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Effect of incubation delay and pollution on the transmission dynamics of infectious disease

Shefali Gupta[1] · **Sumit Kaur Bhatia**[2] · **Naina Arya**[2]

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Abstract
Emerging infectious diseases pose serious threat to human population. Studies suggest that there is correlation between population's pollution status and emerging infectious diseases. We propose a delayed SIS model to examine the effects of environmental contamination on human health, which can lead to the spread of numerous diseases. A threshold parameter called basic reproduction number has been obtained for the system. Within the sight of time delay, stability analysis for equilibrium points has been obtained. The existence of Hopf bifurcation around endemic equilibrium point pertaining to time delay as a critical parameter is observed. Our study suggests that pollution can have detrimental effects on the spread of disease. Analytical results are supported by numerical simulations.

Keywords SIS model · Infectious disease · Time delay · Basic reproduction number · Hopf-bifurcation · Stability analysis

Mathematics Subject Classification 34D20 · 92B05 · 92D25

1 Introduction

Mathematical modelling is a vital tool which has helped to analyze and manage the spread of communicable illness in populace. With the help of mathematical modelling, many researchers have been successful in predicting the effects of the spread of diseases

✉ Sumit Kaur Bhatia
sumit2212@gmail.com

Shefali Gupta
shefaligupta879@gmail.com

Naina Arya
nainaarya55@yahoo.com

[1] Department of Mathematics, Tagore Senior Secondary School, New Delhi, India

[2] Department of Mathematics, AIAS, Amity University, Noida, Uttar Pradesh, India

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in populace [1–7]. Different factors such as economic, geographic conditions, media coverage and so on, have a signinificant role in the transmission of these ailments [8–11]. Every country on the planet is experiencing the uncontrollable spread and effects of certain infections caused by agents such as bacteria, virus, fungi, and others, transmitted through direct or indirect contact with an infected person infected with diseases such as tuberculosis, AIDS, Dengue fever, and Covid-19, among others. The effects of these sicknesses bring about the higher rate of demise. Various strategies like vaccination, isolation, drugs and so forth, have been in regular practice by the doctors and the organizations to manage the unfold of different ailments. In spite of all the methods followed in practice, there are few diseases that are an irreplaceable part of the society, such as lifestyle diseases. Numerous models and demonstrating methods are accessible in writing [12–15] which helps us to analyze the impacts of epidemic diseases on populace.
Environmental pollution continues to be a major cause of health danger around the world. Environmental pollution, which encompasses pollution of the air, water, and soil, inuences the wellbeing of the populace [11, 16–18]. In [18], authors studied epidemic model to study the impact of environmental pollution on spread of infectious diseases. Ecological pressure additionally harms the individuals and they tend to grow more vulnerable to certain ailments. Toxin exposure during pregnancy has an effect on newborns, making them more susceptible to illness. Poisonous synthetic substances existing in the environment as a result of increased contamination cause uncontrollable illnesses [17, 19, 20], which adds to the quick spread of epidemics. Thus, epidemic control becomes challenging for authorities, physicians, and health organizations in the modern society. We investigate a mathematical model that incorporates pollution since environmental contamination is an important aspect that cannot be overlooked while investigating disease dynamics.

Time delays are incorporated in a disease transmission model for a wide range of biological reasons. The delay model is used to describe infectious disease dynamics in an attempt to gain a better understanding of increasingly complex models [2, 21, 22]. The incubation period is defined as the time between host infection and the onset of the symptoms. It is known that disease penetrates and transmits invisibly much before the visible disease symptoms appear, it is difficult to assess health risks and avoid, identify, and control the epidemic growth. As a result, in order to examine disease dynamics, the incubation period must be included in the model's architecture in order to suggest appropriate disease control techniques [6, 23]. Dengue fever, Chikungunya virus, Lyme disease, and malaria are examples of vector-borne diseases that have a seven-day incubation period. Cooke [5] also presented a model for infectious disease transmission involving a susceptible and infected population. To better understand the dynamics, the author in [5] included an incubation period in a disease transmission model. In [18], authors provided detailed dynamical analysis of the disease model. They demonstrated the impact of environmental pollution on the dynamics of disease. But they did not include incubation period in their study. Based on the above literature, we study an SIS model under the effect of pollution incorporating incubation time delay. We have considered the time period in which the infectious pathogen develops in the vector before the infected vector infects susceptible persons. The main objective

of our work is to establish the role of incubation delay and provide the implication of the results on disease dynamics of the model under the effect of pollution.
The paper has been organized as follows: A mathematical model with time delay has been proposed along with its boundedness in Sect. 2. In Sect. 3, the equilibrium points and threshold parameter (R_0) has been obtained. In Sect. 4, we have obtained local stability of equilibria followed by the direction and stability of Hopf-bifurcation. Numerical simulations of the model are presented in Sect. 5. The model system is concluded in Sect. 6.

2 Mathematical model

To examine the effects of contamination and disease we propose an epidemic SIS model. We start with some assumptions, such as total population is categorized into subgroups: susceptible and infected. The population under variable N is sub categorized as S, $\_P$ and I. Here, S is the susceptible population who aren't suffering from pollutants, $\_P$ is the population affected by pollution and I is the class of infected people.
We have considered the following assumptions for our model:

(i) When infectives come into contact with susceptibles, the susceptibles becomes infected at a rate λ rate.
(ii) Because prenatal exposure to pollution has a variety of negative consequences, it is anticipated that a fraction m of all infants will enter to the S class, while the remaining fraction $(1-m)$ will enter into $\_P$ class.
(iii) At a consistent rate θ populace will enter from S into $\_P$.
(iv) Because pollution has a variety of negative effects on stressed people, including a weakened immune system, it is hypothesised that the transition rate to the infected class is higher for pollution-affected persons ($\_P$) than for those who are not (S). Let $\tau > 0$ represent the incubation period of the disease, defined as a fixed time during which the infectious agents develop in the vector, and it is only after that time that the infected vector can infect a susceptible individual. Consequently, proposed mathematical model is as follows :

$$\begin{cases} \frac{dS}{dt} &= aM - \theta S - \lambda SI + \eta\xi I - \mu S \\ \frac{d\_P}{dt} &= (1-a)M + \theta S - \lambda(1+\delta\lambda')\_PI + (1-\eta)\xi I - \mu\_P \\ \frac{dI}{dt} &= \lambda S(t-\tau)I(t-\tau) + \lambda(1+\delta\lambda')\_PI - (\xi+\phi+\mu)I \end{cases} \tag{1}$$

All parameters of the model are assumed to be positive. The initial conditions of the model (1) are as follows:

$$\left.\begin{gathered} S(\beta) = \wp_1(\beta),\ \_P(\beta) = \wp_2(\beta),\ I(\beta) = \wp_3(\beta), \\ \wp_1(\beta) \geq 0,\ \wp_2(\beta) \geq 0,\ \wp_3(\beta) \geq 0,\ \beta \in [-\tau, 0],\ \wp_1(0) > 0,\ \wp_2(0) > 0,\ \wp_3(0) > 0 \end{gathered}\right\} \tag{I.C}$$

Table 1 Parameter and its meaning

Parameter	Meaning
M	Newborns recruitment rate
θ	Susceptible individuals transfer rate into stressed compartment
λ	Disease transmission rate of S
μ	Natural mortality rate
ϕ	Disease induced death rate
ξ	Infected populace recovery rate
δ	Amount which influence transmission rate due to natural contamination
η	Proportion of recovered people returned to S
$(1-\eta)$	Proportion of recovered people returned to $\_P$
λ'	Measures the impact of contamination on λ
$\lambda(1+\delta\lambda')$	Transfer rate to infected class for the people of $\_P$ group

where $(\wp_1(\beta), \wp_2(\beta), \wp_3(\beta)) \epsilon\, C([-\tau, 0], \mathbb{R}^3)$, C is the Banach space of continuous functions.

2.1 Basic properties

This section focuses on the study of positivity and boundedness of solutions, for which we have the following lemma

Lemma 1 *Let S, $\_P$ and I be solutions of the model (1) with initial conditions ($I.C$). Then S, $\_P$ and I are positive for all $t \geq 0$.*

Proof Assuming that one solution of the system (1) is atleast not positive, then we've the subsequent cases: **C**ase I: there exists t_1 such that $S(0) > 0$, $S(t_1) = 0$, $S'(t_1) < 0$, $\_P(t) > 0$, $I(t) > 0$, $0 \leq t < t_1$.
Case II: there exists t_2 such that $\_P(0) > 0$, $\_P(t_2) = 0$, $\_P'(t_2) < 0$, $S(t) > 0$, $I(t) > 0$, $0 \leq t < t_2$.
Case III: there exists t_3 such that $I(0) > 0$, $I(t_3) = 0$, $I'(t_3) < 0$, $S(t) > 0$, $\_P(t) > 0$, $0 \leq t < t_3$.
If Case I holds, then we get $S'(t_1) = M > 0$ which contradicts $S'(t_1) < 0$.
If Case II holds, then we get $\_P'(t_2) = 0$ which contradicts $\_P'(t_2) < 0$.
If Case III holds, then we get $I'(t_3) = 0$ which contradicts $I'(t_3) < 0$.
Therefore, due to arbitrariness of S, $\_P$ and I, all solutions of the system remain positive for all $t \geq 0$. □

Lemma 2 *The feasible region K defined by*

$$K = \left\{ (S, \_P, I) \in \mathbb{R}^3_+ : 0 \leq S + \_P + I \leq \frac{M}{\mu} = A \right\}$$

is positively invariant for system (1) and attracts all the solutions starting in the interior of positive orthant.

Proof Let $W(t) = S(t) + \_P(t) + I(t)$, then

$$\begin{aligned}\frac{dW}{dt} &= M - \mu S - \mu\_P - (\phi + \mu)I - \lambda SI + \lambda S(t-\tau)I(t-\tau)\\ &= M - \mu S - \mu\_P - (\mu + \phi)I\\ &\leq M - \mu W\end{aligned}$$

$$\Rightarrow \lim_{t\to\infty} \sup\ W \leq \frac{M}{\mu}$$

which implies

$$K = \left\{(S, \_P, I) : 0 \leq S + \_P + I \leq \frac{M}{\mu} = A\right\}$$

is a positively invariant set of system (1). Hence, lemma is proved. □

Therefore, in this paper, we consider the dynamics of the model system on the set K.

3 Existence of equilibria and reproduction number

We find the equilibrium points of (1) in this section. There are only two types of equilibrium points namely:

(i) E^0: Disease-Free Equilibrium Point
(ii) E^*: Endemic-Equilibrium Point

(i) Disease-free equilibrium point:
The solution of the following algebraic equations gives disease-free equilibrium point $E^0 = (S^0, \_P^0, I^0)$ for model system (1).

$$aM - \theta S - \lambda SI + \eta\xi I - \mu S = 0 \tag{2}$$
$$(1-a)M + \theta S - \lambda(1 + \delta\lambda')\_PI + (1-\eta)\xi I - \mu\_P = 0 \tag{3}$$
$$\lambda SI + \lambda(1 + \delta\lambda')\_PI - (\xi + \phi + \mu)I = 0 \tag{4}$$

Substituting the value of $I^0 = 0$ in Eq. (2) gives

$$S^0 = \frac{aM}{\theta + \mu}.$$

Now, substitute the value of $I^0 = 0$ in equation (3), we get $(1-a)M + \theta S^0 - \mu\_P^0 = 0$

$$\Rightarrow \_P^0 = \frac{(1-a)M(\theta + \mu) + \theta aM}{\mu(\theta + \mu)}$$

Hence, the disease-free equilibrium point is

$$E^0 = (S^0, \_P^0, 0) = \left(\frac{aM}{\theta + \mu}, \frac{(1-a)M(\theta+\mu) + \theta aM}{\mu(\theta+\mu)}, 0 \right)$$

(ii) Endemic-equilibrium point : $E^* = (S^*, \_P^*, I^*)$,

Before finding endemic equilibrium point E^*, we firstly find the threshold parameter called basic reproduction number. The parameter R_0 helps us to find the number of secondary infections of the infected and characterize the spread of infectious disease. The next generation matrix method [4, 24] is used in calculating R_0. The matrix $\mathbb{F}$ and $\mathbb{V}$ for (1) are given below as:

$$\mathbb{F} = \begin{pmatrix} \lambda SI + \lambda(1+\delta\lambda')\_PI \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbb{V} = \begin{pmatrix} (\xi + \phi + \mu)I \\ -(1-a)M - \theta S + \lambda(1+\delta\lambda')\_PI - (1-\eta)\xi I + \mu\_P \\ -aM + \theta S + \lambda SI - \eta\xi I + \mu S \end{pmatrix}$$

Let F be the jacobian of $\mathbb{F}$ and V be the jacobian of $\mathbb{V}$ at disease free equilibrium point.

$$F = [\lambda S^0 + \lambda(1+\delta\lambda')\_P^0]$$
$$V = [\xi + \phi + \mu]$$

R_0 is the spectral radius of FV^{-1} i.e.,

$$R_0 = \frac{\lambda aM\mu + \lambda(1+\delta\lambda')M[(\theta+\mu) - \mu a]}{\mu(\theta+\mu)(\xi+\phi+\mu)}$$

Now, we evaluate the endemic equilibrium point $E^* = (S^*, \_P^*, I^*)$.
From Eq. (2), we get

$$S^* = \frac{aM + \eta\xi I}{\lambda I + (\theta + \mu)}$$

Similarly, from Eq. (3) we obtain the value of $\_P^*$

$$\_P^* = \frac{(1-a)M + \theta S^* + (1-\eta)\xi I}{\lambda(1+\delta\lambda')I + \mu}$$

Substitute the values of S^* and $\_P^*$ in Eq. (4), we get the quadratic equation of I as:

$$F(I) = AI^2 + BI + C = 0 \tag{5}$$

where $A = -\lambda^2(1+\delta\lambda')(\mu+\phi)$
$B = -[(\lambda(1+\delta\lambda')(\xi+\phi+\mu)(\theta+\mu)+\lambda\mu) - \lambda\eta\mu\xi - \lambda(1+\delta\lambda')[\lambda M - \xi(\theta+\mu(1-\eta))]]$

$C = \lambda a M\mu + \lambda(1+\delta\lambda')M[(\theta+\mu)-\mu a] - \mu(\theta+\mu)(\xi+\phi+\mu) = \mu(\theta+\mu)(\xi+\phi+\mu)(R_0 - 1)$
It is evident from the above expressions that A is always negative. Now it is clear, by Descartes's rule of sign, that Eq. (5) always has a unique positive root whenever $R_0 > 1$. Moreover, Eq. (5) may have more than one positive roots if $R_0 < 1$ and $B > 0$ but in the next section we will show that it is not possible.

4 Stability of equilibria

The objective of this section is to study stability analysis of disease-free equilibrium point as well as of endemic equilibrium point.

4.1 Local stability of disease-free equilibrium point

For the local stability of disease free equilibrium point E^0, we prove the following theorem:

Theorem 3 *E_0 is locally asymptotically stable for $R_0 < 1$ and unstable for $R_0 > 1$ for all $\tau \geq 0$.*

Proof **Analysis for** $R_0 \neq 1$ The characteristics polynomial equation in variable Δ is $(-(\theta+\mu)-\Delta)(-\mu-\Delta)(e^{-\Delta\tau}\lambda S^0 + \lambda(1+\delta\lambda')\_P^0 - (\xi+\phi+\mu)-\Delta) = 0$
Clearly, two of the negative eigenvalues of the above equation are $-(\theta+\mu)$ and $-\mu$ and other root is the solution of $f(\Delta) = 0$, where $f(\Delta)$ is as follows:

$$f(\Delta) = e^{-\Delta\tau}\lambda S^0 + \lambda(1+\delta\lambda')\_P^0 - (\xi+\phi+\mu) = 0$$

Assume that $R_0 < 1$. To obtain a contradiction, suppose that f has a root $\delta_2 \epsilon \mathbb{C}$ such that $Re(\delta_2) \geq 0$. Then, $\Delta_2 = e^{-\Delta\tau}\lambda S^0 + \lambda(1+\delta\lambda')\_P^0 - (\xi+\phi+\mu)$, so
$Re(\Delta_2) \leq \lambda S^0 + \lambda(1+\delta\lambda')\_P^0 - (\xi+\phi+\mu) = R_0 - 1 < 0$
which contradicts our assumption. Therefore, in this case, the eigenvalue has negative real part if $R_0 < 1$.
Now, if $R_0 > 1$, we can see that $f(0) < 0$ and $\lim_{\Delta\to\infty} f(\Delta) = +\infty$ for $\Delta \epsilon \mathbb{R}$ which implies that f has at least one positive root. Hence, E_0 is unstable in this case. Hence, the theorem. □

Analysis for $R_0 = 1$
We will employ centre manifold theory [3] to investigate the equilibrium point's stability behaviour. It is obvious from the value of basic reproduction number that it is proportional to λ, so let the bifurcation parameter be λ. If $R_0 = 1$, then $\lambda = \lambda^* = \dfrac{\mu(\theta+\mu)(\xi+\phi+\mu)}{aM\mu + (1+\delta\lambda')M[(\theta+\mu)-\mu a]}$, and jacobian martix around disease free equilibrium point of the system (1) has one of the characteristic values as 0 and the other characteristic values are negative. Now, Jacobian matrix for the disease-free point for model system (1) is :

$$J = \begin{pmatrix} -(\theta+\mu) & 0 & \eta\xi - \lambda S^0 \\ \theta & -\mu & -\lambda(1+\delta\lambda')\_P^0 + (1-\eta)\xi \\ 0 & 0 & \lambda S^0 + \lambda(1+\delta\lambda')\_P^0 - (\xi+\phi+\mu) \end{pmatrix}$$

For $\lambda = \lambda^*$, Jacobian has characteristic value 0 while remaining characteristic values are negative. Right eigenvector, (v_1, v_2, v_3) is evaluated as

$$v_1 = \frac{1}{(\theta+\mu)}\left[\frac{-a\mu(\xi+\phi+\mu)+\eta\xi[\mu a+(1+\delta\lambda')(\theta+\mu(1-a))]}{\mu a+(1+\delta\lambda')(\theta+\mu(1-a))}\right]$$
$$v_2 = \frac{1}{\mu}$$
$$\left[\frac{\theta v_1+(\xi+\phi+\mu)(1+\delta\lambda')(\theta+\mu(1-a))+(1-\eta)[\mu a+(1+\delta\lambda')(\theta+\mu(1-a))]}{\mu a+(1+\delta\lambda')(\theta+\mu(1-a))}\right]$$
$$v_3 = 1$$

Similarly, left eigenvector (q_1, q_2, q_3) can be evaluated to be $(0, 0, 1)$.

Using theorem 4.1 as given in [3], the coecients c and d can be calculated as:

$$c = \Sigma q_k v_i v_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(E_0, \lambda^*)$$

and

$$d = \Sigma q_k v_i \frac{\partial^2 f_k}{\partial x_i \partial \beta}(E_0, \lambda^*)$$

For model system (1), the values of c and d are as follows:

$$\begin{aligned} c &= q_3 v_3\left[2v_1\frac{\partial^2 f_3}{\partial S_1 \partial I}(E_0,\lambda^*) + 2v_2\frac{\partial^2 f_3}{\partial S_2 \partial I}(E_0,\lambda^*)\right] \\ &= -(\phi+\mu)\left[\mu a(1+\theta)+(1+\delta\lambda')(1+\theta)(\theta+\mu)-\mu a\right] \\ &= -\left[\mu^2 a+\eta\mu\theta-\delta\lambda'\eta\mu^2 a+\delta\lambda'\eta\mu^2\right]\xi\left[\frac{\lambda^*}{\mu(\mu a+(1+\delta\lambda')[(\theta+\mu(1-a)]}\right] \\ &= -\left[\mu^2 a+\eta\mu\theta+\delta\lambda'\eta\mu^2(1-a)\right]\xi\left[\frac{\lambda^*}{\mu(\mu a+(1+\delta\lambda')[\theta+\mu(1-a)]}\right] < 0 \end{aligned}$$

Similarly,

$$\begin{aligned} d &= q_3 v_1\frac{\partial^2 f_3}{\partial S_1 \partial I}(E_0,\lambda^*) + q_3 v_2\frac{\partial^2 f_3}{\partial S_1 \partial I}(E_0,\lambda^*) + q_3 v_3\frac{\partial^2 f_3}{\partial S_1 \partial I}(E_0,\lambda^*) \\ &= \frac{aM\mu+(1+\delta\lambda')(1-a)M(\theta+\mu)+\theta aM}{\mu(\theta+\mu)} > 0 \end{aligned}$$

Thus, we can state the below theorem from [3].

Theorem 4 *The disease-free equilibrium changes its stability from stable to unstable at $R_0 = 1$ and there exists a positive equilibrium as R_0 crosses one. Hence, the system exhibits transcritical bifurcation with bifurcation parameter λ^* at $R_0 = 1$.*

Remark 5 As $c < 0$ and $d > 0$, the existence of backward bifurcation is unfeasible and hence Eq. (5) will not have positive root for $R_0 < 1$.

4.2 Local stability of endemic equilibrium point and Hopf-bifurcation

The system (1) has a positive endemic equilibrium point $E^* = (S^*, \_P^*, I^*)$, as shown in the previous section. Here, we investigate the local stability of E^*.

The Jacobian matrix corresponding to $(S^*, \_P^*, I^*)$ is:

$$Y^* = \begin{pmatrix} \alpha_1 & 0 & \alpha_2 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ 0 & \alpha_8 & \alpha_9 \end{pmatrix} + e^{-\Delta\tau} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha_{11} & 0 & \alpha_{13} \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 & \alpha_2 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ e^{-\Delta\tau}\alpha_{11} & \alpha_8 & \alpha_9 + e^{-\Delta\tau}\alpha_{13} \end{pmatrix}$$

where

$$\begin{aligned} &\alpha_1 = -\theta - \lambda I - \mu, \alpha_2 = -\lambda S + \eta\xi, \alpha_4 = \theta, \alpha_5 = -\lambda(1 + \delta\lambda')I - \mu, \\ &\alpha_6 = -\lambda(1 + \delta\lambda')P + (1 - \eta)\xi, \alpha_8 = \lambda(1 + \delta\lambda')I, \\ &\alpha_9 = \lambda(1 + \delta\lambda')\_P - (\xi + \phi + \mu), \alpha_{11} = \lambda I, \alpha_{12} = \lambda S. \end{aligned}$$

The characteristic equation corresponding to the endemic equilibrium point is:

$$\Delta^3 + a_2\Delta^2 + a_1\Delta + a_0 + (b_2\Delta^2 + b_1\Delta + b_0)e^{-\Delta\tau} = 0 \tag{6}$$

where

$$\begin{aligned} a_2 &= \alpha_1 + \alpha_5 + \alpha_9, \\ a_1 &= -(\alpha_5\alpha_9 + \alpha_1\alpha_9 + \alpha_1\alpha_5 - \alpha_8\alpha_6), \\ a_0 &= \alpha_1\alpha_5\alpha_9 - \alpha_1\alpha_8\alpha_6 + \alpha_2\alpha_4\alpha_8. \\ b_2 &= \alpha_{13} \\ b_1 &= \alpha_2\alpha_{11} + \alpha_1\alpha_{13} + \alpha_5\alpha_{13} \\ b_0 &= \alpha_1\alpha_3\alpha_{13} + \alpha_2\alpha_5\alpha_{11} \end{aligned}$$

Case 1 When $\tau = 0$
Put $\tau = 0$ in Eq. (6), we get

$$\Delta^3 + c_2\Delta^2 + c_1\Delta + c_0 = 0 \tag{7}$$

where$c_2 = b_2 + a_2$, $c_1 = b_1 + a_1$, $c_0 = b_0 + a_0$

All roots of Eq. (7) must have negative real parts according to the Routh-Hurwitz criteria if $c_2 > 0, c_1 > 0, c_0 > 0$ and $c_2c_1 > c_0$ holds.

By the above analysis we have the following theorem:

Theorem 6 *The endemic point E^* is locally stable for $\tau = 0$ if following inequalities hold:*

$$(D1) c_2 > 0, c_1 > 0, c_0 > 0 and c_2c_1 > c_0.$$

Case 2 When $\tau > 0$

We obtain conditions under which the roots of the Eq. (6) will have negative real parts. Let $\Delta = \varphi(\tau) + i\varrho(\tau)$ be the eigen value of characteristic Eq. (6),where $\varphi(\tau)$ and $\varrho(\tau)$ depends on delay τ. We have already shown that the endemic equilibrium point E^* is stable when $\tau = 0$, which implies that $\varphi(\tau) < 0$ for sufficiently small τ. By increasing τ, the real part of the root of the Eq. (6) reaches the value zero at $\tau = \tau^*$, i.e., $\varphi(\tau^*) = 0$, which implies $\Delta = i\varrho(\tau^*)$ for a specific value of $\tau^* > 0$. So, we get $\Delta = i\varrho(\tau^*)$ which is purely imaginary root which means that the Eq. (6) will have root with positive real part and then E^* becomes unstable. If such τ^* does not exist then it would mean that Eq. (6) will not have a purely imaginary root for all delay and E^* will always be stable. Assume $\Delta = i\varrho$ to be root of the Eq. (6) with $\varrho > 0$. Put $\Delta = i\varrho$ in (6), separating real and imaginary parts, we get:

$$-a_2(\tau)\varrho^2 + a_0(\tau) = (b_2(\tau)\varrho^2 - b_0(\tau))\cos(\varrho\tau) - b_1(\tau)\varrho\sin(\varrho\tau) \quad (8)$$

$$-\varrho^3 + a_1(\tau)\varrho = -b_1(\tau)\varrho\cos(\varrho\tau) - (b_2(\tau)\varrho^2 - b_0(\tau))\sin(\varrho\tau) \quad (9)$$

Now, squaring and adding Eqs. (8) and (9), we get

$$\varrho^6 + (a_2^2(\tau) - 2a_1(\tau) - b_2^2(\tau))\varrho^4 + (a_1^2(\tau) - 2a_2(\tau)a_0(\tau) + 2b_2(\tau)b_0(\tau) - b_1^2(\tau))\varrho^2 + (a_0^2(\tau) - b_0^2(\tau)) = 0$$

We can rewrite the above equation as:

$$\varrho^6 + f_2(\tau)\varrho^4 + f_1(\tau)\varrho^2 + f_0(\tau) = 0 \quad (10)$$

where

$$f_2(\tau) = a_2^2(\tau) - 2a_1(\tau) - b_2^2(\tau)$$
$$f_1(\tau) = a_1^2(\tau) - 2a_2a_0(\tau) + 2b_2b_0(\tau) - b_1^2(\tau)$$
$$f_0(\tau) = a_0^2(\tau) - b_0^2(\tau)$$

Putting $k = \varrho^2$, in Eq. (10), reduces it to the following:

$$c(k) = k^3 + f_2(\tau)k^2 + f_1(\tau)k + f_0(\tau) = 0 \quad (11)$$

The discussion of the roots of Eq. (11) is similar to the discussion in [21], and we put forward the following lemma:

Lemma 7 *Following results hold for the Eq. (11):*
***D2**: If $f_0(\tau) \geq 0$, and $\Delta = f_2^2(\tau) - 3f_1(\tau) \leq 0$ holds, then Eq. (11) has no positive roots.*
***D3**: If $f_0(\tau) \geq 0$, and $\Delta = f_2^2(\tau) - 3f_1(\tau) > 0$ holds, then Eq. (11) has positive root if and only if $k^* = \frac{-f_2(\tau) + \sqrt{\Delta}}{3}$ and $c(k^*) \leq 0$*
***D4**: If $f_0(\tau) < 0$ holds, then Eq. (11) has atleast one positive root.*

Furthermore, let us assume that Eq. (11) has positive roots. Without the loss of generality, we suppose that the Eq. (11) has three positive solutions, say k_1, k_2 and k_3 respectively. Then Eq. (10) has three positive solution $\varrho_n = \sqrt{k_n}$, where $n = 1, 2, 3$. Each positive solution $\varrho(\tau)$ of (10) is also defined as the solution of Eqs. (12) and (13) which are given below. We obtain Eqs. (12) and (13) from Eq. (6) as follows:

$$\begin{aligned}\sin(\varrho\tau) &= Im\left(\frac{E(\varrho\iota,\tau)}{F(\varrho\iota,\tau)}\right)\\ &= \frac{b_2(\tau)\rho^5 + (a_2(\tau)b_1(\tau) - b_2(\tau)a_1(\tau) - b_0(\tau))\varrho^3 + (a_1(\tau)b_0(\tau) - b_1(\tau)a_0(\tau))\varrho}{b_2(\tau)\varrho^2 - b_0(\tau)^2 + (b_1\varrho)^2}\end{aligned} \tag{12}$$

$$\begin{aligned}\cos(\varrho\tau) &= -Re(\frac{E(\varrho\iota,\tau)}{F(\varrho\iota,\tau)})\\ &= \frac{(b_1(\tau) - a_2(\tau)b_2(\tau))\varrho^4 + (a_0(\tau)b_2(\tau) + a_1(\tau)b_1(\tau))\varrho^2 - a_0(\tau)b_0(\tau)}{b_2(\tau)\varrho^2 - b_0(\tau)^2 + (b_1\varrho)^2}\end{aligned} \tag{13}$$

From (13), we get

$$\tau_n^{(i)} = \frac{1}{\varrho^n}\cos^{-1}\left(\frac{(b_1(\tau) - a_2(\tau)b_2(\tau))\varrho^4 + (a_0(\tau)b_2(\tau) + b_0(\tau)a_2(\tau) - a_1(\tau)b_1(\tau))\varrho^2 - a_0(\tau)b_0(\tau)}{(b_2(\tau)\varrho^2 - b_0(\tau))^2 + (b_1\varrho)^2}\right) + 2j\pi$$

for $n = 1, 2, 3$; $i = 0, 1, 2, 3, \ldots$
where b_j's and a_j's for $j = 0, 1, 2$ are considered to be bounded functions of τ, then $\pm i\varrho$ are imaginary roots of Eq. (6) for $\tau = \tau_n$
Let us take $\tau^* = min\{\tau_n, \varrho_0 = \varrho_n, n = 1, 2, 3\}$ at $\tau = \tau^*$. Let $\lambda = \varphi(\tau) + i\varrho(\tau)$ be solution of Eq. (6), satisfying $\varphi(\tau^*) = 0, \varrho(\tau^*) = \varrho_0$. Now, we have subsequent transversality conditions.

Lemma 8 ***(D5)** Let us take $UR - VZ > 0$, where*

$$U = (3\varrho^2 - a_1(\tau))(a_0(\tau) - a_2(\tau)\varrho^2) - 2a_2(\tau)\varrho(a_1(\tau)\varrho$$

$$
\begin{aligned}
&-\varrho^3) + b_1(\tau)(b_0(\tau) - b_2(\tau)\varrho^2) \\
&+2b_1(\tau)b_2\varrho^2 - \tau((b_0(\tau) - b_2(\tau)\varrho^2)^2 + (b_1(\tau)\varrho)^2) \qquad (14)\\
V &= 2b_2(\tau)\varrho(b_0(\tau) - b_2(\tau)\varrho^2) - 2a_2(\tau)\varrho(a_0(\tau) - a_2(\tau)\varrho^2) \\
&-(3\varrho^2 - a_1\varrho(\tau))(a_1(\tau)\varrho - \varrho^3) - b_1^2(\tau)\varrho \qquad (15)\\
R &= b_2'(\tau)\varrho^2(b_0(\tau) - b_2\varrho^2) + b_1(\tau)b_1'(\tau)\varrho^2 - (a_0(\tau) \\
&-a_2(\tau)\varrho^2)a_2'(\tau)\varrho^2 + a_1'(\tau)\varrho - \varrho^3) \qquad (16)\\
Z &= \varrho((b_0(\tau) - b_2(\tau)\varrho^2)^2 + (b_1(\tau)\varrho^2) + b_1'(\tau)\varrho(b_0(\tau) \\
&-b_2(\tau)\varrho^2) - b_1(\tau)b_2'(\tau)\varrho^3 + a_1'(\tau)\varrho(a_0(\tau) \\
&-a_2(\tau)\varrho^2) \qquad (17)\\
&+a_2'(\tau)\varrho^2(a_1(\tau)\varrho - \varrho^3)
\end{aligned}
$$

then $\frac{dRe(\lambda)}{d\tau}|_{\tau=\tau^*}$ *and* $UR - VZ$ *have same sign and* $\frac{dRe(\lambda)}{d\tau}|_{\tau=\tau^*} > 0$.

Proof Differentiate Eq. (6) w.r.t. τ we get,

$$
\begin{aligned}
&((3\lambda^2 + 2a_2(\tau)\lambda + a_1(\tau)) + (2b_2(\tau)\lambda + b_1(\tau))e^{-\lambda\tau} \\
&-\tau(b_2(\tau)\lambda^2 + b_1\lambda + b_0(\tau))e^{\lambda\tau})\frac{d\lambda}{d\tau} \\
&+(a_2'(\tau)\lambda^2 + a_1'(\tau)\lambda) + (b_2'(\tau)\lambda^2 + b_1'(\tau)\lambda)e^{-\lambda\tau} \\
&-\lambda(b_2(\tau)\lambda^2 + b_1(\tau))e^{-\lambda\tau} = 0
\end{aligned}
$$

which implies

$$
\begin{aligned}
\left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{(3\lambda^2 + 2a_2(\tau)\lambda + a_1)e^{\lambda\tau} + (2b_2(\tau)\lambda + b_1(\tau)) - \tau(b_2(\tau)\lambda^2 + b_1(\tau)\lambda + b_0(\tau))}{\lambda(b_2(\tau)\lambda^2 + b_1(\tau)\lambda + b_0(\tau)) - ((a_2'(\tau)\lambda^2 + a_1'(\tau)\lambda)e^{\lambda\tau} + (b_2'(\tau)\lambda^2 + b_1'(\tau)\lambda)} \\
&= \frac{-(\frac{3\lambda^2 + 2a_2(\tau)\lambda + a_1(\tau)}{\lambda^3 + a_2(\tau)\lambda^2 + a_1(\tau)\lambda + a_0(\tau)}) + \frac{2b_2(\tau)\lambda + b_1(\tau)}{\lambda(b_2(\tau)\lambda^2 + b_1(\tau)\lambda + b_0(\tau))} - \tau}{\lambda - \frac{b_2'(\tau)\lambda^2 + b_1'(\tau)\lambda}{b_2(\tau)\lambda^2 + b_1(\tau)\lambda + b_0(\tau)} + \frac{a_2'(\tau)\lambda^2 + a_1'(\tau)\lambda}{\lambda^3 + a_2(\tau)\lambda^2 + a_1(\tau)\lambda + a_0(\tau)}}
\end{aligned}
$$

Put $\lambda = i\varrho$ in the above expression and simplifying, we get,

$$
\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{\frac{(3\varrho^2 - a_1(\tau)) - i(2a_2(\tau)\varrho)}{(a_0(\tau) - a_2(\tau)\varrho^2) + i(a_1(\tau)\varrho - \varrho^3)} + \frac{b_1(\tau) + i(2b_2(\tau)\varrho)}{\lambda(b_0(\tau) - b_2(\tau)\varrho^2) + i(b_1(\tau)\varrho)} - \tau}{i\varrho + \frac{b_2'(\tau)\varrho^2 + ib_1'(\tau)\varrho}{(b_0(\tau) - b_2(\tau)\varrho^2) + i(b_1(\tau)\varrho)} + \frac{-a_2'(\tau)\varrho^2 + ia_1'(\tau)\varrho}{(a_0(\tau) - a_2(\tau)\varrho^2) + i(a_1(\tau)\varrho - \varrho^3)}}
$$

Now, we rationalise each term in numerator and denominator and using $A(\varrho, \tau) = |E(i\varrho, \tau)|^2 - |F(i\varrho, \tau)|^2 = 0$, we get,

$$
\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{U + iV}{R + iZ}
$$

where U, V, R and Z are mentioned above in lemma 5 by Eqs. (14), (15), (16) and (17). Thus, we get,

$$sign\left\{\frac{dRe(\lambda)}{d\tau}|\tau=\tau^*\right\} = sign\left\{Re[\frac{d\lambda}{d\tau}]^{-1}_{\tau=\tau^*}\right\} = sign\{UR - VZ\}$$

As $UR - VZ \neq 0$, then $\frac{dRe(\lambda)}{d\tau} \neq 0$ for $\tau = \tau^*$. Let us assume that if $\frac{dRe(\lambda)}{d\tau} < 0$, then characteristic equation has positive real parts when $\tau < \tau^*$ which contradicts the local stability of the endemic equilibrium point. Thus, $\frac{dRe(\lambda)}{d\tau} > 0$ is satisfied. Because of continuity, real part of $\Delta(\tau)$ becomes positive for $\tau > \tau^*$ and accordingly stable state converts to unstable state. Since the loss of stability relates to the root $\Delta = i\varrho$ of the characteristic equation therefore there will be periodic solutions. Using the previous analysis, we get the subsequent theorem. □

Theorem 9 *Relating to model system (1), if (D1) holds then we have the following:*

(i) The positive equilibrium point $(S^*, \_P^*, I^*)$ *is locally stable for all* $\tau \geq 0$, *if (D2) holds.*

(ii) The positive equilibrium point $(S^*, \_P^*, I^*)$ *is stable for all* $\tau \in [0, \tau^*)$ *and unstable for* $\tau > \tau^*$, *if (D3), (D4) and (D5) hold. Also, for* $\tau = \tau^*$, *the system (1) undergoes a Hopf-bifurcation at the positive equilibrium point,* $(S^*, \_P^*, 1^*)$.

4.3 Direction and stability of Hopf-bifurcation

According to the analysis done in the previous section, we obtained certain conditions under which a given system undergoes Hopf-bifurcation, with time lag (τ) being a critical parameter. In this section, we will use the normal form and the center manifold theory established in [21, 22, 25] to check the direction of Hopf bifurcation and the stability of the bifurcation periodic solutions. Throughout this section, we will assume that either (D3) or (D4) holds and $UR - VZ \neq 0$.
If we take $v_1 = S - S^*$, $v_2 = \_P - \_P^*$ and $v_3 = I - I^*$, then the Taylor expansion up to second order terms for system at E^* becomes

$$\begin{aligned}\frac{dv_1(t)}{dt} &= a_1v_1(t) + a_2v_3(t)\\ &\quad + a_3v_1(t)v_3(t) \qquad (18)\\ \frac{dv_2(t)}{dt} &= a_4v_1(t) + a_5v_2(t) + a_6v_3(t)\\ &\quad + a_7v_2(t)v_3(t) \qquad (19)\\ \frac{dv_3(t)}{dt} &= a_8v_2(t) + a_9v_3(t) + a_{10}v_2(t)v_3(t)\\ &\quad + a_{11}v_1(t-\tau) + a_{13}v_3(t-\tau) + a_{14}v_1(t-\tau)v_3(t-\tau) \qquad (20)\end{aligned}$$

where

$$a_1 = -\theta - \lambda I - \mu, a_2 = -\lambda S + \eta\xi, a_3 = -\lambda, a_4 = \theta, a_5 = -\lambda(1+\delta\lambda')I - \mu$$
$$a_6 = -\lambda(1+\delta\lambda')\_P + (1-\eta)\xi, a_7 = -\lambda(1+\delta\lambda'), a_8 = \lambda(1+\delta\lambda')I,$$
$$a_9 = \lambda(1+\delta\lambda')P - (\xi + \phi + \mu), a_{10} = \lambda(1+\delta\lambda'), a_{11} = \lambda I, a_{13} = \lambda S$$

Also, we will write $v(t) = (v_1(t), v_2(t), v_3(t))^T$, $\tau = \tau^* + \zeta$ and $v_t(\omega) = v(t+\omega)$ for $\omega \epsilon [-\tau, 0]$. Now, denoted C as $C \to R^3$ with $|\gamma| = \sup_{\omega\epsilon[-\tau,0]} |\gamma|$. Let

$$Q_1 = \begin{pmatrix} a_1 & 0 & a_2 \\ a_4 & a_5 & a_6 \\ 0 & a_8 & a_9 \end{pmatrix}$$
$$Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{11} & a_{12} & 0 \end{pmatrix}$$

For $\gamma = (\gamma_1, \gamma_2, \gamma_3)^T \epsilon$ C, define

$$O_\zeta(\gamma) = Q_1\gamma(0) + Q_2\gamma(-\tau) \tag{21}$$

Also, Eqs. (18)–(20) can be rewritten as

$$v(t) = O_\zeta(v_t) + F(\zeta, v_t) \tag{22}$$

where

$$F(\zeta, \gamma) = \begin{pmatrix} a_3\gamma_1(0)\gamma_3(0) \\ a_7\gamma_2(0)\gamma_3(0) \\ a_{14}\gamma_1(-\tau)\gamma_3(-\tau) \end{pmatrix}.$$

Obviously, O_ζ is a one-parameter family of bounded linear operators on C. According to the Riesz representation theorem, for $\omega\ \varepsilon\ [-\tau, 0]$ there is a matrix function of 3×3 and $\rho(\omega, \zeta)$ of bounded variation s.t.

$$O_\zeta(\gamma) = \int_{-\tau}^{0} d\rho(\omega, \zeta)\gamma(\omega) \tag{23}$$

for all $\gamma \epsilon$ C. Now, we can take $\rho(\omega, \zeta) = Q_1\kappa(\omega) - Q_2\kappa(\omega + \tau)$ where Dirac delta function is κ.
Now, $\gamma \epsilon C^1 \to R^3$. Define

$$Z(\zeta)\gamma = \begin{cases} \gamma(\omega), & \omega\epsilon[-\tau, 0]. \\ \int_{-\tau}^{0} d\rho(q, \zeta)\gamma(q), & \omega = 0. \end{cases} \tag{24}$$

$$R(\zeta)\gamma = \begin{cases} 0, & \omega\epsilon[-\tau, 0]. \\ F(\zeta, \gamma), & \omega = 0. \end{cases} \tag{25}$$

Then Eq. (22) can be written as

$$v_t = Z(\zeta)v_t + R(\zeta)v_t \tag{26}$$

For $\chi \epsilon C^1$, the adjoint operator Z^* can be defined as

$$Z^*\chi(q) = \begin{cases} -\chi(q), & q\epsilon[0, \tau]. \\ \int_{-\tau}^{0} d\rho^T(t, 0)\chi(-t), & q = 0. \end{cases} \tag{27}$$

and for $\chi, \phi \epsilon C^1$. Define the form as

$$(\chi, \phi) = \bar{\chi}^T(0)\phi(0) - \int_{\omega=-\tau}^{0}\int_{q=0}^{\omega} \bar{\chi}^T(q-\omega)d\rho(\omega)\phi(q)dq \tag{28}$$

where $\rho(\omega) = \rho(\omega, 0)$. It can be verified that Z^* and Z(0) are adjoint operators with respect to this bilinear form,, and $\pm i\sigma_0$ are eigenvalues of Z(0), also they are eigenvalues value of Z^*.
Let $p \epsilon C^1$ be an eigenvector of Z(0) associated with $i\sigma_0$ such that $p(\omega) = (p_1, 1, p_2)^T e^{i\sigma_0\omega}$. At that point $Z(0)p(\omega) = i\sigma_0 p(\omega)$ for all $\omega \epsilon [-\tau, 0]$. When $\omega = 0$, we can use (21), (23), (24) and (25) to get

$$Q_1 p(0) + Q_2 p(-\tau^*) = i\sigma_0 p(0)$$

which gives

$$p_2 = \frac{(i\sigma_0 - a_5)(i\sigma_0 - a_1)}{1 + a_6(i\sigma_0 - a_1)}$$

Similarly, assume that the eigenvector p^* of Z^* comparing to $-i\sigma_0$ takes the form $p^*(q) = \frac{1}{D}(p_1^*, 1, p_2^*)^T e^{i\sigma_0\omega}$. Then

$$Q_1^T p^*(0) + Q_2^T p^*(\tau^*) = -i\sigma_0 p^*(0)$$

Solving the above equation gives

$$p_2^* = \frac{-a_6(i\sigma_0 + a_1)}{(a_8 + a_{13}i\sigma_0\tau^* + i\sigma_0)(i\sigma_0 + a_1) - a_4a_7a_3 - a_3a_{11}e^{i\sigma_0\tau^*}}$$

Now, we will find the value of D to certify that $(p^*, p) = 1$. By (28), we have

$$
\begin{aligned}
(p^*, p) &= \bar{p^*}^T(0)p(0) - \int_{\omega=-\tau^*}^{0}\int_{q=0}^{\omega} \bar{p^*}^T(q-\omega)d\rho(\omega)p(q)dq \\
&= \frac{1}{\bar{D}}(1 + p_1\bar{p_1^*} + p_2\bar{p_2^*}) \\
&\quad - \int_{\omega=-\tau^*}^{0}\int_{q=0}^{\omega} \frac{1}{\bar{D}}(\bar{p_1^*}, 1, \bar{p_2^*})e^{-i\sigma_0(q-\omega)}d\rho(\omega)(p_1, 1, p_2)^T e^{i\sigma_0 q}dq \\
&= \frac{1}{\bar{D}}[1 + p_2\bar{p_2^*} \\
&\quad - \int_{\omega=-\tau^*}^{0}(\bar{p_1^*}, 1, \bar{p_2^*})\omega e^{i\sigma_0\omega}d\rho(\omega)(p_1, 1, p_2)^T] \\
&= \frac{1}{\bar{D}}[1 + p_1\bar{p_1^*} + p_2\bar{p_2^*} \\
&\quad + \tau^* e^{-i\sigma_0\tau^*}(\bar{p_1^*}, 1, \bar{p_2^*})Q_2(p_1, 1, p_2)^T] \\
&= \frac{1}{\bar{D}}[1 + p_1\bar{p_1^*} + p_2\bar{p_2^*} \\
&\quad + \tau^* e^{-i\sigma_0\tau^*}(a_1p_1\bar{p_1^*} + a_3p_1\bar{p_2^*} \\
&\quad + a_4\bar{p_1^*}a_5 + a_6\bar{p_2^*} + a_7p_1\bar{p_1^*} + a_8p_2\bar{p_2^*})]
\end{aligned}
$$

Hence, for $\bar{D} = 1 + p_1\bar{p_1^*} + p_2\bar{p_2^*} + \tau^* e^{i\sigma_0\tau^*}(a_1p_1\bar{p_1^*} + a_3p_1\bar{p_2^*} + a_4\bar{p_1^*}a_5 + a_6\bar{p_2^*} + a_7p_1\bar{p_1^*} + a_8p_2\bar{p_2^*})$, we get $(p^*, p) = 1$. Now, we will calculate the center manifold coordinates C_0 at $\zeta = 0$. So let us take v_t to be the solution of Eq. (26) for $\zeta = 0$. Let us define

$$z(t) = (p^*, v_t) \tag{29}$$

$$X(t, \omega) = v_t - zp - \bar{z}\bar{p} = v_t - 2Re(z(t)p(\omega)) \tag{30}$$

For center manifold C_0,

$$X(t, \omega) = X(z(t), \bar{z}(t), \omega) \tag{31}$$

where

$$X(z, \bar{z}, \omega) = X_{20}(\omega)\frac{z^2}{2} + X_{11}(\omega)z\bar{z} + X_{02}(\omega)\frac{\bar{z}^2}{2} + \cdots \tag{32}$$

and z, $\bar{z}$ are the center manifold coordinates C_0 in the ways of p^* and $\bar{p^*}$ respectively where X is real if v_t is real. From (29) and (30), we get

$$(p^*, X) = (p^*, v_t) - z(t)(p^*, p) - \bar{z}(t)(p^*, \bar{p}) = 0$$

For a solution $v_t \in C_0$ of Eqs. (26), (24), (25) and (27) with $\zeta = 0$ gives

$$\begin{aligned}\bar{z}(t) &= (p^*, \bar{v_t}) = (p^*, Z(0)v_t + R(0)v_t) \\ &= (Z^* p^*, v_t) + \bar{p^*}^T(0)F(0, v_t) = i\sigma_0 z(t) + \bar{p^*}^T(0) f_0(z, \bar{z})\end{aligned} \tag{33}$$

where $f_0(z, \bar{z}) = F(0, X(z, \bar{z}, \omega))$. Equation (33) is also written as

$$\dot{z}(t) = i\sigma_0 z(t) + h(z, \bar{z}) \tag{34}$$

with

$$h(z, \bar{z}) = h_{20}\frac{z^2}{2} + h_{11} z\bar{z} + h_{02}\frac{\bar{z}^2}{2} + h_{21}\frac{z^2\bar{z}}{2} + \cdots \tag{35}$$

Put (26) and (33) into $\dot{X} = \dot{u_t} - \dot{z}p - \dot{\bar{z}}p$, we have

$$\dot{X} = \begin{cases} ZX - 2Re(\bar{p^*}(0) f_0(z, \bar{z}) p(\omega)), & \omega\epsilon[-\tau, 0]. \\ ZX - 2Re(\bar{p^*}(0) f_0(z, \bar{z}) p(\omega)) + f_0(z, \bar{z}), & \omega = 0. \end{cases} \tag{36}$$

that is,

$$\dot{X} = ZX + G(z, \bar{z}, \omega)$$

where

$$G(z, \bar{z}, \omega) = G_{20}(\omega)\frac{z^2}{2} + G_{11}(\omega) z\bar{z} + G_{02}\frac{\bar{z}^2}{2} + \cdots \tag{37}$$

Again, we have $\dot{X} = x_z \dot{z} + x_{\bar{z}}\dot{\bar{z}}$ on C_0. Put Eqs. (29), (30) and (34) into (37) and comparing coefficients with Eq. (36), we get

$$(Z - 2i\sigma_0) X_{20}(\omega) = -G_{20}(\omega) \tag{38}$$

$$Z X_{11} = -G_{11}(\omega) \tag{39}$$

$$(Z + 2i\sigma_0) X_{02}(\omega) = -G_{02}(\omega) \tag{40}$$

As

$$v_t = u(t + \omega) = x(z, \bar{z}, \omega) + zp + \bar{z}\bar{p},$$

we have

$$\begin{aligned} v_1(t + \omega) = {} & p_1 z e^{i\sigma\omega} + \bar{p}_1 \bar{z}_1 e^{-i\sigma\omega} + X_{20}^{(1)}(0)\frac{z^2}{2} + X_{11}^{(1)}(0) z\bar{z} \\ & + X_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + X_{11}^{(1)}(0) z^2 \bar{z} p_1 \end{aligned}$$

$$v_2(t+\omega) = ze^{i\sigma\omega} + \bar{z_1}e^{-i\sigma\omega} + X_{20}^{(2)}(0)\frac{z^2}{2} + X_{11}^{(2)}(0)z\bar{z} + X_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + X_{11}^{(2)}(0)z^2\bar{z}$$

$$v_3(t+\omega) = p_2 ze^{i\sigma\omega} + \bar{p_2}\bar{z_1}e^{-i\sigma\omega} + X_{20}^{(3)}(0)\frac{z^2}{2} + X_{11}^{(3)}(0)z\bar{z} + X_{02}^{(3)}(0)\frac{\bar{z}^2}{2} + X_{11}^{(3)}(0)z^2\bar{z}p_2$$

and

$$f_0(z,\bar{z}) = \begin{pmatrix} U_{11}z^2 + U_{12}z\bar{z} + U_{13}\bar{z}^2 + U_{14}z^2\bar{z} \\ U_{21}z^2 + U_{22}z\bar{z} + U_{23}\bar{z}^2 + U_{24}z^2\bar{z} \\ U_{31}z^2 + U_{32}z\bar{z} + U_{33}\bar{z}^2 + U_{34}z^2\bar{z} \end{pmatrix} + \cdots$$

where

$$\begin{aligned}
U_{11} &= a_3 p_1 \\
U_{12} &= a_3(p_1 + \bar{p_1}) \\
U_{13} &= a_3\bar{p_1} \\
U_{14} &= a_3(X_{11}^{(3)}(0)p_1 + X_{11}^{(1)}(0)) \\
U_{21} &= a_7 p_2 \\
U_{22} &= a_7(p_2 + \bar{p_2}) \\
U_{23} &= a_7\bar{p_1} \\
U_{24} &= a_7(X_{11}^{(3)}(0) + X_{11}^{(0)}(-\tau)p_2) \\
U_{31} &= a_{14}p_1 p_2 e^{-2i\sigma\tau^*} \\
U_{32} &= a_{14}(p_1\bar{p_2} + p_2\bar{p_1}) \\
U_{33} &= a_{14}\bar{p_1}\bar{p_2}e^{-2i\sigma\tau^*} \\
U_{34} &= a_{14}(X_{11}^{(3)}(-\tau)p_1 + X_{11}^{(1)}(-\tau)p_2)
\end{aligned}$$

As $p^*(0) = \frac{1}{\bar{D}}(p_1^*, 1, p_2^*)^T$, we get

$$\begin{aligned}
g(z,\bar{z}) &= \bar{p^*}(0)^T f_0(z,\bar{z}) \\
&= \frac{1}{\bar{D}}(\bar{p_1^*}, 1, \bar{p_2^*}) \begin{pmatrix} U_{11}z^2 + U_{12}z\bar{z} + U_{13}\bar{z}^2 + U_{14}z^2\bar{z} \\ U_{21}z^2 + U_{22}z\bar{z} + U_{23}\bar{z}^2 + U_{24}z^2\bar{z} \\ U_{31}z^2 + U_{32}z\bar{z} + U_{33}\bar{z}^2 + U_{34}z^2\bar{z} \end{pmatrix} \\
&= \frac{1}{\bar{D}}[(\bar{p_1^*}U_{11} + U_{21} + U_{31}\bar{p_2^*})z^2 + (\bar{p_1^*}U_{12} + U_{22} + U_{23}\bar{p_2^*})z\bar{z} + (\bar{p_1^*}U_{13} + U_{23} \\
&\quad + U_{33}\bar{p_2^*})\bar{z}^2 + (\bar{p_1^*}U_{14} + U_{24} + U_{34}\bar{p_2^*})z^2\bar{z}].
\end{aligned}$$

Now, we compare the coefficients with Eq. (35), we obtain

$$h_{20} = \frac{2}{\bar{D}}(\bar{p_1^*}U_{11} + U_{21} + U_{31}\bar{p_2^*}),$$
$$h_{11} = \frac{1}{D}(\bar{p_1^*}U_{12} + U_{22} + U_{23}\bar{p_2^*}),$$
$$h_{02} = \frac{2}{\bar{D}}(\bar{p_1^*}U_{13} + U_{23} + U_{33}\bar{p_2^*}),$$
$$h_{21} = \frac{2}{\bar{D}}(\bar{p_1^*}U_{14} + U_{24} + U_{34}\bar{p_2^*}).$$

Now, we follow the same procedure as in [22] and we get

$$X_{20}(\omega) = \frac{ih_{20}}{\sigma_0}p(0)e^{i\sigma_0\omega} + \frac{i\bar{h_{02}}}{3\sigma_0}\bar{p}(0)e^{-i\sigma_0\omega} + O_1e^{2i\sigma_0\omega}$$

and

$$X_{11}(\omega) = \frac{-ih_{11}}{\sigma_0}p(0)e^{i\sigma_0\omega} + \frac{i\bar{h_{11}}}{3\sigma_0}\bar{q}(0)e^{-i\sigma_0\omega} + O_2$$

where O_1 and O_2 are both two dimensional vectors such that

$$(2i\sigma_0 I_2 - \int_{-\tau_0}^{0} e^{2i\sigma_0\omega}d\rho(\omega))O_1 = (U_{11}, U_{21}, U_{31})^T \text{ and } \int_{-\tau_0}^{0} d\rho(\omega)O_2 = -(U_{12}, U_{22}, U_{23})^T$$

where

$$O_1 = (O_1^{(1)}, O_1^{(2)}, O_1^{(3)}),\ O_2 = (O_2^{(1)}, O_2^{(2)}, O_2^{(3)})$$

and Y_n denotes the n × n identity matrix. Hence,

$$\begin{bmatrix} -2i\sigma_0 - a_1 & 0 & -a_2 \\ -a_4 & -2i\sigma_0 - a_5 & -a_6 \\ -a_{11}e^{-2i\sigma_0\tau^*} & -a_8 & -2i\sigma_0 - a_9 - a_{13}e^{-2i\sigma_0\tau_q^*} \end{bmatrix}$$
$$\begin{bmatrix} O_1^{(1)} \\ O_1^{(2)} \\ O_1^{(3)} \end{bmatrix} = \begin{bmatrix} U_{11} \\ U_{21} \\ U_{31} \end{bmatrix} \tag{41}$$
$$\begin{bmatrix} a_1 & 0 & a_2 \\ a_4 & a_5 & a_8 \\ a_{11} & a_8 & a_9 + a_{13} \end{bmatrix} \begin{bmatrix} O_2^{(1)} \\ O_2^{(2)} \\ O_2^{(3)} \end{bmatrix} = -\begin{bmatrix} U_{12} \\ U_{22} \\ U_{32} \end{bmatrix} \tag{42}$$

and the specific articulations for O_1 and O_2 are acquired by solving the above linear equation.

Based on the above analysis, we can write each h_{ij} in terms of the parameters of the system, and we can thus compute the following quantities:

$$C_1(0) = \frac{i}{2\sigma_0}(h_{20}h_{11} - 2|h_{11}|^2 - \frac{1}{3}|h_{02}|^2) + \frac{h_{21}}{2}$$
$$W_2 = -\frac{Re[C_1]}{Re[\nu'(\tau^*)]},$$
$$T_2 = -\frac{Im[C_1(0)] + W_2 Im[\nu'(\tau^*)]\tau}{\sigma_0},$$
$$\beta_2 = 2Re[C_1(0)]$$

From [21], we can form the subsequent conclusions.

Theorem 10 *Suppose that at least one of (D3) or (D4) holds and $UR - VZ \neq 0$. Then,*

(i) The sign of W_2 determines the direction of the Hopf bifurcation: if $W_2 > 0$ ($W_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tau^*$ ($\tau < \tau^*$).

(ii) The sign of β_2 determines the stability of the bifurcating periodic solutions: if $\beta_2 < 0$ ($\beta_2 > 0$), then the bifurcating periodic solutions are stable (unstable).

(iii) The sign of T_2 determines the period of the bifurcating periodic solutions: if $T_2 > 0$ ($T_2 < 0$), then the period increases (decreases).

5 Numerical simulation

To observe and understand the dynamical conduct, we perform numerical simulations for the system (1) with the help of MATLAB. We investigate to explore the effect that natural contamination may have on the spread of illness. Consequently, we conduct a mathematical investigation to check the effect of pollution associated parameters on the disease dynamics of system (1).

Figure 1 presents the disease-free equilibrium point for the system (1). The parametric value set considered in the simulation is: M = 10, $a = 0.9832$, $\theta = 0.00009$, $\mu = 0.1$, $\lambda = 0.0001$, $\eta = 0.0001$, $\delta = 0.001$, $\lambda' = 0.1785$, $\xi = 0.9875$, $\phi = 0.9541$. It can be seen from Fig. 1 that the trajectories approach E^0 ($S_0 = 98.23$, $\_P_0 = 1.768$, $I_0 = 0$). It can easily be calculated that $R_0 = 0.0049 < 1$, for the set of parameters, which also satisfies the condition of Theorem 3.

Let $\theta = 0.7$, $\eta = 0.9$, $\delta = 0.1$, $\xi = 0.1$ and $\mu = 0.0001$. For these set of parametric values, we obtain $R_0 = 9.6552 > 1$ and the trajectories approach the endemic equilibrium point, which is, ($S^* = 58.83$, $\_P^* = 15.34$, $I^* = 10.45$), as represented in Fig. 2. In order to check the effect of pollution on the disease dynamics of the system (1), we increase δ from 0.1 to 0.4. It can be seen that for $\delta = 0.4$, $R_0 = 10.1631$, which is more in comparison with the case when $\delta = 0.1$. Also, the endemic equilibrium point for $\delta = 0.4$ is $S^* = 14.71$, $\_P^* = 15.36$, $I^* = 10.47$, as shown in Fig. 3. So it can be concluded that as pollution increases, the number of stressed and infectives also increases. Therefore, it is important to control pollution in the system.

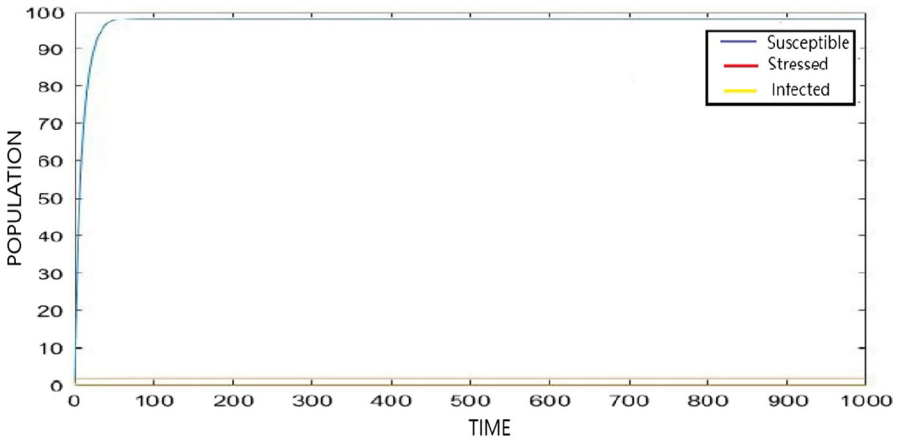


Fig. 1 Time series solution of the model when $R_0 < 1$

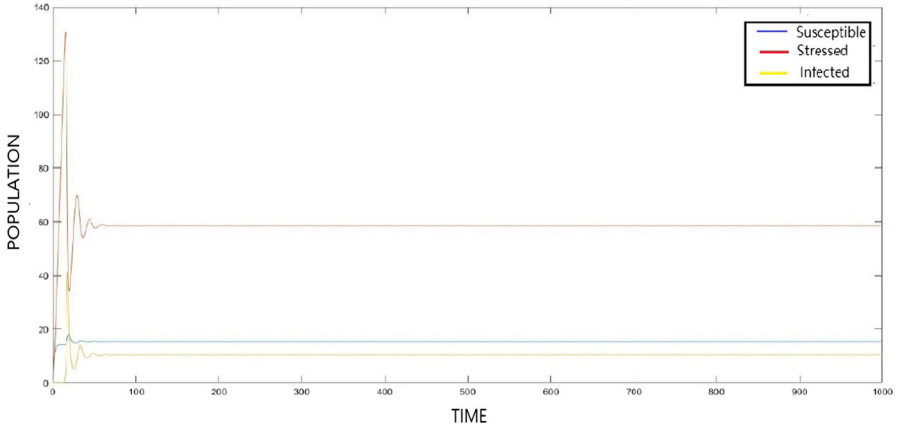


Fig. 2 Time series solution of the model when $R_0 > 1$ and $\delta = 0.1$

We will now take the parameters to be, M = 250, $\theta = 0.059865$, $\mu = 0.01$. Figure 5a, b shows that the endemic equilibrium point looses its stability and undergoes Hopf-bifurcation at $\tau = \tau^* = 28$. Since, we know that the Hopf-bifurcation is supercritical, there exists periodic solutions for $\tau > 28$ and endemic equilibrium point is unstable. Now, for the same set of parameters, if we take $\tau < 28$, we observe that the system is stable around endemic equilibrium point, as shown in Fig. 4a, b. It can be easily checked that parameters satisfy the conditions of Theorem 9. Again to check the effect of pollution on the disease dynamics of the model (1), we increase δ. Increase in transmission rate due to natural contamination (δ) stabilizes the system around endemic equilibrium point, as represented in Fig. 6a, b but the number of infected and stressed individuals increases and overall population size decreases. Also, we notice that by decreasing λ (disease transmission rate), the system stabilizes around endemic equilibrium point, as shown in Fig. 7a, b. Thus, it is very important to keep check on

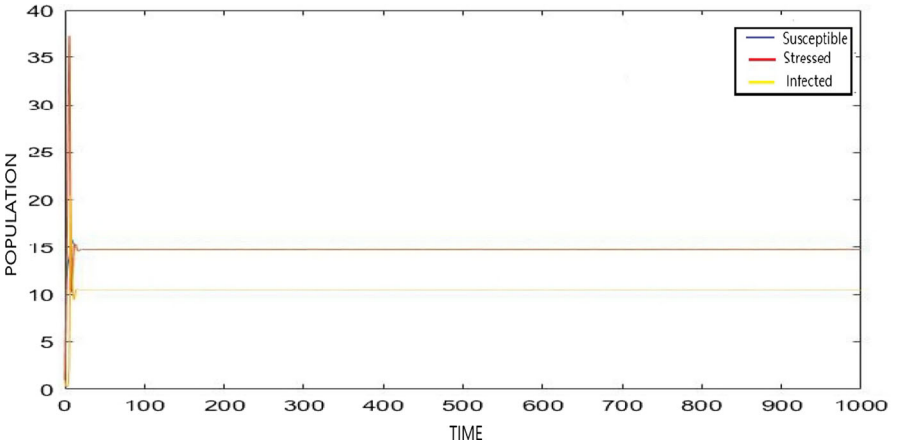
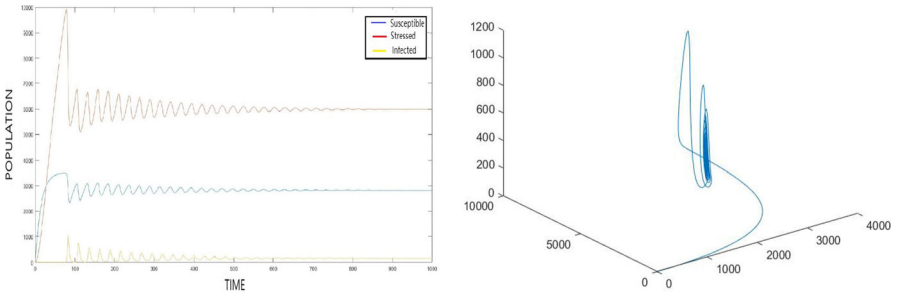


Fig. 3 Time series solution of the model when $R_0 > 1$ and $\delta = 0.4$



(a) **(b)**

Fig. 4 Endemic equilibrium point of the system is stable for $\tau = 25$, $\delta = 0.001$, $\lambda = 0.0001$

transmission rate due to natural contamination and disease transmission rate for the survival and better recovery of the population.

6 Conclusion

An SIS model is developed to contemplate the impact of environmental contamination on the spread of epidemic illness. We categorized total populace into three divisions, which are susceptible populace, stressed populace, and infected populace. Analysis of the model indicates that there are two types of equilibrium points: (a) disease-free equilibrium point, and (b) endemic equilibrium point. The disease-free equilibrium point's local stability has been proved in Theorem 3, which shows that the system is stable around disease free equilibrium point if $R_0 < 1$. The existence of endemic-equilibrium point has been established. We discussed stability analysis for endemic equilibrium point (E^*) for $\tau \geq 0$. We have shown that the time delay plays a crucial role in shaping the dynamics of the system. When $\tau < \tau^*$ the system is stable around

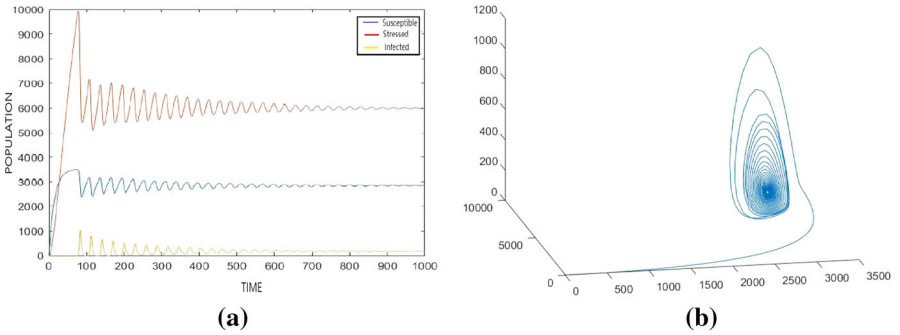


Fig. 5 Endemic equilibrium point of the system undergoes Hopf-bifurcation for $\tau = 28$, $\delta = 0.001$, $\lambda = 0.0001$

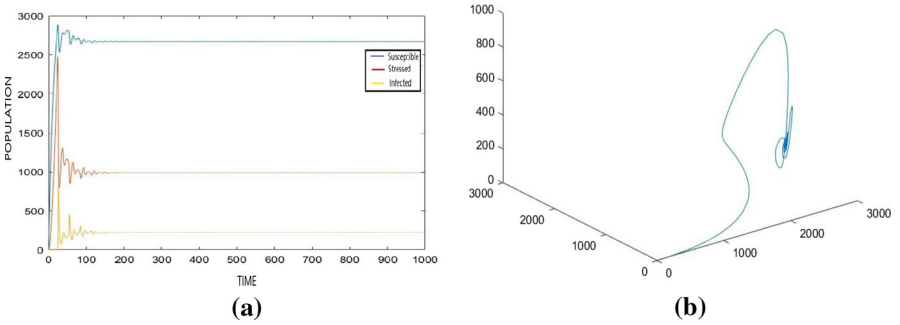


Fig. 6 Endemic equilibrium point of the system is stable for $\tau = 28$, $\delta = 0.009$, $\lambda = 0.0001$

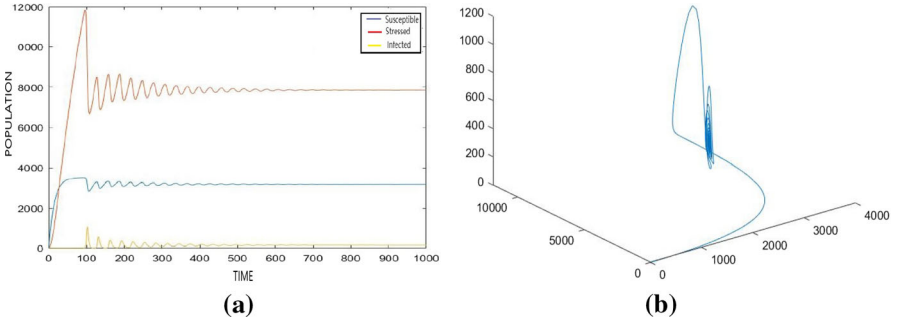


Fig. 7 Endemic equilibrium point of the system is stable for $\tau = 28$, $\delta = 0.001$, $\lambda = 0.00005$

endemic equilibrium point but it loses its stability and a Hopf-bifurcation occurs at a threshold value τ^* as shown in Theorem 9. The endemic equilibrium point become unstable for $\tau > \tau^*$. Our model predicts that with the increase in the transmission rate due to natural contamination (δ), the system becomes stable around endemic equilibrium point. However, this leads in a population drop and an increase in diseased persons, which is not beneficial for our society. As a result, our model suggests that

a healthy community requires a check on the transmission rate produced by natural contamination (δ) and disease transmission rate (λ).

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