - S)\left(1-\frac{S_0}{S}\right) + k\left(1-\frac{S_0}{S}\right)(V_2 - V_{20}) - c_3V_2 - \varepsilon V_{20}\frac{V_1}{V_2} \\ & + c_3V_{20} + \frac{p\varepsilon}{c_2}S - \frac{p\varepsilon V_{10}S}{c_2V_1} + \frac{\varepsilon\mu qA}{c_2} - \frac{\varepsilon V_{10}\mu qA}{c_2V_1} + \varepsilon V_{10}. \end{aligned}$$

For simplicity, denote $x_1 = \frac{S}{S_0}$, $x_2 = \frac{V_1}{V_{10}}$, $x_3 = \frac{V_2}{V_{20}}$, then

$$\begin{aligned} H(x_1, x_2, x_3) = {} & (2S_0c_1 + \mu V_{20} + 2\varepsilon V_{10} - \frac{p\varepsilon S_0}{c_2}) - (c_1 - \frac{p\varepsilon}{c_2})S_0x_1 - (c_1S_0 - kV_{20})\frac{1}{x_1} \\ & - kV_{20}\frac{x_3}{x_1} - \mu V_{20}x_3 - \frac{\varepsilon V_{10}x_2}{x_3} - (\varepsilon V_{10} - \frac{p\varepsilon}{c_2}S_0)\frac{1}{x_2} - \frac{p\varepsilon S_0x_1}{c_2x_2}. \end{aligned}$$

Using the algebraic approach provided by the reference [8], we will prove the function $H(x_1, x_2, x_3) \leq 0$. Firstly, we can get five groups

$$\{x_1, \frac{1}{x_1}\}, \{x_3, \frac{x_2}{x_3}, \frac{1}{x_2}\}, \{\frac{1}{x_1}, x_3, \frac{x_1}{x_2}, \frac{x_2}{x_3}\}, \{x_1, \frac{x_3}{x_1}, \frac{1}{x_2}, \frac{x_2}{x_3}\}, \{\frac{x_3}{x_1}, \frac{x_1}{x_2}, \frac{x_2}{x_3}\}$$

and the product of all functions within each group is one, then we have

$$\begin{aligned} H_1(x_1, x_2, x_3) = {} & b_1(2 - x_1 - \frac{1}{x_1}) + b_2(3 - x_3 - \frac{x_2}{x_3} - \frac{1}{x_2}) + b_3(3 - \frac{x_3}{x_1} - \frac{x_1}{x_2} - \frac{x_2}{x_3}) \\ & + b_4(4 - \frac{1}{x_1} - x_3 - \frac{x_1}{x_2} - \frac{x_2}{x_3}) + b_5(4 - x_1 - \frac{x_3}{x_1} - \frac{1}{x_2} - \frac{x_2}{x_3}) \end{aligned}$$

Since $H(x_1, x_2, x_3) = H_1(x_1, x_2, x_3)$, we can get

$$\begin{cases} b_1 = c_1 S_0 - kV_{20} - b_4 \\ b_2 = \mu V_{20} - b_4 \\ b_5 = kV_{20} - \frac{p\varepsilon S_0}{c_2} + b_4 \\ b_3 = \frac{p\varepsilon S_0}{c_2} - b_4 \end{cases}$$

As the nonnegativity of $b_i (i = 1, 2 \ldots 5)$, b_4 must satisfy the following condition

$$\max\{0, \frac{p\varepsilon S_0}{c_2} - kV_{20}\} \leq b_4 \leq \min\{\mu(1-q)A, \mu V_{20}, \frac{p\varepsilon S_0}{c_2}\},$$

It is easy to prove the existence of the positive number b_4. So $H(x_1, x_2, x_3) \leq 0$ and $H(x_1, x_2, x_3) = 0$ if and only if $x_1 = x_2 = x_3 = 1$. In summary, when $R_0 < 1$ we have $L_1' < 0$, and when $R_0 = 1$, we get $L_1' \leq 0$, and $L_1' = 0$ if and only if $S = S_0 V_1 = V_{10} V_2 = V_{20}$. The largest invariant set for (2) on the set $\{(S, V_1, V_2, I) \in \Omega : S = S_0, V_1 = V_{10}, V_2 = V_{20}\}$ is $\{P_0\}$. Using the literature [10], we can prove the theorem.

4 Numerical Simulation

The numerical simulations on system (2) were carried out. We can see that if $R_0 \leq 1$, then $P_0(S_0, V_{10}, V_{20}, 0)$ is global stable (Fig. 1) and P^* is globally stable when $R_0 > 1$ (Fig. 2).

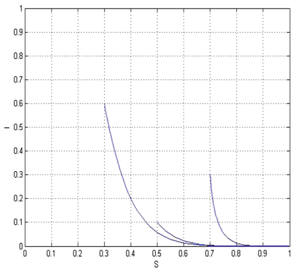

Fig. 1.

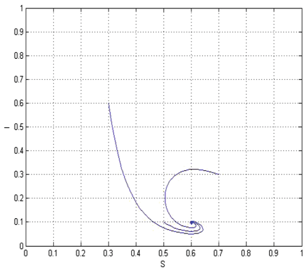

Fig. 2.

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