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Gradient estimates for a nonlinear elliptic equation on smooth metric measure spaces and applications

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ABSTRACT

In this paper local and global gradient estimates are obtained for positive solutions to the following nonlinear elliptic equation

$$\Delta_f u + p(x)u + q(x)u^\alpha = 0,$$

on complete smooth metric measure spaces $(M^N, g, e^{-f}dv)$ with ∞-Bakry-Émery Ricci tensor bounded from below, where α is an arbitrary real constant, $p(x)$ and $q(x)$ are smooth functions. As an application, Liouville-type theorems for various special cases of the equation are recovered. Furthermore, we discuss nonexistence of smooth solution to Yamabe type problem on $(M^N, g, e^{-f}dv)$ with nonpositive weighted scalar curvature.

1. Introduction

1.1. Introduction

This paper is concerned with positive smooth solutions to the following nonlinear elliptic equation

$$\Delta_f u + p(x)u + q(x)u^\alpha = 0, \tag{1.1}$$

on a complete smooth metric measure spaces $(M^N, g, e^{-f}dv)$, otherwise known as a weighted Riemmanian manifold. Here α is an arbitrary real constant, $p(x)$ and $q(x)$ are smooth functions. If $p(x)$ and $q(x)$ are zeros, then (1.1) reduces to the f-harmonic equation

$$\Delta_f u = 0, \tag{1.2}$$

which is known not to admit any bounded solution different from constant on nonnegative Bakry-Émery Ricci tensor [1]. Suppose one chooses specific values for the functions $p(x)$ and $q(x)$ and constant α, (1.1) becomes Yamabe problem for f being a constant [2] (see next section and [3] for further discussion). Another special case of (1.1) when $p(x) \equiv 0$ and α restricted to $1 \le \alpha < (N+2)/(N-2)$ was studied by Gidas and Spruck [4]. They proved that any nonnegative solution to $\Delta_f u + q(x)u^\alpha = 0$ for $1 \le \alpha < (N+2)/(N-2)$ is identically zero. Li [5] proved this result under mild assumption on $q(x)$ for $1 < \alpha < N/(N-2)$, $N \ge 4$. Physical applications of (1.1) are found in the theory of stellar structure in Astrophysics ($N = 3$) and Yang Mills' problem for $N = 4$ and $\alpha = (N+2)/(N-3)$ in Physics (see [4, 6]). See also [4, Appendix B] for abstract examples.

This paper aims at presenting improved local and gradient estimates on positive solutions to (1.1), and applying the estimates so obtained to establish various Liouville type theorems and to discuss nonexistence of positive solution of a special case (Yamabe type problem) on complete smooth metric measure spaces with Bakry-Émery Ricci tensor bounded from below.

Recently, there have been many interesting results on gradient estimates and Liouville type theorems on either Riemannian manifolds or smooth metric measure spaces. The pioneering work on gradient estimates can be traced back to Li and Yau [7] where they derived gradient estimates on positive solutions to the heat equation on manifolds with Ricci tensor bounded from below. They built on [8] which first proved a gradient estimate for harmonic functions via the maximum principle. This estimate was applied to obtain a Liouville theorem. A Liouville theorem says that a bounded positive solution to the harmonic equation is constant. Then, Hamilton [9] proved an elliptic type gradient estimate for the heat equation. But this Hamilton type of estimates is a global result which requires the heat equation defined on closed manifolds. Souplet and Zhang [10] later proved a localized version of Hamilton type gradient estimate by combining Li-Yau's Harnack inequality [7] and Hamilton's gradient estimate [9]. For recent results

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on smooth metric measure spaces see for examples [1, 11, 12, 13] and references therein (see also [21, 22] when f is constant). In particular, Brighton [1] proved an elliptic gradient estimate for positive weighted-harmonic functions by applying Yau's idea to function u^ϵ $(0<\epsilon<1)$ instead of $\log u$ used in [14], and hence obtained a Liouville theorem for positive bounded weighted harmonic functions with nonnegative ∞-Bakry-Émery Ricci tensor. Inspired by the idea in [1] and [12], gradient estimates without any restriction on $|\nabla f|$ are proved in this paper.

1.2. Main result

Let $(M^N, g, e^{-f}dv)$ be an N-dimensional smooth metric measure space. Fix a point x_0 and denote by $r(x)=d(x,x_0)$, a distance function from x_0 to x with respect to g. Denote by $B(x_0,R)$ a ball of radius $R>0$ and centred at x_0. Denote the gradient operator by ∇ and the norm with respect to g by $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(B(x_0,R))} =: \sup_{(B(x_0,R))}|\cdot|$. Our main result on the local gradient estimates is stated below.

Theorem 1.1. *Let $(M^N, g, e^{-f}dv)$ be an N-dimensional complete smooth metric measure space with $Ric_f(B(x_0,2R)) \geq -(N-1)K$ for some $K\geq 0$ and $R>1$. Suppose that $u(x)\leq A$ is a positive smooth solution to (1.1) in $B(x_0,2R)$, $x_0\in M$ is fixed. Then the following inequality holds on $B(x_0,2R)$*

$$|\nabla u(x)|^2 \leq A^2\Bigg[C_1\Big(\sup_{B(x_0,2R)}\{u^{\epsilon+\alpha-1}\}\|\nabla q^+\|_\infty + \sup_{B(x_0,2R)}\{u^\epsilon\}\|\nabla p^+\|_\infty\Big)^2 + C_2\Big([(\epsilon+\alpha-1)q]^+\sup_{B(x_0,2R)}\{u^{\alpha-1}\}+\epsilon p^+ + (N-1)K\Big) + C_3\Big(\frac{1+|\mu|}{R}\Big)\Bigg], \tag{1.3}$$

where $p^+=\max\{p(x),0\}$, $q^+=\max\{q(x),0\}$, $\mu := \max_{\{x|d(x,x_0)=1\}}\Delta_f r(x)$, $r(x)$ is the distance from a fixed point x_0 to point x in M, $\epsilon\in(0,1)$ and C_1, C_2, C_3 are positive constants depending on N.

Letting $R\to\infty$, leads to the following global estimates on complete noncompact smooth metric measure spaces.

Corollary 1.2. *Let $(M^N, g, e^{-f}dv)$ be an N-dimensional complete noncompact smooth metric measure space with $Ric_f \geq -(N-1)K$ for some $K\geq 0$. Suppose that $u(x)\leq A$ is a positive smooth solution to (1.1). Then the following inequality holds*

$$|\nabla u(x)|^2 \leq A^2\Bigg[C_1\Big(\sup_{M^N}\{u^{\epsilon+\alpha-1}\}\|\nabla q^+\|_\infty + \sup_{M^N}\{u^\epsilon\}\|\nabla p^+\|_\infty\Big)^2 + C_2\Big([(\epsilon+\alpha-1)q]^+\sup_{M^N}\{u^{\alpha-1}\}+\epsilon p^+ + (N-1)K\Big)\Bigg], \tag{1.4}$$

where $p^+=\max\{p(x),0\}$, $q^+=\max\{q(x),0\}$ $\epsilon\in(0,1)$ and C_1 and C_2 are positive constants depending on N.

In order to fix other notations appearing in the results, background information about smooth metric measure spaces and Yamabe type problem are discussed in Section 2. The proof of main results and the applications are given in Section 3 and 4, respectively.

2. Background

2.1. Smooth metric measure spaces

Let (M^N, g) be an N-dimensional complete manifold with the Riemannian metric tensor g, dv be volume element and f be a C^∞ real-valued function on M. A smooth metric measure space is defined by the triple $(M^N, g, e^{-f}dv)$, where $e^{-f}dv$ is the weighted measure. The weighted Laplacian, $\Delta_f := \Delta - \langle\nabla f, \nabla\cdot\rangle$, where Δ is the Laplace-Beltrami operator, is defined on $(M^N, g, e^{-f}dv)$. The m-Bakry-Émery tensor is defined by

$$Ric_f^m := Ric + \nabla^2 f - \frac{1}{m}df\otimes df$$

for some constant $m>0$, where Ric is the Ricci tensor of the manifold and ∇^2 is the Hessian with respect to the metric g. When m is infinite we have the ∞-Bakry-Émery tensor

$$Ric_f = Ric + \nabla^2 f.$$

This tensor is related to the gradient Ricci soliton $Ric_f = \lambda g$, where λ is a real constant. A gradient Ricci soliton is said to be shrinking, steady or expanding, if λ is positive, zero or negative, respectively. Ricci solitons play an imporatant role in the theory of singularities for the Ricci flow [15]. The weighted Laplacian and the Bakry-Émery tensor are related by Bochner formula

$$\frac{1}{2}\Delta_f(|\nabla u|^2) = |\nabla^2 u|^2 + \langle\nabla\Delta_f u, \nabla u\rangle + Ric_f(\nabla u, \nabla u).$$

2.2. Yamabe type problem

Suppose that f is a constant, the equation

$$\Delta u + p(x)u + q(x)u^\alpha = 0, \quad \alpha>1 \tag{2.1}$$

is equivalent to Yamabe problem on noncompact Riemannian manifold [16] by conformal deformation of the scalar curvature. Clearly, setting $\widetilde{g} = u^{4/N-2}g$, $N\geq 3$, $u>0$, then for $\mathcal{R}(x)$, the scalar curvature of g and $\widetilde{\mathcal{R}}\in C^\infty(M)$, the scalar curvature of $\widetilde{g}$, we have the relation

$$\Delta u - \frac{N-2}{4(N-1)}\mathcal{R}(x)u + \frac{N-2}{4(N-1)}\widetilde{\mathcal{R}}(x)u^{\frac{N+2}{N-2}} = 0, \tag{2.2}$$

which is of the form (2.1). Yamabe problem demands the existence of a positive everywhere defined solution of (2.2). In the case of compact M^N and constant $\mathcal{R}$, the existence of u in (2.2) has been completely determined by Schoen [17]. Indeed, the existence and uniqueness of such solution depends on the geometry of the underlying manifold. For further discussions on existence, uniqueness and a priori estimates of (2.1) (resp. Yamabe-type equation) see [18].

Similarly, for nonconstant f and a special α, (1.1) is related to the Euler-Lagrange equation for the weighted Yamabe quotient on compact smooth metric measure spaces

$$\Delta_f u - \frac{m+N-2}{4(m+N-1)}\mathcal{R}_f^m u - c_1(\Lambda)u^{\frac{m+N}{m+N-2}}e^{\frac{f}{m}} + c_2(\Lambda)u^{\frac{m+N+2}{m+N-2}} = 0, \quad m>0, \tag{2.3}$$

where $\mathcal{R}_f^m$ is the weighted scalar curvature defined by

$$\mathcal{R}_f^m := \mathcal{R} + 2\Delta f - \frac{m+1}{m}|\nabla f|^2$$

and

$$\Lambda(g, e^{-f}dv) = \inf_{0\neq u\in C^\infty(M)}\left\{\frac{\left(\int|\nabla u|^2 + \frac{m+N-2}{4(m+N-1)}\mathcal{R}_f^m u^2\right)\left(\int|u|^{\frac{2(m+N-1)}{m+N-2}}e^{\frac{f}{m}}\right)^{\frac{2m}{n}}}{\left(\int|u|^{\frac{2(m+N)}{m+N-2}}\right)^{\frac{2m+N-2}{N}}}\right\}$$

which is called the weighted Yamabe constant. Here, all integrals are with respect to the weighted measure. Thus, the weighted volume can be conformally deformed. In fact, setting

$$(M, \widetilde{g}, e^{-\widetilde{f}}d\widetilde{v}, m) = (M, e^{\frac{2\sigma}{m+N-2}}g, e^{\frac{(m+N)\sigma}{m+N-2}}e^{-f}dv)$$

for some $\sigma\in C^\infty(M)$, then the weighted Yamabe quotient is conformally invariant. The situation where $\Lambda=0$ implies $c_1(\Lambda)=c_2(\Lambda)=0$ in (2.3) which is related to situation when $q(x)\equiv 0$ in (1.1). Case [19] also shows that Yamabe-type problem on $(M, g, e^{-f}dv)$ interpolates between Yamabe problem and the problem of finding minimizers for Perelman's ν-entropy.

3. Results

Suppose u is a positive solution to (1.1) with $u \le A$ for some positive constant A. Scaling $u \to \widetilde{u} = u/A$, then $0 < \widetilde{u} \le 1$ and $\widetilde{u}$ solves

$$\Delta_f \widetilde{u} + p(x)\widetilde{u} + \widetilde{q}(x)\widetilde{u} = 0$$

with $\widetilde{q} = A^{\alpha-1}q$. Owing to this, assume $0 < u \le 1$ without loss of generality and let $h = u^{\varepsilon}$ for some constant $\varepsilon \in (0,1)$ to be determined.

3.1. Basic lemma

We now state and prove a basic lemma that will play fundamental role in the proof of the main results. This lemma can be viewed as an extension of [1].

Lemma 3.1. *Let $u(x) \le A$ be a positive smooth solution to (1.1) in $B(x_0, 2R)$ with $Ric_f(B(x_0,2R)) \ge -(N-1)K$ for some $K \ge 0$ and $R > 1$. Denote $\widetilde{u} = u/A$ and $h = \widetilde{u}^{\varepsilon}$ for $\varepsilon \in (0,1)$. Then there exists a positive constant δ satisfying*

$$\frac{2(\varepsilon-1)}{N\varepsilon\delta} + \frac{1}{N} \ge 0 \tag{3.1}$$

such that the following inequality

$$\begin{aligned}\frac{1}{2}\Delta_f(|\nabla h|^2) \ge{}& \left(\frac{(\varepsilon-1)^2}{N\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon} + \frac{2\delta(\varepsilon-1)}{N\varepsilon}\right)\frac{|\nabla h|^4}{h^2} + \frac{\varepsilon-1}{\varepsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)\\ &- \left([(\varepsilon+\alpha-1)q]^+ h^{\frac{\alpha-1}{\varepsilon}} + \varepsilon p^+ + (N-1)K\right)|\nabla h|^2 \\ &- \varepsilon h\left(h^{\frac{\alpha-1}{\varepsilon}}\langle\nabla h, \nabla q\rangle + \langle \nabla h, \nabla p\rangle\right)\end{aligned} \tag{3.2}$$

holds, where $p^+ = \max\{p(x), 0\}$ and $q^+ = \max\{q(x), 0\}$.

Proof. Let $h = u^{\varepsilon}$, where $\varepsilon \in (0,1)$. Notice that tilde on u is dropped for convinience sake. Direct computation gives $\varepsilon^2 \frac{|\nabla u|^2}{u^2} = \frac{|\nabla h|^2}{h^2}$. Applying the Bochner formula to h and using the inequality $|\nabla^2 h| \ge \frac{1}{N}(\Delta h)^2$ (obtained by Cauchy-Schwarz inequality) yields

$$\frac{1}{2}\Delta_f(|\nabla h|^2) \ge \frac{1}{N}(\Delta h)^2 + \langle \nabla \Delta_f h, \nabla h\rangle + Ric_f(\nabla h, \nabla h). \tag{3.3}$$

Now compute

$$\begin{aligned}\Delta_f h &= \varepsilon(\varepsilon-1)u^{\varepsilon}\frac{|\nabla u|^2}{u^2} + \varepsilon u^{\varepsilon-1}\Delta_f u\\ &= \frac{(\varepsilon-1)}{\varepsilon}\frac{|\nabla h|^2}{h} - \varepsilon q h^{1+\frac{\alpha-1}{\varepsilon}} - \varepsilon p h,\end{aligned} \tag{3.4}$$

where (1.1) was used to obtain the last equality.

$$\begin{aligned}\langle \nabla h, \nabla(\Delta_f h)\rangle &= \left\langle \nabla h, \nabla\left(\frac{(\varepsilon-1)}{\varepsilon}\frac{|\nabla h|^2}{h} - \varepsilon q h^{1+\frac{\alpha-1}{\varepsilon}} - \varepsilon p h\right)\right\rangle\\ &= \frac{(\varepsilon-1)}{\varepsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2) - \frac{(\varepsilon-1)}{\varepsilon}\frac{|\nabla h|^4}{h^2}\\ &\quad - \varepsilon h^{1+\frac{\alpha-1}{\varepsilon}}\langle \nabla h, \nabla q\rangle\\ &\quad - (\varepsilon+\alpha-1)h^{\frac{\alpha-1}{\varepsilon}} q|\nabla h|^2 - \varepsilon h\langle \nabla h, \nabla p\rangle - \varepsilon p|\nabla h|^2.\end{aligned} \tag{3.5}$$

Also

$$\begin{aligned}\frac{1}{N}(\Delta h)^2 &= \frac{1}{N}(\Delta_f h + \langle \nabla f, \nabla h\rangle)^2\\ &= \frac{1}{N}\left(\frac{(\varepsilon-1)}{\varepsilon}\frac{|\nabla h|^2}{h} - \varepsilon q h^{1+\frac{\alpha-1}{\varepsilon}} - \varepsilon p h + \langle \nabla f, \nabla h\rangle\right)^2\\ &= \frac{(\varepsilon-1)^2}{N\varepsilon^2}\frac{|\nabla h|^4}{h^2}\\ &\quad + \frac{2(\varepsilon-1)}{N\varepsilon}\frac{|\nabla h|^2}{h}(\langle \nabla f, \nabla h\rangle - \varepsilon q h^{1+\frac{\alpha-1}{\varepsilon}} - \varepsilon p h)\\ &\quad + \frac{1}{N}(\langle \nabla f, \nabla h\rangle - \varepsilon q h^{1+\frac{\alpha-1}{\varepsilon}} - \varepsilon p h)^2.\end{aligned} \tag{3.6}$$

Substituting (3.4)–(3.6) and the condition $Ric_f \ge -(N-1)K, K > 0$ into (3.3) yields

$$\begin{aligned}\frac{1}{2}\Delta_f(|\nabla h|^2) \ge{}& \left(\frac{(\varepsilon-1)^2}{N\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}\right)\frac{|\nabla h|^4}{h^2} + \frac{2(\varepsilon-1)}{N\varepsilon}\frac{|\nabla h|^2}{h}\Big(\langle \nabla f, \nabla h\rangle\\ &- \varepsilon q h^{1+\frac{\alpha-1}{\varepsilon}} - \varepsilon p h) + \frac{1}{N}\left(\langle \nabla f, \nabla h\rangle - \varepsilon q h^{1+\frac{\alpha-1}{\varepsilon}} - \varepsilon p h\right)^2\\ &+ \frac{(\varepsilon-1)}{\varepsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)\\ &- \left[(\varepsilon+\alpha-1)q h^{\frac{\alpha-1}{\varepsilon}} + \varepsilon p + (N-1)K\right]|\nabla h|^2\\ &- \varepsilon\left[h^{1+\frac{\alpha-1}{\varepsilon}}\langle \nabla h, \nabla q\rangle + h\langle \nabla h, \nabla p\rangle\right].\end{aligned} \tag{3.7}$$

There are two cases to examine here. First, for any fixed point x_0, if there exists a positive constant δ such that $\langle \nabla f, \nabla h\rangle - \varepsilon q h^{1+\frac{\alpha-1}{\varepsilon}} - \varepsilon p h \le \delta \frac{|\nabla h|^2}{h}$ in $B(x, 2R)$, $R > 1$, then

$$\frac{2(\varepsilon-1)}{N\varepsilon}\frac{|\nabla h|^2}{h}\left(\langle \nabla f, \nabla h\rangle - \varepsilon q h^{1+\frac{\alpha-1}{\varepsilon}} - \varepsilon p h\right) \ge \frac{2(\varepsilon-1)}{N\varepsilon}\frac{|\nabla h|^2}{h}\left(\delta\frac{|\nabla h|^2}{h}\right)$$

and (3.7) then implies

$$\begin{aligned}\frac{1}{2}\Delta_f(|\nabla h|^2) \ge{}& \left(\frac{(\varepsilon-1)^2}{N\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}\right)\frac{|\nabla h|^4}{h^2} + \frac{2(\varepsilon-1)}{N\varepsilon}\frac{|\nabla h|^2}{h}\left(\delta\frac{|\nabla h|^2}{h}\right)\\ &+ \frac{1}{N}(\langle \nabla f, \nabla h\rangle - \varepsilon q h^{1+\frac{\alpha-1}{\varepsilon}} - \varepsilon p h)^2 + \frac{(\varepsilon-1)}{\varepsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)\\ &- \left[(\varepsilon+\alpha-1)q h^{\frac{\alpha-1}{\varepsilon}} + \varepsilon p + (N-1)K\right]|\nabla h|^2\\ &- \varepsilon\left[h^{1+\frac{\alpha-1}{\varepsilon}}\langle \nabla h, \nabla q\rangle + h\langle \nabla h, \nabla p\rangle\right].\end{aligned}$$

Now suppose on the contrary that $\langle \nabla f, \nabla h\rangle - \varepsilon q h^{1+\frac{\alpha-1}{\varepsilon}} - \varepsilon p h \ge \delta \frac{|\nabla h|^2}{h}$ at the point x_0. Then we have

$$\begin{aligned}&\frac{2(\varepsilon-1)}{N\varepsilon}\frac{|\nabla h|^2}{h}\left(\langle \nabla f, \nabla h\rangle - \varepsilon q h^{1+\frac{\alpha-1}{\varepsilon}} - \varepsilon p h\right)\\ &\qquad \ge \frac{2(\varepsilon-1)}{N\varepsilon}\frac{|\nabla h|^2}{h}\frac{1}{\delta}\left(\langle \nabla f, \nabla h\rangle - \varepsilon q h^{1+\frac{\alpha-1}{\varepsilon}} - \varepsilon p h\right)^2\end{aligned}$$

and then (3.7) implies

$$\begin{aligned}\frac{1}{2}\Delta_f(|\nabla h|^2) \ge{}& \left(\frac{(\varepsilon-1)^2}{N\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}\right)\frac{|\nabla h|^4}{h^2} + \left(\frac{1}{N} + \frac{2(\varepsilon-1)}{N\varepsilon\delta}\right)(\langle \nabla f, \nabla h\rangle\\ &- \varepsilon q h^{1+\frac{\alpha-1}{\varepsilon}} - \varepsilon p h)^2 + \frac{(\varepsilon-1)}{\varepsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2) - \Big[(\varepsilon+\alpha-1)q h^{\frac{\alpha-1}{\varepsilon}}\\ &+ \varepsilon p + (N-1)K\Big]|\nabla h|^2 - \varepsilon\left[h^{1+\frac{\alpha-1}{\varepsilon}}\langle \nabla h, \nabla q\rangle + h\langle \nabla h, \nabla p\rangle\right]\\ \ge{}& \left(\frac{(\varepsilon-1)^2}{N\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon} + \frac{2\delta(\varepsilon-1)}{N\varepsilon}\right)\frac{|\nabla h|^4}{h^2} + \frac{(\varepsilon-1)}{\varepsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)\\ &- \left[(\varepsilon+\alpha-1)q h^{\frac{\alpha-1}{\varepsilon}} + \varepsilon p + (N-1)K\right]|\nabla h|^2\\ &- \varepsilon\left[h^{1+\frac{\alpha-1}{\varepsilon}}\langle \nabla h, \nabla q\rangle + h\langle \nabla h, \nabla p\rangle\right]\end{aligned}$$

since $\frac{1}{N} + \frac{2(\varepsilon-1)}{N\varepsilon\delta} \ge 0$.

Therefore in the two cases (3.7) yields

$$\begin{aligned}\frac{1}{2}\Delta_f(|\nabla h|^2) \ge{}& \left(\frac{(\varepsilon-1)^2}{N\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon} + \frac{2\delta(\varepsilon-1)}{N\varepsilon}\right)\frac{|\nabla h|^4}{h^2} + \frac{(\varepsilon-1)}{\varepsilon}\frac{\nabla h}{h}\nabla(|\nabla h|^2)\\ &- \left[[(\varepsilon+\alpha-1)q]^+ h^{\frac{\alpha-1}{\varepsilon}} + \varepsilon p^+ + (N-1)k\right]|\nabla h|^2\\ &- \varepsilon\left[h^{1+\frac{\alpha-1}{\varepsilon}}\langle \nabla h, \nabla q\rangle + h\langle \nabla h, \nabla p\rangle\right]\end{aligned}$$

where $p^+ = \max\{p(x), 0\}$, and the estimates holds on all of $B(x_0, 2R)$.

This completes the proof. □

3.2. Proof of Theorem 1.1

In order to prove Theorem 1.1, the maximum principle will be applied on (3.2) and bound will be obtained on $|\nabla h|$. It is convinient to choose δ such that the coefficient of $\frac{|\nabla h|^4}{h^2}$ in (3.2) is positive, that is,

$$\frac{(\varepsilon-1)^2}{N\varepsilon^2}-\frac{\varepsilon-1}{\varepsilon}+\frac{2\delta(\varepsilon-1)}{N\varepsilon}>0.$$

Recall that $\varepsilon\in(0,1)$ and $\delta>0$. In particular, choosing $\varepsilon=\frac{4}{N+4}$ and lettting $\delta\to\frac{N}{2}$, then (3.1) holds and (3.2) becomes

$$\begin{aligned}\frac{1}{2}\Delta_f(|\nabla h|^2)\geq{}&\frac{N}{16}\frac{|\nabla h|^4}{h^2}-\frac{N}{4}\frac{\nabla h}{h}\nabla(|\nabla h|^2)\\&-\varepsilon h\Big[h^{\frac{\alpha-1}{\varepsilon}}\langle\nabla h,\nabla q\rangle+\langle\nabla h,\nabla p\rangle\Big]\\&-\Big([(\varepsilon+\alpha-1)q]^+\sup_{B(x,2R)}\{u^{\alpha-1}\}+\varepsilon p^++(N-1)K\Big)|\nabla h|^2.\end{aligned}\tag{3.8}$$

Cut-off function. Define a C^2 cut-off function ψ on $\mathbb{R}^+=[0,+\infty)$ [7], (see also [1]) such that $\psi(t)=1$ for $t\in[0,R]$, $\psi(t)=0$ for $t\in[2R,+\infty)$ and $\psi(t)\in[0,1]$ satisfying

$$0\geq\frac{\psi'(t)}{\sqrt{\psi(t)}}\geq-\frac{C}{R}\quad\text{and}\quad|\psi''(t)|\leq\frac{C}{R^2}$$

for some positive constant C.

Denote by $r(x):=d(x,x_0)$ the distance function. Let

$$\varphi:=\psi(r(x)).$$

By the argument of Calabi-Yau [8] one can assume without loss of generality that φ is a smoothly supported function in $B(x_0,2R)$. Hence, we have

$$\frac{|\nabla\varphi|^2}{\varphi}\leq\frac{C}{R^2}\tag{3.9}$$

and $\Delta_f\varphi=\psi'\Delta_f r+\psi''|\nabla r|^2$ in $B(x_0,2R)$. By the weighted Laplacian comparison theorem [20] $\Delta_f r(x)\leq\mu+(N-1)K(2R-1)$, where $\mu:=\max_{x|d(x,x_0)=1}\Delta_f r(x)$. Hence at x, we have

$$\Delta_f\varphi\geq-\frac{C}{R^2}-\frac{C[\mu+(N-1)K(2R-1)]}{R},\tag{3.10}$$

where C is a positive constant.

We begin the proof of Theorem 1.1. Set $G=\varphi|\nabla h|^2$. Suppose G achieves its maximum at the point $x_1\in B(x_0,2R)$ and assume $G(x_1)>0$, otherwise the proof will be trivial. Then at x_1, it holds that $\nabla G=0$ which implies

$$\nabla(|\nabla h|^2)=-\frac{|\nabla h|^2}{\varphi}\nabla\varphi\tag{3.11}$$

and

$$\Delta_f G\leq 0.\tag{3.12}$$

Now, by application of (3.8), (3.11) and (3.12)

$$\begin{aligned}0\geq{}&\Delta_f G\\={}&\varphi\Delta_f(|\nabla h|^2)+|\nabla h|^2\Delta_f\varphi+2\nabla\varphi\nabla(|\nabla h|^2)\\={}&\varphi\Delta_f(|\nabla h|^2)+\frac{\Delta_f\varphi}{\varphi}G-2\frac{|\nabla\varphi|^2}{\varphi^2}G\\\geq{}&2\varphi\Bigg[\frac{N}{16}\frac{|\nabla h|^4}{h^2}-\frac{N}{4}\frac{\nabla h}{h}\nabla(|\nabla h|^2)-\frac{4}{N+4}\Big[h^{1+\frac{\alpha-1}{\varepsilon}}\langle\nabla h,\nabla q\rangle+h\langle\nabla h,\nabla p\rangle\Big]\\&-\Big([(\varepsilon+\alpha-1)q]^+\sup_{B(x_0,2R)}\{u^{\alpha-1}\}+\varepsilon p^++(N-1)K\Big)|\nabla h|^2\Bigg]\\&+\frac{\Delta_f\varphi}{\varphi}G-2\frac{|\nabla\varphi|^2}{\varphi^2}G\\={}&\frac{N}{8}\frac{G^2}{\varphi h^2}+\frac{N}{2}\frac{\nabla h}{h}\nabla\varphi\frac{G}{\varphi}-\frac{8}{N+4}\Big[h^{1+\frac{\alpha-1}{\varepsilon}}\langle\nabla h,\nabla q\rangle+h\langle\nabla h,\nabla p\rangle\Big]\varphi\\&-2\Big([(\varepsilon+\alpha-1)q]^+\sup_{B(x_0,2R)}\{u^{\alpha-1}\}+\varepsilon p^++(N-1)K\Big)G\\&+\frac{\Delta_f\varphi}{\varphi}G-2\frac{|\nabla\varphi|^2}{\varphi^2}G.\end{aligned}$$

Multiplying both sides by $\frac{\varphi}{G}$, we obtain

$$\begin{aligned}\frac{N}{8}\frac{G}{h^2}\leq{}&-\frac{N}{2}\frac{\nabla h}{h}\nabla\varphi\\&+2\Big([(\varepsilon+\alpha-1)q]^+\sup_{B(x_0,2R)}\{u^{\alpha-1}\}+\varepsilon p^++(N-1)K\Big)\varphi\\&+\frac{8}{N+4}\Big[h^{1+\frac{\alpha-1}{\varepsilon}}\langle\nabla h,\nabla q\rangle+h\langle\nabla h,\nabla p\rangle\Big]\frac{\varphi^2}{G}-\Delta_f\varphi+2\frac{|\nabla\varphi|^2}{\varphi}.\end{aligned}\tag{3.13}$$

There is need to control the first and third terms on the right hand side of (3.13). Note by the Cauchy-Schwarz inequality with $\beta\in(0,1)$

$$\begin{aligned}-\frac{N}{2}\frac{\nabla h}{h}\nabla\varphi&\leq\frac{N}{2}\frac{|\nabla h|}{h}|\nabla\varphi|\\&\leq\frac{N}{4\beta}\frac{|\nabla\varphi|^2}{\varphi}+\frac{N\beta}{4}\varphi\frac{|\nabla h|^2}{h^2}=\frac{N}{4\beta}\frac{|\nabla\varphi|^2}{\varphi}+\frac{N\beta}{4h^2}G\end{aligned}$$

and by elementary inequality

$$\begin{aligned}&\frac{8}{N+4}\Big[h^{1+\frac{\alpha-1}{\varepsilon}}\langle\nabla h,\nabla q\rangle+h\langle\nabla h,\nabla p\rangle\Big]\frac{\varphi^2}{G}\\&\leq\frac{8}{N+4}\Big[h^{1+\frac{\alpha-1}{\varepsilon}}|\nabla q|+h|\nabla p|\Big]|\nabla h|\frac{\varphi^2}{G}\\&\leq\frac{8}{N+4}\Big[h^{1+\frac{\alpha-1}{\varepsilon}}\|\nabla q^+\|_\infty+h\|\nabla p^+\|_\infty\Big]^2|\nabla h|^2\frac{\varphi^2}{G}\\&\leq\frac{8}{N+4}\Big[\sup_{B(x_0,2R)}\{u^{\varepsilon+\alpha-1}\}\|\nabla q^+\|_\infty+\sup_{B(x_0,2R)}\{u^{\varepsilon}\}\|\nabla p^+\|_\infty\Big]^2\varphi,\end{aligned}$$

where $\|\cdot\|_\infty=\|\cdot\|_{L^\infty(B(x_0,R))}=:\sup_{(B(x_0,R))}|\cdot|$ is the norm with respect to g. Substituting the last two inequalities into (3.13) yields

$$\begin{aligned}\Big(\frac{(1-2\beta)N}{8}\Big)\frac{G}{h^2}\leq{}&2\Big([(\varepsilon+\alpha-1)q]^+\sup_{B(x_0,2R)}\{u^{\alpha-1}\}+\varepsilon p^++(N-1)K\Big)\varphi\\&+\frac{8}{N+4}\Big[\sup_{B(x_0,2R)}\{u^{\varepsilon+\alpha-1}\}\|\nabla q^+\|_\infty+\sup_{B(x_0,2R)}\{u^{\varepsilon}\}\|\nabla p^+\|_\infty\Big]^2\varphi\\&-\Delta_f\varphi+\Big(\frac{N+8\beta}{4\beta}\Big)\frac{|\nabla\varphi|^2}{\varphi}.\end{aligned}\tag{3.14}$$

In particular, choosing $\beta=\frac{1}{4}$ in (3.14) and using the estimates (3.9) and (3.10) we obtain

$$\begin{aligned}\frac{N}{16}\frac{G}{h^2}\leq{}&2\Big([(\varepsilon+\alpha-1)q]^+\sup_{B(x_0,2R)}\{u^{\alpha-1}\}+\varepsilon p^++(N-1)K\Big)\varphi\\&+\frac{8}{N+4}\Big[\sup_{B(x_0,2R)}\{u^{\varepsilon+\alpha-1}\}\|\nabla q^+\|_\infty+\sup_{B(x_0,2R)}\{u^{\varepsilon}\}\|\nabla p^+\|_\infty\Big]^2\varphi\\&+\frac{C[\mu+(N-1)K(2R-1)]}{R}+\frac{\widetilde{C}}{R^2},\end{aligned}\tag{3.15}$$

where $\widetilde{C}$ is a positive constant depending on N. Hence, for $x\in B(x_0,R)$, $R\geq 1$ it follows from (3.14) that

$$\begin{aligned}\frac{N}{16}G(x)\leq{}&\frac{N}{16}G(x_1)\\\leq{}&h^2(x_1)\Bigg[C_4\Big([(\varepsilon+\alpha-1)q]^+\sup_{B(x_0,2R)}\{u^{\alpha-1}\}\\&+\varepsilon p^++(N-1)K\Big)\\&+C_5\Big[\sup_{B(x_0,2R)}\{u^{\varepsilon+\alpha-1}\}\|\nabla q^+\|_\infty+\sup_{B(x_0,2R)}\{u^{\varepsilon}\}\|\nabla p^+\|_\infty\Big]^2\\&+C_6\Big(\frac{|\mu|+1}{R}\Big)\Bigg]\varphi.\end{aligned}\tag{3.16}$$

Using the definition of h, i.e., $h = u^\epsilon$, $u \le A$ and $G = \varphi|\nabla h|^2$, we find that $G = \varphi\epsilon^2 h^2 \frac{|\nabla u|^2}{u^2}$, $\frac{N}{16}G = c(N)\varphi h^2 \frac{|\nabla u|^2}{u^2}$ since ϵ is a positive real number, and

$$|\nabla u(x)|^2 \le C(N)A^2\Bigg[C_4\Big([(\epsilon+\alpha-1)q]^+ \sup_{B(x_0,2R)}\{u^{\alpha-1}\} + \epsilon p^+ + (N-1)K\Big) + C_5\Big[\sup_{B(x_0,2R)}\{u^{\epsilon+\alpha-1}\}\|\nabla q^+\|_\infty + \sup_{B(x_0,2R)}\{u^\epsilon\}\|\nabla p^+\|_\infty\Big]^2 + C_6\Big(\frac{|\mu|+1}{R}\Big)\Bigg],$$

where $c(N), C(N)$ are constants depending on N. This finishes the proof of Theorem 1.1. □

Remark 3.2. Suppose $p(x) \equiv 0 \equiv q(x)$ in (1.1), clearly, the estimate in (1.3) of Theorem 1.1 becomes

$$|\nabla u| \le A\sqrt{\frac{c(N,\mu) + c(N)RK}{R}}$$

which is the Brighotn [1] gradient estimate on the positive f-harmonic function in geodesic ball $B(x,R)$ of $(M, g, e^{-f}dv)$ with $Ric_f \ge (N-1)K$, $K \ge 0$. If in addition to $p = 0 = q$, f is a constant function, then (1.3) reduces to

$$|\nabla u| \le C(N)A\Big(\frac{1+R\sqrt{K}}{R}\Big)$$

which is equivalent to the classical Yau [14] gradient estimate on the positive harmonic function in geodesic ball $B(x,R)$ of Riemannian manifolds with $Ric \ge (N-1)K$, $K \ge 0$.

3.3. Proof of Corollary 1.2

Letting $R \to \infty$, leads to the global estimates (1.4) on complete noncompact smooth metric measure spaces. This completes the proof of Corollary 1.2. □

4. Discussion

4.1. Liouville type theorem

Consider (1.1) and suppose $q(x) \equiv 0$. If $p(x) \not\equiv 0$ with $p^+ = \max\{p(x), 0\} > 0$ being a constant, then by Corollary 1.2

$$|\nabla u|^2 \le C_7 A^2(p^+ + K), \tag{4.1}$$

where C_7 is a positive constant depending only on N. On the other hand, if $p^+ = 0$ (including the case $p \le -K$, $K \ge 0$), then $|\nabla u| \le \widetilde{C}_7 A\sqrt{K}$. Thus, the condition $K = 0$ is required to obtain $|\nabla u| \le 0$ whenever u is a bounded positive solution, (and the solution u would be a constant). Hence the following Liouville type theorem can be obtained immediately from Corollary 1.2.

Proposition 4.1. *Let $(M^N, g, e^{-f}dv)$ be an N-dimensional complete noncompact smooth metric measure space with $Ric_f \ge -(N-1)K, K \ge 0$. Suppose that u is a bounded positive solution to (1.1) with $q(x) \equiv 0$ and $p < 0$ is a constant. Then $u \equiv 1$ is a constant.*

On the other hand, supposing $p^+ = \max\{p(x), 0\} = 0$ when $q(x) \equiv 0$, we obtain the following

Proposition 4.2. *Let $(M^N, g, e^{-f}dv)$ be an N-dimensional complete noncompact smooth metric measure space with $Ric_f \ge -(N-1)K, K \ge 0$. Suppose that u is a bounded positive solution to*

$$\Delta_f u = 0,$$

then

$$|\nabla u| \le C_8 A\sqrt{K}, \tag{4.2}$$

where C_8 is a positive constant depending only on N.

In particular, if $Ric_f \ge 0$, then any bounded positive solution to (4.2) must be a constant. Combining Propositions 4.1 and 4.2 with $Ric_f \ge 0$, we obtain the following.

Corollary 4.3. *Let $(M^N, g, e^{-f}dv)$ be an N-dimensional complete noncompact smooth metric measure space with $Ric_f \ge 0$. If u is a bounded positive solution to (1.1) with $q(x) \equiv 0$ and $p(x) \le 0$ is a constant, then u is a constant.*

Remark 4.4. Suppose $p(x) \equiv 0$ and $q(x) \ge 0$ with $\alpha < 1$. There does not exist any positive solution to

$$\Delta_f u + q(x)u^\alpha = 0, \quad \alpha < 1 \text{ and } q \ge 0.$$

The last claim was recently proved in [11] using a different approach.

4.2. Nonexistence of Yamabe minimizer

First, we remark that Theorem 1.1 (resp. Corollary 1.2) still holds on the condition that $Ric_f^m \ge -(m+N-1)K, K \ge 0$, though the proof may require little modification but without further assumption.

Now consider (1.1) again with $q(x) \equiv 0$ and $p(x) \le 0$. Indeed, choosing

$$p(x) = -\frac{m+N-2}{4(m+N-1)}\mathcal{R}_f^m(x)$$

then (1.1) reads

$$\Delta_f u(x) - \frac{m+N-2}{4(m+N-1)}\mathcal{R}_f^m u = 0 \tag{4.3}$$

which is exactly (2.3) for $\Lambda = 0$ (zero weighted Yamabe constant). Suppose $R_f^m(x)$ (weighted scalar curvature) is a nonpositive constant, one can compare (4.3) to the case $p(x)$ is a nonnegative constant, $\alpha = \frac{m+N+2}{m+N-2}$ and $q(x) \equiv 0$, in which case the equation $\Delta_f u + p(x)u = 0$ does not admit any nonconstant positive solution by Proposition 4.1. By this we obtain the following result (which has also been proved in [11] using parabolic gradient estimates).

Theorem 4.5. *Let $(M^N, g, e^{-f}dv)$ be an N-dimensional ($N \ge 3$) complete noncompact smooth metric measure space with $Ric_f^m \ge 0$. Suppose $\mathcal{R}_f^m \le 0$ is a constant, then there does not exist a positive volume normalized minimizer u such that the weighted Yamabe constant, Λ is zero.*

Proof. It suffices to check the nonexistence of positive smooth solutions to (4.3) since $\Lambda = 0$ results to the equation. Note that the assumption of the theorem that $\mathcal{R}_f^m \le 0$ implies that

$$-\frac{m+N-2}{4(m+N-1)}\mathcal{R}_f^m \ge 0$$

and by the preceeding explanation, the conclusion that there does not exist a positive solution to the equation follows. □

Declarations

Author contribution statement

A. Abolarinwa: Conceived and designed the analysis; Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper.

S.O. Salawu, C.A. Onate: Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper.

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