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The tripartite quantum-memory-assisted entropic uncertainty relation and upper bound on shareability of quantum discord

Hazhir Dolatkhah^{1,2}, Abolhassan Mohammadi^{2✉} & Soroush Haseli³

Quantum discord and quantum uncertainty are two important features of the quantum world. In this work, the relation between entropic uncertainty relation and the shareability of quantum discord is studied. By using tripartite quantum-memory-assisted entropic uncertainty relation, an upper bound for the shareability of quantum discord among different parties of a composite system is obtained. It is also shown that, for a specific class of tripartite states, the obtained relation could be expressed as monogamy of quantum discord. Moreover, it is illustrated that the relation could be generalized and an upper bound for the shareability of quantum discord for multipartite states is derived.

The uncertainty principle plays a crucial role in the field of quantum mechanics and it is known to be one of the fundamental concepts of the quantum world¹. In quantum information theory, the uncertainty principle could be expressed in terms of the Shannon entropy. The entropy was used by Deutsch, as a criterion of uncertainty, which led to the formulation of the most famous form of the entropic uncertainty relation (EUR)². The Deutsch's uncertainty bound was modified by Kraus³, and a year later, it was proved by Maassen and Uffink⁴. The relation states that for two incompatible observables X and Z , the following EUR will hold

$$H(X) + H(Z) \geq \log_2 \frac{1}{c} \equiv q_{MU}, \quad (1)$$

in which $H(Q) = -\sum_k p_k \log_2 p_k$ is the Shannon entropy of the measurable $Q \in \{X, Z\}$, p_k stands for the probability of the outcome k , and the parameter c is defined as $c = \max_{\{\mathbb{X}, \mathbb{Z}\}} |\langle x_i | z_j \rangle|^2$, where $\mathbb{X} = \{|x_i\rangle\}$ and $\mathbb{Z} = \{|z_j\rangle\}$ are the eigenstates of the observables X and Z , respectively. Also, q_{MU} is addressed as the incompatibility measure.

Expanding and modifying the relation is one of the main purposes in the field of quantum information, which is being pursued by many researchers^{5–28}. In⁵, it was found that using the memory particle, the entropic uncertainty could be decreased. It resulted in a new uncertainty relation known as bipartite quantum-memory-assisted entropic uncertainty relation (QMA-EUR). The relation is read as

$$S(X|B) + S(Z|B) \geq q_{MU} + S(A|B), \quad (2)$$

in which $S(A|B)$ is the conditional von-Neumann entropy of ρ_{AB} , and $S(\mathcal{O}|B) = S(\rho_{\mathcal{O}B}) - S(\rho_B)$, $\mathcal{O} \in \{X, Z\}$ are the conditional von-Neumann entropies of the post-measurement states after measuring X and Z on the part A ,

$$\rho_{XB} = \sum_i (|x_i\rangle\langle x_i|_A \otimes I_B) \rho_{AB} (|x_i\rangle\langle x_i|_A \otimes I_B), \quad (3)$$

$$\rho_{ZB} = \sum_j (|z_j\rangle\langle z_j|_A \otimes I_B) \rho_{AB} (|z_j\rangle\langle z_j|_A \otimes I_B). \quad (4)$$

¹RCQI, Institute of physics, Slovak Academy of Sciences, Dúbravská Cesta 9, 84511 Bratislava, Slovakia. ²Department of Physics, University of Kurdistan, Sanandaj, P. O. Box 66177-15175, Iran. ³Faculty of Physics, Urmia University of Technology, Urmia, Iran. ✉email: abolhassanm@gmail.com

The bipartite QMA-EUR could be extended to the tripartite QMA-EUR^{5,6}, where the quantum memories are played by two extra particles B and C . In tripartite QMA-EUR, a quantum state ρ_{ABC} is shared by Alice, Bob, and Charlie, so that Alice, Bob, and Charlie have access to parts A , B , and C , respectively. Then, Alice carries the measurement X or Z on her quantum system. Suppose that Alice measures X . Then, it is Bob's job to minimize his uncertainty about X . On the other hand, if Alice measures Z , then it would be Charlie's task to minimize his uncertainty about Z . The tripartite QMA-EUR is given by^{5,6},

$$S(X|B) + S(Z|C) \geq q_{MU}. \quad (5)$$

Some efforts have been put into modifying and improving the bound presented in Eq. (5)^{29,30}. In³⁰, the lower bound of the tripartite QMA-EUR is improved by adding two additional terms to the lower bound of the relation as

$$S(X|B) + S(Z|C) \geq q_{MU} + \frac{S(A|B) + S(A|C)}{2} + \max\{0, \delta\}, \quad (6)$$

where

$$\delta = \frac{1}{2}[I(A : B) + I(A : C)] - [I(X : B) + I(Z : C)],$$

in which $I(A : B)$ and $I(P : B)$ respectively are mutual information and Holevo quantity, given by

$$I(A : B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \quad (7)$$

$$I(P : B) = S(\rho_B) - \sum_i p_i S(\rho_{B|i}), \quad (8)$$

and the observable P is $P \in \{X, Z\}$. Note that, as the observable P on the part A is measured by Alice, the i -th outcome is obtained with probability $p_i = \text{Tr}_{AB}(\Pi_i^A \rho_{AB} \Pi_i^A)$ and the part B is left in the corresponding state $S(\rho_{B|i}) = \frac{\text{Tr}_A(\Pi_i^A \rho_{AB} \Pi_i^A)}{p_i}$. Recently, it is shown that this lower bound, Eq. (6), is tighter than the bounds that have been introduced^{30,31}.

EURs and QMA-EURs with two observables are the topics we have discussed so far, but QMA-EURs can be generalized to more than two observables. This has been the main subject of many research studies and up to now, many QMA-EURs for more than two observables have been introduced^{28,32–41}. For instance, new QMA-EUR for multipartite systems has been proposed in⁴², where the memory is divided into multiple parts, as follows

$$\sum_{m=1}^N S(M_m|X_m) \geq -\log_2(b) + \frac{N-1}{N} \sum_{m=1}^N S(A|X_m) + \max\{0, \delta^N\}, \quad (9)$$

in which

$$b = \max_{i_N} \left\{ \sum_{i_2 \sim i_{N-1}} \max_{i_1} [|\langle u_{i_1}^1 | u_{i_2}^2 \rangle|^2] \prod_{m=2}^{N-1} |\langle u_{i_m}^m | u_{i_{m+1}}^{m+1} \rangle|^2 \right\}, \quad (10)$$

where $|u_{i_m}^m\rangle$ is the i -th eigenvector of M_m , and $\delta^N = \frac{N-1}{N} \sum_{m=1}^N I(A : X_m) - \sum_{m=1}^N I(M_m : X_m)$. M_m indicates the different incompatible observables and X_m stands for the memory particles for m -th measurement. In this uncertainty game, a multipartite quantum state $\rho_{AX_1 \dots X_N}$ is shared by Alice and the others. Now, Alice measures one of the observables M_m ($m = 1, 2, \dots, N$) on her quantum system. As Alice measures the observable M_m , the X_m 's task will be minimization his uncertainty about M_m .

The QMA-EUR has been realized to have potential applications in various quantum information processing tasks, such as quantum key distribution^{5,43}, quantum metrology⁴⁴, quantum cryptography^{45,46}, quantum randomness^{47,48}, entanglement witness^{49,50}, EPR steering^{51,52}, and so on.

Additionally, several authors have attempted to find relations between quantum correlations and EURs^{53–76}. On the other hand, the monogamy of quantum correlation has broad application in quantum information^{77–83}. In a recent study, Hu and Fan could obtain a new upper bound on quantum discord (QD) through bipartite QMA-EUR⁵³. They also could extract an upper bound on shareability of QD.

In this paper, inspiring from⁵³ and by using tripartite QMA-EUR, an upper bound on shareability of QD will be found. In the beginning, new relations for tripartite QMA-EUR are introduced. Then, it is shown that by using these relations, one could obtain a new upper bound for the shareability of QD. Also, it is shown that for specific states, the obtained relation could be considered as monogamy of QD. Finally, it is exhibited that the above procedure could be generalized to a multipartite system, in which an upper bound for the shareability of QD in a multipartite system is derived.

The paper has been organized as follows: In “Quantum discord” section, the QD will be defined as one of the measures of quantum correlation. In “Tripartite QMA-EUR and shareability of QD” section, the new relation for the tripartite QMA-EUR is expressed and also an upper bound for the shareability of QD is extracted. The results will be summarized in “Conclusion” section..

Quantum discord

QD is another important concept within the field of quantum information. Considerable attention has been paid to QD due to its potential connection with other aspects of quantum information and beyond, including quantum communication, quantum computation, many-body physics, and open quantum dynamics (see⁸⁴ for further details).

The concept of QD of a bipartite quantum system is defined in several ways which could be classified into two wide categories. One of these categories is based on measurement in any one of the subsystems, which will be used in our discussion.

QD is the difference between the total and the classical correlations^{85,86}, namely,

$$D_A(\rho_{AB}) = I(\rho_{AB}) - J_A(\rho_{AB}), \quad (11)$$

in which the subscript of $D_A(\rho_{AB})$ denotes that the measurement has been performed on the subsystem A . The total correlations in state ρ_{AB} measured by the quantum mutual information (7) and the classical correlation $J_A(\rho_{AB})$, which is defined as

$$J_A(\rho_{AB}) = S(\rho_B) - \min_{\Pi_i^A} S(\rho_{B|\Pi_i^A}), \quad (12)$$

where $S(\rho_{B|\Pi_i^A}) = \sum_i p_i S(\rho_{B|i})$ and the minimization is taken over all quantum measurements, Π_i^A , performed on the system A .

Recently, Hu and Fan have investigated a relation between QD and bipartite QMA-EUR⁵³. Their consideration led to an improvement on the upper bounds for QD⁵³. They also considered the effects of the bipartite QMA-EUR on the shearability of quantum correlation among different subsystems. With the use of the bipartite QMA-EUR, Hu and Fan found an upper bound on the shearability of QD among different parties of a composite system, which is given by⁵³

$$D_A(\rho_{AB}) + D_A(\rho_{AC}) \leq S(\rho_A) + \delta_T, \quad (13)$$

in which $\delta_T = S(X|B) + S(Z|B) - q_{MU} - S(A|B)$. They showed that for any tripartite state ρ_{ABC} with $S(\rho_A) = -S(A|BC)$, the above relation can be written as:

$$D_A(\rho_{AB}) + D_A(\rho_{AC}) \leq D_A(\rho_{A:BC}) + \delta_T. \quad (14)$$

This equation can be considered as the released version of the monogamy relation of QD. It applies to all tripartite pure states as well as to extended classes of mixed states⁵³.

Tripartite QMA-EUR and shareability of QD

In this section, inspired by Hu and Fan⁵³, who obtained an upper bound on the shareability of QD among the constituent parties by using bipartite QMA-EUR, we are going to introduce a new upper bound on the shareability of QD by utilizing tripartite QMA-EUR.

New lower bound for the tripartite QMA-EUR. Here, we introduce new tripartite QMA-EURs, which depend on the incompatibility of two quantum measurements, the strong subadditivity (SSA) inequality, the QD, and the classical correlations of a state shared between the observed system and quantum memories.

Theorem 1 For any tripartite state, the following equations hold

$$S(X|B) + S(Z|C) \geq q_{MU} + \frac{1}{2} [S(A|B) + S(A|C)] + \max\{O, \delta'^3\}, \quad (15)$$

$$S(X|B) + S(Z|C) \geq q_{MU} + \frac{1}{2} [S(A|B) + S(A|C)] + \max\{O, \delta''^3\}, \quad (16)$$

where

$$\delta'^3 = \frac{1}{2} \{D_A(\rho_{AB}) + D_A(\rho_{AC}) - J_A(\rho_{AB}) - J_A(\rho_{AC})\}, \quad (17)$$

$$\delta''^3 = \left\{ D_A(\rho_{AB}) + D_A(\rho_{AC}) - \frac{1}{2} [I(A : B) + I(A : C)] \right\}. \quad (18)$$

Proof The theorem is proved using the definition of classical correlation, QD, and tripartite QMA-EUR, Eq. (6). Regarding Eq. (6), one obtains

$$\begin{aligned} S(X|B) + S(Z|C) &\geq q_{MU} + \frac{1}{2} [S(A|B) + S(A|C)] + \frac{1}{2} [I(A : B) + I(A : C)] - I(X : B) - I(Z : C) \\ &\geq q_{MU} + \frac{1}{2} [S(A|B) + S(A|C)] + \frac{1}{2} [I(A : B) + I(A : C) - 2J_A(\rho_{AB}) - 2J_A(\rho_{AC})] \\ &= q_{MU} + \frac{1}{2} [S(A|B) + S(A|C)] + \frac{1}{2} \{D_A(\rho_{AB}) + D_A(\rho_{AC}) - J_A(\rho_{AB}) - J_A(\rho_{AC})\}. \end{aligned} \quad (19)$$

Note that in the second row of the above relation we have applied the definition of the classical correlation, $J_A(\rho_{AY}) = \max_{\Pi^A} I(P : Y)$, where $Y \in \{B, C\}$, and the fact that observables X and Z may not be necessarily the maximizing quantum measurements for $J_A(\rho_{AY})$, so that $J_A(\rho_{AB}) \geq I(X : B)$, and also $J_A(\rho_{AC}) \geq I(Z : C)$. In the last line of the above proof, the definition of QD has been used as well. The other equation of the theorem is proved by following the same procedure. \square

Also, one can utilize above procedure to obtain a relation for QMA-EUR for multipartite system.

Corollary 1.1 For any multipartite state, The equation below is hold

$$\sum_{i=1}^N S(M_i|X_i) \geq -\log_2(b) + \frac{N-1}{N} \sum_{i=1}^N S(A|X_i) + \max\{0, \delta'^N\}, \quad (20)$$

where $\delta'^N = \sum_{i=1}^N D_A(\rho_{AX_i}) - \frac{1}{N} \sum_{i=1}^N I(A : X_i)$.

Proof Regarding Eq. (9) and following the same method used in Theorem. 1, one can arrive at Eq.(20). \square

Bounds on shareability of QD from the tripartite QMA-EUR. Applying the resulted obtained in the previous subsection and following the same approach presented in⁵³, one can obtain an upper bound for the shareability of QD among different subsystems.

Theorem 2 For any tripartite state ρ_{ABC} , we have

$$\Delta_1 + \Delta_2 + S(\rho_A) \geq D_A(\rho_{AB}) + D_A(\rho_{AC}), \quad (21)$$

where

$$\Delta_1 = S(X|B) + S(Z|C) - q_{MU} - \frac{1}{2} [S(A|B) + S(A|C)],$$

and

$$\Delta_2 = \frac{-1}{2} [S(A|B) + S(A|C)].$$

Proof From Eq. (16), one arrives at

$$S(X|B) + S(Z|C) - q_{MU} - \frac{1}{2} [S(A|B) + S(A|C)] + \frac{1}{2} [I(A : B) + I(A : C)] \geq D_A(\rho_{AB}) + D_A(\rho_{AC}). \quad (22)$$

Substituting the following relation

$$\frac{1}{2} [I(A : B) + I(A : C)] = S(\rho_A) - \frac{1}{2} [S(A|B) + S(A|C)], \quad (23)$$

in Eq.(22), one comes to

$$S(X|B) + S(Z|C) - q_{MU} - \frac{1}{2} [S(A|B) + S(A|C)] + S(\rho_A) - \frac{1}{2} [S(A|B) + S(A|C)] \geq D_A(\rho_{AB}) + D_A(\rho_{AC}). \quad (24)$$

Therefore, the theorem has been proved. \square

This is the main result of this work. As can be seen from Eq. (21), our relation contains three terms: $S(\rho_A)$ which implies the entropy of the subsystem A , Δ_1 that depends on the tripartite QMA-EUR, and Δ_2 that is related to the SSA inequality. From the tripartite EUR, we can find $\Delta_1 \geq 0$, and from the SSA inequality, it is realized that $\Delta_2 \leq 0$.

Let us now compare our upper bound (21) with Hu and Fan upper bound (13). Due to the fact that the following two relations

$$S(X|B) + S(Z|B) \geq q_{MU} + S(A|B),$$

and

$$S(X|B) + S(Z|C) \geq q_{MU},$$

are equivalent⁵, it is realized that our bound has one extra term as $-(S(A|B) + S(A|C))$. From SSA inequality, one finds that the term is negative, i.e. $-(S(A|B) + S(A|C)) \leq 0$. Therefore, our bound is tighter than that of Hu and Fang. However, for states that SSA inequality is saturated, the upper bound of Eq. (21) is equivalent to the bound of Eq. (13) obtained by Hu and Fan⁵³.

It is interesting to note that for all tripartite pure and some special mixed tripartite states, our upper bound is a released version of the monogamy relation of QD⁸⁷⁻⁹⁰.

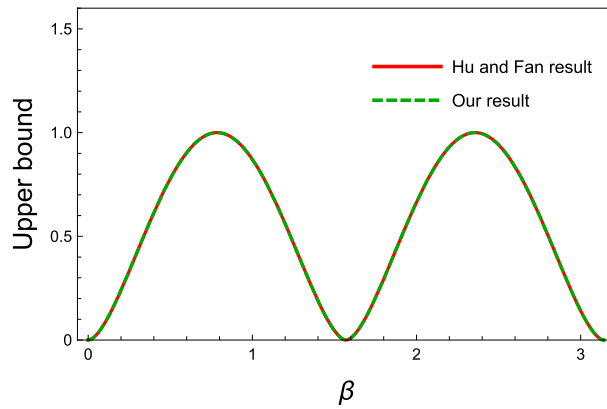


Figure 1. Different upper bounds on the shareability of QD for the state in Eq. (26), versus the state’s parameter β . The figure vividly shows that our upper bound and that of Hu and Fan will be the same. It is due to this point that for any pure state, as Eq.(26), the difference between our upper bound and Hu and Fan upper bound will be eliminated.

Corollary 2.1 For any tripartite state ρ_{ABC} with $S(A) = -S(A|BC)$, one has

$$D_A(\rho_{AB}) + D_A(\rho_{AC}) \leq D_A(\rho_{A:BC}) + \Delta_1 + \Delta_2. \tag{25}$$

Proof The outline of the proof is similar to what we have in⁵³. Due to the fact that $D_A(\rho_{A:BC}) = S(\rho_A)$ and $S(\rho_A) = -S(A|BC)$, it is realized that Eq. (25) is valid for all tripartite pure states. As stated in⁹¹, under a specific condition, the relation $S(\rho_A) = -S(A|BC)$ is reliable even for a mixed state ρ_{ABC} . The relation is true for a mixed state if and only if for the Hilbert space \mathcal{H}_{BC} we have a factorization $\mathcal{H}_{BC} = \mathcal{H}_{(BC)^L} \otimes \mathcal{H}_{(BC)^R}$ in which $\rho_{ABC} = |\psi\rangle_{A(BC)^L} \langle \psi| \otimes \rho_{(BC)^R}$. For this case, it is obtained that $D_A(\rho_{A:BC}) = D_A(|\psi\rangle_{A(BC)^L}) = S(\rho_A)$. \square

Our results should have several useful applications in the areas of quantum information theory. One of the consequences of our inequality (25) is that, if for tripartite pure state $|\psi\rangle_{ABC}$ one finds two observables X and Z that saturate $S(X|B) + S(Z|C) \geq q_{MU}$, then it could be stated that we have the sufficient condition for the monogamy QD. The generalized Greenberger-Horne-Zeilinger (GHZ) state could be implied as one of the examples of the situation.

Examples. To clarify the above-mentioned results, four examples are considered. For the first two examples, we consider pure states, however for the next two examples, mixed states are investigated. In these examples, the observables that are measured on the part A of quantum states are assumed to be the Pauli matrices $X = \sigma_1$ and $Z = \sigma_3$.

Generalized GHZ state. First, let us consider the generalized GHZ states which have the form

$$|gGHZ\rangle = \cos \beta |000\rangle + \sin \beta |111\rangle, \tag{26}$$

where $\beta \in [0, 2\pi)$. In Fig. 1, different upper bounds of the shareability of QD for these states are plotted versus the parameter β . As it was expected, the obtained upper bound (21) coincides with Hu and Fan upper bound (13).

Generalized W state. As the second example, consider the following generalized W state:

$$|gW\rangle = \sin \theta \cos \phi |100\rangle + \sin \theta \sin \phi |010\rangle + \cos \theta |001\rangle, \tag{27}$$

where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. Same as the previous case, it is realized that for this state, the obtained upper bound (21) is exactly the same as that of Hu and Fan (13); shown in Fig. 2.

Werner-GHZ state. As an another example, we consider Werner-GHZ state, defined as

$$\rho_w = (1 - p)|GHZ\rangle\langle GHZ| + \frac{p}{8}\mathbf{I}_{ABC}, \tag{28}$$

where $|GHZ\rangle = 1/\sqrt{2}(|000\rangle + |111\rangle)$ is the GHZ state, and $0 \leq p \leq 1$. In Fig. 3, the upper bounds of the shareability of QD for this state are plotted versus the parameter p . As can be seen, Hu and Fan upper bound (13) is constant as a function of the parameter p , whereas our upper bound (21) is tighter and also it decreases by enhancement of p , and reaches zero at $p = 1$. From physical point of view, this is an acceptable result because at $p = 1$ we have the maximally mixed state and there is no quantum correlation. This physical feature is illustrated properly in our upper bound, however, the Hu and Fan bound does not exhibit such a feature.

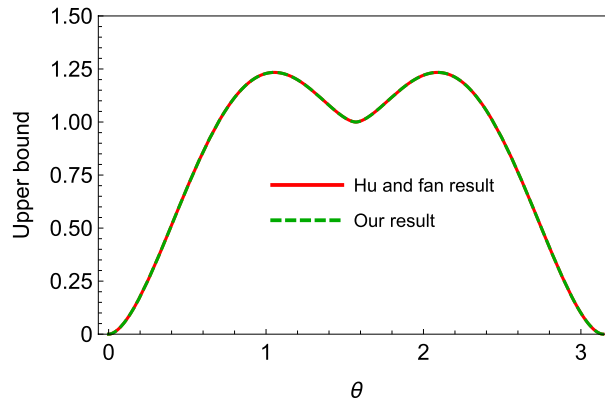


Figure 2. Different upper bounds on the shareability of QD for the state in Eq. (27), versus the state’s parameter θ , where $\phi = \pi/4$. Since the state is pure, our upper bound will be the same as the upper bound of Hu and Fan, which is plotted in the figure.

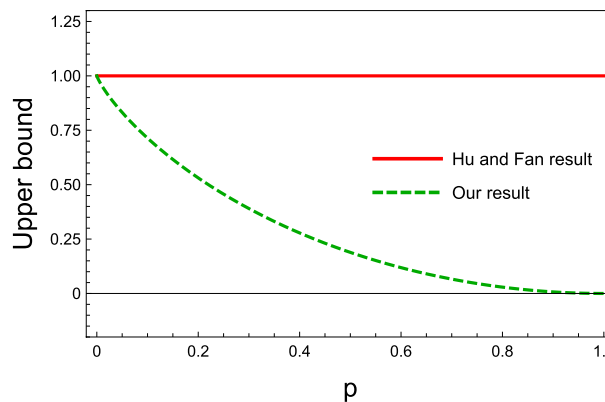


Figure 3. Different upper bounds on the shareability of QD for the state in Eq. (28), versus the state’s parameter p , where $0 \leq p \leq 1$. The solid-red curve stands for Hu and Fan upper bound and the dashed-green curve indicates our upper bound. At $p = 0$, the state is pure, and our bound coincides with Hu and Fan upper bound; as it was expected for a pure state. It is realized that for $0 < p < 1$, our upper bound is tighter than that of Hu and Fan, and for $p = 1$ it reaches zero which shows that at this point there is no quantum correlation.

A mixed three-qubit state. As the last example, let us consider a state of the following form

$$\rho = (1 - p)|GHZ\rangle\langle GHZ| + p|W\rangle\langle W|, \tag{29}$$

where $0 \leq p \leq 1$ is a real number and the usual $|W\rangle$ state is defined as

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle).$$

In Fig. 4, the upper bounds of the shareability of QD for the state in Eq. (29) are plotted versus the parameter p . According to the figure, it is realized that at $p = 0$ and $p = 1$, our upper bound coincides with the upper bound of Hu and Fan because there are pure states. However, for $0 < p < 1$ where the states are mixed, our upper bound is tighter than that of Hu and Fan.

Generalization. An implication of the presented method and results is that they could be generalized to obtain a constraint on the shareability of the QD among different parties of a $(N + 1)$ -partite states. By utilizing the multipartite uncertainty relation with quantum memory, it is possible to find an upper bound for the shareability of multipartite QD. This will be presented in the following theorem.

Theorem 3 For any $N + 1$ -partite state, we have

$$\Delta_1^N + \Delta_2^N + S(\rho_A) \geq \sum_{i=1}^N D_A(\rho_{AX_i}), \tag{30}$$

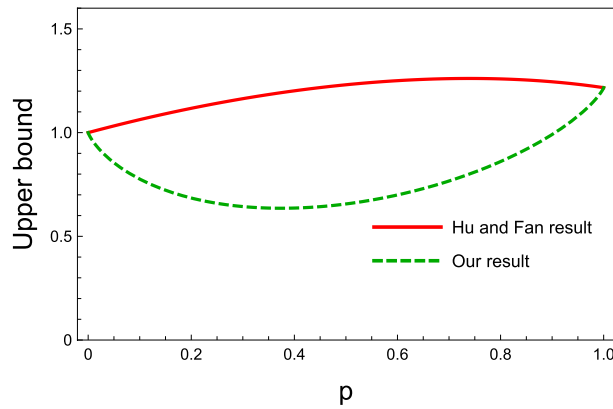


Figure 4. Different upper bounds on the shareability of QD for the state in Eq. (29), versus the state’s parameter p , where $0 \leq p \leq 1$. The solid-red curve stands for Hu and Fan upper bound and the dashed-green curve indicates our upper bound. Since for $p = 0$ and $p = 1$, there is a pure state, our bound completely coincides with that of Hu and Fan. However, the curves illustrate that for $0 < p < 1$, our obtained upper bound is tighter than that of Hu and Fan.

in which

$$\Delta_1^N = \sum_{i=1}^N S(M_i|X_i) + \log_2(b) - \frac{N-1}{N} \sum_{i=1}^N S(A|X_i), \tag{31}$$

$$\Delta_2^N = \frac{-1}{N} \sum_{i=1}^N S(A|X_i). \tag{32}$$

Proof Regarding the Eq. (20), one has

$$\sum_{i=1}^N S(M_i|X_i) + \log_2(b) - \frac{N-1}{N} \sum_{i=1}^N S(A|X_i) + \frac{1}{N} \sum_{i=1}^N I(A : X_i) \geq \sum_{i=1}^N D_A(\rho_{AX_i}), \tag{33}$$

Applying the relation below

$$S(\rho_A) = \frac{1}{N} \left[\sum_{i=1}^N S(A|X_i) + \sum_{i=1}^N I(A : X_i) \right], \tag{34}$$

one comes to

$$\sum_{i=1}^N S(M_i|X_i) + \log_2(b) - \frac{N-1}{N} \sum_{i=1}^N S(A|X_i) + S(\rho_A) - \frac{1}{N} \sum_{i=1}^N S(A|X_i) \geq \sum_{i=1}^N D_A(\rho_{AX_i}). \tag{35}$$

The above equation could be rewritten as

$$\Delta_1^N + \Delta_2^N + S(\rho_A) \geq \sum_{i=1}^N D_A(\rho_{AX_i}). \tag{36}$$

□

Now, let us consider the above result for a four-partite state, i.e. $N = 3$. For this case, Eq. (30) is rewritten as

$$\Delta_1^3 + \Delta_2^3 + S(\rho_A) \geq D_A(\rho_{AB}) + D_A(\rho_{AC}) + D_A(\rho_{AD}), \tag{37}$$

where the quantity Δ_1^3 is given by

$$\Delta_1^3 = S(M_1|B) + S(M_2|C) + S(M_3|D) + \log_2(b') - \frac{2}{3} [S(A|B) + S(A|C) + S(A|D)], \tag{38}$$

and

$$b' = \max_k \left\{ \sum_j \max_i [|\langle u_i^1 | u_j^2 \rangle|^2] |\langle u_j^3 | u_k^3 \rangle|^2 \right\}, \quad (39)$$

in which $|u_i^1\rangle$, $|u_j^2\rangle$, and $|u_k^3\rangle$ are the eigenstates of the three observables M_1 , M_2 , and M_3 , respectively. The other quantity Δ_2^3 is read as

$$\Delta_2^3 = \frac{-1}{3} [S(A|B) + S(A|C) + S(A|D)]. \quad (40)$$

Assume there is a four-partite state ρ_{ABCD} , where the particles A, B, C, and D are respectively sent to Alice, Bob, Charlie, and David. Then, Alice does a measurement on her quantum system with one of the three observables M_m (where $m = 1, 2, 3$) and informs the other about her choice of measurement. If Alice measures M_1 , it is Bob's duty to minimize his uncertainty about M_1 . If M_2 is measured by Alice, it is Charlie's task to minimize his uncertainty about M_2 . And for the last case, if M_3 is measured by Alice, it is David's task to minimize his uncertainty about M_3 .

Conclusion

There are many applications for the tripartite QMA-EUR in quantum information theory; quantum key distribution could be addressed as one of these applications. In the presented work, we introduced another application of tripartite QMA-EUR. It was discussed that using tripartite QMA-EUR, one could obtain an upper bound for the shareability of QD. Our bound includes three terms in which one is related to the entropy of the subsystem that is being measured. The second term is related to the tripartite QMA-EUR, and the third term implies the SSA inequality. In another word, our bound relates tripartite QMA-EUR, SSA inequality, and QD which are known as three important features of quantum information. A comparison was made between the obtained upper bound and Hu and Fan upper bound for some states such as the Werner-GHZ state and a mixed three-qubit state. Results show that our upper bound is tighter than that obtained by Hu and Fan⁵³.

The obtained bound could be applicable in the field of quantum information. This relation can be converted into the monogamy of QD for certain tripartite quantum states. The result indicates that for a tripartite pure state if one can find two observables X and Z that saturate $S(X|B) + S(Z|C) \geq q_{MU}$, then, a sufficient condition for the monogamy of QD is provided.

Furthermore, the work can be generalized to obtain an upper bound on the shareability of QD for multipartite states, indicating that quantum correlations cannot be freely shared.

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Author contributions

H.D. has put forward the main idea. H.D., A.M., and S.H. all contributed to the development and completion of the idea. The calculation and analysing were performed by H.D. and A.M. The final results were discussed by H.D., A.M., and S.H. The first draft of the manuscript was prepared by H.D. and A.M., and then it was reviewed by all authors.

Competing interests

The authors declare no competing interests.

Additional information

Correspondence and requests for materials should be addressed to A.M.

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