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A comprehensive study of upward fuzzy preference relation based fuzzy rough set models: Properties and applications in treatment of coronavirus disease

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Abstract

In this paper, we first introduce a new type of rough sets called α-upward fuzzified preference rodownward fuzzy preferenceugh sets using upward fuzy preference relation. Thereafter on the basis of α-upward fuzzified preference rough sets, we propose approximate precision, rough degree, approximate quality and their mutual relationships. Furthermore, we presented the idea of new types of fuzzy upward β-coverings, fuzzy upward β-neighborhoods and fuzzy upward complement β-neighborhoods and some relavent properties are discussed. Hereby, we formulate a new type of upward lower and upward upper approximations by applying an upward β-neighborhoods. After employing the upward β-neighborhoods based upward rough set approach to it any times, we can only get the six different sets at most. That is to say, every rough set in a universe can be approximated by only six sets, where the lower and upper approximations of each set in the six sets are still lying among these six sets. The relationships among these six sets are established. Subsequently, we presented the idea to combine the fuzzy implicator and t-norm to introduce multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy upward rough set applying fuzzy upward β-covering and some relative properties are discussed.

Finally we presented a new technique for the selection of medicine for treatment of coronavirus disease (COVID-19) using multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy upward rough sets.

KEYWORDS

fuzzy preference relation, fuuzy upward β-coverings

1 | INTRODUCTION

Multiattribute group decision-making problems has always been an important direction of modern decision making sciences. Based on multiattribute group decision making systems with different natures, researchers not only developed a wealth of decision-making scheme, but also solved various practical problems such as Kreyea et al.[1] developed a new approach of group decision making problem to manage their application in logistics, Mou et al.[2] introduced group decision-making technique based on graph approach under the intuitionistic fuzzy (fuzzy preference relation) information and applied to energy related problem. Ishizaka and Nemery[3] introduced a new approach to multicriteria decision analysis and discussed their application in safety management. Alcantud[4] developed a group decision making technique to handle a problem related to facility location. Inan et al.[5] initiated a multiattribute group decision approach for the comparison of firms occupational health and safety management. Ishizaka and Nemery[6] put forward a new idea of decision making technique to solve a problems related to supplier selection. Aldape-Perez et al.[7] defined a novel approach to perform pattern classification tasks for medical decision support systems. Arsene et al.[8] applied an expert system under the framework of software agents for medicine diagnosis. Azar and Metwally[9] presented a decision tree classifier for automated medical diagnosis problem, Esfandiari et al.[10] presented data mining application in medicine. The whole of the world is engulfed with the spread of COVID-19 and it is very painstakingly difficult for the denizens of the world to live a peaceful life. The epidemic is viral and the attack is so severe that the World Health Organization (WHO) is compelled to announce a global emergency. In the last quarter of 2019, some cases of the disease were reported in Wuhan city China, which after the diagnosis was found as coronavirus (COVID-19). In the wake of this incident, the virus is circulated in the entire world and became the sole cause of the demise of thousands and thousands of people in the whole of the world. The word "coronavirus" is derived from the Latin word "corona" which means a "crown i.e., a circle of light or nimbus." The virus promptly affects the lungs. It has similar symptoms as those of influenza and pneumonia. In the very outset, it was found that the people who worked or shopped at the seafood market in Wuhan became the victims of this virus. After that it pervaded universally through import, export, travelling, and social contacting with infected people. Several researchers investigated and developed different methods to address the problem. In decision making, there have been a lot of uncertainties, imprecise, and vague information, whose representation and management are always the central issues. Health professionals and healthcare administrators are working to reduce clinical and maintenance costs for the prevention and management of disease. It is concluded that the coronavirus is a burning issue and needs mathematical formulation/technique for selection of medicine for treatment of the disease.

The main objective of this study is to establish decision making with the help of fuzzy rough set theory. We developed a technique for the selection of medicine to treat the coronavirus disease (COVID-19) using fuzzy rough sets.

1.1 | Rough set theory and decision making problems

The theory of rough sets introduced by Pawlak[11] provides a new mathematical approach to extend classical set theory. In rough set theory, creating a pair of approximation operators called lower and upper approximation operators is important. The classical approximation operators are described with the help of an equivalent relation in the universe. Rough set methodology is the fundamental method of solving uncertain knowledge and their application in various fields such as, the field of expert systems, pattern recognition, image processing, decision analysis, artificial intelligence, and so on. Accordingly, to enhance the utilization rate of information in information systems, several authors have extended the definition of rough set approximation through applying general relations, such as Bonikowski introduced algebraic structures of rough sets,[12] Liu and Zhu[13] further generalized the work of Bonikowski by introducing algebraic structures of generalized rough sets, Slowinski and Stefanowski[14] discussed the solution of medical information systems problems based on rough fuzzy hybridized structure. Dubois and Prade[15] combined the notion of rough sets and fuzzy sets to form fuzzy rough sets and rough fuzzy sets. Then the upper and lower approximation operators of approximate space in fuzzy environment are widely used.[16] After the formation of rough set,[11] number of generalizations have been presented in terms of various demands. Dubois and Prade[15] initiated the idea of rough set theory based on set valued mapping. Zhu[17] integrated the idea of generalized rough sets using general relation. Yao and Yao[18] initiated the idea of rough sets based on covering approach and their applications. Zhu[19] presented the idea of covering rough sets based on topological properties and their applications. Zhu and Wang[20] studied covering rough sets based on reduction of attributes and their applications. Further, Zhu and Wang discussed three different types of rough sets based on covering and their mutual relationships.[21] Based on topological approach, Zhao[22] introduced various types of rough sets based on covering and their applications. Deng and Yao[23] studied fuzzy environment based three way approximation with decision theoretic fuzzy rough sets and applied to group decision making problems. Sun et al.[24] integrated the idea of rough fuzzy sets based on decision theoretic approach and applied to multiattribute group decision-making problems. Ziarko[25] introduced another rough set model called variable precision rough sets model, which is the generalized form of Pawlak's rough set model. Yao[26] initiated three way decisions analysis with the help of probabilistic rough sets and discussed their applications in multiattribute group decision making problem. Greco et al.[27-29] initiated rough approximations based on dominance relations and their applications in various multicriteria group decision making problems. Qian et al.[30,31] generalized the Pawlak's single granulation rough set model to a multigranulation rough set model for finding two terminologies called optimistic/pessimistic multigranulation rough set models and disclosed their applications in decision making process. Qian et al.[32] further extended multigranulation methodology to decision theoretic rough sets and applied them to multicriteria group decision making problem. Lin et al.[33] initiated covering based multigranulation rough sets and applied them in decision making problems. Ali et al.[34]

originated multigranulation rough sets approaches based on dominance relations and their application in labor management negotiations in conflict analysis problems. Rehman et al.[35] applied soft preference relation for the construction of soft optimistic/soft pessimistic multigranulation rough sets and presented their application in conflict problems. Different researchers have confabulated the applications of rough sets in medical sciences such as, Cheng and Liu[36] argued wavelet packet based rough set technique for Identify brain disease.

1.2 | Covering based fuzzy rough set theory

The original definition of a fuzzy covering is given in Reference [37]. Let $\mathcal{U} = \{q_1, q_2, q_3\}$ be the set of three alternatives/medicines which are evaluated by different attributes/tests, where $\mathrm{C} = \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3\}$ is the collection of different attributes/tests. For these attributes/tests $\{C(\mathcal{K}_1), C(\mathcal{K}_2), C(\mathcal{K}_3)\}$, where

$$C(\mathcal{K}_1) = \left\{\frac{0.5}{q_1}, \frac{0.5}{q_2}, \frac{0.3}{q_3}\right\}, C(\mathcal{K}_2) = \left\{\frac{0.6}{q_1}, \frac{0.4}{q_2}, \frac{0.4}{q_3}\right\}, C(\mathcal{K}_3) = \left\{\frac{0.7}{q_1}, \frac{0.6}{q_2}, \frac{0.3}{q_3}\right\},$$

where $C(\mathcal{K}_i)(q_j)$ denotes the efficiency of the medicine q_j for the test $\mathcal{K}_i$. Ma defined fuzzy β-covering which is the generalized form of fuzzy covering to replace 1 by a parameter $\beta \in (0, 1]$.[38] Subsequently, Ma[38] presented two new types of rough set models based on fuzzy covering by applying the concept of fuzzy β-neighborhood. Further, Ma[38] defined two types of rough set models based on fuzzy covering and presented their applications in fuzzy lattice theory. Yang and Hu[39] initiated various types of rough set models using fuzzy covering approach and applied them to medical diagnosis problems. Zhan et al.[40] combined the fuzzy implicator and t-norm to introduce multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy rough set models based on covering and further studied their application in assessment of appointing a system analysis engineer. Based on fuzzy implicator and t-norm, Jiang et al.[41] presented variable precision $(\mathcal{I}, \mathcal{T})$-fuzzy rough sets using covering methodology and presented their application to supplier selection problems. Zhang et al.[42] integrated the idea of fuzzy rough sets applying fuzzy soft covering methodology and discussed their applications to select athletes for table tennis team. Zhang and Zhan,[43] introduced a new hybridized structure called fuzzy rough sets using fuzzy soft β-covering model and its application in decision making problem.

1.3 | Goals of this study

There are some limitation, for instance, the above example is fuzzy β -covering model for $\beta \in (0, 0.4]$. If the required critical value $\beta = 0.5$, then how is it possible to make β-covering model for $\beta \in (0, 0.5]$? Hu et al.[44] adopted the well-known logis transfer function to compute the fuzzy preference degree of the feasible alternatives. Pan et al.[45] pointed out that the transfer fuzzy preference degree of Hu et al. is not additive consistent and suggested another transfer function. The motive of this paper is first to point out that the transfer function for computing the fuzzy preference degree of Pan et al.[45] for the construction of upward/downward fuzzy preference relations are not additive consistent. The appropriate counterexample is given and their modified versions are presented. Furthermore, we construct upward consistency matrices of experts which satisfy the upward additive consistency and the upward order consistency simultaneously. Subsequently, we introduced a new type of rough sets called α-upward fuzzified preference rough sets using upward fuzzy preference relation. On the basis of α-upward fuzzified preference rough sets, we introduced approximate precision, rough degree, approximate quality and their mutual relationships. Furthermore, we presented the idea of new types of fuzzy upward β-coverings, fuzzy upward β-neighborhoods and fuzzy upward complement β-neighborhoods and related properties are discussed. Hereby, we propose a new type of upward lower and upward upper approximations by employing an upward β-neighborhoods. It is worth mentioning by applying an upward β-neighborhoods based upward rough set approach to it any times, we can only get the six different sets at most. That is to say, every rough set in a universe can be approximated by only six sets, where the lower and upper approximations of each set in the six sets are still lying among these six sets. The relationships among these six sets are established. Afterwords, we presented the idea to combine the fuzzy implicator and t-norm to introduce multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy upward rough set applying fuzzy upward β-covering and some related properties are discussed. Finally we presented a new technique for the selection of medicine to treat the coronavirus disease (COVID-19) using multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy upward rough sets.

1.4 | The structure of this paper

The remainder of this manuscript is as follows: Section 2 recalls preliminary notions concerning fuzzy preference relation, fuzzy additive consistency and logis sigmoid transfer function. In Section 3, we construct upward/downward fuzzy preference relations which are additive consistent. Furthermore, we construct upward consistency matrices of experts which satisfy the upward additive consistency and the upward order consistency. Section 4, introduce a new type of roughness called α-upward fuzzified preference rough sets using upward fuzzy preference relation. A new type of upward lower and upward upper approximations by applying an upward β-neighborhoods and after employing an upward β-neighborhoods based upward rough sets approach to it any times, we get the six different sets at most are discussed in Section 5. In Section 6, we presented the idea to combine the fuzzy implicator and t-norm to introduce optimistic/pessimistic multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy upward rough sets using fuzzy upward β-covering approach. Section 7, confabulated the algorithm to handle the uncertainty problems using multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy upward rough sets. Section 8 highlights the applications of the proposed model in prescription of medicine for treatment of coronavirus disease (COVID-19). In Section 9, we focus our attention on comparison of various models with the proposed technique. The paper is concluded in Section 10.

2 | PRELIMINARIES

In this section, some basic notations of fuzzy preference relation, fuzzy additive consistency and logis sigmoid transfer function have been discussed.

Definition 1 (Herrera-Viedma et al.[46]). A fuzzy preference relation $\mathcal{R}$ is a fuzzy set in $\mathcal{U} \times \mathcal{U}$, which is a membership function $\mu_{\mathcal{R}}: \mathcal{U} \times \mathcal{U} \to [0, 1]$, $\mathcal{U} = \{q_1, q_2, ..., q_n\}$ is the set of feasible alternatives. The fuzzy preference relation can also be represented by an $n \times n$ matrix $(f_{ij})_{n\times n}$ as:

$$\mathcal{R} = (f_{ij})_{n\times n} = \begin{array}{c} \\ q_1 \\ q_2 \\ \vdots \\ q_n \end{array} \begin{array}{c} \begin{array}{cccc} q_1 & q_2 & \cdots & q_n \end{array} \\ \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{pmatrix} \end{array}$$

where f_{ij} shows that the preference degree of q_i over $q_j, f_{ij} \in [0, 1], f_{ij} + f_{ji} = 1$, for all $i, j \in \{1, 2, ..., n\}$. Especially, $f_{ij} = 0.5$ means that the behavior of q_i and q_j are same in fuzzy information system; $f_{ij} > 0.5$ shows that the behavior of q_i is better than the behavior of q_j; $f_{ij} = 1$ means that the behavior of q_i is absolutely better than the behavior of q_j; the $f_{ij} < 0.5$ depicts that the behavior of q_j is better than the behavior of q_i; $f_{ij} = 0$ means that the behavior of q_j is absolutely better than the behavior of q_i.

Definition 2. A fuzzy preference relation $\mathcal{R} = (f_{ij})_{n\times n}$ is called additive consistent, if $f_{ij} = f_{ik} - f_{kj} + 0.5$, for all $i, j, k \in \{1, 2, ..., n\}$.

In the above definition, the fuzzy preference relation is considered, f_{ij} merely presents that the degree of preference of feasible alternative q_i is before the feasible alternative q_j. However, in practical applications, we need to show the degree of q_i is poorer than q_j. To satisfy both the cases, we call the fuzzy preference relation as an upward fuzzy preference relation and the other downward fuzzy preference relation. The upward fuzzy preference relation is denoted as $\mathcal{R}^{\Uparrow} = \left(f_{ij}^{\Uparrow}\right)_{n\times n}$ and downward fuzzy preference relation as $\mathcal{R}^{\Downarrow} = \left(f_{ij}^{\Downarrow}\right)_{n\times n}$. In general, $f_{ij}^{\Uparrow} + f_{ij}^{\Downarrow} = 1$. For downward fuzzy preference relation, $f_{ij}^{\Downarrow} = 0.5$ means that the behavior of q_i and q_j are same in fuzzy information system; $f_{ij}^{\Downarrow} > 0.5$ depicts that the behavior of q_i is poorer than the behavior of q_j; $f_{ij}^{\Downarrow} = 1$ means the behavior of q_i is absolutely poorer than the behavior of q_j; $f_{ij}^{\Downarrow} < 0.5$ shows that the behavior of q_j is poorer than the behavior of q_i; $f_{ij}^{\Downarrow} = 0$ means that behavior of q_j is absolutely poor than the behavior of q_i.

Hu et al.[44] used the well-known logis sigmoid transfer function $\frac{1}{1+e^{k(g(q_i,c)+g(q_j,c))}}$ for the construction of fuzzy preference degree of the feasible alternative q_i to the feasible alternative q_j as

$$f_{ij}^{\Uparrow} = \frac{1}{1 + e^{-k(g(q_i,c)+g(q_j,c))}} \tag{1}$$

$$f_{ij}^{\Downarrow} = \frac{1}{1 + e^{k(g(q_i,c)+g(q_j,c))}} \tag{2}$$

where k is a positive constant. Pan et al.[45] showed that the fuzzy preference degree based on logis sigmoid transfer function is not additive consistent and they suggested another transfer function to compute the fuzzy preference degree. The fuzzy preference degree of q_i to q_j is given by

$$f_{ij}^{\Uparrow} = 0.5 \times \left(\frac{g(q_i, c) - \wedge_{i=1}^{n} g(q_i, c)}{\vee_{i=1}^{n} g(q_i, c) - \wedge_{i=1}^{n} g(q_i, c)} - \frac{g(q_j, c) - \wedge_{i=1}^{n} g(q_i, c)}{\vee_{i=1}^{n} g(q_i, c) - \wedge_{i=1}^{n} g(q_i, c)} + 1 \right) \tag{3}$$

$$f_{ij}^{\Downarrow} = 0.5 \times \left(\frac{g(q_j, c) - \wedge_{i=1}^{n} g(q_i, c)}{\vee_{i=1}^{n} g(q_i, c) - \wedge_{i=1}^{n} g(q_i, c)} - \frac{g(q_i, c) - \wedge_{i=1}^{n} g(q_i, c)}{\vee_{i=1}^{n} g(q_i, c) - \wedge_{i=1}^{n} g(q_i, c)} + 1 \right) \tag{4}$$

where $g(q_i, c) \in [0, 1]$.

3 | PROPOSED UPWARD/DOWNWARD FUZZY PREFERENCE RELATION

We observed that in the case when the value of feasible alternatives on any criterion are different then the technique of Pan et al. works, but on a larger domain of equal values of feasible alternatives on some criteria Pan et al.'s technique fails. Further, in the current manuscript we pointed out that the transfer function for computing the fuzzy preference degree of 45 for the construction of upward/downward fuzzy preference relations are not additive consistent as seen in the following example.

Example 1. It is difficult to come to the exact number of births every day since not all births are registered or recorded. The UNICEF estimates that an average of 353,000 babies are born each day around the world. The crude birth rate is 18.9 births per 1000 population or 255 births globally per minute or 4.3 births every second (as of Dec 2013 estimate). Based on the initial record (birth information, weight, age, etc. of the babies), of the babies birth's taken from various hospitals, we have the following example. Let $A = \{q_j : j = 1, 2, ..., 9\}$ be the set of (feasible alternatives) babies born in a same minute in the world, $C = \{c_1, c_2, c_3\}$ be the set criteria, where c_1, shows the weight of the babies, c_2, represent the ages of babies in the same minute and c_3 shows baby birth on normal delivery. In this study, the value of information function $g(q_j, c_i)$ is belonging to $[0, 1]$ and $g(q_j, c_i)$ describe the fuzzy evaluation of q_j on criterion c_i. The information system is given in the following Table 1.

Based on criterion c_1 and using Equation (1), to calculate the upward fuzzy preference degree of $q_i (i = 1, 2, ..., 9)$ to $q_j (j = 1, 2, ..., 9)$, one can acquire

$$\mathcal{R}_{c_1}^{\Uparrow}(q_i, q_j) = \begin{pmatrix} 0.5000 & 0.9167 & 1 & 0.6667 & 0.8333 & 1 & 0.9167 & 0.9167 & 0.9167 \\ 0.0833 & 0.5000 & 0.5833 & 0.2500 & 0.4167 & 0.5833 & 0.5000 & 0.5000 & 0.5000 \\ 0 & 0.4167 & 0.5000 & 0.1667 & 0.3333 & 0.5000 & 0.4167 & 0.4167 & 0.4167 \\ 0.3333 & 0.7500 & 0.8333 & 0.5000 & 0.6667 & 0.8333 & 0.7500 & 0.7500 & 0.7500 \\ 0.1667 & 0.5833 & 0.6667 & 0.3333 & 0.5000 & 0.6667 & 0.5833 & 0.5833 & 0.5833 \\ 0 & 0.4167 & 0.5000 & 0.1667 & 0.3333 & 0.5000 & 0.4167 & 0.4167 & 0.4167 \\ 0.0833 & 0.5000 & 0.5888 & 0.2500 & 0.4167 & 0.5833 & 0.5000 & 0.5000 & 0.5000 \\ 0.0833 & 0.5000 & 0.5888 & 0.2500 & 0.4167 & 0.5833 & 0.5000 & 0.5000 & 0.5000 \\ 0.0833 & 0.5000 & 0.5888 & 0.2500 & 0.4167 & 0.5833 & 0.5000 & 0.5000 & 0.5000 \end{pmatrix}.$$

TABLE 1 Fuzzy information system

	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9
c_1	0.8	0.3	0.2	0.6	0.4	0.2	0.3	0.3	0.3
c_2	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4
c_3	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6

But with criterion c_2, one can derive

$$f_{11}^{\Uparrow} = 0.5 \times \left(\frac{g(q_1, c_2) - \wedge_{i=1}^{n} g(q_i, c_2)}{\vee_{i=1}^{n} g(q_i, c_2) - \wedge_{i=1}^{n} g(q_i, c_2)} - \frac{g(q_1, c_2) - \wedge_{i=1}^{n} g(q_i, c_2)}{\vee_{i=1}^{n} g(q_i, c_2) - \wedge_{i=1}^{n} g(q_i, c_2)} + 1 \right)$$
$$= 0.5 \times \left(\frac{0.4 - 0.4}{0.4 - 0.4} - \frac{0.4 - 0.4}{0.4 - 0.4} + 1 \right) = 0.5 \times \left(\frac{0}{0} - \frac{0}{0} + 1 \right) = \, ? \, .$$

Similarly we can see that the values of $f_{ij}^{\Uparrow}$ do not exist for all $i, j = 1, 2, ..., 9$. Thus the upward fuzzy preference relations $\mathcal{R}_{c_2}^{\Uparrow}(q_i, q_j)$ cannot be obtained based on the Pan et al. technique. Hereby, we present another transfer function for the construction of upward/downward fuzzy preference degree of q_i to q_j is

$$f_{ij}^{\Uparrow} = 0.5 \times \left(\frac{g(q_i, c) - \wedge_{j=1}^{n} g(q_j, c)}{\vee_{j=1}^{n} g(q_j, c) + \wedge_{j=1}^{n} g(q_j, c) + 0.5} - \frac{g(q_j, c) - \wedge_{j=1}^{n} g(q_j, c)}{\vee_{j=1}^{n} g(q_j, c) + \wedge_{j=1}^{n} g(q_j, c) + 0.5} + 1 \right) \tag{5}$$

$$f_{ij}^{\Downarrow} = 0.5 \times \left(\frac{g(q_j, c) - \wedge_{i=1}^{n} g(q_i, c)}{\vee_{j=1}^{n} g(q_j, c) + \wedge_{j=1}^{n} g(q_j, c) + 0.5} - \frac{g(q_i, c) - \wedge_{j=1}^{n} g(q_j, c)}{\vee_{j=1}^{n} g(q_j, c) + \wedge_{j=1}^{n} g(q_j, c) + 0.5} + 1 \right). \tag{6}$$

Based on criterion c_1 and using Equation (5), to calculate the upward fuzzy preference degree of $q_i (i = 1, 2, ..., 9)$ to $q_j (j = 1, 2, ..., 9)$, one can acquire

$$\mathcal{R}_{c_1}^{\Uparrow}(q_i, q_j) = \begin{pmatrix} 0.50000 & 0.66667 & 0.70000 & 0.56667 & 0.63333 & 0.70000 & 0.66667 & 0.66667 & 0.66667 \\ 0.33333 & 0.50000 & 0.53333 & 0.40000 & 0.46667 & 0.53333 & 0.50000 & 0.50000 & 0.50000 \\ 0.30000 & 0.46667 & 0.50000 & 0.36667 & 0.43333 & 0.50000 & 0.46667 & 0.46667 & 0.46667 \\ 0.43333 & 0.60000 & 0.63333 & 0.50000 & 0.56667 & 0.63333 & 0.60000 & 0.60000 & 0.60000 \\ 0.36667 & 0.53333 & 0.56667 & 0.43333 & 0.50000 & 0.56667 & 0.53333 & 0.53333 & 0.53333 \\ 0.30000 & 0.46667 & 0.50000 & 0.36667 & 0.43333 & 0.50000 & 0.46667 & 0.46667 & 0.46667 \\ 0.33333 & 0.50000 & 0.53333 & 0.40000 & 0.46667 & 0.53333 & 0.50000 & 0.50000 & 0.50000 \\ 0.33333 & 0.50000 & 0.53333 & 0.40000 & 0.46667 & 0.53333 & 0.50000 & 0.50000 & 0.50000 \\ 0.33333 & 0.50000 & 0.53333 & 0.40000 & 0.46667 & 0.53333 & 0.50000 & 0.50000 & 0.50000 \end{pmatrix}.$$

Further based on criterion c_2 and using Equation (5), to calculate the upward fuzzy preference degree of $q_i (i = 1, 2, ..., 9)$ to $q_j (j = 1, 2, ..., 9)$, one can get

$$\mathcal{R}_{c_2}^{\Uparrow}(q_i, q_j) = \begin{pmatrix} 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 \\ 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 \\ 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 \\ 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 \\ 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 \\ 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 \\ 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 \\ 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 \\ 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 \end{pmatrix}.$$

Finally based on criterion c_3 and using Equation (5), to calculate the upward fuzzy preference degree of $q_i(i = 1, 2, ..., 9)$ to $q_j(j = 1, 2, ..., 9)$, one can acquire

$$\mathcal{R}^{\Uparrow}_{c_3}(q_i, q_j) = \begin{pmatrix} 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 \\ 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 \\ 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 \\ 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 \\ 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 \\ 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 \\ 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 \\ 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 \\ 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 & 0.50000 \end{pmatrix}.$$

The above example consists of two types of attributes (criteria) first like c_1 (where all/some values of the feasible alternatives are different) and second like c_2 or c_3 (where all values of the feasible alternatives are same). On criterion like c_1, the Pan et al.'s transfer function and the proposed transfer function work to yield the upward fuzzy preference relation. On the other hand, on criterion like c_2 or c_3, the transfer function of Pan et al. fails to obtain the upward fuzzy preference relation, but the proposed transfer function works to obtain the upward fuzzy preference relation.

In real world problems, the information system depends on the behavior of the decision maker(s), so such type of attributes (criteria) like c_2 and/or c_3 may or may not exist. If attributes (criteria) like c_2 and/or c_3 are exist, then we do not have any technique to handle the situation. Existence of criterion like c_2 or c_3 in a fuzzy information system has its own importance. Regarding the reducing a criterion from the information system by a decision maker using his/her own technique arises the question that whether the technique works on this criterion or not? If c_2 or c_3 to be reduced from the information system to apply Pan et al. technique for the construction of upward fuzzy preference relation and develop a technique based on Pan et. al. technique, then as seen in above example, c_2 or c_3 cannot be reduced because Pan et. al. technique fails to handle c_2 or c_3. The proposed technique presents the only way to handle the failure situations.

Further we prove that our constructed upward fuzzy preference relation is upward fuzzy additive consistent we can get

(i) $$f^{\Uparrow}_{ii} = 0.5 \times \left(\frac{g(q_i, c) - \wedge_{i=1}^{n} g(q_i, c)}{\vee_{i=1}^{n} g(q_i, c) + \wedge_{i=1}^{n} g(q_i, c) + 0.5} - \frac{g(q_i, c) - \wedge_{i=1}^{n} g(q_i, c)}{\vee_{i=1}^{n} g(q_i, c) + \wedge_{i=1}^{n} g(q_i, c) + 0.5} + 1 \right)$$
$$= 0.5 \times (0 + 1) = 0.5.$$

(ii) $$\begin{aligned} f^{\Uparrow}_{ij} + f^{\Uparrow}_{ji} &= 0.5 \times \left(\frac{g(q_i, c) - \wedge_{j=1}^{n} g(q_j, c)}{\vee_{j=1}^{n} g(q_j, c) + \wedge_{j=1}^{n} g(q_j, c) + 0.5} - \frac{g(q_j, c) - \wedge_{j=1}^{n} g(q_j, c)}{\vee_{j=1}^{n} g(q_j, c) + \wedge_{j=1}^{n} g(q_j, c) + 0.5} + 1 \right) \\ &\quad + 0.5 \times \left(\frac{g(q_j, c) - \wedge_{i=1}^{n} g(q_i, c)}{\vee_{j=1}^{n} g(q_j, c) + \wedge_{j=1}^{n} g(q_j, c) + 0.5} - \frac{g(q_i, c) - \wedge_{j=1}^{n} g(q_j, c)}{\vee_{j=1}^{n} g(q_j, c) + \wedge_{j=1}^{n} g(q_j, c) + 0.5} + 1 \right) \\ &= 0.5 \times \left(\begin{array}{l} \frac{g(q_i, c) - \wedge_{j=1}^{n} g(q_j, c)}{\vee_{j=1}^{n} g(q_j, c) + \wedge_{j=1}^{n} g(q_j, c) + 0.5} - \frac{g(q_j, c) - \wedge_{j=1}^{n} g(q_j, c)}{\vee_{j=1}^{n} g(q_j, c) + \wedge_{j=1}^{n} g(q_j, c) + 0.5} + 1 \\ + \frac{g(q_j, c) - \wedge_{j=1}^{n} g(q_j, c)}{\vee_{j=1}^{n} g(q_j, c) + \wedge_{j=1}^{n} g(q_j, c) + 0.5} - \frac{g(q_i, c) - \wedge_{i=1}^{n} g(q_i, c)}{\vee_{j=1}^{n} g(q_j, c) + \wedge_{j=1}^{n} g(q_j, c) + 0.5} + 1 \end{array} \right) \\ &= 0.5 \times (1 + 1) = 1. \end{aligned}$$

(iii) $$f_{ij}^{\Uparrow}+f_{jk}^{\Uparrow}=0.5\times\left(\frac{g(q_i,c)-\wedge_{j=1}^{n}g(q_j,c)}{\vee_{j=1}^{n}g(q_j,c)+\wedge_{j=1}^{n}g(q_j,c)+0.5}-\frac{g(q_j,c)-\wedge_{j=1}^{n}g(q_j,c)}{\vee_{j=1}^{n}g(q_j,c)+\wedge_{j=1}^{n}g(q_j,c)+0.5}+1\right)$$
$$+0.5\times\left(\frac{g(q_j,c)-\wedge_{j=1}^{n}g(q_j,c)}{\vee_{j=1}^{n}g(q_j,c)+\wedge_{j=1}^{n}g(q_j,c)+0.5}-\frac{g(q_k,c)-\wedge_{j=1}^{n}g(q_j,c)}{\vee_{j=1}^{n}g(q_j,c)+\wedge_{j=1}^{n}g(q_j,c)+0.5}+1\right)$$
$$=0.5\times\begin{pmatrix}\frac{g(q_i,c)-\wedge_{j=1}^{n}g(q_j,c)}{\vee_{j=1}^{n}g(q_j,c)+\wedge_{j=1}^{n}g(q_j,c)+0.5}-\frac{g(q_j,c)-\wedge_{j=1}^{n}g(q_j,c)}{\vee_{j=1}^{n}g(q_j,c)+\wedge_{j=1}^{n}g(q_j,c)+0.5}+1\\+\frac{g(q_j,c)-\wedge_{j=1}^{n}g(q_j,c)}{\vee_{j=1}^{n}g(q_j,c)+\wedge_{j=1}^{n}g(q_j,c)+0.5}-\frac{g(q_k,c)-\wedge_{j=1}^{n}g(q_j,c)}{\vee_{j=1}^{n}g(q_j,c)+\wedge_{j=1}^{n}g(q_j,c)+0.5}+1\end{pmatrix}$$
$$=0.5\times\left(\frac{g(q_i,c)-\wedge_{j=1}^{n}g(q_j,c)}{\vee_{j=1}^{n}g(q_j,c)+\wedge_{j=1}^{n}g(q_j,c)+0.5}-\frac{g(q_k,c)-\wedge_{j=1}^{n}g(q_j,c)}{\vee_{j=1}^{n}g(q_j,c)+\wedge_{j=1}^{n}g(q_j,c)+0.5}+2\right)$$
$$=0.5\times\left(\frac{g(q_i,c)-\wedge_{j=1}^{n}g(q_j,c)}{\vee_{j=1}^{n}g(q_j,c)+\wedge_{j=1}^{n}g(q_j,c)+0.5}-\frac{g(q_k,c)-\wedge_{j=1}^{n}g(q_j,c)}{\vee_{j=1}^{n}g(q_j,c)+\wedge_{j=1}^{n}g(q_j,c)+0.5}+1\right)+0.5$$
$$=f_{ik}^{\Uparrow}+0.5.$$

Theorem 1. *Let $\mathcal{R}^{\Uparrow}=\left(f_{ij}^{\Uparrow}\right)_{n\times n}$ be the upward fuzzy preference relation which satisfy the upward additive consistency condition. Then based on $\mathcal{R}^{\Uparrow}$, we can derive the upward additive consistency matrix for $\overline{\mathcal{R}^{\Uparrow}}$ which satisfies the additive consistency, where*

$$\overline{\mathcal{R}^{\Uparrow}}=\left(\overline{f_{ik}^{\Uparrow}}\right)_{n\times n}=\left(\frac{1}{2n}\sum_{j=1}^{n}\left(f_{ij}^{\Uparrow}-f_{ji}^{\Uparrow}+f_{jk}^{\Uparrow}-f_{kj}^{\Uparrow}\right)+0.5\right)_{n\times n}.$$

Proof. The proof is straightforward. □

Theorem 2. *The upward consistency matrix $\overline{\mathcal{R}^{\Uparrow}}=\left(\overline{f_{ik}^{\Uparrow}}\right)_{n\times n}$ satisfies the upward additive consistency condition and the upward order consistency condition as follows*:

(i) $\overline{f_{ik}^{\Uparrow}}+\overline{f_{ki}^{\Uparrow}}=1$;
(ii) $\overline{f_{ii}^{\Uparrow}}=0.5$;
(iii) $\overline{f_{ik}^{\Uparrow}}=\overline{f_{is}^{\Uparrow}}+\overline{f_{sk}^{\Uparrow}}-0.5$;
(iv) $\overline{f_{ik}^{\Uparrow}}\leq\overline{f_{is}^{\Uparrow}}$ *for all* $i\in\{1,2,\ldots,n\}$, *where* $k\in\{1,2,\ldots,n\}$ *and* $s\in\{1,2,\ldots,n\}$;
(v) $\overline{f_{ik}^{\Uparrow}}-\overline{f_{is}^{\Uparrow}}=\overline{f_{tk}^{\Uparrow}}-\overline{f_{ts}^{\Uparrow}}$ *for all* $i\in\{1,2,\ldots,n\}$ *and* $t\in\{1,2,\ldots,n\}$, *where* $k\in\{1,2,\ldots,n\}$ *and* $s\in\{1,2,\ldots,n\}$.

Proof. The proof is straightforward. □

4 | α-UPWARD FUZZIFIED PREFERENCE ROUGH SET

Preference relations are very useful in expressing decision maker's preference information in ordinal decision problems. Fuzzy preference relation is first proposed by Orlovsky (1978) to represent an expert's opinion about a set of alternatives. The fuzzy preference relation not only can reflect that one alternative is before another alternative, but also can show the preference degree. The Pawlak's rough set model and fuzzy rough set are not able to receive and extract the information of ordinal structure and cannot be used to analyze the information with preference relations. Pawlak discussed this problem in Reference [47]. Greco et al. proposed a novel rough set model for preference analysis and constructed dominance relation based on the decision preference.[27-29] Hu et al. proposed a type of fuzzy preference relation rough sets model in Reference [44]. Hu et al.[44] adopted the well-known logis transfer function to compute the fuzzy preference degree of the feasible alternatives for the construction of fuzzy preference relation and proposed a new type of fuzzy preference relation rough set model. Pan et al.[45] pointed out that the transfer fuzzy preference degree of Hu et al. is not additive consistent and they suggested another transfer function to modify the fuzzy preference relation rough set model of Hu et al. As mentioned earlier that the transfer function for computing the fuzzy preference degree in Pan et al.[45] for the construction of upward/downward fuzzy preference relations are not additive consistent. Less effort has been made to explore the structure of these fuzzy preference relation rough sets using fuzzy preference relations. In current literature the researchers utilized the idea of fuzzy preference relation to find the fuzzy approximations. However the scholars were unable/incapable to find the crisp approximations with the help of fuzzy preference relations. Naturally question arises that whether we can find the crisp approximations with the help of fuzzy preference relations? The affirmative answer to this question has led the present authors to the introduction of α-upward fuzzified preference rough sets. Moreover, the approximation defined based on α-upward fuzzified preference rough sets play a bridging role between fuzzy preference relation and crisp set. Furthermore, the approximation defined based on α-upward fuzzified preference rough sets are useful in different uncertainties such as approximate precision, rough degree, and approximate quality and their mutual relationships. Similarly the very same concept can be applied for linguistic/ordinal information systems.

Definition 3. The upward and downward fuzzy preference classes $\mathcal{A}_i^{\mathcal{R}^{\Uparrow}}$ and $\mathcal{A}_i^{\mathcal{R}^{\Downarrow}}$ of q_i induced by the upward and downward additive fuzzy preference relations $\mathcal{R}^{\Uparrow}$ and $\mathcal{R}^{\Downarrow}$ are defined by

$$\mathcal{A}_i^{\mathcal{R}^{\Uparrow}} = \frac{f_{i1}^{\Uparrow}}{q_1} + \frac{f_{i2}^{\Uparrow}}{q_2} + \cdots + \frac{f_{in}^{\Uparrow}}{q_n}$$
$$\mathcal{A}_i^{\mathcal{R}^{\Downarrow}} = \frac{f_{i1}^{\Downarrow}}{q_1} + \frac{f_{i2}^{\Downarrow}}{q_2} + \cdots + \frac{f_{in}^{\Downarrow}}{q_n},$$

where "+" means the union operation. The upward fuzzy preference relation and downward fuzzy preference relation from a family of fuzzy information granules from the universe which composes the upward fuzzy preference granular structure and downward fuzzy preference granular structure given by

$$\mathcal{P}(\mathcal{R}^{\Uparrow}) = \left\{\mathcal{A}_1^{\mathcal{R}^{\Uparrow}}, \mathcal{A}_2^{\mathcal{R}^{\Uparrow}}, \ldots, \mathcal{A}_n^{\mathcal{R}^{\Uparrow}}\right\},$$

and

$$\mathcal{P}(\mathcal{R}^{\Downarrow}) = \left\{\mathcal{A}_1^{\mathcal{R}^{\Downarrow}}, \mathcal{A}_2^{\mathcal{R}^{\Downarrow}}, \ldots, \mathcal{A}_n^{\mathcal{R}^{\Downarrow}}\right\}.$$

Definition 4. Let $(\mathcal{U}, \mathcal{P}(\mathcal{R}^{\Uparrow}))$ be an upward fuzzy preference approximation space, where $\mathcal{U}$ is an arbitrary universe and $\mathcal{P}(\mathcal{R}^{\Uparrow})$ an upward additive fuzzy preference granular structure. For any $\alpha \in [0.5, 1)$, the upward fuzzified preference lower and upper approximations for a given set $\mathcal{X}_1 \subseteq \mathcal{U}$ are defined as

$$\left(\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\bigtriangledown}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow} = \left\{q_i \in \mathcal{U}: f_{ij}^{\Uparrow} < 1 - \alpha \quad \text{for all} \quad q_j \in \mathcal{X}_1^c\right\}$$

and

$$\left(\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow} = \left\{q_i \in \mathcal{U}: f_{ij}^{\Uparrow} \geq 1 - \alpha \quad \text{for some} \quad q_j \in \mathcal{X}_1\right\}.$$

The pair $\left(\left(\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow}, \left(\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow}\right)$ is referred as an α-upward fuzzified preference rough set. The positive, negative and boundary regions of $\mathcal{X}_1 \subseteq \mathcal{U}$ for any $\alpha \in [0.5, 1)$ are defined and denoted as:

$$POS_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\Uparrow} = \left(\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow}$$
$$NEG_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\Uparrow} = \left(\left(\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow}\right)^c$$
$$BND_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\Uparrow} = \left(\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow} - \left(\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow}.$$

Theorem 3. *Let $(\mathcal{U}, \mathcal{P}(\mathcal{R}^{\Uparrow}))$ be an upward fuzzy preference approximation space and $\alpha_1 \leq \alpha_2$, where $\alpha_1, \alpha_2 \in [0.5, 1)$. Then*

(i) $\left(\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{X}_1)^{\alpha_2}\right)^{\Uparrow} \subseteq \left(\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{X}_1)^{\alpha_1}\right)^{\Uparrow}$;
(ii) $\left(\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{X}_1)^{\alpha_1}\right)^{\Uparrow} \subseteq \left(\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{X}_1)^{\alpha_2}\right)^{\Uparrow}$.

Proof.

(i) For any $q_i \in (\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{X}_1)^{\alpha_2})^{\Uparrow}, f_{ij}^{\Uparrow} < 1 - \alpha_2$ for all $q_j \in \mathcal{X}_1^c$. But $1 - \alpha_2 \leq 1 - \alpha_1$, we have $f_{ij}^{\Uparrow} < 1 - \alpha_1$ for all $q_j \in \mathcal{X}_1^c$. Thus $q_i \in (\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{X}_1)^{\alpha_1})^{\Uparrow}$ showing that $(\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{X}_1)^{\alpha_2})^{\Uparrow} \subseteq (\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{X}_1)^{\alpha_1})^{\Uparrow}$.
(ii) For any $q_i \in (\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{X}_1)^{\alpha_1})^{\Uparrow}, f_{ij}^{\Uparrow} \geq 1 - \alpha_1$ for some $q_j \in \mathcal{X}_1$. But $1 - \alpha_2 \leq 1 - \alpha_1$, we have $f_{ij}^{\Uparrow} \geq 1 - \alpha_2$ for some $q_j \in \mathcal{X}_1$. Thus $q_i \in (\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{X}_1)^{\alpha_2})^{\Uparrow}$ showing that $(\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{X}_1)^{\alpha_1})^{\Uparrow} \subseteq (\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{X}_1)^{\alpha_2})^{\Uparrow}$. □

Theorem 4. *Let $(\mathcal{U}, \mathcal{P}(\mathcal{R}^{\Uparrow}))$ be an upward fuzzy preference approximation space and $\alpha \in [0.5, 1)$. Then*

(i) $(\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{X}_1)^{\alpha})^{\Uparrow} \subseteq \mathcal{X}_1 \subseteq (\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{X}_1)^{\alpha})^{\Uparrow}$,
(ii) $(\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\varnothing)^{\alpha})^{\Uparrow} = \varnothing = (\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\varnothing)^{\alpha})^{\Uparrow}$ *and* $(\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{U})^{\alpha})^{\Uparrow} = \mathcal{U} = (\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{U})^{\alpha})^{\Uparrow}$,
(iii) $(\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{X}_1^c)^{\alpha})^{\Uparrow} = ((\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{X}_1)^{\alpha})^{\Uparrow})^c$ *and* $(\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{X}_1^c)^{\alpha})^{\Uparrow} = ((\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{X}_1)^{\alpha})^{\Uparrow})^c$,
(iv) *if* $\mathcal{X}_1 \subseteq \mathcal{Y}_1$, *then* $(\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{X}_1)^{\alpha})^{\Uparrow} \subseteq (\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{Y}_1)^{\alpha})^{\Uparrow}$ *and* $(\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{X}_1)^{\alpha})^{\Uparrow} \subseteq (\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{Y}_1)^{\alpha})^{\Uparrow}$,
(v) *if* $\mathcal{R}_1^{\Uparrow} \subseteq \mathcal{R}_2^{\Uparrow}$, *then* $(\mathcal{L}\mathrm{A}_{\mathcal{R}_2^{\Uparrow}}^{\triangledown}(\mathcal{X}_1)^{\alpha})^{\Uparrow} \subseteq (\mathcal{L}\mathrm{A}_{\mathcal{R}_1^{\Uparrow}}^{\triangledown}(\mathcal{X}_1)^{\alpha})^{\Uparrow}$ *and* $(\mathcal{UA}_{\mathcal{R}_1^{\Uparrow}}^{\triangle}(\mathcal{X}_1)^{\alpha})^{\Uparrow} \subseteq (\mathcal{UA}_{\mathcal{R}_2^{\Uparrow}}^{\triangle}(\mathcal{X}_1)^{\alpha})^{\Uparrow}$,
(vi) $(\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{X}_1 \cup \mathcal{Y}_1)^{\alpha})^{\Uparrow} \supseteq (\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{X}_1)^{\alpha})^{\Uparrow} \cup (\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{Y}_1)^{\alpha})^{\Uparrow}$,
(vii) $(\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{X}_1 \cap \mathcal{Y}_1)^{\alpha})^{\Uparrow} \subseteq (\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{X}_1)^{\alpha})^{\Uparrow} \cap (\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{Y}_1)^{\alpha})^{\Uparrow}$,
(viii) $(\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{X}_1 \cap \mathcal{Y}_1)^{\alpha})^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{X}_1)^{\alpha})^{\Uparrow} \cap (\mathcal{L}\mathrm{A}_{\mathcal{R}^{\Uparrow}}^{\triangledown}(\mathcal{Y}_1)^{\alpha})^{\Uparrow}$,
(ix) $(\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{X}_1 \cup \mathcal{Y}_1)^{\alpha})^{\Uparrow} = (\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{X}_1)^{\alpha})^{\Uparrow} \cup (\mathcal{UA}_{\mathcal{R}^{\Uparrow}}^{\triangle}(\mathcal{Y}_1)^{\alpha})^{\Uparrow}$.

Proof.

(i) and (ii) straightforward.

(iii) Let $q \in (\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1^c)^{\alpha})^{\Uparrow}$. Then $f_{ij}^{\Uparrow} < 1 - \alpha$ for all $q_j \in (\mathcal{X}_1^c)^c = \mathcal{X}_1$. This implies that $f_{ij}^{\Uparrow} \geq 1 - \alpha$ for some $q_j \in \mathcal{X}_1$. This implies that $q \notin (\mathcal{UA}^{\bigtriangleup}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha})^{\Uparrow}$. Therefore $q \in ((\mathcal{UA}^{\bigtriangleup}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha})^{\Uparrow})^c$. On the other hand, for any $q \in ((\mathcal{UA}^{\bigtriangleup}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha})^{\Uparrow})^c$, then $q \notin (\mathcal{UA}^{\bigtriangleup}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha})^{\Uparrow}$. This implies that $f_{ij}^{\Uparrow} \geq 1 - \alpha$ for some $q_j \in \mathcal{X}_1$, we can get $f_{ij}^{\Uparrow} < 1 - \alpha$ for all $q_j \in \mathcal{X}_1$. Thus $q \in (\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1^c)^{\alpha})^{\Uparrow}$. Therefore $(\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1^c)^{\alpha})^{\Uparrow} = ((\mathcal{UA}^{\bigtriangleup}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha})^{\Uparrow})^c$. Similarly we can get $(\mathcal{UA}^{\bigtriangleup}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1^c)^{\alpha})^{\Uparrow} = ((\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha})^{\Uparrow})^c$.

(iv) Let $q \in (\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha})^{\Uparrow}$. Then $f_{ij}^{\Uparrow} < 1 - \alpha$ for all $q_j \in \mathcal{X}_1^c$. But $\mathcal{Y}_1^c \subseteq \mathcal{X}_1^c$, this implies that $f_{ij}^{\Uparrow} < 1 - \alpha$ for all $q_j \in \mathcal{Y}_1^c$. Hence $q \in (\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)^{\alpha})^{\Uparrow}$. Therefore $(\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha})^{\Uparrow} \subseteq (\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)^{\alpha})^{\Uparrow}$. Similarly we can get $(\mathcal{UA}^{\bigtriangleup}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha})^{\Uparrow} \subseteq (\mathcal{UA}^{\bigtriangleup}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)^{\alpha})^{\Uparrow}$.

(v) Let $q_i \in (\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}_2^{\Uparrow}}(\mathcal{X}_1)^{\alpha})^{\Uparrow}$. Then $f_{2ij}^{\Uparrow} < 1 - \alpha$ for all $q_j \in \mathcal{X}_1^c$. But $f_1^{\Uparrow} \leq f_2^{\Uparrow}$, this implies that $f_{1ij}^{\Uparrow} < 1 - \alpha$ for all $q_j \in \mathcal{X}_1^c$. Thus $q \in (\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}_1^{\Uparrow}}(\mathcal{X}_1)^{\alpha})^{\Uparrow}$. Therefore $(\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}_2^{\Uparrow}}(\mathcal{X}_1)^{\alpha})^{\Uparrow} \subseteq (\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}_1^{\Uparrow}}(\mathcal{X}_1)^{\alpha})^{\Uparrow}$. Similarly $(\mathcal{UA}^{\bigtriangleup}_{\mathcal{R}_1^{\Uparrow}}(\mathcal{X}_1)^{\alpha})^{\Uparrow} \subseteq (\mathcal{UA}^{\bigtriangleup}_{\mathcal{R}_2^{\Uparrow}}(\mathcal{X}_1)^{\alpha})$.

The proof process of (vi) and (vii) is similar to the proof of (iv).

(viii) By using (iv), we can write $(\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1 \cap \mathcal{Y}_1)^{\alpha})^{\Uparrow} \subseteq (\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha})^{\Uparrow}$ and $(\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1 \cap \mathcal{Y}_1)^{\alpha})^{\Uparrow} \subseteq (\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)^{\alpha})^{\Uparrow}$. This implies that $(\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1 \cap \mathcal{Y}_1)^{\alpha})^{\Uparrow} \subseteq (\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha})^{\Uparrow} \cap (\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)^{\alpha})^{\Uparrow}$. For the reverse inclusion, let $q_i \in (\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha})^{\Uparrow} \cap (\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)^{\alpha})^{\Uparrow}$. Then $q_i \in (\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha})^{\Uparrow}$ and $q_i \in (\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)^{\alpha})^{\Uparrow}$. This implies that $f_{ij}^{\Uparrow} < 1 - \alpha$ for all $q_j \in \mathcal{X}_1^c$ and $f_{ik}^{\Uparrow} < 1 - \alpha$ for all $q_k \in \mathcal{Y}_1^c$. This implies that $f_{il}^{\Uparrow} < 1 - \alpha$ for all $q_l \in \mathcal{X}_1^c \cup \mathcal{Y}_1^c = (\mathcal{X}_1 \cap \mathcal{Y}_1)^c$. Thus $q_i \in (\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1 \cap \mathcal{Y}_1)^{\alpha})^{\Uparrow}$. Therefore $(\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1 \cap \mathcal{Y}_1)^{\alpha})^{\Uparrow} = (\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha})^{\Uparrow} \cap (\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)^{\alpha})^{\Uparrow}$.

(ix) The proof process is similar to the proof of (viii). □

Definition 5. Let $(\mathcal{U}, \mathcal{P}(\mathcal{R}^{\Uparrow}))$ be an upward fuzzy preference approximation space and $\alpha \in [0.5, 1)$. The approximate precision $\rho^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\Uparrow}$ of $\mathcal{X}_1$ is defined by:

$$\rho^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\Uparrow} = \frac{\left|\left(\mathcal{L}\mathrm{A}^{\bigtriangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow}\right|}{\left|\left(\mathcal{UA}^{\bigtriangleup}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow}\right|},$$

where $\mathcal{X}_1$ is a nonempty subset of $\mathcal{U}$ and $|\cdot|$ denotes the cardinality of a set. Let $\mu^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\Uparrow} = 1 - \rho^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\Uparrow}$. Then $\mu^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\Uparrow}$ is called the rough degree of $\mathcal{X}_1$, where $\mu^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\Uparrow}, \rho^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\Uparrow} \in [0, 1]$.

The following theorem describes the relationship of the rough degree $\mu^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\Uparrow}$ and the approximate precision $\rho^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\Uparrow}$ for the union and intersection of subsets $\mathcal{X}_1$ and $\mathcal{Y}_1$ of the universe $\mathcal{U}$.

Theorem 5. *Let* $(\mathcal{U}, \mathcal{P}(\mathcal{R}^{\Uparrow}))$ *be an upward fuzzy preference approximation space and* $\alpha \in [0.5, 1)$. *Then the* approximate precision *and* rough degree *of the subsets* $\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_1 \cup \mathcal{Y}_1$ *and* $\mathcal{X}_1 \cap \mathcal{Y}_1$ *of the universe* $\mathcal{U}$ *satisfy the following relations.*

(i)

$$\begin{aligned}&\mu^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1 \cup \mathcal{Y}_1)\left|\left(\mathcal{UA}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow} \cup \left(\mathcal{UA}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)^{\alpha}\right)^{\Uparrow}\right|\\ &\leq \mu^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)\left|\left(\mathcal{UA}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow}\right| + \mu^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)\left|\left(\mathcal{UA}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)^{\alpha}\right)^{\Uparrow}\right|\\ &- \mu^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1 \cap \mathcal{Y}_1)\left|\left(\mathcal{UA}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow} \cap \left(\mathcal{UA}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)^{\alpha}\right)^{\Uparrow}\right|.\end{aligned}$$

(ii)

$$\begin{aligned}&\rho^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\Uparrow}(\mathcal{X}_1 \cup \mathcal{Y}_1)\left|\left(\mathcal{UA}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow} \cup \left(\mathcal{UA}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)^{\alpha}\right)^{\Uparrow}\right|\\ &\geq \rho^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\Uparrow}\left|\left(\mathcal{UA}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow}\right| + \rho^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\Uparrow}\left|\left(\mathcal{UA}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)^{\alpha}\right)^{\Uparrow}\right|\\ &- \rho^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\Uparrow}(\mathcal{X}_1 \cap \mathcal{Y}_1)\left|\left(\mathcal{UA}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow} \cap \left(\mathcal{UA}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)^{\alpha}\right)^{\Uparrow}\right|.\end{aligned}$$

Proof. The proof is straightforward. □

Definition 6. Let $(\mathcal{U}, \mathcal{P}(\mathcal{R}^{\Uparrow}))$ be an upward fuzzy preference approximation space and $\alpha \in [0.5, 1)$. The *approximate quality* $\gamma^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)$ of $\mathcal{X}_1$ is defined by:

$$\gamma^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1) = \frac{\left|\left(\mathcal{L}\mathrm{A}^{\triangledown}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow}\right|}{|\mathcal{U}|},$$

where $\mathcal{X}_1$ is a nonempty subset of $\mathcal{U}$ and $\gamma^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1) \in [0, 1]$.

The following theorem describes the relationship of the rough degree $\mu^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)$ and the approximate quality $\gamma^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)$ for the union and intersection of subsets $\mathcal{X}_1$ and $\mathcal{Y}_1$ of the universe $\mathcal{U}$.

Theorem 6. *Let* $(\mathcal{U}, \mathcal{P}(\mathcal{R}^{\Uparrow}))$ *be an upward fuzzy preference approximation space and* $\alpha \in [0.5, 1)$. *Then the* rough degree *and* approximate quality *for all subsets* $\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_1 \cup \mathcal{Y}_1$ *and* $\mathcal{X}_1 \cap \mathcal{Y}_1$ *of the universe* $\mathcal{U}$ *satisfy the following relation*:

$$\begin{aligned}&\mu^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1 \cup \mathcal{Y}_1)\left|\left(\mathcal{UA}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow} \cup \left(\mathcal{UA}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)^{\alpha}\right)^{\Uparrow}\right|\\ &\leq \left|\left(\mathcal{UA}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow}\right| + \left|\left(\mathcal{UA}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)^{\alpha}\right)^{\Uparrow}\right| - \left|\mathcal{U}\right|\left\{\gamma^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1) + \gamma^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)\right\}\\ &- \mu^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1 \cap \mathcal{Y}_1)\left|\left(\mathcal{UA}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow} \cap \left(\mathcal{UA}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)^{\alpha}\right)^{\Uparrow}\right|.\end{aligned}$$

Proof. The proof is straightforward. □

The following theorem highlights the relationship between *approximate precision* and *approximate quality* for the union and intersection of two sets.

Theorem 7. *Let* $(\mathcal{U}, \mathcal{P}(\mathcal{R}^{\Uparrow}))$ *be an upward fuzzy preference approximation space and* $\alpha \in [0.5, 1)$. *Then the* approximate quality *and* approximate precision *for all subsets* $\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_1 \cup \mathcal{Y}_1$ *and* $\mathcal{X}_1 \cap \mathcal{Y}_1$ *of the universe* $\mathcal{U}$ *satisfy the following relation*:

$$\begin{aligned}
&\rho^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1 \cup \mathcal{Y}_1)\left|\left(\mathcal{U}\mathcal{A}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow} \cup \left(\mathcal{U}\mathcal{A}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)^{\alpha}\right)^{\Uparrow}\right| \\
\geq\ &\left|\mathcal{U}\right|\left\{\gamma^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1) + \gamma^{\alpha}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)\right\} \\
&-\rho^{\alpha}_{\mathcal{R}^{\Uparrow}}\left(\mathcal{X}_1 \cap \mathcal{Y}_1\right)\left|\left(\mathcal{U}\mathcal{A}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{X}_1)^{\alpha}\right)^{\Uparrow} \cap \left(\mathcal{U}\mathcal{A}^{\triangle}_{\mathcal{R}^{\Uparrow}}(\mathcal{Y}_1)^{\alpha}\right)^{\Uparrow}\right|.
\end{aligned}$$

Proof. The proof is straightforward. □

5 | β-NEIGHBORHOOD BASED UPWARD ROUGH SETS

In this section, we first find upward β-neighborhood in the fuzzy upward covering approximation space and then present upward rough sets and discussed some of their properties.

Definition 7. Let $\mathcal{U}$ be an arbitrary universal set and $\mathcal{P}(\mathcal{R}^{\Uparrow})$ be an upward additive fuzzy preference granular structure. Then for each $\beta \in (0, 1]$, $\mathcal{P}(\mathcal{R}^{\Uparrow})$ is a fuzzy upward β-covering of $\mathcal{U}$, if $\left(\bigcup_{i=1}^{n} \mathcal{A}_i^{\mathcal{R}\Uparrow}\right)(q) \geq \beta$ for each $q \in \mathcal{U}$. The pair $(\mathcal{U}, \mathcal{P}(\mathcal{R}^{\Uparrow}))$ is called fuzzy upward covering approximation space.

Definition 8. Let $(\mathcal{U}, \mathcal{P}(\mathcal{R}^{\Uparrow}))$ be a fuzzy upward covering approximation space, where $\mathcal{P}(\mathcal{R}^{\Uparrow})$ is a fuzzy upward β-covering of $\mathcal{U}$. For each $q \in \mathcal{U}$, we define

(i) the fuzzy β-upward neighborhood $\mathcal{N}_q^{\Uparrow\beta}$ of q as

$$\mathcal{N}_q^{\Uparrow\beta} = \bigcap\left\{\mathcal{A}_i^{\mathcal{R}^{\Uparrow}} \in \mathcal{P}(\mathcal{R}^{\Uparrow}) \colon \mathcal{A}_i^{\mathcal{R}^{\Uparrow}}(q) \geq \beta\right\}.$$

(ii) the fuzzy complementary β-upward neighborhood $\mathcal{M}_q^{\Uparrow\beta}$ of q as

$$\mathcal{M}_q^{\Uparrow\beta}(y) = \mathcal{N}_y^{\Uparrow\beta}(q) \quad \text{for all} \quad y \in \mathcal{U}.$$

Proposition 1. *For each* $q \in \mathcal{U}, \mathcal{N}_q^{\Uparrow\beta}(q) \geq \beta$.

Proof. Let $q \in \mathcal{U}$. Then it follows that

$$\mathcal{N}_q^{\Uparrow\beta}(q) = \left(\bigcap_{\mathcal{A}_i^{\mathcal{R}^{\Uparrow}}(q)\geq\beta} \mathcal{A}_i^{\mathcal{R}^{\Uparrow}}\right)(q) = \bigwedge_{\mathcal{A}_i^{\mathcal{R}^{\Uparrow}}(q)\geq\beta} \mathcal{A}_i^{\mathcal{R}^{\Uparrow}}(q) \geq \beta.$$ □

Proposition 2. *For all* $q, y, z \in \mathcal{U}$, *if* $\mathcal{N}_q^{\Uparrow\beta}(y) \geq \beta$ *and* $\mathcal{N}_y^{\Uparrow\beta}(z) \geq \beta$, *then* $\mathcal{N}_q^{\Uparrow\beta}(z) \geq \beta$.

Proof. For $\mathcal{N}_q^{\Uparrow\beta}(y) \geq \beta$ and for each $i \in I = \{1, 2, ..., n\}$, if $\mathcal{A}_i^{\mathcal{R}^\Uparrow}(q) \geq \beta$, then $\mathcal{A}_i^{\mathcal{R}^\Uparrow}(y) \geq \beta$. Similarly if $\mathcal{N}_y^{\Uparrow\beta}(z) \geq \beta$ which implies $\mathcal{A}_i^{\mathcal{R}^\Uparrow}(y) \geq \beta$, and thus $\mathcal{A}_i^{\mathcal{R}^\Uparrow}(z) \geq \beta$. Hence, for each $i \in I$, $\mathcal{A}_i^{\mathcal{R}^\Uparrow}(q) \geq \beta$ implies $\mathcal{A}_i^{\mathcal{R}^\Uparrow}(z) \geq \beta$. Therefore $\mathcal{N}_q^{\Uparrow\beta}(z) \geq \beta$. □

Proposition 3. *If $\beta_1 \leq \beta_2$, then $\mathcal{N}_q^{\Uparrow\beta_1} \subseteq \mathcal{N}_q^{\Uparrow\beta_2}$ for all $q \in \mathcal{U}$.*

Proof. Let $\beta_1 \leq \beta_2$ for each $q \in \mathcal{U}$. $\{\mathcal{A}_i^{\mathcal{R}^\Uparrow}: \mathcal{A}_i^{\mathcal{R}^\Uparrow}(q) \geq \beta_1\} \supseteq \{\mathcal{A}_i^{\mathcal{R}^\Uparrow}: \mathcal{A}_i^{\mathcal{R}^\Uparrow}(q) \geq \beta_2\}$. Hence $\mathcal{N}_q^{\Uparrow\beta_1} = \bigcap\{\mathcal{A}_i^{\mathcal{R}^\Uparrow}: \mathcal{A}_i^{\mathcal{R}^\Uparrow}(q) \geq \beta_1\} \subseteq \bigcap\{\mathcal{A}_i^{\mathcal{R}^\Uparrow}: \mathcal{A}_i^{\mathcal{R}^\Uparrow}(q) \geq \beta_2\} = \mathcal{N}_q^{\Uparrow\beta_2}$. □

Definition 9. Let $(\mathcal{U}, \mathcal{P}(\mathcal{R}^\Uparrow))$ be a fuzzy upward covering approximation space, where $\mathcal{P}(\mathcal{R}^\Uparrow)$ is a fuzzy upward β-covering of $\mathcal{U}$ for some $\beta \in (0, 1]$. Then for each $q \in \mathcal{U}$, we define the upward β-neighborhood $N_q^{\Uparrow\beta}$ of q as:

$$N_q^{\Uparrow\beta} = \left\{y \in \mathcal{U}: \mathcal{N}_q^{\Uparrow\beta}(y) \geq \beta\right\}.$$

Definition 10. Let $(\mathcal{U}, \mathcal{P}(\mathcal{R}^\Uparrow))$ be a fuzzy upward covering approximation space, where $\mathcal{P}(\mathcal{R}^\Uparrow)$ is a fuzzy upward β-covering of $\mathcal{U}$ for some $\beta \in (0, 1]$. We define the upward lower approximation $(\mathcal{L}\mathrm{A}_\beta^\bigtriangledown(\mathcal{X}_1))^\Uparrow$ and the upward upper approximation $(\mathcal{U}\mathcal{A}_\beta^\bigtriangleup(\mathcal{X}_1))^\Uparrow$ of $\mathcal{X}_1$ as:

$$\left(\mathcal{L}\mathrm{A}_\beta^\bigtriangledown(\mathcal{X}_1)\right)^\Uparrow = \cup\left\{N_q^{\Uparrow\beta}: N_q^{\Uparrow\beta} \subseteq \mathcal{X}_1\right\};$$
$$\left(\mathcal{U}\mathcal{A}_\beta^\bigtriangleup(\mathcal{X}_1)\right)^\Uparrow = \left(\left(\mathcal{L}\mathrm{A}_\beta^\bigtriangledown\mathcal{X}_1^c\right)^\Uparrow\right)^c.$$

Further, $\mathcal{X}_1$ is called a definable set on upward approximation space if $(\mathcal{L}\mathrm{A}_\beta^\bigtriangledown(\mathcal{X}_1))^\Uparrow = (\mathcal{U}\mathcal{A}^\bigtriangleup(\mathcal{X}_1))^\Uparrow$. Otherwise, the pair $((\mathcal{L}\mathrm{A}_\beta^\bigtriangledown(\mathcal{X}_1))^\Uparrow, (\mathcal{U}\mathcal{A}_\beta^\bigtriangleup(\mathcal{X}_1))^\Uparrow)$ is called upward rough set, where $\mathcal{X}_1^c$ denote the complementary set $\mathcal{U} - \mathcal{X}_1$ of $\mathcal{X}_1 \subseteq \mathcal{U}$.

Theorem 8. *Let $(\mathcal{U}, \mathcal{P}(\mathcal{R}^\Uparrow))$ be a fuzzy upward covering approximation space, where $\mathcal{P}(\mathcal{R}^\Uparrow)$ is a fuzzy upward β-covering of $\mathcal{U}$ for some $\beta \in (0, 1]$. Then*

(i) $(\mathcal{U}\mathcal{A}_\beta^\bigtriangleup(\mathcal{U}\mathcal{A}_\beta^\bigtriangleup(\mathcal{X}_1)))^\Uparrow = (\mathcal{U}\mathcal{A}_\beta^\bigtriangleup(\mathcal{X}_1))^\Uparrow$,
(ii) $(\mathcal{L}\mathrm{A}_\beta^\bigtriangledown(\mathcal{L}\mathrm{A}_\beta^\bigtriangledown(\mathcal{X}_1)))^\Uparrow = (\mathcal{L}\mathrm{A}_\beta^\bigtriangledown(\mathcal{X}_1))^\Uparrow$.

In classical rough set theory, if the lower approximation $R(\underline{\mathcal{X}_1})$ and upper approximation $R(\overline{\mathcal{X}_1})$ of the set $\mathcal{X}_1$ are equal to $\mathcal{X}_1$, then $\mathcal{X}_1$ is called definable, otherwise $\mathcal{X}_1$ is considered a rough set. So both $R(\underline{\mathcal{X}_1})$ and $R(\overline{\mathcal{X}_1})$ of the set $\mathcal{X}_1$ in Pawlak's rough set model are definable sets, that is, $R(\underline{R(\mathcal{X}_1)}) = R(\underline{\mathcal{X}_1}) = R(\overline{R(\underline{\mathcal{X}_1})})$, $R(\underline{R(\overline{\mathcal{X}_1})}) = R(\overline{\mathcal{X}_1}) = R(\overline{R(\overline{\mathcal{X}_1})})$. But in upward β-neighborhood based upward rough sets, the upward lower approximation $(\mathcal{L}\mathrm{A}_\beta^\bigtriangledown(\mathcal{X}_1))^\Uparrow$ and the upward upper approximation $(\mathcal{U}\mathcal{A}_\beta^\bigtriangleup(\mathcal{X}_1))^\Uparrow$ of $\mathcal{X}_1$ are hardly definable sets. In general, they are still rough sets, that is,$(\mathcal{L}\mathrm{A}_\beta^\bigtriangledown(\mathcal{L}\mathrm{A}_\beta^\bigtriangledown(\mathcal{X}_1)))^\Uparrow = (\mathcal{L}\mathrm{A}_\beta^\bigtriangledown(\mathcal{X}_1))^\Uparrow \neq (\mathcal{U}\mathcal{A}_\beta^\bigtriangleup(\mathcal{L}\mathrm{A}_\beta^\bigtriangledown(\mathcal{X}_1)))^\Uparrow$, $(\mathcal{U}\mathcal{A}_\beta^\bigtriangleup(\mathcal{U}\mathcal{A}_\beta^\bigtriangleup(\mathcal{X}_1)))^\Uparrow = (\mathcal{U}\mathcal{A}_\beta^\bigtriangleup(\mathcal{X}_1))^\Uparrow \neq (\mathcal{L}\mathrm{A}_\beta^\bigtriangledown(\mathcal{U}\mathcal{A}_\beta^\bigtriangleup(\mathcal{X}_1)))^\Uparrow$. We now give an example to show this fact.

Example 2. Let $\mathcal{U} = \{q_j: j = 1, 2, ..., 9\}$ be the set of actions. Consider the fuzzy upward β-neighborhoods as follows:

$$\mathcal{N}_{q_1}^{\Uparrow 0.5} = \frac{0.5000}{q_1} + \frac{0.9167}{q_2} + \frac{1.0000}{q_3} + \frac{0.6667}{q_4} + \frac{0.8333}{q_5} + \frac{1.0000}{q_6} + \frac{0.9167}{q_7} + \frac{0.9167}{q_8} + \frac{0.9167}{q_9};$$

$$\mathcal{N}_{q_2}^{\Uparrow 0.5} = \frac{0.0833}{q_1} + \frac{0.5000}{q_2} + \frac{0.5833}{q_3} + \frac{0.2500}{q_4} + \frac{0.4167}{q_5} + \frac{0.5833}{q_6} + \frac{0.5000}{q_7} + \frac{0.5000}{q_8} + \frac{0.5000}{q_9};$$

$$\mathcal{N}_{q_3}^{\Uparrow 0.5} = \frac{0.0000}{q_1} + \frac{0.4167}{q_2} + \frac{0.5000}{q_3} + \frac{0.1667}{q_4} + \frac{0.3333}{q_5} + \frac{0.5000}{q_6} + \frac{0.4167}{q_7} + \frac{0.4167}{q_8} + \frac{0.4167}{q_9};$$

$$\mathcal{N}_{q_4}^{\Uparrow 0.5} = \frac{0.3333}{q_1} + \frac{0.7500}{q_2} + \frac{0.8333}{q_3} + \frac{0.5000}{q_4} + \frac{0.6667}{q_5} + \frac{0.8333}{q_6} + \frac{0.7500}{q_7} + \frac{0.7500}{q_8} + \frac{0.7500}{q_9};$$

$$\mathcal{N}_{q_5}^{\Uparrow 0.5} = \frac{0.1667}{q_1} + \frac{0.5833}{q_2} + \frac{0.6667}{q_3} + \frac{0.3333}{q_4} + \frac{0.5000}{q_5} + \frac{0.6667}{q_6} + \frac{0.5833}{q_7} + \frac{0.5833}{q_8} + \frac{0.5833}{q_9};$$

$$\mathcal{N}_{q_6}^{\Uparrow 0.5} = \frac{0.0000}{q_1} + \frac{0.4167}{q_2} + \frac{0.5000}{q_3} + \frac{0.1667}{q_4} + \frac{0.3333}{q_5} + \frac{0.5000}{q_6} + \frac{0.4167}{q_7} + \frac{0.4167}{q_8} + \frac{0.4167}{q_9};$$

$$\mathcal{N}_{q_7}^{\Uparrow 0.5} = \frac{0.0833}{q_1} + \frac{0.5000}{q_2} + \frac{0.5833}{q_3} + \frac{0.2500}{q_4} + \frac{0.4167}{q_5} + \frac{0.5833}{q_6} + \frac{0.5000}{q_7} + \frac{0.5000}{q_8} + \frac{0.5000}{q_9};$$

$$\mathcal{N}_{q_8}^{\Uparrow 0.5} = \frac{0.0833}{q_1} + \frac{0.5000}{q_2} + \frac{0.5833}{q_3} + \frac{0.2500}{q_4} + \frac{0.4167}{q_5} + \frac{0.5833}{q_6} + \frac{0.5000}{q_7} + \frac{0.5000}{q_8} + \frac{0.5000}{q_9};$$

$$\mathcal{N}_{q_9}^{\Uparrow 0.5} = \frac{0.0833}{q_1} + \frac{0.5000}{q_2} + \frac{0.5833}{q_3} + \frac{0.2500}{q_4} + \frac{0.4167}{q_5} + \frac{0.5833}{q_6} + \frac{0.5000}{q_7} + \frac{0.5000}{q_8} + \frac{0.5000}{q_9}.$$

Also

$N_{q_1}^{\Uparrow 0.5} = \{q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9\}$, $N_{q_2}^{\Uparrow 0.5} = \{q_2, q_3, q_6, q_7, q_8, q_9\}$, $N_{q_3}^{\Uparrow 0.5} = \{q_3, q_6\}$, $N_{q_4}^{\Uparrow 0.5}$. $= \{q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9\}$, $N_{q_5}^{\Uparrow 0.5} = \{q_1, q_2, q_3, q_5, q_6, q_7, q_8, q_9\}$, $N_{q_6}^{\Uparrow 0.5}$ $= \{q_3, q_6\}$, $N_{q_7}^{\Uparrow 0.5} = \{q_2, q_3, q_6, q_7, q_8, q_9\}$, $N_{q_8}^{\Uparrow 0.5} = \{q_2, q_3, q_6, q_7, q_8, q_9\}$, $N_{q_9}^{\Uparrow 0.5}$ $= \{q_2, q_3, q_6, q_7, q_8, q_9\}$

Now let $\mathcal{X}_1 = \{q_3, q_6\}$. Then $(\mathcal{L}\mathrm{A}_\beta^{\bigtriangledown}(\mathcal{L}\mathrm{A}_\beta^{\bigtriangledown}(\mathcal{X}_1)))^{\Uparrow} = (\mathcal{L}\mathrm{A}_\beta^{\bigtriangledown}(\mathcal{X}_1))^{\Uparrow} = \{q_3, q_6\}$ and $(\mathcal{U}\mathcal{A}_\beta^{\bigtriangleup}(\mathcal{L}\mathrm{A}_\beta^{\bigtriangledown}(\mathcal{X}_1)))^{\Uparrow} = \{q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9\}$. Thus $(\mathcal{L}\mathrm{A}_\beta^{\bigtriangledown}(\mathcal{L}\mathrm{A}_\beta^{\bigtriangledown}(\mathcal{X}_1)))^{\Uparrow} =$

$(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1))^{\Uparrow} \neq (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1)))^{\Uparrow}$. Similarly we can get $(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1)))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1))^{\Uparrow} \neq (\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1)))^{\Uparrow}$.

From the above example we see that $(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1))^{\Uparrow}$ and $(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1))^{\Uparrow}$ are still rough sets, if we apply the lower or upper approximation operations over and over again to a subset $\mathcal{X}_1$, we obtain six different sets at most. These sets are

$$(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1))^{\Uparrow}, (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1)))^{\Uparrow}, (\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1))))^{\Uparrow}, (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1))^{\Uparrow},$$
$$(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1)))^{\Uparrow}, (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1))))^{\Uparrow}.$$

Theorem 9. *Let $(\mathcal{U}, \mathcal{P}(\mathcal{R}^{\Uparrow}))$ be a fuzzy upward covering approximation space, where $\mathcal{P}(\mathcal{R}^{\Uparrow})$ is a fuzzy upward β-covering of $\mathcal{U}$ for some $\beta \in (0, 1]$. For any crisp subset $\mathcal{X}_1$ of $\mathcal{U}$, the following properties hold*:

(i) $(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1)))))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1)))^{\Uparrow}$;

(ii) $(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1)))))^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1)))^{\Uparrow}$.

Proof.

(i) As we know that $(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1))^{\Uparrow} \subseteq (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1)))^{\Uparrow}$. This implies that $(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1)))^{\Uparrow} \subseteq (\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1))))^{\Uparrow}$. Therefore $(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1))^{\Uparrow} \subseteq (\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1))))^{\Uparrow}$. Also $(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1)))^{\Uparrow} \subseteq (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1)))))^{\Uparrow}$. Now $(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1))))^{\Uparrow} \subseteq (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1)))^{\Uparrow}$. This implies that $(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1)))))^{\Uparrow} \subseteq (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1))))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1)))^{\Uparrow}$.
Therefore $(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1)))))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1)))^{\Uparrow}$.

(ii) Now $(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1)))^{\Uparrow} \subseteq (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1))^{\Uparrow}$. Then $(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1))))^{\Uparrow} \subseteq (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1)))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1))^{\Uparrow}$. This implies that $(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1)))))^{\Uparrow} \subseteq (\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1)))^{\Uparrow}$.
Moreover, it follows from that $(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1)))^{\Uparrow} \subseteq (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1))))^{\Uparrow}$.
Also $(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1))))^{\Uparrow} \subseteq (\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1)))))^{\Uparrow}$. This implies that $(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1)))^{\Uparrow} \subseteq (\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1)))))^{\Uparrow}$.
Therefore $(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1)))))^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1)))^{\Uparrow}$. □

The following theorem gives the relationship between the aforesaid six sets.

Theorem 10. *Let $(\mathcal{U}, \mathcal{P}(\mathcal{R}^{\Uparrow}))$ be a fuzzy upward covering approximation space, where $\mathcal{P}(\mathcal{R}^{\Uparrow})$ is a fuzzy upward β-covering of $\mathcal{U}$ for some $\beta \in (0, 1]$. For any crisp subset $\mathcal{X}_1$ of $\mathcal{U}$, the following properties hold*:

(i) $(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1)))))^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1)))^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1))))^{\Uparrow}$;
(ii) $(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1)))))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1)))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1))))^{\Uparrow}$;
(iii) *If* $(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1))^{\Uparrow}$, *then* $(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1))^{\Uparrow} = \mathcal{X}_1 = (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1))^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1)))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{X}_1))))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1)))^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{UA}_{\beta}^{\bigtriangleup}(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mathcal{X}_1))))^{\Uparrow}$;

(iv) *If* $(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{X}_1))^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{X}_1))))^{\Uparrow}$, *then* $(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{X}_1))^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{X}_1)))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{X}_1))))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{X}_1)))^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{X}_1))))^{\Uparrow}$;

(v) *If* $(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{X}_1))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{X}_1))))^{\Uparrow}$, *then* $(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{X}_1))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{X}_1)))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{X}_1))))^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{X}_1)))^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{X}_1))))^{\Uparrow}$;

(vi) *If* $(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{X}_1))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{X}_1)))^{\Uparrow}$, *then* $(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{X}_1))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{X}_1)))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{X}_1))))^{\Uparrow}$ *and* $(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{X}_1)))^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{X}_1))))^{\Uparrow}$;

(vii) *If* $(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{X}_1))^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{X}_1)))^{\Uparrow}$, *then* $(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{X}_1))^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{X}_1)))^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{X}_1))))^{\Uparrow}$ *and* $(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{X}_1)))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{X}_1))))^{\Uparrow}$;

(viii) *If* $(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{X}_1))))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{X}_1))))^{\Uparrow}$, *then* $(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{X}_1)))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{X}_1))))^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{X}_1)))^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{X}_1))))^{\Uparrow}$;

(ix) *If* $(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{X}_1)))^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{X}_1)))^{\Uparrow}$, *then* $(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{X}_1)))^{\Uparrow} = (\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{X}_1))))^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{X}_1)))^{\Uparrow} = (\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{UA}_{\beta}^{\triangle}(\mathcal{L}\mathrm{A}_{\beta}^{\triangledown}(\mathcal{X}_1))))^{\Uparrow}$.

6 | MULTIGRANULATION $(\mathcal{I}, \mathcal{T})$-FUZZY UPWARD ROUGH SETS APPLYING FUZZY UPWARD $\boldsymbol{\beta}$-COVERING

Qian et al. generalized the Pawlak's single granulation rough set model to a multigranulation rough set model for finding two terminologies called optimistic/pessimistic multigranulation rough set models and discussed their applications in decision making process.[30,31] Qian et al.[32] further extended multigranulation methodology to decision theoretic rough sets and applied them to multi-criteria group decision making problem. In this section, we presented the idea to combine the fuzzy implicator and t-norm to introduce optimistic/pessimistic multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy upward rough sets using fuzzy upward β-covering approach and some relative properties are discussed.

Definition 11 (Radzikowska and Kerre[48]). Let $\mathcal{T}$: $[0, 1] \times [0, 1] \to [0, 1]$ be an increasing, associative and commutative mapping which satisfy the boundary condition, for all $q \in [0, 1]$, $\mathcal{T}(q, 1) = q$. Then $\mathcal{T}$ is called t-norm.

Similarly if $\mathcal{S}$: $[0, 1] \times [0, 1] \to [0, 1]$ is an increasing, associative and commutative mapping which satisfy the boundary condition, for all $q \in [0, 1]$, $\mathcal{S}(q, 0) = q$, then $\mathcal{S}$ is called t-conorm.

Let $\mathcal{I}$: $[0, 1] \times [0, 1] \to [0, 1]$ be a mapping which satisfy the conditions: $\mathcal{I}$ is left monotonic decreasing and right monotonic increasing; $\mathcal{I}(1, 0) = 0, \mathcal{I}(1, 1) = \mathcal{I}(0, 1) = \mathcal{I}(0, 0) = 1$. Then $\mathcal{I}$ is called the implicator.

Generally speaking, the main popular continuous t-norms and t-conorms are:

(i) $\mathcal{T}_M(q, y) = \min\{q, y\}$;
(ii) $\mathcal{S}_M(q, y) = \max\{q, y\}$;
(iii) $\mathcal{T}_P(q, y) = q * y$;
(iv) $\mathcal{S}_P(q, y) = q + y - q * y$;
(v) $\mathcal{T}_L(q, y) = \max\{0, q + y - 1\}$;
(vi) $\mathcal{S}_L(q, y) = \min\{1, q + y\}$.

Definition 12. Let $(\mathcal{U}, \mathcal{P}(\mathcal{R}^{\Uparrow}))$ be a fuzzy upward covering approximation space. Then for every fuzzy subset μ of $\mathcal{U}$, the lower approximation $(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\mu)^{\Uparrow}$ and the upper approximation $(\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}\mu)^{\Uparrow}$ of μ are defined by:

$$\left(\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\mu\right)^{\Uparrow}(q) = \bigwedge_{y\in\mathcal{U}}\left\{\left(1-\mathcal{N}_q^{\Uparrow\beta}(y)\right)\vee\mu(y)\right\},\quad q\in\mathcal{U},$$

$$\left(\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}\mu\right)^{\Uparrow}(q) = \bigvee_{y\in\mathcal{U}}\left\{\mathcal{N}_q^{\Uparrow\beta}(y)\wedge\mu(y)\right\},\quad q\in\mathcal{U}.$$

Definition 13. Let $(\mathcal{U}, \mathcal{P}(\mathbb{R}^{\Uparrow}))$ be a fuzzy upward covering approximation space and $\mathcal{P}(\mathbb{R}^{\Uparrow})$=$\{\mathcal{P}(\mathcal{R}_{c_1}^{\Uparrow}), \mathcal{P}(\mathcal{R}_{c_2}^{\Uparrow}), \mathcal{P}(\mathcal{R}_{c_3}^{\Uparrow}), \ldots, \mathcal{P}(\mathcal{R}_{c_l}^{\Uparrow})\}$ be l fuzzy upward β-covering of $\mathcal{U}$ with $\beta\in(0,1]$, where $\mathcal{P}(\mathcal{R}_{c_t}^{\Uparrow}) = \{\mathcal{A}_1^{\mathcal{R}_{c_t}^{\Uparrow}}, \mathcal{A}_2^{\mathcal{R}_{c_t}^{\Uparrow}}, \ldots, \mathcal{A}_n^{\mathcal{R}_{c_t}^{\Uparrow}}\}$ for all $t=1,2,\ldots,l$. Assume that ${}_{c_t}\mathcal{N}_q^{\Uparrow\beta}$ is a fuzzy upward β-neighborhood of q in $\mathcal{U}$ induced by c_t, $t=1,2,\ldots,l$. Then for all fuzzy set μ of the universe $\mathcal{U}$, we define the optimistic multigranulation $(\mathcal{I},\mathcal{T})$-fuzzy upward lower approximation $(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu))_1^{\mathcal{I}(o)\Uparrow}$ and the optimistic multigranulation $(\mathcal{I},\mathcal{T})$-fuzzy upward upper approximation $\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu)\right)_1^{\mathcal{T}(o)\Uparrow}$ by

$$\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_1^{\mathcal{I}(o)\Uparrow}(q) = \bigvee_{t=1}^{l}\ \bigwedge_{y\in\mathcal{U}}\ \mathcal{I}\left\{{}_{c_t}\mathcal{N}_q^{\Uparrow\beta}(y),\mu(y)\right\},\quad q\in\mathcal{U}$$

and

$$\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}\mu\right)_1^{\mathcal{T}(o)\Uparrow}(q) = \bigwedge_{t=1}^{l}\ \bigvee_{y\in\mathcal{U}}\ \mathcal{T}\left\{{}_{c_t}\mathcal{N}_q^{\Uparrow\beta}(y),\mu(y)\right\},\quad q\in\mathcal{U}.$$

The pair $\left(\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_1^{\mathcal{I}(o)\Uparrow}, \left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu)\right)_1^{\mathcal{T}(o)\Uparrow}\right)$ is called optimistic multigranulation $(\mathcal{I},\mathcal{T})$-fuzzy upward rough set of the fuzzy subset μ if $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_1^{\mathcal{I}(o)\Uparrow} \neq \left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu)\right)_1^{\mathcal{T}(o)\Uparrow}$. Otherwise μ is called a definable on multigranulation $(\mathcal{I},\mathcal{T})$-fuzzy upward approximation space.

(i) If $\mathcal{I}$ and $\mathcal{T}$ are the Kleene–Dienes implicator $\mathcal{I}_{KD}$ and standard min operator $\mathcal{T}_M$ based on $\mathcal{N}_S$ and $\mathcal{S}_M$, respectively, then the above definition become

$$\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_1^{\mathcal{I}(o)\Uparrow}(q) = \bigvee_{t=1}^{l}\ \bigwedge_{y\in\mathcal{U}}\ \left\{\left(1-{}_{c_t}\mathcal{N}_q^{\Uparrow\beta}(y)\right)\vee\mu(y)\right\},\quad q\in\mathcal{U}$$

and

$$\left(\sum_{t=1}^{l}\mathcal{UA}_{\beta}^{\triangle}(\mu)\right)_{1}^{\mathcal{T}(o)\Uparrow}(q)=\bigwedge_{t=1}^{l}\bigvee_{y\in\mathcal{U}}\left\{{}_{c_t}\mathcal{N}_{q}^{\Uparrow\beta}(y)\wedge\mu(y)\right\},\quad q\in\mathcal{U}.$$

This means that $\left(\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_{1}^{\mathcal{I}(o)\Uparrow},\left(\sum_{t=1}^{l}\mathcal{UA}_{\beta}^{\triangle}(\mu)\right)_{1}^{\mathcal{T}(o)\Uparrow}\right)$ is the optimistic multi-granulation fuzzy upward rough set of μ.

(ii) If ${}_{c_t}\mathcal{N}_{q}^{\Uparrow\beta}(y)=\mathcal{N}_{\mathbb{C}_i(q)}^{\beta}(y)$, then the above expression become:

$$\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_{1}^{\mathcal{I}(o)\Uparrow}(q)=\bigvee_{t=1}^{l}\bigwedge_{y\in\mathcal{U}}\mathcal{I}\left\{\mathcal{N}_{\mathbb{C}_i(q)}^{\beta}(y),\mu(y)\right\},\quad q\in\mathcal{U}$$

and

$$\left(\sum_{t=1}^{l}\mathcal{UA}_{\beta}^{\triangle}(\mu)\right)_{1}^{\mathcal{T}(o)\Uparrow}(q)=\bigwedge_{t=1}^{l}\bigvee_{y\in\mathcal{U}}\mathcal{T}\left\{\mathcal{N}_{\mathbb{C}_i(q)}^{\beta}(y),\mu(y)\right\},\quad q\in\mathcal{U}.$$

This means that $\left(\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_{1}^{\mathcal{I}(o)\Uparrow},\left(\sum_{t=1}^{l}\mathcal{UA}_{\beta}^{\triangle}(\mu)\right)_{1}^{\mathcal{T}(o)\Uparrow}\right)$ is the optimistic multi-granulation fuzzy rough set of μ as proposed by Zhan et al.[40]

(iii) If $c=c_1=c_2=\cdots=c_l$ or $m=1$, then the above definition as follows:

$$\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_{1}^{\mathcal{I}(o)\Uparrow}(q)=\bigwedge_{y\in\mathcal{U}}\left\{\left(1-{}_{c}\mathcal{N}_{q}^{\Uparrow\beta}(y)\right)\vee\mu(y)\right\},\quad q\in\mathcal{U}$$

and

$$\left(\sum_{t=1}^{l}\mathcal{UA}_{\beta}^{\triangle}(\mu)\right)_{1}^{\mathcal{T}(o)\Uparrow}(q)=\bigvee_{y\in\mathcal{U}}\left\{{}_{c}\mathcal{N}_{q}^{\Uparrow\beta}(y)\wedge\mu(y)\right\},\quad q\in\mathcal{U}.$$

This means that $\left(\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_{1}^{\mathcal{I}(o)\Uparrow},\left(\sum_{t=1}^{l}\mathcal{UA}_{\beta}^{\triangle}(\mu)\right)_{1}^{\mathcal{T}(o)\Uparrow}\right)$ is the fuzzy upward rough set of μ.

(iv) If ${}_{c}\mathcal{N}_{q}^{\Uparrow\beta}(y)=\mathcal{N}_{\mathbb{C}(q)}^{\beta}(y)$, then the above expression become as follows:

$$\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_{1}^{\mathcal{I}(o)\Uparrow}(q)=\bigwedge_{y\in\mathcal{U}}\left\{\left(1-\mathcal{N}_{\mathbb{C}(q)}^{\beta}(y)\right)\vee\mu(y)\right\},\quad q\in\mathcal{U}$$

and

$$\left(\sum_{t=1}^{l}\mathcal{UA}_{\beta}^{\triangle}(\mu)\right)_{1}^{\mathcal{T}(o)\Uparrow}(q)=\bigvee_{y\in\mathcal{U}}\left\{\mathcal{N}_{\mathbb{C}(q)}^{\beta}(y)\wedge\mu(y)\right\},\quad q\in\mathcal{U}.$$

This means that $\left(\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_{1}^{\mathcal{I}(o)\Uparrow},\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu)\right)_{1}^{\mathcal{T}(o)\Uparrow}\right)$ is the fuzzy rough set of μ as proposed by Ma.[38]

Remark 1. Let $\mathcal{I}$ be an implicator $\mathcal{T}_1$ and $\mathcal{T}_2$ be t-norms and $\mathcal{S}$ be a t-conorm. Then the following hold:

(C_1) $\mathcal{I}(q,\mathcal{T}_1(y,z)) \geq \mathcal{T}_1(\mathcal{I}(q,y),\mathcal{I}(q,z))\,\forall q,y,z \in [0,1]$. If $\mathcal{T}_1=\mathcal{T}_M$ and $\mathcal{I}$ is right monotonic, then the inequality will be equality.
(C_2) $\mathcal{T}_1(q,\mathcal{T}_2(y,z)) \geq \mathcal{T}_2(\mathcal{T}_1(q,y),\mathcal{T}_1(q,z))\,\forall q,y,z \in [0,1]$. If $\mathcal{T}_1=\mathcal{T}_M$, then the inequality will be equality.
(C_3) $\mathcal{I}(q,\mathcal{S}(y,z)) \geq \mathcal{S}(\mathcal{I}(q,y),\mathcal{I}(q,z))\,\forall q,y,z \in [0,1]$. If $\mathcal{S}=\mathcal{S}_M$ and $\mathcal{I}$ is right monotonic, then the inequality will be equality.

Theorem 11. *Let $(\mathcal{U},\mathcal{P}(\mathbb{R}^{\Uparrow}))$ be a fuzzy upward covering approximation space and $\mathcal{P}(\mathbb{R}^{\Uparrow})=\{\mathcal{P}(\mathcal{R}_{c_1}^{\Uparrow}),\mathcal{P}(\mathcal{R}_{c_2}^{\Uparrow}),\mathcal{P}(\mathcal{R}_{c_3}^{\Uparrow}),\ldots,\mathcal{P}(\mathcal{R}_{c_l}^{\Uparrow})\}$ be l fuzzy upward β-covering of $\mathcal{U}$ with $\beta\in(0,1]$, where $\mathcal{P}(\mathcal{R}_{c_t}^{\Uparrow})=\{\mathcal{A}_1^{\mathcal{R}_{c_t}^{\Uparrow}},\mathcal{A}_2^{\mathcal{R}_{c_t}^{\Uparrow}},\ldots,\mathcal{A}_n^{\mathcal{R}_{c_t}^{\Uparrow}}\}$ for all $t=1,2,\ldots,l$. Assume that ${}_{c_t}\mathcal{N}_q^{\Uparrow\beta}$ is a fuzzy upward β-neighborhood of q in $\mathcal{U}$ induced by c_t, $t=1,2,\ldots,l$. If μ,λ are fuzzy sets in $\mathcal{U}$, then the following hold:*

(i) *If $\mu\subseteq\lambda$ and $\mathcal{I}$ is right monotonic, then* $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_{1}^{\mathcal{I}(o)\Uparrow}\subseteq\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\lambda)\right)_{1}^{\mathcal{I}(o)\Uparrow}$;

(ii) *If $\mu\subseteq\lambda$, then* $\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu)\right)_{1}^{\mathcal{T}(o)\Uparrow}\subseteq\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\lambda)\right)_{1}^{\mathcal{T}(o)\Uparrow}$;

(iii) *If $\mathcal{I}$ and $\mathcal{T}_1$ satisfy the condition (C_1), then* $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_{1}^{\mathcal{I}(o)\Uparrow}\cap_{\mathcal{T}_1}\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\lambda)\right)_{1}^{\mathcal{I}(o)\Uparrow}\subseteq\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu\cap_{\mathcal{T}_1}\lambda)\right)_{1}^{\mathcal{I}(o)\Uparrow}$;

(iv) *If $\mathcal{I}$ is right monotonic, then* $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_{1}^{\mathcal{I}(o)\Uparrow}\cap\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\lambda)\right)_{1}^{\mathcal{I}(o)\Uparrow}=\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu\cap\lambda)\right)_{1}^{\mathcal{I}(o)\Uparrow}$;

(v) *If $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy the condition (C_2), then* $\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu\cap_{\mathcal{T}_2}\lambda)\right)_{1}^{\mathcal{T}_1(o)\Uparrow}\subseteq\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu)\right)_{1}^{\mathcal{T}_1(o)\Uparrow}\cap_{\mathcal{T}_2}\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\lambda)\right)_{1}^{\mathcal{T}_1(o)\Uparrow}$;

(vi) *If $\mathcal{I}$ and $\mathcal{S}$ satisfy condition (C_3), then* $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_{1}^{\mathcal{I}(o)\Uparrow}\cup_{\mathcal{S}}\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\lambda)\right)_{1}^{\mathcal{I}(o)\Uparrow}\subseteq\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\left(\mu\cup_{\mathcal{S}}\lambda\right)\right)_{1}^{\mathcal{I}(o)\Uparrow}$;

(vii) *If $\mathcal{T}_1$ and $\mathcal{S}$ satisfy the weak distributivity laws, then* $\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu\cup_{\mathcal{S}}\lambda)\right)_{1}^{\mathcal{T}_1(o)\Uparrow}\subseteq\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu)\right)_{1}^{\mathcal{T}_1(o)\Uparrow}\cup_{\mathcal{S}}\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\lambda)\right)_{1}^{\mathcal{T}_1(o)\Uparrow}$.

Definition 14. Let $(\mathcal{U},\mathcal{P}(\mathbb{R}^{\Uparrow}))$ be a fuzzy upward covering approximation space and $\mathcal{P}(\mathbb{R}^{\Uparrow})=\{\mathcal{P}(\mathcal{R}_{c_1}^{\Uparrow}),\mathcal{P}(\mathcal{R}_{c_2}^{\Uparrow}),\mathcal{P}(\mathcal{R}_{c_3}^{\Uparrow}),\ldots,\mathcal{P}(\mathcal{R}_{c_l}^{\Uparrow})\}$ be l fuzzy upward β-covering of $\mathcal{U}$ with $\beta\in(0,1]$, where $\mathcal{P}(\mathcal{R}_{c_t}^{\Uparrow})=\{\mathcal{A}_1^{\mathcal{R}_{c_t}^{\Uparrow}},\mathcal{A}_2^{\mathcal{R}_{c_t}^{\Uparrow}},\ldots,\mathcal{A}_n^{\mathcal{R}_{c_t}^{\Uparrow}}\}$ for all $t=1,2,\ldots,l$. Assume that ${}_{c_t}\mathcal{N}_q^{\Uparrow\beta}$

is a fuzzy upward β-neighborhood of q in $\mathcal{U}$ induced by c_t, $t = 1, 2, ..., l$. Then for all fuzzy set μ in the universe $\mathcal{U}$, we define the pessimistic multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy upward lower approximation $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_1^{\mathcal{I}(p)\Uparrow}$ and the pessimistic multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy upward upper approximation $\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\triangle}(\mu)\right)_1^{\mathcal{T}(p)\Uparrow}$ of μ as:

$$\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_1^{\mathcal{I}(p)\Uparrow}(q) = \bigwedge_{t=1}^{l}\bigwedge_{y\in\mathcal{U}} \mathcal{I}\left\{{}_{c_t}\mathcal{N}_q^{\Uparrow\beta}(y), \mu(y)\right\}, \quad q \in \mathcal{U}$$

and

$$\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\triangle}(\mu)\right)_1^{\mathcal{T}(p)\Uparrow}(q) = \bigvee_{t=1}^{l}\bigvee_{y\in\mathcal{U}} \mathcal{T}\left\{{}_{c_t}\mathcal{N}_q^{\Uparrow\beta}(y), \mu(y)\right\}, \quad q \in \mathcal{U}.$$

The pair $\left(\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_1^{\mathcal{I}(p)\Uparrow}, \left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\triangle}(\mu)\right)_1^{\mathcal{T}(p)\Uparrow}\right)$ is called pessimistic multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy upward rough set of the fuzzy subset μ if $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_1^{\mathcal{I}(p)\Uparrow} \neq \left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\triangle}(\mu)\right)_1^{\mathcal{T}(p)\Uparrow}$. Otherwise μ is called definable on multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy upward approximation space.

(i) If $\mathcal{I}$ and $\mathcal{T}$ are the Kleene–Dienes implicator $\mathcal{I}_{KD}$ and standard min operator $\mathcal{T}_M$ based on $\mathcal{N}_S$ and $\mathcal{S}_M$, respectively, then from the above definition it follows that

$$\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_1^{\mathcal{I}(p)\Uparrow}(q) = \bigwedge_{t=1}^{l}\bigwedge_{y\in\mathcal{U}} \left\{\left(1 - {}_{c_t}\mathcal{N}_q^{\Uparrow\beta}(y)\right) \vee \mu(y)\right\}, \quad q \in \mathcal{U}$$

and

$$\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\triangle}(\mu)\right)_1^{\mathcal{T}(p)\Uparrow}(q) = \bigvee_{t=1}^{l}\bigvee_{y\in\mathcal{U}} \left\{{}_{c_t}\mathcal{N}_q^{\Uparrow\beta}(y) \wedge \mu(y)\right\}, \quad q \in \mathcal{U}.$$

This means that $\left(\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_1^{\mathcal{I}(p)\Uparrow}, \left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\triangle}(\mu)\right)_1^{\mathcal{T}(p)\Uparrow}\right)$ becomes the pessimistic multigranulation fuzzy upward rough set of μ.

If ${}_{c_t}\mathcal{N}_q^{\Uparrow\beta}(y) = \mathcal{N}_{\mathbb{C}_i(q)}^{\beta}(y)$, then the above expression becomes as

$$\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_1^{\mathcal{I}(p)\Uparrow}(q) = \bigwedge_{t=1}^{l}\bigwedge_{y\in\mathcal{U}} \mathcal{I}\left\{\mathcal{N}_{\mathbb{C}_i(q)}^{\beta}(y), \mu(y)\right\}, \quad q \in \mathcal{U}$$

and

$$\left(\sum_{t=1}^{l}\mathcal{UA}_{\beta}^{\triangle}(\mu)\right)_1^{\mathcal{T}(p)\Uparrow}(q)=\bigvee_{t=1}^{l}\bigvee_{y\in\mathcal{U}}\mathcal{T}\left\{\mathcal{N}_{\mathbb{C}_i(q)}^{\beta}(y),\mu(y)\right\},\quad q\in\mathcal{U}.$$

This means that $\left(\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_1^{\mathcal{I}(p)\Uparrow},\left(\sum_{t=1}^{l}\mathcal{UA}_{\beta}^{\triangle}(\mu)\right)_1^{\mathcal{T}(p)\Uparrow}\right)$ becomes the pessimistic multigranulation fuzzy rough set of μ as proposed by Zhan et al.[40]

Theorem 12. *Let $(\mathcal{U},\mathcal{P}(\mathbb{R}^{\Uparrow}))$ be a fuzzy upward covering approximation space and $\mathcal{P}(\mathbb{R}^{\Uparrow})=\{\mathcal{P}(\mathcal{R}_{c_1}^{\Uparrow}),\mathcal{P}(\mathcal{R}_{c_2}^{\Uparrow}),\mathcal{P}(\mathcal{R}_{c_3}^{\Uparrow}),\ldots,\mathcal{P}(\mathcal{R}_{c_l}^{\Uparrow})\}$ be l fuzzy upward β-covering of $\mathcal{U}$ with $\beta\in(0,1]$, where $\mathcal{P}(\mathcal{R}_{c_t}^{\Uparrow})=\{\mathcal{A}_1^{\mathcal{R}_{c_t}^{\Uparrow}},\mathcal{A}_2^{\mathcal{R}_{c_t}^{\Uparrow}},\ldots,\mathcal{A}_n^{\mathcal{R}_{c_t}^{\Uparrow}}\}$ for all $t=1,2,\ldots,l$. Assume that ${}_{c_t}\mathcal{N}_q^{\Uparrow\beta}$ is a fuzzy upward β-neighborhood of q in $\mathcal{U}$ induced by c_t, $t=1,2,\ldots,l$. If μ and λ are fuzzy sets in $\mathcal{U}$, then the following hold:*

(i) *If $\mu\subseteq\lambda$ and $\mathcal{I}$ is right monotonic, then* $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_1^{\mathcal{I}(p)\Uparrow}\subseteq\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\lambda)\right)_1^{\mathcal{I}(p)\Uparrow}$;

(ii) *If $\mu\subseteq\lambda$, then* $\left(\sum_{t=1}^{l}\mathcal{UA}_{\beta}^{\triangle}(\mu)\right)_1^{\mathcal{T}(p)\Uparrow}\subseteq\left(\sum_{t=1}^{l}\mathcal{UA}_{\beta}^{\triangle}(\lambda)\right)_1^{\mathcal{T}(p)\Uparrow}$;

(iii) *If $\mathcal{I}$ and $\mathcal{T}_1$ satisfied condition (C_1), then* $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_1^{\mathcal{I}(p)\Uparrow}\cap_{\mathcal{T}_1}\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\lambda)\right)_1^{\mathcal{I}(p)\Uparrow}\subseteq\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\left(\mu\cap_{\mathcal{T}_1}\lambda\right)\right)_1^{\mathcal{I}(p)\Uparrow}$;

(iv) *If $\mathcal{I}$ is right monotonic, then* $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_1^{\mathcal{I}(p)\Uparrow}\cap\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\lambda)\right)_1^{\mathcal{I}(p)\Uparrow}=\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu\cap\lambda)\right)_1^{\mathcal{I}(p)\Uparrow}$;

(v) *If $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy the condition (C_2), then* $\left(\sum_{t=1}^{l}\mathcal{UA}_{\beta}^{\triangle}(\mu\cap_{\mathcal{T}_2}\lambda)\right)_1^{\mathcal{T}_1(p)\Uparrow}\subseteq\left(\sum_{t=1}^{l}\mathcal{UA}_{\beta}^{\triangle}(\mu)\right)_1^{\mathcal{T}_1(p)\Uparrow}\cap_{\mathcal{T}_2}\left(\sum_{t=1}^{l}\mathcal{UA}_{\beta}^{\triangle}(\lambda)\right)_1^{\mathcal{T}_1(p)\Uparrow}$;

(vi) *If $\mathcal{I}$ and $\mathcal{S}$ satisfied condition (C_3), then* $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_1^{\mathcal{I}(p)\Uparrow}\cup_{\mathcal{S}}\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\lambda)\right)_1^{\mathcal{I}(p)\Uparrow}\subseteq\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu\cup_{\mathcal{S}}\lambda)\right)_1^{\mathcal{I}(p)\Uparrow}$;

(vii) If $\mathcal{T}_1$ and $\mathcal{S}$ satisfy the weak distributivity laws, then $\left(\sum_{t=1}^{l}\mathcal{UA}_{\beta}^{\triangle}(\mu\cup_{\mathcal{S}}\lambda)\right)_1^{\mathcal{T}_1(p)\Uparrow}\subseteq\left(\sum_{t=1}^{l}\mathcal{UA}_{\beta}^{\triangle}(\mu)\right)_1^{\mathcal{T}_1(p)\Uparrow}\cup_{\mathcal{S}}\left(\sum_{t=1}^{l}\mathcal{UA}_{\beta}^{\triangle}(\lambda)\right)_1^{\mathcal{T}_1(p)\Uparrow}$.

Definition 15. Let $(\mathcal{U},\mathcal{P}(\mathbb{R}^{\Uparrow}))$ be a fuzzy upward covering approximation space and $\mathcal{P}(\mathbb{R}^{\Uparrow})=\{\mathcal{P}(\mathcal{R}_{c_1}^{\Uparrow}),\mathcal{P}(\mathcal{R}_{c_2}^{\Uparrow}),\mathcal{P}(\mathcal{R}_{c_3}^{\Uparrow}),\ldots,\mathcal{P}(\mathcal{R}_{c_l}^{\Uparrow})\}$ be l fuzzy upward β-covering of $\mathcal{U}$ with $\beta\in(0,1]$, where $\mathcal{P}(\mathcal{R}_{c_t}^{\Uparrow})=\{\mathcal{A}_1^{\mathcal{R}_{c_t}^{\Uparrow}},\mathcal{A}_2^{\mathcal{R}_{c_t}^{\Uparrow}},\ldots,\mathcal{A}_n^{\mathcal{R}_{c_t}^{\Uparrow}}\}$, for all $t=1,2,\ldots,l$. Assume that the fuzzy complementary β-upward neighborhood $\mathcal{M}_q^{\Uparrow\beta}$ of q in $\mathcal{U}$ induced by c_t, $t=1,2,\ldots,l$. Then for all fuzzy set μ in the universe $\mathcal{U}$, define the optimistic multigranulation $(\mathcal{I},\mathcal{T})$-fuzzy upward lower approximation $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\mu\right)_2^{\mathcal{I}(o)\Uparrow}$ and the optimistic multigranulation $(\mathcal{I},\mathcal{T})$-fuzzy upward upper approximation $\left(\sum_{t=1}^{l}\mathcal{UA}_{\beta}^{\triangle}\mu\right)_2^{\mathcal{T}(o)\Uparrow}$ of μ by:

$$\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\mu\right)_{2}^{\mathcal{I}(o)\Uparrow}(q)=\bigvee_{t=1}^{l}\bigwedge_{y\in\mathcal{U}}\mathcal{I}\left\{_{c_t}\mathcal{M}_{q}^{\Uparrow\beta}(y),\mu(y)\right\},\quad q\in\mathcal{U}$$

and

$$\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}\mu\right)_{2}^{\mathcal{T}(o)\Uparrow}(q)=\bigwedge_{t=1}^{l}\bigvee_{y\in\mathcal{U}}\mathcal{T}\left\{_{c_t}\mathcal{M}_{q}^{\Uparrow\beta}(y),\mu(y)\right\},\quad q\in\mathcal{U}.$$

The pair $\left(\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_{2}^{\mathcal{I}(o)\Uparrow},\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu)\right)_{2}^{\mathcal{T}(o)\Uparrow}\right)$ is called optimistic multigranulation $(\mathcal{I},\mathcal{T})$-fuzzy upward rough set of the fuzzy set μ if $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_{2}^{\mathcal{I}(o)\Uparrow}\neq\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu)\right)_{2}^{\mathcal{T}(o)\Uparrow}$. Otherwise μ is called a definable on multigranulation $(\mathcal{I},\mathcal{T})$-fuzzy upward approximation space.

Theorem 13. *Let $(\mathcal{U},\mathcal{P}(\mathbb{R}^{\Uparrow}))$ be a fuzzy upward covering approximation space and $\mathcal{P}(\mathbb{R}^{\Uparrow})=\{\mathcal{P}(\mathcal{R}_{c_1}^{\Uparrow}),\mathcal{P}(\mathcal{R}_{c_2}^{\Uparrow}),\mathcal{P}(\mathcal{R}_{c_3}^{\Uparrow}),\ldots,\mathcal{P}(\mathcal{R}_{c_l}^{\Uparrow})\}$ be l fuzzy upward β-covering of $\mathcal{U}$ with $\beta\in(0,1]$, where $\mathcal{P}(\mathcal{R}_{c_t}^{\Uparrow})=\{\mathcal{A}_1^{\mathcal{R}_{c_t}^{\Uparrow}},\mathcal{A}_2^{\mathcal{R}_{c_t}^{\Uparrow}},\ldots,\mathcal{A}_n^{\mathcal{R}_{c_t}^{\Uparrow}}\}$, for all $t=1,2,\ldots,l$. Assume that the fuzzy complementary β-upward neighborhood $\mathcal{M}_q^{\Uparrow\beta}$ of q in $\mathcal{U}$ induced by c_t, $t=1,2,\ldots,l$. If μ,λ are fuzzy sets in $\mathcal{U}$, then the following hold:*

(i) *If $\mu\subseteq\lambda$ and $\mathcal{I}$ is right monotonic, then $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\mu\right)_{2}^{\mathcal{I}(o)\Uparrow}\subseteq\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\lambda\right)_{2}^{\mathcal{I}(o)\Uparrow}$;*

(ii) *If $\mu\subseteq\lambda$, then $\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}\mu\right)_{2}^{\mathcal{T}(o)\Uparrow}\subseteq\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}\lambda\right)_{2}^{\mathcal{T}(o)\Uparrow}$;*

(iii) *If $\mathcal{I}$ and $\mathcal{T}_1$ satisfied condition (C_1), then $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\mu\right)_{2}^{\mathcal{I}(o)\Uparrow}\cap_{\mathcal{T}_1}\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\lambda\right)_{2}^{\mathcal{I}(o)\Uparrow}\subseteq\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu\cap_{\mathcal{T}_1}\lambda)\right)_{2}^{\mathcal{I}(o)\Uparrow}$;*

(iv) *If $\mathcal{I}$ is right monotonic, then $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\mu\right)_{2}^{\mathcal{I}(o)\Uparrow}\cap\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\lambda\right)_{2}^{\mathcal{I}(o)\Uparrow}=\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu\cap\lambda)\right)_{2}^{\mathcal{I}(o)\Uparrow}$;*

(v) *If $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy the condition (C_2), then $\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu\cap_{\mathcal{T}_2}\lambda)\right)_{2}^{\mathcal{T}_1(o)\Uparrow}\subseteq\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu)\right)_{2}^{\mathcal{T}_1(o)\Uparrow}\cap_{\mathcal{T}_2}\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\lambda)\right)_{2}^{\mathcal{T}_1(o)\Uparrow}$;*

(vi) *If $\mathcal{I}$ and $\mathcal{S}$ satisfied condition (C_3), then $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\mu\right)_{2}^{\mathcal{I}(o)\Uparrow}\cup_{\mathcal{S}}\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\lambda\right)_{2}^{\mathcal{I}(o)\Uparrow}\subseteq\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu\cup_{\mathcal{S}}\lambda)\right)_{2}^{\mathcal{I}(o)\Uparrow}$;*

(vii) If $\mathcal{T}_1$ and $\mathcal{S}$ satisfy the weak distributive laws, then $\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu\cup_{\mathcal{S}}\lambda)\right)_{2}^{\mathcal{T}_1(o)\Uparrow}\subseteq\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu)\right)_{2}^{\mathcal{T}_1(o)\Uparrow}\cup_{\mathcal{S}}\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\lambda)\right)_{2}^{\mathcal{T}_1(o)\Uparrow}$.

Definition 16. Let $(\mathcal{U},\mathcal{P}(\mathbb{R}^{\Uparrow}))$ be a fuzzy upward covering approximation space and $\mathcal{P}(\mathbb{R}^{\Uparrow})=\{\mathcal{P}(\mathcal{R}_{c_1}^{\Uparrow}),\mathcal{P}(\mathcal{R}_{c_2}^{\Uparrow}),\mathcal{P}(\mathcal{R}_{c_3}^{\Uparrow}),\ldots,\mathcal{P}(\mathcal{R}_{c_l}^{\Uparrow})\}$ be l fuzzy upward β-covering of $\mathcal{U}$ with

$\beta \in (0, 1]$, where $\mathcal{P}(\mathcal{R}^{\Uparrow}_{c_t}) = \{\mathcal{A}_1^{\mathcal{R}^{\Uparrow}_{c_t}}, \mathcal{A}_2^{\mathcal{R}^{\Uparrow}_{c_t}}, ..., \mathcal{A}_n^{\mathcal{R}^{\Uparrow}_{c_t}}\}$, for all $t = 1, 2, ..., l$. Assume that the fuzzy complementary β-upward neighborhood $\mathcal{M}_q^{\Uparrow\beta}$ of q in $\mathcal{U}$ induced by c_t, $t = 1, 2, ..., l$. Then for all fuzzy set μ in the universe $\mathcal{U}$, define the pessimistic multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy upward lower approximation $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\mu\right)_2^{\mathcal{I}(p)\Uparrow}$ and the pessimistic multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy upward upper approximation $\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}\mu\right)_2^{\mathcal{T}(p)\Uparrow}$ of μ by:

$$\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\mu\right)_2^{\mathcal{I}(p)\Uparrow}(q) = \bigwedge_{t=1}^{l}\bigwedge_{y\in\mathcal{U}}\mathcal{I}\left\{{}_{c_t}\mathcal{M}_q^{\Uparrow\beta}(y), \mu(y)\right\}, \quad q\in\mathcal{U}$$

and

$$\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}\mu\right)_2^{\mathcal{T}(p)\Uparrow}(q) = \bigvee_{t=1}^{l}\bigvee_{y\in\mathcal{U}}\mathcal{T}\left\{{}_{c_t}\mathcal{M}_q^{\Uparrow\beta}(y), \mu(y)\right\}, \quad q\in\mathcal{U}.$$

The pair $\left(\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_2^{\mathcal{I}(p)\Uparrow}, \left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu)\right)_2^{\mathcal{T}(p)\Uparrow}\right)$ is called pessimistic multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy upward rough set of the fuzzy subset μ if $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu)\right)_2^{\mathcal{I}(p)\Uparrow} \neq \left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu)\right)_2^{\mathcal{T}(p)\Uparrow}$. Otherwise μ is called definable on multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy upward approximation space.

Theorem 14. *Let $(\mathcal{U}, \mathcal{P}(\mathbb{R}^{\Uparrow}))$ be a fuzzy upward covering approximation space and $\mathcal{P}(\mathbb{R}^{\Uparrow})=\{\mathcal{P}(\mathcal{R}^{\Uparrow}_{c_1}), \mathcal{P}(\mathcal{R}^{\Uparrow}_{c_2}), \mathcal{P}(\mathcal{R}^{\Uparrow}_{c_3}), ..., \mathcal{P}(\mathcal{R}^{\Uparrow}_{c_l})\}$ be l fuzzy upward β-covering of $\mathcal{U}$ with $\beta \in (0, 1]$, where $\mathcal{P}(\mathcal{R}^{\Uparrow}_{c_t}) = \{\mathcal{A}_1^{\mathcal{R}^{\Uparrow}_{c_t}}, \mathcal{A}_2^{\mathcal{R}^{\Uparrow}_{c_t}}, ..., \mathcal{A}_n^{\mathcal{R}^{\Uparrow}_{c_t}}\}$, for all $t = 1, 2, ..., l$. Assume that the fuzzy complementary β-upward neighborhood $\mathcal{M}_q^{\Uparrow\beta}$ of q in $\mathcal{U}$ induced by c_t, $t = 1, 2, ..., l$. If μ and λ are fuzzy sets in $\mathcal{U}$, the following hold:*

(i) *If $\mu \subseteq \lambda$ and $\mathcal{I}$ is right monotonic, then $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\mu\right)_2^{\mathcal{I}(p)\Uparrow} \subseteq \left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\lambda\right)_2^{\mathcal{I}(p)\Uparrow}$;*

(ii) *If $\mu \subseteq \lambda$, then $\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}\mu\right)_2^{\mathcal{T}(p)\Uparrow} \subseteq \left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}\lambda\right)_2^{\mathcal{T}(p)\Uparrow}$;*

(iii) *If $\mathcal{I}$ and $\mathcal{T}_1$ satisfied condition (C_1), then $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\mu\right)_2^{\mathcal{I}(p)\Uparrow} \cap_{\mathcal{T}_1} \left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\lambda\right)_2^{\mathcal{I}(p)\Uparrow} \subseteq \left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu \cap_{\mathcal{T}_1} \lambda)\right)_2^{\mathcal{I}(p)\Uparrow}$;*

(iv) *If $\mathcal{I}$ is right monotonic, then $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\mu\right)_2^{\mathcal{I}(p)\Uparrow} \cap \left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\lambda\right)_2^{\mathcal{I}(p)\Uparrow} = \left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}(\mu \cap \lambda)\right)_2^{\mathcal{I}(p)\Uparrow}$;*

(v) *If $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy the condition (C_2), then $\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu \cap_{\mathcal{T}_2} \lambda)\right)_2^{\mathcal{T}_1(p)\Uparrow} \subseteq \left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu)\right)_2^{\mathcal{T}_1(p)\Uparrow} \cap_{\mathcal{T}_2} \left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\lambda)\right)_2^{\mathcal{T}_1(p)\Uparrow}$;*

(vi) *If $\mathcal{I}$ and $\mathcal{S}$ satisfied condition* (C_3), *then* $\left(\sum_{t=1}^{l} \mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\mu\right)_2^{\mathcal{I}(p)\Uparrow} \cup_{\mathcal{S}} \left(\sum_{t=1}^{l} \mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\lambda\right)_2^{\mathcal{I}(p)\Uparrow} \subseteq \left(\sum_{t=1}^{l} \mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\left(\mu \cup_{\mathcal{S}} \lambda\right)\right)_2^{\mathcal{I}(p)\Uparrow}$;

(vii) If $\mathcal{T}_1$ and $\mathcal{S}$ satisfy the weak distributive laws, then $\left(\sum_{t=1}^{l} \mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu \cup_{\mathcal{S}} \lambda)\right)_2^{\mathcal{T}_1(p)\Uparrow} \subseteq \left(\sum_{t=1}^{l} \mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\mu)\right)_2^{\mathcal{T}_1(p)\Uparrow} \cup_{\mathcal{S}} \left(\sum_{t=1}^{l} \mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}(\lambda)\right)_2^{\mathcal{T}_1(p)\Uparrow}$.

7 | ALGORITHM FOR DECISION MAKING PROBLEM

Over the years, numerous decision making procedures have been introduced in the literature, of which technique for order preference by similarity to ideal solution (TOPSIS) is one of the extensively and efficiently used famous methods. Hwang and Yoon[49] presented the TOPSIS to deal multiattribute decision making problems. According to which the alternative is the smallest distance from the positive ideal solution and the furthest distance from the negative ideal solution in decision making problems is the best alternative.

Some advantages of fuzzy TOPSIS are:

(i) When the fuzzy analytic hierarchy process (AHP) and fuzzy TOPSIS methods are compared with respect to the amount of computations, fuzzy AHP requires more complex computations than fuzzy TOPSIS.
(ii) Pairwise comparisons for criteria, sub criteria and alternatives are made in fuzzy AHP, while there is no pairwise comparison in fuzzy TOPSIS which are in fact based on their relative distances to positive ideal solution and negative ideal solutions.
(iii) TOPSIS has been proved to be one of the best methods addressing rank reversal issue that is the change in the ranking of the alternatives when a nonoptimal alternative is introduced.
(iv) Zhan et al. proposed TOPSIS method based on generalized fuzzy rough sets. While comparing their proposed model with other models they achieved more correct ranking of the alternatives than other models (See tables 15 and 16 of Reference [50]).

Now we present an approach for solving multiattributes decision making problem of medicine selections under the environment of multigranulation $(\mathcal{I}, \mathcal{T})$ fuzzy upward rough sets. For this, let $\mathcal{U} = \{q_j: j = 1, 2, ..., n\}$ be the set of n alternatives which are evaluated by the different attributes/tests, where $\mathrm{C} = \{c_1, c_2, ..., c_l\}$ is the collection of different attributes/tests. The unknown weight vector of l attributes is denoted by $W = (\omega_1, \omega_2, ..., \omega_l)^T$ with subject to $\omega_i \geq 0$ for $i = 1, 2, ..., l$ such that $\sum_{i=1}^{l} \omega_i = 1$. Let E be a finite set of the domain for the information function $g(q_i, c_t)$, where $g(q_i, c_t) \in [0, 1]$. Here we present the fuzzy information system $(\mathcal{U}, \mathrm{C}, W, E)$.

7.1 | The steps of the decision making model

To find the most suitable medicine among the given ones, we initiate an algorithm based on the proposed multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy upward rough sets applying fuzzy upward β-covering approach and their corresponding steps are compiled as follows:

Input: Given fuzzy information system $(\mathcal{U}, \mathrm{C}, W, E)$;

Step 1: Using the proposed transfer function for the construction of $\mathcal{R}_{c_t}^{\Uparrow}$, where $t = 1, 2, \ldots, m$;

Step 2: Compute ${}_{c_t}\mathcal{N}_{q_j}^{\Uparrow\beta}$ of q_j with respect to c_t;

Step 3: Apply the principle of fuzzy TOPSIS method to compute the individual's best and worst fuzzy decision making objects ${}_{c_t}\mathcal{X}_1^{+}$ and ${}_{c_t}\mathcal{X}_1^{-}$ where

$$ {}_{c_t}\mathcal{X}_1^{+} = \max_{1\le j\le n}\left\{\mathcal{A}_i^{\mathcal{R}_{c_t}\Uparrow}(q_j) : i = 1, 2, \ldots, n\right\} \tag{7}$$

and

$$ {}_{c_t}\mathcal{X}_1^{-} = \min_{1\le j\le n}\left\{\mathcal{A}_i^{\mathcal{R}_{c_t}\Uparrow}(q_j) : i = 1, 2, \ldots, n\right\}, \tag{8}$$

where $t = 1, 2, \ldots, l$.

Step 4: Integrate the following approximations $(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}({}_{c_t}\mathcal{X}_1^{+}))_1^{\mathcal{I}(o)\Uparrow}$, $(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}({}_{c_t}\mathcal{X}_1^{+}))_1^{\mathcal{I}(o)\Uparrow}$; $(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}({}_{c_t}\mathcal{X}_1^{-}))_1^{\mathcal{I}(o)\Uparrow}$and $(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}({}_{c_t}\mathcal{X}_1^{-}))_1^{\mathcal{I}(o)\Uparrow}$;

Step 5: Find the ranking function $\delta_k(q_j)$, where

$$\delta_k(q_j) = \left\{\frac{1}{2}\left\{\begin{array}{l}\left\{\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\left({}_{c_t}\mathcal{X}_1^{+}\right)\right)_1^{\mathcal{I}(o)\Uparrow}(q_j) - \left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\left({}_{c_t}\mathcal{X}_1^{-}\right)\right)_1^{\mathcal{I}(o)\Uparrow}(q_j)\right\}^2 \\ +\left\{\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}\left({}_{c_t}\mathcal{X}_1^{+}\right)\right)_1^{\mathcal{I}(o)\Uparrow}(q_j) - \left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}\left({}_{c_t}\mathcal{X}_1^{-}\right)\right)_1^{\mathcal{I}(o)\Uparrow}(q_j)\right\}^2\end{array}\right\}\right\}^{\frac{1}{2}}; \tag{9}$$

Step 6: Constructed the optimal index function $\delta(q_j)$, where

$$\delta(q_j) = \sum_{t=1}^{l}\omega_k\delta_k(q_j) \tag{10}$$

and determine the weight vectors of every attributes/tests according to $\delta_k(q_j)$ by

$$\omega_k = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\delta_k(q_i) - \delta_k(q_j)\right|}{\sum_{t=1}^{l}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\delta_k(q_i) - \delta_k(q_j)\right|}; \tag{11}$$

Step 7: Rank the alternatives/medicines by the value of the overall ranking function $\delta(q_j)$ and make the decision.

Output: A ranking result of all the alternatives/medicines.

7.2 | Algorithm with pseudo code

Begin

for $i = 1$ to n and $t = 1$ to l **do**

compute $\mathcal{R}_{c_t}^{\Uparrow}$, where $t = 1$ to l//according to Equation (5)

end

for $j = 1$ to n and $t = 1$ to l **do**

compute ${}_{c_t}\mathcal{N}_{q_j}^{\Uparrow\beta}$ of q_j with respect to c_t//according to Definition 8

end
for $i = 1$ to $n, j = 1$ to n and $t = 1$ to l **do**
compute ${}_{c_t}\mathcal{X}_1^+$ and ${}_{c_t}\mathcal{X}_1^-$//according to Equation (7) and Equation (8)
end
for $t = 1$ to l **do**
compute $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\nabla}({}_{c_t}\mathcal{X}_1^+)\right)_1^{\mathcal{I}(o)\Uparrow}$, $\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\triangle}({}_{c_t}\mathcal{X}_1^+)\right)_1^{\mathcal{I}(o)\Uparrow}$, $\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\nabla}({}_{c_t}\mathcal{X}_1^-)\right)_1^{\mathcal{I}(o)\Uparrow}$ and $\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\triangle}({}_{c_t}\mathcal{X}_1^-)\right)_1^{\mathcal{I}(o)\Uparrow}$//according to Definition 13
end
for $k = 1$ to m and $j = 1$ to n **do**
calculate $\delta_k(q_j)$//according to Equation (9)
end
for $k = 1$ to $m, i = 1$ to n and $j = 1$ to n **do**
calculate $\delta(q_j)$//according to Equation (10)
end
End

8 | APPLICATIONS: AN ILLUSTRATIVE EXAMPLE

The doctors usually combine some kinds of medicines to treat the coronavirus disease, denoted by A. Let $\mathcal{U} = \{q_i : i = 1, 2, ..., n\}$ be the universe of n kinds of medicines $\mathcal{V} = \{y_p : p = 1, 2, ..., k\}$ be k most common symptoms (e.g., dry cough, fever, tiredness, etc.) of the coronavirus disease A, and E be a finite set of the domain for the information function $g(q_i, c)$, where $g(q_i, c) \in [0, 1]$. Now $g(q_i, c)$ shows the degree of recommendation of medicine q_i by the doctor c. $\mathcal{A}_i^{\mathcal{R}_c\Uparrow}(q_j)$ denote the efficacy value of the medicine q_i for the symptom $y_i (i = 1, 2, ..., n, p = 1, 2, ..., k)$. For a critical value β suppose that for each medicine $q_i \in \mathcal{U}$, there is at least one symptoms $y_p \in \mathcal{V}$ such that the efficacy value of the medicine q_j for the symptom y_i is not less than β, and $\mathcal{P}(\mathcal{R}_c^{\Uparrow})$ is a fuzzy upward β-covering of $\mathcal{U}$. Then the fuzzy upward β-neighborhood ${}_c\mathcal{N}_{q_j}^{\Uparrow\beta}$ of q_j with respect to c is a fuzzy set given by

$${}_c\mathcal{N}_{q_j}^{\Uparrow\beta}(q_t) = \left[\bigcap\left\{\mathcal{A}_i^{\mathcal{R}_c\Uparrow} : \mathcal{A}_i^{\mathcal{R}_c\Uparrow}(q_j) \geq \beta\right\}\right](q_t), \quad t = 1, 2, ..., l,$$

which denotes the minimum value among all the efficacy values of each medicine q_k for treating the symptoms. If a fuzzy set μ denotes the ability of all medicines in $\mathcal{U}$ to cure the coronavirus disease A, since the inaccuracy of μ, then we can take it approximate evaluation according to the lower and upper approximation of μ. Let $\mathcal{U} = \{q_i : i = 1, 2, ..., 9\}$ be the set of medicines and c_i be the criteria. Then the evaluation of $\mathcal{U}$ by the c, is given in Table 2.

TABLE 2 Multicriteria decision making table

	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9
c_1	0.8	0.3	0.2	0.6	0.4	0.2	0.3	0.3	0.3
c_2	0.1	0.5	0.1	0.3	0.4	0.3	0.3	0.4	0.2
c_3	0.2	0.2	0.6	0.5	0.3	0.5	0.6	0.3	0.4

TABLE 3 Comparison of different methods when β is 0.5

	Methods	Ranking of alternatives
1	Ma[38]	No ranking
2	Yang & Hu[39]	No ranking
3	Proposed	$q_8 \succ q_4 \succ q_9 \succ q_5 \succ q_7 \succ q_3 \succ q_6 \succ q_1 \succ q_2$

Based on criterion c_1 and using Equation (5), to compute the fuzzy preference degree of q_i $(i = 1, 2, ..., 9)$ to $q_j(j = 1, 2, ..., 9)$, one can derive

$$\mathcal{R}_{c_1}^{\Uparrow}(q_i, q_j) = \begin{pmatrix} 0.50000 & 0.66667 & 0.70000 & 0.56667 & 0.63333 & 0.70000 & 0.66667 & 0.66667 & 0.66667 \\ 0.33333 & 0.50000 & 0.53333 & 0.40000 & 0.46667 & 0.53333 & 0.50000 & 0.50000 & 0.50000 \\ 0.30000 & 0.46667 & 0.50000 & 0.36667 & 0.43333 & 0.50000 & 0.46667 & 0.46667 & 0.46667 \\ 0.43333 & 0.60000 & 0.63333 & 0.50000 & 0.56667 & 0.63333 & 0.60000 & 0.60000 & 0.60000 \\ 0.36667 & 0.53333 & 0.56667 & 0.43333 & 0.50000 & 0.56667 & 0.53333 & 0.53333 & 0.53333 \\ 0.30000 & 0.46667 & 0.50000 & 0.36667 & 0.43333 & 0.50000 & 0.46667 & 0.46667 & 0.46667 \\ 0.33333 & 0.50000 & 0.53333 & 0.40000 & 0.46667 & 0.53333 & 0.50000 & 0.50000 & 0.50000 \\ 0.33333 & 0.50000 & 0.53333 & 0.40000 & 0.46667 & 0.53333 & 0.50000 & 0.50000 & 0.50000 \\ 0.33333 & 0.50000 & 0.53333 & 0.40000 & 0.46667 & 0.53333 & 0.50000 & 0.50000 & 0.50000 \end{pmatrix}.$$

Based on criterion c_2 and using Equation (5), to compute the fuzzy preference degree of q_i $(i = 1, 2, ..., 9)$ to $q_j(j = 1, 2, ..., 9)$, one can acquire

$$\mathcal{R}_{c_2}^{\Uparrow}(q_i, q_j) = \begin{pmatrix} 0.50000 & 0.31818 & 0.50000 & 0.40909 & 0.36364 & 0.40909 & 0.40909 & 0.36364 & 0.45455 \\ 0.68182 & 0.50000 & 0.68182 & 0.59091 & 0.54545 & 0.59091 & 0.59091 & 0.54545 & 0.63636 \\ 0.50000 & 0.31818 & 0.50000 & 0.40909 & 0.36364 & 0.40909 & 0.40909 & 0.36364 & 0.45455 \\ 0.59091 & 0.40909 & 0.59091 & 0.50000 & 0.45455 & 0.50000 & 0.50000 & 0.45455 & 0.54555 \\ 0.63636 & 0.45455 & 0.63636 & 0.54545 & 0.50000 & 0.54545 & 0.54545 & 0.50000 & 0.59091 \\ 0.59091 & 0.40919 & 0.59091 & 0.50000 & 0.45455 & 0.50000 & 0.50000 & 0.45455 & 0.54545 \\ 0.59091 & 0.40909 & 0.59091 & 0.50000 & 0.45455 & 0.50000 & 0.50000 & 0.45455 & 0.54545 \\ 0.63636 & 0.45455 & 0.63636 & 0.54545 & 0.50000 & 0.54545 & 0.54545 & 0.50000 & 0.59091 \\ 0.54545 & 0.36364 & 0.54545 & 0.45455 & 0.40909 & 0.45455 & 0.45455 & 0.40909 & 0.50000 \end{pmatrix}.$$

Based on criterion c_3 and using Equation (5), to compute the fuzzy preference degree of q_i $(i = 1, 2, ..., 9)$ to $q_j(j = 1, 2, ..., 9)$, one can derive

$$\mathcal{R}_{c_3}^{\Uparrow}(q_i, q_j) = \begin{pmatrix} 0.50000 & 0.50000 & 0.34615 & 0.38462 & 0.46154 & 0.38462 & 0.34615 & 0.46154 & 0.42308 \\ 0.50000 & 0.50000 & 0.34615 & 0.38462 & 0.46154 & 0.38462 & 0.34615 & 0.46154 & 0.42308 \\ 0.65385 & 0.65385 & 0.50000 & 0.53846 & 0.61538 & 0.53846 & 0.50000 & 0.61538 & 0.57692 \\ 0.61538 & 0.61538 & 0.46154 & 0.50000 & 0.57692 & 0.50000 & 0.46154 & 0.57692 & 0.53846 \\ 0.53846 & 0.53846 & 0.38462 & 0.42308 & 0.50000 & 0.42308 & 0.38462 & 0.50000 & 0.46154 \\ 0.61538 & 0.61538 & 0.46154 & 0.50000 & 0.57692 & 0.50000 & 0.46154 & 0.57692 & 0.53846 \\ 0.65385 & 0.65385 & 0.50000 & 0.53846 & 0.61538 & 0.53846 & 0.50000 & 0.61538 & 0.57692 \\ 0.53846 & 0.53846 & 0.38462 & 0.42308 & 0.50000 & 0.42308 & 0.38462 & 0.50000 & 0.46154 \\ 0.57692 & 0.57692 & 0.42308 & 0.46154 & 0.53846 & 0.46154 & 0.42308 & 0.53846 & 0.50000 \end{pmatrix}.$$

The upward fuzzy preference classes $\mathcal{A}_i^{\mathcal{R}_{c1}\Uparrow}$ are given by:

$$\mathcal{A}_1^{\mathcal{R}_{c1}\Uparrow}=\frac{0.50000}{q_1}+\frac{0.66667}{q_2}+\frac{0.70000}{q_3}+\frac{0.56667}{q_4}+\frac{0.63333}{q_5}+\frac{0.70000}{q_6}+\frac{0.66667}{q_7}+\frac{0.66667}{q_8}+\frac{0.66667}{q_9};$$

$$\mathcal{A}_2^{\mathcal{R}_{c1}\Uparrow}=\frac{0.33333}{q_1}+\frac{0.50000}{q_2}+\frac{0.53333}{q_3}+\frac{0.40000}{q_4}+\frac{0.46667}{q_5}+\frac{0.53333}{q_6}+\frac{0.50000}{q_7}+\frac{0.50000}{q_8}+\frac{0.50000}{q_9};$$

$$\mathcal{A}_3^{\mathcal{R}_{c1}\Uparrow}=\frac{0.30000}{q_1}+\frac{0.46667}{q_2}+\frac{0.50000}{q_3}+\frac{0.36667}{q_4}+\frac{0.43333}{q_5}+\frac{0.50000}{q_6}+\frac{0.46667}{q_7}+\frac{0.46667}{q_8}+\frac{0.46667}{q_9};$$

$$\mathcal{A}_4^{\mathcal{R}_{c1}\Uparrow}=\frac{0.43333}{q_1}+\frac{0.60000}{q_2}+\frac{0.63333}{q_3}+\frac{0.50000}{q_4}+\frac{0.56667}{q_5}+\frac{0.63333}{q_6}+\frac{0.60000}{q_7}+\frac{0.60000}{q_8}+\frac{0.60000}{q_9};$$

$$\mathcal{A}_5^{\mathcal{R}_{c1}\Uparrow}=\frac{0.36667}{q_1}+\frac{0.53333}{q_2}+\frac{0.56667}{q_3}+\frac{0.43333}{q_4}+\frac{0.50000}{q_5}+\frac{0.56667}{q_6}+\frac{0.53333}{q_7}+\frac{0.53333}{q_8}+\frac{0.53333}{q_9};$$

$$\mathcal{A}_6^{\mathcal{R}_{c1}\Uparrow}=\frac{0.30000}{q_1}+\frac{0.46667}{q_2}+\frac{0.50000}{q_3}+\frac{0.36667}{q_4}+\frac{0.43333}{q_5}+\frac{0.50000}{q_6}+\frac{0.46667}{q_7}+\frac{0.46667}{q_8}+\frac{0.46667}{q_9};$$

$$\mathcal{A}_7^{\mathcal{R}_{c1}\Uparrow}=\frac{0.33333}{q_1}+\frac{0.50000}{q_2}+\frac{0.53333}{q_3}+\frac{0.40000}{q_4}+\frac{0.46667}{q_5}+\frac{0.53333}{q_6}+\frac{0.50000}{q_7}+\frac{0.50000}{q_8}+\frac{0.50000}{q_9};$$

$$\mathcal{A}_8^{\mathcal{R}_{c1}\Uparrow}=\frac{0.33333}{q_1}+\frac{0.50000}{q_2}+\frac{0.53333}{q_3}+\frac{0.40000}{q_4}+\frac{0.46667}{q_5}+\frac{0.53333}{q_6}+\frac{0.50000}{q_7}+\frac{0.50000}{q_8}+\frac{0.50000}{q_9};$$

$$\mathcal{A}_9^{\mathcal{R}_{c1}\Uparrow}=\frac{0.33333}{q_1}+\frac{0.50000}{q_2}+\frac{0.53333}{q_3}+\frac{0.40000}{q_4}+\frac{0.46667}{q_5}+\frac{0.53333}{q_6}+\frac{0.50000}{q_7}+\frac{0.50000}{q_8}+\frac{0.50000}{q_9}.$$

We see that $\mathcal{P}\left(\mathcal{R}_{c_1}^{\Uparrow}\right)=\left\{\mathcal{A}_1^{\mathcal{R}_{c1}\Uparrow},\mathcal{A}_2^{\mathcal{R}_{c1}\Uparrow},\mathcal{A}_3^{\mathcal{R}_{c1}\Uparrow},\mathcal{A}_4^{\mathcal{R}_{c1}\Uparrow},\mathcal{A}_5^{\mathcal{R}_{c1}\Uparrow},\mathcal{A}_6^{\mathcal{R}_{c1}\Uparrow},\mathcal{A}_7^{\mathcal{R}_{c1}\Uparrow},\mathcal{A}_8^{\mathcal{R}_{c1}\Uparrow},\mathcal{A}_9^{\mathcal{R}_{c1}\Uparrow}\right\}$ is a fuzzy upward β-covering of $\mathcal{U}(0<\beta\leq 0.5)$. The upward fuzzy preference classes $\mathcal{A}_i^{\mathcal{R}_{c2}\Uparrow}$ are given by:

$$\mathcal{A}_1^{\mathcal{R}_{c2}\Uparrow} = \frac{0.50000}{q_1} + \frac{0.31818}{q_2} + \frac{0.50000}{q_3} + \frac{0.40909}{q_4} + \frac{0.36364}{q_5} + \frac{0.40909}{q_6} + \frac{0.40909}{q_7} + \frac{0.36364}{q_8} + \frac{0.45455}{q_9};$$

$$\mathcal{A}_2^{\mathcal{R}_{c2}\Uparrow} = \frac{0.68182}{q_1} + \frac{0.50000}{q_2} + \frac{0.68182}{q_3} + \frac{0.59091}{q_4} + \frac{0.54545}{q_5} + \frac{0.59091}{q_6} + \frac{0.59091}{q_7} + \frac{0.54545}{q_8} + \frac{0.63636}{q_9};$$

$$\mathcal{A}_3^{\mathcal{R}_{c2}\Uparrow} = \frac{0.50000}{q_1} + \frac{0.31818}{q_2} + \frac{0.50000}{q_3} + \frac{0.40909}{q_4} + \frac{0.36364}{q_5} + \frac{0.40909}{q_6} + \frac{0.40909}{q_7} + \frac{0.36364}{q_8} + \frac{0.45455}{q_9};$$

$$\mathcal{A}_4^{\mathcal{R}_{c2}\Uparrow} = \frac{0.59091}{q_1} + \frac{0.40909}{q_2} + \frac{0.59091}{q_3} + \frac{0.50000}{q_4} + \frac{0.45455}{q_5} + \frac{0.50000}{q_5} + \frac{0.50000}{q_7} + \frac{0.45455}{q_8} + \frac{0.54555}{q_9};$$

$$\mathcal{A}_5^{\mathcal{R}_{c2}\Uparrow} = \frac{0.63636}{q_1} + \frac{0.45455}{q_2} + \frac{0.63636}{q_3} + \frac{0.54545}{q_4} + \frac{0.50000}{q_5} + \frac{0.54545}{q_6} + \frac{0.54545}{q_7} + \frac{0.50000}{q_8} + \frac{0.59091}{q_9};$$

$$\mathcal{A}_6^{\mathcal{R}_{c2}\Uparrow} = \frac{0.59091}{q_1} + \frac{0.40919}{q_2} + \frac{0.59091}{q_3} + \frac{0.50000}{q_4} + \frac{0.45455}{q_5} + \frac{0.50000}{q_6} + \frac{0.50000}{q_7} + \frac{0.45455}{q_8} + \frac{0.54545}{q_9};$$

$$\mathcal{A}_7^{\mathcal{R}_{c2}\Uparrow} = \frac{0.59091}{q_1} + \frac{0.40919}{q_2} + \frac{0.59091}{q_3} + \frac{0.50000}{q_4} + \frac{0.45455}{q_5} + \frac{0.50000}{q_6} + \frac{0.50000}{q_7} + \frac{0.45455}{q_8} + \frac{0.54545}{q_9};$$

$$\mathcal{A}_8^{\mathcal{R}_{c2}\Uparrow} = \frac{0.63636}{q_1} + \frac{0.45455}{q_2} + \frac{0.63636}{q_3} + \frac{0.54545}{q_4} + \frac{0.50000}{q_5} + \frac{0.54545}{q_6} + \frac{0.54545}{q_7} + \frac{0.50000}{q_8} + \frac{0.59091}{q_9};$$

$$\mathcal{A}_9^{\mathcal{R}_{c2}\Uparrow} = \frac{0.54545}{q_1} + \frac{0.36364}{q_2} + \frac{0.54545}{q_3} + \frac{0.45455}{q_4} + \frac{0.40909}{q_5} + \frac{0.45455}{q_6} + \frac{0.45455}{q_7} + \frac{0.40909}{q_8} + \frac{0.50000}{q_9}.$$

We see that $\mathcal{P}\left(\mathcal{R}_{c_2}^{\Uparrow}\right) = \left\{\mathcal{A}_1^{\mathcal{R}_{c2}\Uparrow}, \mathcal{A}_2^{\mathcal{R}_{c2}\Uparrow}, \mathcal{A}_3^{\mathcal{R}_{c2}\Uparrow}, \mathcal{A}_4^{\mathcal{R}_{c2}\Uparrow}, \mathcal{A}_5^{\mathcal{R}_{c2}\Uparrow}, \mathcal{A}_6^{\mathcal{R}_{c2}\Uparrow}, \mathcal{A}_7^{\mathcal{R}_{c2}\Uparrow}, \mathcal{A}_8^{\mathcal{R}_{c2}\Uparrow}, \mathcal{A}_9^{\mathcal{R}_{c2}\Uparrow}\right\}$ is a fuzzy upward β-covering of $\mathcal{U}(0 < \beta \leq 0.5)$. The upward fuzzy preference classes $\mathcal{A}_i^{\mathcal{R}_{c3}\Uparrow}$ are given by:

$$\mathcal{A}_1^{\mathcal{R}_{c3}\Uparrow} = \frac{0.50000}{q_1} + \frac{0.50000}{q_2} + \frac{0.34615}{q_3} + \frac{0.38462}{q_4} + \frac{0.46154}{q_5} + \frac{0.38462}{q_6} + \frac{0.34615}{q_7} + \frac{0.46154}{q_8} + \frac{0.42308}{q_9};$$

$$\mathcal{A}_2^{\mathcal{R}_{c3}\Uparrow} = \frac{0.50000}{q_1} + \frac{0.50000}{q_2} + \frac{0.34615}{q_3} + \frac{0.38462}{q_4} + \frac{0.46154}{q_5} + \frac{0.38462}{q_6} + \frac{0.34615}{q_7} + \frac{0.46154}{q_8} + \frac{0.42308}{q_9};$$

$$\mathcal{A}_3^{\mathcal{R}_{c3}\Uparrow} = \frac{0.65385}{q_1} + \frac{0.65385}{q_2} + \frac{0.50000}{q_3} + \frac{0.53846}{q_4} + \frac{0.61538}{q_5} + \frac{0.53846}{q_6} + \frac{0.50000}{q_7} + \frac{0.61538}{q_8} + \frac{0.57692}{q_9};$$

$$\mathcal{A}_4^{\mathcal{R}_{c3}\Uparrow} = \frac{0.61538}{q_1} + \frac{0.61538}{q_2} + \frac{0.46154}{q_3} + \frac{0.50000}{q_4} + \frac{0.57692}{q_5} + \frac{0.50000}{q_6} + \frac{0.46154}{q_7} + \frac{0.57692}{q_8} + \frac{0.53846}{q_9};$$

$$\mathcal{A}_5^{\mathcal{R}_{c3}\Uparrow} = \frac{0.53846}{q_1} + \frac{0.53846}{q_2} + \frac{0.38462}{q_3} + \frac{0.42308}{q_4} + \frac{0.50000}{q_5} + \frac{0.42308}{q_6} + \frac{0.38462}{q_7} + \frac{0.50000}{q_8} + \frac{0.46154}{q_9};$$

$$\mathcal{A}_6^{\mathcal{R}_{c3}\Uparrow} = \frac{0.61538}{q_1} + \frac{0.61538}{q_2} + \frac{0.46154}{q_3} + \frac{0.50000}{q_4} + \frac{0.57692}{q_5} + \frac{0.50000}{q_6} + \frac{0.46154}{q_7} + \frac{0.57692}{q_8} + \frac{0.53846}{q_9};$$

$$\mathcal{A}_7^{\mathcal{R}_{c3}\Uparrow} = \frac{0.65385}{q_1} + \frac{0.65385}{q_2} + \frac{0.50000}{q_3} + \frac{0.53846}{q_4} + \frac{0.61538}{q_5} + \frac{0.53846}{q_6} + \frac{0.50000}{q_7} + \frac{0.61538}{q_8} + \frac{0.57692}{q_9};$$

$$\mathcal{A}_8^{\mathcal{R}_{c3}\Uparrow} = \frac{0.53846}{q_1} + \frac{0.53846}{q_2} + \frac{0.38462}{q_3} + \frac{0.42308}{q_4} + \frac{0.50000}{q_5} + \frac{0.42308}{q_6} + \frac{0.38462}{q_7} + \frac{0.50000}{q_8} + \frac{0.46154}{q_9};$$

$$\mathcal{A}_9^{\mathcal{R}_{c3}\Uparrow} = \frac{0.57692}{q_1} + \frac{0.57692}{q_2} + \frac{0.42308}{q_3} + \frac{0.46154}{q_4} + \frac{0.53846}{q_5} + \frac{0.46154}{q_6} + \frac{0.42308}{q_7} + \frac{0.53846}{q_8} + \frac{0.50000}{q_9}.$$

It follows that

$$ {}_{c_1}\mathcal{N}_{q_1}^{\Uparrow 0.5} = \frac{0.50000}{q_1} + \frac{0.66667}{q_2} + \frac{0.70000}{q_3} + \frac{0.56667}{q_4} + \frac{0.63333}{q_5} + \frac{0.70000}{q_6} + \frac{0.66667}{q_7} + \frac{0.66667}{q_8} + \frac{0.66667}{q_9}; $$

$$ {}_{c_1}\mathcal{N}_{q_2}^{\Uparrow 0.5} = \frac{0.33333}{q_1} + \frac{0.50000}{q_2} + \frac{0.53333}{q_3} + \frac{0.40000}{q_4} + \frac{0.46667}{q_5} + \frac{0.53333}{q_6} + \frac{0.50000}{q_7} + \frac{0.50000}{q_8} + \frac{0.50000}{q_9}; $$

$$ {}_{c_1}\mathcal{N}_{q_3}^{\Uparrow 0.5} = \frac{0.30000}{q_1} + \frac{0.46667}{q_2} + \frac{0.50000}{q_3} + \frac{0.36667}{q_4} + \frac{0.43333}{q_5} + \frac{0.50000}{q_6} + \frac{0.46667}{q_7} + \frac{0.46667}{q_8} + \frac{0.46667}{q_9}; $$

$$ {}_{c_1}\mathcal{N}_{q_4}^{\Uparrow 0.5} = \frac{0.43333}{q_1} + \frac{0.60000}{q_2} + \frac{0.63333}{q_3} + \frac{0.50000}{q_4} + \frac{0.56667}{q_5} + \frac{0.63333}{q_6} + \frac{0.60000}{q_7} + \frac{0.6000}{q_8} + \frac{0.60000}{q_9}; $$

$$ {}_{c_1}\mathcal{N}_{q_5}^{\Uparrow 0.5} = \frac{0.36667}{q_1} + \frac{0.53333}{q_2} + \frac{0.56667}{q_3} + \frac{0.43333}{q_4} + \frac{0.50000}{q_5} + \frac{0.56667}{q_6} + \frac{0.53333}{q_7} + \frac{0.53333}{q_8} + \frac{0.53333}{q_9}; $$

$$ {}_{c_1}\mathcal{N}_{q_6}^{\Uparrow 0.5} = \frac{0.30000}{q_1} + \frac{0.46667}{q_2} + \frac{0.50000}{q_3} + \frac{0.36667}{q_4} + \frac{0.43333}{q_5} + \frac{0.50000}{q_6} + \frac{0.46667}{q_7} + \frac{0.46667}{q_8} + \frac{0.46667}{q_9}; $$

$$ {}_{c_1}\mathcal{N}_{q_7}^{\Uparrow 0.5} = \frac{0.33333}{q_1} + \frac{0.50000}{q_2} + \frac{0.53333}{q_3} + \frac{0.40000}{q_4} + \frac{0.46667}{q_5} + \frac{0.53333}{q_6} + \frac{0.50000}{q_7} + \frac{0.50000}{q_8} + \frac{0.50000}{q_9}; $$

$$ {}_{c_1}\mathcal{N}_{q_8}^{\Uparrow 0.5} = \frac{0.33333}{q_1} + \frac{0.50000}{q_2} + \frac{0.53333}{q_3} + \frac{0.40000}{q_4} + \frac{0.46667}{q_5} + \frac{0.53333}{q_6} + \frac{0.50000}{q_7} + \frac{0.50000}{q_8} + \frac{0.50000}{q_9}; $$

$$ {}_{c_1}\mathcal{N}_{q_9}^{\Uparrow 0.5} = \frac{0.33333}{q_1} + \frac{0.50000}{q_2} + \frac{0.53333}{q_3} + \frac{0.40000}{q_4} + \frac{0.46667}{q_5} + \frac{0.53333}{q_6} + \frac{0.50000}{q_7} + \frac{0.50000}{q_8} + \frac{0.50000}{q_9}. $$

$$
\begin{aligned}
{}_{c_2}\mathcal{N}_{q_1}^{\Uparrow 0.5} = {} & \frac{0.50000}{q_1} + \frac{0.31818}{q_2} + \frac{0.50000}{q_3} + \frac{0.40909}{q_4} + \frac{0.36364}{q_5} + \frac{0.40909}{q_6} \\
& + \frac{0.40909}{q_7} + \frac{0.36364}{q_8} + \frac{0.45455}{q_9}; \\
{}_{c_2}\mathcal{N}_{q_2}^{\Uparrow 0.5} = {} & \frac{0.68182}{q_1} + \frac{0.50000}{q_2} + \frac{0.68182}{q_3} + \frac{0.59091}{q_4} + \frac{0.54545}{q_5} + \frac{0.59091}{q_6} \\
& + \frac{0.59091}{q_7} + \frac{0.54545}{q_8} + \frac{0.63636}{q_9}; \\
{}_{c_2}\mathcal{N}_{q_3}^{\Uparrow 0.5} = {} & \frac{0.50000}{q_1} + \frac{0.31818}{q_2} + \frac{0.50000}{q_3} + \frac{0.40909}{q_4} + \frac{0.36364}{q_5} + \frac{0.40909}{q_6} \\
& + \frac{0.40909}{q_7} + \frac{0.36364}{q_8} + \frac{0.45455}{q_9}; \\
{}_{c_2}\mathcal{N}_{q_4}^{\Uparrow 0.5} = {} & \frac{0.59091}{q_1} + \frac{0.40919}{q_2} + \frac{0.59091}{q_3} + \frac{0.50000}{q_4} + \frac{0.45455}{q_5} + \frac{0.50000}{q_6} \\
& + \frac{0.50000}{q_7} + \frac{0.45455}{q_8} + \frac{0.54545}{q_9}; \\
{}_{c_2}\mathcal{N}_{q_5}^{\Uparrow 0.5} = {} & \frac{0.63636}{q_1} + \frac{0.45455}{q_2} + \frac{0.63636}{q_3} + \frac{0.54545}{q_4} + \frac{0.50000}{q_5} + \frac{0.54545}{q_6} \\
& + \frac{0.54545}{q_7} + \frac{0.50000}{q_8} + \frac{0.59091}{q_9}; \\
{}_{c_2}\mathcal{N}_{q_6}^{\Uparrow 0.5} = {} & \frac{0.59091}{q_1} + \frac{0.40919}{q_2} + \frac{0.59091}{q_3} + \frac{0.50000}{q_4} + \frac{0.45455}{q_5} + \frac{0.50000}{q_6} \\
& + \frac{0.50000}{q_7} + \frac{0.45455}{q_8} + \frac{0.54545}{q_9}; \\
{}_{c_2}\mathcal{N}_{q_7}^{\Uparrow 0.5} = {} & \frac{0.59091}{q_1} + \frac{0.40919}{q_2} + \frac{0.59091}{q_3} + \frac{0.50000}{q_4} + \frac{0.45455}{q_5} + \frac{0.50000}{q_6} \\
& + \frac{0.50000}{q_7} + \frac{0.45455}{q_8} + \frac{0.54545}{q_9}; \\
{}_{c_2}\mathcal{N}_{q_8}^{\Uparrow 0.5} = {} & \frac{0.63636}{q_1} + \frac{0.45455}{q_2} + \frac{0.63636}{q_3} + \frac{0.54545}{q_4} + \frac{0.50000}{q_5} + \frac{0.54545}{q_6} \\
& + \frac{0.54545}{q_7} + \frac{0.50000}{q_8} + \frac{0.59091}{q_9}; \\
{}_{c_2}\mathcal{N}_{q_9}^{\Uparrow 0.5} = {} & \frac{0.54545}{q_1} + \frac{0.36364}{q_2} + \frac{0.54545}{q_3} + \frac{0.45455}{q_4} + \frac{0.40909}{q_5} + \frac{0.45455}{q_6} \\
& + \frac{0.45455}{q_7} + \frac{0.40909}{q_8} + \frac{0.50000}{q_9}.
\end{aligned}
$$

$$
\begin{aligned}
{}_{c_3}\mathcal{N}_{q_1}^{\Uparrow 0.5} &= \frac{0.50000}{q_1} + \frac{0.50000}{q_2} + \frac{0.34615}{q_3} + \frac{0.38462}{q_4} + \frac{0.46154}{q_5} + \frac{0.38462}{q_6} \\
&\quad + \frac{0.34615}{q_7} + \frac{0.46154}{q_8} + \frac{0.42308}{q_9}; \\
{}_{c_3}\mathcal{N}_{q_2}^{\Uparrow 0.5} &= \frac{0.50000}{q_1} + \frac{0.50000}{q_2} + \frac{0.34615}{q_3} + \frac{0.38462}{q_4} + \frac{0.46154}{q_5} + \frac{0.38462}{q_6} \\
&\quad + \frac{0.34615}{q_7} + \frac{0.46154}{q_8} + \frac{0.42308}{q_9}; \\
{}_{c_3}\mathcal{N}_{q_3}^{\Uparrow 0.5} &= \frac{0.65385}{q_1} + \frac{0.65385}{q_2} + \frac{0.50000}{q_3} + \frac{0.53846}{q_4} + \frac{0.61538}{q_5} + \frac{0.53846}{q_6} \\
&\quad + \frac{0.50000}{q_7} + \frac{0.61538}{q_8} + \frac{0.57692}{q_9}; \\
{}_{c_3}\mathcal{N}_{q_4}^{\Uparrow 0.5} &= \frac{0.61538}{q_1} + \frac{0.61538}{q_2} + \frac{0.46154}{q_3} + \frac{0.50000}{q_4} + \frac{0.57692}{q_5} + \frac{0.50000}{q_6} \\
&\quad + \frac{0.46154}{q_7} + \frac{0.57692}{q_8} + \frac{0.53846}{q_9}; \\
{}_{c_3}\mathcal{N}_{q_5}^{\Uparrow 0.5} &= \frac{0.53846}{q_1} + \frac{0.53846}{q_2} + \frac{0.38468}{q_3} + \frac{0.42308}{q_4} + \frac{0.50000}{q_5} + \frac{0.42308}{q_6} \\
&\quad + \frac{0.38462}{q_7} + \frac{0.50000}{q_8} + \frac{0.46154}{q_9}; \\
{}_{c_3}\mathcal{N}_{q_6}^{\Uparrow 0.5} &= \frac{0.61538}{q_1} + \frac{0.61538}{q_2} + \frac{0.46154}{q_3} + \frac{0.50000}{q_4} + \frac{0.57692}{q_5} + \frac{0.50000}{q_6} \\
&\quad + \frac{0.46154}{q_7} + \frac{0.57692}{q_8} + \frac{0.53846}{q_9}; \\
{}_{c_3}\mathcal{N}_{q_7}^{\Uparrow 0.5} &= \frac{0.65385}{q_1} + \frac{0.65385}{q_2} + \frac{0.50000}{q_3} + \frac{0.53846}{q_4} + \frac{0.61538}{q_5} + \frac{0.53846}{q_6} \\
&\quad + \frac{0.50000}{q_7} + \frac{0.61538}{q_8} + \frac{0.57692}{q_9}; \\
{}_{c_3}\mathcal{N}_{q_8}^{\Uparrow 0.5} &= \frac{0.53846}{q_1} + \frac{0.53846}{q_2} + \frac{0.38468}{q_3} + \frac{0.42308}{q_4} + \frac{0.50000}{q_5} + \frac{0.42308}{q_6} \\
&\quad + \frac{0.38462}{q_7} + \frac{0.50000}{q_8} + \frac{0.46154}{q_9}; \\
{}_{c_3}\mathcal{N}_{q_9}^{\Uparrow 0.5} &= \frac{0.61538}{q_1} + \frac{0.61538}{q_2} + \frac{0.46154}{q_3} + \frac{0.50000}{q_4} + \frac{0.57692}{q_5} + \frac{0.50000}{q_6} \\
&\quad + \frac{0.46154}{q_7} + \frac{0.57692}{q_8} + \frac{0.53846}{q_9}.
\end{aligned}
$$

By using the principle of fuzzy TOPSIS method to acquire the best and worst optimal fuzzy decision making objects we have

$$
\begin{aligned}
{}_{c_1}\mathcal{X}_1^{+} &= \frac{0.70000}{q_1} + \frac{0.53333}{q_2} + \frac{0.50000}{q_3} + \frac{0.63333}{q_4} + \frac{0.56667}{q_5} + \frac{0.50000}{q_6} \\
&\quad + \frac{0.53000}{q_7} + \frac{0.53333}{q_8} + \frac{0.53333}{q_9}; \\
{}_{c_1}\mathcal{X}_1^{-} &= \frac{0.50000}{q_1} + \frac{0.33333}{q_2} + \frac{0.30000}{q_3} + \frac{0.43333}{q_4} + \frac{0.36667}{q_5} + \frac{0.30000}{q_6} \\
&\quad + \frac{0.33333}{q_7} + \frac{0.33333}{q_8} + \frac{0.33333}{q_9}; \\
{}_{c_2}\mathcal{X}_1^{+} &= \frac{0.50000}{q_1} + \frac{0.68182}{q_2} + \frac{0.50000}{q_3} + \frac{0.59091}{q_4} + \frac{0.63636}{q_5} + \frac{0.59091}{q_6} \\
&\quad + \frac{0.59091}{q_7} + \frac{0.63636}{q_8} + \frac{0.54545}{q_9}; \\
{}_{c_2}\mathcal{X}_1^{-} &= \frac{0.31818}{q_1} + \frac{0.50000}{q_2} + \frac{0.36364}{q_3} + \frac{0.40909}{q_4} + \frac{0.45455}{q_5} + \frac{0.40919}{q_6} \\
&\quad + \frac{0.40909}{q_7} + \frac{0.45455}{q_8} + \frac{0.36364}{q_9}; \\
{}_{c_3}\mathcal{X}_1^{+} &= \frac{0.50000}{q_1} + \frac{0.50000}{q_2} + \frac{0.65385}{q_3} + \frac{0.61538}{q_4} + \frac{0.53846}{q_5} + \frac{0.61538}{q_6} \\
&\quad + \frac{0.65385}{q_7} + \frac{0.53846}{q_8} + \frac{0.57692}{q_9}; \\
{}_{c_3}\mathcal{X}_1^{-} &= \frac{0.34615}{q_1} + \frac{0.34615}{q_2} + \frac{0.50000}{q_3} + \frac{0.46154}{q_4} + \frac{0.38462}{q_5} + \frac{0.46154}{q_6} \\
&\quad + \frac{0.50000}{q_7} + \frac{0.38462}{q_8} + \frac{0.42308}{q_9}.
\end{aligned}
$$

Fix $\mathcal{I} = \mathcal{I}_{KD}$ based on $\mathcal{S}_M$ and $\mathcal{N}_S$ and $\mathcal{T} = \mathcal{T}_M$ it follows that:

$$
\begin{aligned}
\left(\sum_{t=1}^{l} \mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\left({}_{c_1}\mathcal{X}_1^{+}\right)\right)^{\mathcal{I}(o)\Uparrow} &= \frac{0.53333}{q_1} + \frac{0.53333}{q_2} + \frac{0.50000}{q_3} + \frac{0.50000}{q_4} + \frac{0.53333}{q_5} \\
&\quad + \frac{0.50000}{q_6} + \frac{0.50000}{q_7} + \frac{0.53333}{q_8} + \frac{0.50000}{q_9}; \\
\left(\sum_{t=1}^{l} \mathcal{U}\mathcal{A}_{\beta}^{\triangle}\left({}_{c_1}\mathcal{X}_1^{+}\right)\right)^{\mathcal{T}(o)\Uparrow} &= \frac{0.50000}{q_1} + \frac{0.50000}{q_2} + \frac{0.50000}{q_3} + \frac{0.56667}{q_4} + \frac{0.53333}{q_5} \\
&\quad + \frac{0.50000}{q_6} + \frac{0.50000}{q_7} + \frac{0.50000}{q_8} + \frac{0.50000}{q_9}; \\
\left(\sum_{t=1}^{l} \mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\left({}_{c_1}\mathcal{X}_1^{-}\right)\right)^{\mathcal{I}(o)\Uparrow} &= \frac{0.50000}{q_1} + \frac{0.46667}{q_2} + \frac{0.50000}{q_3} + \frac{0.40909}{q_4} + \frac{0.46154}{q_5} \\
&\quad + \frac{0.50000}{q_6} + \frac{0.46667}{q_7} + \frac{0.46667}{q_8} + \frac{0.46667}{q_9}; \\
\left(\sum_{t=1}^{l} \mathcal{U}\mathcal{A}_{\beta}^{\triangle}\left({}_{c_1}\mathcal{X}_1^{-}\right)\right)^{\mathcal{T}(o)\Uparrow} &= \frac{0.50000}{q_1} + \frac{0.36667}{q_2} + \frac{0.50000}{q_3} + \frac{0.43333}{q_4} + \frac{0.43333}{q_5} \\
&\quad + \frac{0.36667}{q_6} + \frac{0.40000}{q_7} + \frac{0.40000}{q_8} + \frac{0.40000}{q_9};
\end{aligned}
$$

$$\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\left({}_{c_2}\mathcal{X}_1^{+}\right)\right)^{\mathcal{I}(o)\Uparrow}=\frac{0.50000}{q_1}+\frac{0.50000}{q_2}+\frac{0.50000}{q_3}+\frac{0.50000}{q_4}+\frac{0.50000}{q_5}+\frac{0.50000}{q_6}+\frac{0.50000}{q_7}+\frac{0.50000}{q_8}+\frac{0.50000}{q_9};$$

$$\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}\left({}_{c_2}\mathcal{X}_1^{+}\right)\right)^{\mathcal{T}(o)\Uparrow}=\frac{0.50000}{q_1}+\frac{0.50000}{q_2}+\frac{0.50000}{q_3}+\frac{0.54545}{q_4}+\frac{0.53846}{q_5}+\frac{0.50000}{q_6}+\frac{0.53333}{q_7}+\frac{0.53333}{q_8}+\frac{0.50000}{q_9};$$

$$\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\left({}_{c_2}\mathcal{X}_1^{-}\right)\right)^{\mathcal{I}(o)\Uparrow}=\frac{0.50000}{q_1}+\frac{0.50000}{q_2}+\frac{0.50000}{q_3}+\frac{0.40909}{q_4}+\frac{0.46154}{q_5}+\frac{0.50000}{q_6}+\frac{0.46667}{q_7}+\frac{0.46667}{q_8}+\frac{0.46667}{q_9};$$

$$\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}\left({}_{c_2}\mathcal{X}_1^{-}\right)\right)^{\mathcal{T}(o)\Uparrow}=\frac{0.40909}{q_1}+\frac{0.50000}{q_2}+\frac{0.40909}{q_3}+\frac{0.54545}{q_4}+\frac{0.45455}{q_5}+\frac{0.45455}{q_6}+\frac{0.45455}{q_7}+\frac{0.45455}{q_8}+\frac{0.40909}{q_9};$$

$$\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\left({}_{c_3}\mathcal{X}_1^{+}\right)\right)^{\mathcal{I}(o)\Uparrow}=\frac{0.50000}{q_1}+\frac{0.53846}{q_2}+\frac{0.53333}{q_3}+\frac{0.50000}{q_4}+\frac{0.53846}{q_5}+\frac{0.53333}{q_6}+\frac{0.50000}{q_7}+\frac{0.50000}{q_8}+\frac{0.50000}{q_9};$$

$$\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}\left({}_{c_3}\mathcal{X}_1^{+}\right)\right)^{\mathcal{T}(o)\Uparrow}=\frac{0.50000}{q_1}+\frac{0.50000}{q_2}+\frac{0.50000}{q_3}+\frac{0.53846}{q_4}+\frac{0.50000}{q_5}+\frac{0.50000}{q_6}+\frac{0.53333}{q_7}+\frac{0.50000}{q_8}+\frac{0.50000}{q_9};$$

$$\left(\sum_{t=1}^{l}\mathcal{L}\mathrm{A}_{\beta}^{\bigtriangledown}\left({}_{c_3}\mathcal{X}_1^{-}\right)\right)^{\mathcal{I}(o)\Uparrow}=\frac{0.50000}{q_1}+\frac{0.50000}{q_2}+\frac{0.40909}{q_3}+\frac{0.50000}{q_4}+\frac{0.50000}{q_5}+\frac{0.50000}{q_6}+\frac{0.46667}{q_7}+\frac{0.36364}{q_8}+\frac{0.46667}{q_9};$$

$$\left(\sum_{t=1}^{l}\mathcal{U}\mathcal{A}_{\beta}^{\bigtriangleup}\left({}_{c_3}\mathcal{X}_1^{-}\right)\right)^{\mathcal{T}(o)\Uparrow}=\frac{0.42308}{q_1}+\frac{0.46154}{q_2}+\frac{0.50000}{q_3}+\frac{0.50000}{q_4}+\frac{0.50000}{q_5}+\frac{0.46154}{q_6}+\frac{0.50000}{q_7}+\frac{0.42308}{q_8}+\frac{0.46154}{q_9}.$$

$$\delta_1(q_j)=\frac{0.02357}{q_1}+\frac{0.05270}{q_2}+\frac{0.00000}{q_3}+\frac{0.11411}{q_4}+\frac{0.08705}{q_5}+\frac{0.09428}{q_6}+\frac{0.07453}{q_7}+\frac{0.07453}{q_8}+\frac{0.07453}{q_9}$$

$$\delta_2(q_j)=\frac{0.06428}{q_1}+\frac{0.00000}{q_2}+\frac{0.06428}{q_3}+\frac{0.06428}{q_4}+\frac{0.06384}{q_5}+\frac{0.03214}{q_6}+\frac{0.06049}{q_7}+\frac{0.06049}{q_8}+\frac{0.06847}{q_9}$$

$$\delta_3(q_j)=\frac{0.05439}{q_1}+\frac{0.03846}{q_2}+\frac{0.08785}{q_3}+\frac{0.02720}{q_4}+\frac{0.02720}{q_5}+\frac{0.03599}{q_6}+\frac{0.03333}{q_7}+\frac{0.11070}{q_8}+\frac{0.03599}{q_9}.$$

The weight of every attributes are as follows, respectively

$$\omega_1 = 0.30279, \quad \omega_2 = 0.32236, \quad \omega_3 = 0.37485.$$

Now, we are ready to apply optimal index formula for alternative q_j,

$$\delta(q_j) = \frac{0.04825}{q_1} + \frac{0.03037}{q_2} + \frac{0.05365}{q_3} + \frac{0.06547}{q_4} + \frac{0.05713}{q_5} + \frac{0.05240}{q_6} + \frac{0.05456}{q_7} + \frac{0.08356}{q_8} + \frac{0.05813}{q_9}.$$

Finally, we can see that the ranking of the nine alternatives is

$$q_8 \succ q_4 \succ q_9 \succ q_5 \succ q_7 \succ q_3 \succ q_6 \succ q_1 \succ q_2.$$

This ranking shows that the medicine q_8 is the most important for the treatment of the coronavirus disease A.

9 | COMPARISON AND DISCUSSION

A comparative analysis among the methods of Ma[38] and Yang and Hu[39] with our proposed method is discussed in this section. On one hand, in light of the numerical example of the previous section, we compare the methods of Ma, Yang, and Hu, with our proposed method. On the other hand, for the drawbacks of the above mentioned methods that cannot make a decision in some situations for example when $\beta = 0.5$, we find that proposed method can make up for this defect. In the study of multiple attributes decision making problems with fuzzy information, there are many decision making methods based on a fuzzy binary relation. However, not all multiple attributes decision making problems can be characterized by a fuzzy binary relation. For this reason, we set methods to solve multiple attributes decision making problems with fuzzy information based on the optimistic multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy upward rough set based on fuzzy upward β-covering. Furthermore, by comparative analysis, we find that our proposed method is more widely used than the above mentioned methods based on a fuzzy binary relation.

9.1 | Advantages/significance and suitability

(i) The proposed transfer function is suitable for computing the fuzzy preference degree of Pan et al.[45] for the construction of upward/downward fuzzy preference relations.

(ii) Another novelty of the proposed method is that it can be applied to form fuzzy preference relation from simple fuzzy set rather to be a preassumed one. Further the very same method can be applied to construct a fuzzy β-coverings called fuzzy upward β-coverings.

(iii) The proposed method more suitable to construct the upward consistency matrices of experts which satisfy the upward additive consistency and the upward order consistency based on upward fuzzy preference relations.

(iv) The advantage of the proposed model is that, it can be used for ranking of the feasible alternatives with auto adjustable weights.

(v) The present work is more technically advanced as compared with other existing techniques. As stated in Definitions 7, 12, 13, and 14 other existing literature/techniques become the particular cases of the present technique.

10 | CONCLUSION

In this study, we have discussed a new type of rough sets called α-upward fuzzified preference rough sets using upward fuzzy preference relation. On the basis of α-upward fuzzified preference rough sets, we introduced approximate precision, rough degree, approximate quality, and their mutual relationships. Furthermore, we presented the idea of new types of fuzzy upward β-coverings, fuzzy upward β-neighborhoods and fuzzy upward complement β-neighborhoods and some relative properties are discussed. We further proposed a new type of upward lower and upward upper approximations by applying an upward β-neighborhoods. After employing an upward β-neighborhoods based upward rough set approach. We observed that every rough set in a universe can be approximated by only six sets, where the lower and upper approximations of each set in the six sets are still lying among these six sets. The relationships among these six sets are established. Subsequently, we presented the idea to combined the fuzzy implicator and t-norm to introduce multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy upward rough set applying fuzzy upward β-covering and some relative properties are discussed. Finally we presented a new technique/algorithm for the selection of medicine using multigranulation $(\mathcal{I}, \mathcal{T})$-fuzzy upward rough set.

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