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MHD Flow of Sodium Alginate-Based Casson Type Nanofluid Passing Through A Porous Medium With Newtonian Heating

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Casson nanofluid, unsteady flow over an isothermal vertical plate with Newtonian heating (NH) is investigated. Sodium alginate (base fluid)is taken as counter example of Casson fluid. MHD and porosity effects are considered. Effects of thermal radiation along with heat generation are examined. Sodium alginate with Silver, Titanium oxide, Copper and Aluminum oxide are added as nano particles. Initial value problem with physical boundary condition is solved by using Laplace transform method. Exact results are obtained for temperature and velocity fields. Skin-friction and Nusselt number are calculated. The obtained results are analyzed graphically for emerging flow parameters and discussed. It is bring into being that temperature and velocity profile are decreasing with increasing nano particles volume fraction.

The fluid is a particular kind of matter which have no fixed shape and deforms easily due to external pressure[1]. Fluids are mainly of two type's i.e Newtonian and non-Newtonian. Non-Newtonian fluids have numerous industrial applications[2,3]. Furthermore, its application with magnetohydrodynamic (MHD) flow in a porous medium can widely be seen in irrigation problem, biological system, petroleum, textile, polymer industries. More investigations have been published on numerous aspects of MHD non-Newtonian fluid passes over a porous medium[4–7]. The entropy analysis for nanofluid with different type of nano particles and water type base fluid for unsteady MHD flow was studied by[8]. The impact of magnetic field on free convection of nanofluid in a porous medium is presented by[9]. The effects of heat transfer on MHD nanofluid in a porous semi annulus has investigated by[10] using numerical methods. Sheikholeslami *et al.*[11] examined the influence of free convection in a semi annulus enclosure for ferrofluid flow in the presence of magnetic source with the consideration of thermal radiation. The observation of non-uniform magnetic field and variable magnetic field on forced convection heat is investigated by[12,13]. The observation of MHD on fluid flow with heat transfer is studded by[14–16]. Recently[17,18] investigated the nanofluid transportation in a in the presence of magnetic source and porous cavity using CuO nano particles. The influence of external magnetic field for nanofluid as water is a base fluid of free convection flow is studied in[19]. Sheikholeslami and Ganji[20] have investigated the effect of convective heat transfer for the nanofluid by semi analytical and numerical approaches. The same author has also investigated the influence of heat transfer for nanofluid between parallel plates in[21]. The influence of Lorentz forces and convection nanofluid flow is investigated by[22–24]. Dissimilar types of nano particles with water based fluid are studied by[25,26]. The influence of melting heat for nanofluid is studied by[27]. The transportation of nanofluid in porous media is investigated by[28]. The influence of magnetic field for nanofluid with entropy generation is analysed by[29–31].

Nanotechnology is that kind of technology which provides the materials with size less than 100 nm called nanomaterials. On the basis of the structure and their properties, nanomaterials are divided into four categories[32]. Carbon based nano materials, metal based nano materials, Dendrimers and composite. The terminology of

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nanofluid was first investigated by Choi[33]. He defined that the fluids occupying the sizes of particles less than 100 nm is called nanofluid. The categorieswith different attitude of nano particles are particle material, Base fluid, size and concentration, of the nanofluid. Suspend these nano particles into any type of conventional fluid like oil, water, ethylene glycol to make nanofluids. The reason why nano size particles are preferred over micro size particles has been explained by[34]. Nano particles over micro particles, good improvement have seen in thermo physical properties. Nanofluids have various applications such as in air conditioning cooling, automotive, power plant cooling, improving diesel generator efficiency etc.[35]. Usually water, ethylene glycol are utilized as heat transfer base fluids. Different substances are used for the production of nanoparticles, which are generally divided into metallic i.e. copper[36], metal-oxide i.e. CuO[37], chalcogenides sulphides, selenides and telluride's, mentioned[38] and different particles, such like carbon nanotubes[39]. In literature the size of one particle is in between 20 nm[40] and 100 nm[41].

Casson fluid model was first presented by Casson in 1959. Casson fluids in tubes was first studied by Oka[42]. Examples of Casson fluids are honey, blood, soup, jelly, stuffs, slurries, artificial fibers etc. Cassonnanofluid flow with Newtonian heatingpresented by[43]. Sarojamma *et al.*[44] investigated Casson nanofluid past over perpendicular cylinder in the occurrence of a transverse magnetic field with internal heat generation or absorption.

Khalid *et al.*[45] examined unsteady MHD Casson fluid withfree convection flow in a porous medium. Bhattacharyya *et al.*[46] studied systematically magnetohydrodynamic Casson fluid flow over a stretching shrinking sheet with wall mass transfer. Arthur *et al.*[47] studied Casson fluid flow in excess of a perpendicular porous surface, chemical reaction in the existence of magnetic field. Recently, Fetecau *et al.*[48] has investigated fractional nanofluids for natural convection flow over an isothermal perpendicular plate with thermal radiation. Hussanan *et al.*[49] investigates the unsteady heat transfer flow of a non-Newtonian Casson fluid over an oscillating perpendicular plate with Newtonian heating. Recently, Imran *et al.*[50] analyzed the effect of Newtonian heating with slip condition on MHD flow of Casson fluid. MHD flow of Casson fluid with heat transfer and Newtonian heating is analyzed by Hussanan *et al.*[51]. The effect of Newtonian heating for nanofluid is recently investigated by[43,52]. But no work is done until now on heat transfer enhancement in Sodium alginate fluid with additional effects of NH, MHD, porosity, heat generation, and thermal radiation. Silver (Ag), Titanium oxide (TiO_2), Copper (Cu) and Aluminum oxide (Al_2O_3) are nano particles suspended in base fluid. Problem is solved and interpreted graphically with some conclusions.

Mathematical Modeling and solution of the Problem

Sodium alginate with Silver (Ag), Titanium oxide (TiO_2), Copper (Cu) and Aluminum oxide (Al_2O_3) nano particles is considered. Heat transfer, thermal radiation and heat generation are taken. Unsteady flow is over an infinite vertical plate ($\xi > 0$) embedded in a saturated porous medium. MHD effect with uniform magnetic field B of strength B_0 and small magnetic Reynolds number. Initially both the plate and fluid are at rest with constant temperature Θ_∞. At time $t = 0^+$ the plate originates oscillation in its plane $\xi = 0$ according to condition

$$u = UH(t)\cos(\omega t)i;\ \text{ or u} = U\ \sin(\omega t)i;\ \ t > 0 \tag{1}$$

After some time, plate temperature is raised to Θ_w. The fluid is electrically conducting. Therefore, by Maxwell equations

$$\text{div}\mathbf{B} = \mathbf{0},\ \ \text{Curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},\ \ \text{Curl}\mathbf{B} = \mu_e \mathbf{J}. \tag{2}$$

By using Ohm's law

$$\mathbf{J} = \sigma_{nf}(\mathbf{E} + \mathbf{V} \times \mathbf{B}), \tag{3}$$

The quantities ρ_{nf}, μ_e and σ are assumed constants. Magnetic field $\mathbf{B}$ is normal to $\mathbf{V}$. The Reynolds number is so small that flow is laminar. Hence,

$$\frac{1}{\rho_{nf}}\mathbf{J} \times \mathbf{B} = \frac{\sigma_{nf}}{\rho_{nf}}[(\mathbf{V} \times \mathbf{B}_0) \times \mathbf{B}_0] = -\frac{\sigma_{nf} B_0^2 \mathbf{V}}{\rho_{nf}}. \tag{4}$$

Equation for an incompressible Casson fluid flow[53–55]

$$\tau = \tau_0 + \mu\gamma^{\bullet} \tag{5}$$

Or

$$\tau_{ab} = \begin{cases} 2\left(\mu_\eta + \dfrac{p_\lambda}{\sqrt{2\pi}}\right)e_{ab}, & \pi > \pi_c \\ 2\left(\mu_\eta + \dfrac{p_\lambda}{\sqrt{2\pi_c}}\right)e_{ab}, & \pi > \pi_c \end{cases}, \tag{6}$$

where $\pi = e_{ab}e_{ab}$ and e_{ab} is the $(a, b)^{ah}$ factor of the deformation rate, π is represent the product of the factor of deformation rate with itself, π_c is represent the critical value of this product based on the non-Newtonian model, μ_η is represent the plastic dynamic viscosity of the non- Newtonian fluid and P_λ is yield stress of fluid. Under these

	ρ (kgm^{-3})	c_p ($kg^{-1}k^{-1}$)	k ($Wm^{-1}k^{-1}$)	$\beta \times 10^{-5}$ (k^{-1})
$C_6H_9NaO_7(SA)$	989	4175	0.613	0.99
Al_2O_3	3970	765	40	0.85
Cu	8933	385	401	1.67
TiO_2	4250	686.2	8.9528	0.9
Ag	10500	235	429	1.89

Table 1. Thermophysical properties of nanofluids[58–60].

conditions alongside with the assumption that the viscous dissipation term in the energy equation is neglected, we get the following system[56]:

$$\rho_{nf}(u_t) = \left(1 + \frac{1}{\gamma}\right)\mu_{nf}(u_{\xi\xi}) - \left(\sigma_{nf}B_0^2 + \left(1 + \frac{1}{\gamma}\right)\frac{\mu_{nf}\psi}{k}\right)u + g(\rho\beta)_{nf}[\Theta - \Theta_\infty];\ t,\ \xi > 0, \tag{7}$$

$$(\rho c_p)_{nf}\Theta_t = k_{nf}\left(1 + \frac{16\sigma^*\Theta_\infty^3}{3k_{nf}k^*}\right)T_{\xi\xi} + Q_0(\Theta - \Theta_\infty);\ \xi,\ t > 0, \tag{8}$$

$$\left.\begin{array}{l} u = 0,\ \Theta = \Theta_\infty;\ \xi \geq 0,\ t < 0 \\ u = UH(t)\cos(\omega t) \text{ or } u = U\sin(\omega t),\ \dfrac{\partial\Theta}{\partial\xi} = -h_s\Theta;\ t \geq 0,\ \xi = 0 \\ u \to 0,\ \Theta \to \Theta_\infty \text{ as } \xi \to \infty \end{array}\right\}, \tag{9}$$

where k^* is absorption coefficient and σ^* is Stefan-Boltzmann constant. Where Q_0 is the heat generation term, ρ_{nf} is the density of nanofluids, μ_{nf} is the dynamic viscosity, u is the fluid velocity in the x-axis perpendicular direction, γ is the Casson fluid parameter, $\psi(0 < \psi < 1)$, $K > 0$, ψ is the porous medium and K is the permeability of porous medium, h_s is a constant heat transfer coefficient, Θ_w is the constant plate temperature ($\Theta_w < \Theta_\infty$, $\Theta_w > \Theta_\infty$ due to the cooled or heated plate, respectively), g is the acceleration due to gravity, and β_{nf} is the thermal expansion coefficient of the nanofluid.

Expressions for $(\rho c_p)_{nf}$, $(\rho\beta)_{nf}$, μ_{nf}, ρ_{nf}, σ_{nf}, k_{nf} are given by[24]:

$$\begin{aligned} \rho_{nf} &= (1 - \phi)\rho_f + \phi\rho_s,\ \mu_{nf} = \frac{\mu_f}{(1 - \phi)^{2.5}},\ \sigma = \frac{\sigma_f}{\sigma_s}, \\ (\rho\beta)_{nf} &= (1 - \phi)(\rho\beta)_f + \phi(\rho\beta)_s,\ (\rho c_p)_{nf} = (1 - \phi)(\rho c_p)_f + \phi(\rho c_p)_s, \\ k_{nf} &= k_f\left(\frac{k_s + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + \phi(k_f - k_s)}\right),\ \sigma_{nf} = \sigma_f\left(1 + \frac{3(\sigma - 1)\phi}{(\sigma + 2) - (\sigma - 1)\phi}\right), \end{aligned} \tag{10}$$

where ϕ the volume fraction of nano particles, ρ_f and ρ_s is represent the density of base fluid and particle respectively, and c_p is specific heat on constant pressure. k_{nf}, k_f, and k_s are the thermal conductivities of the nanofluid, the base-fluid, and the solid particles, respectively. The expressions of Eq. (10) are classified to nano particles[57]. For supplementary nano particles with unlike thermal conductivity, dynamic viscosity, see to Table 1[58–60].

the dimensionless variables are[56],

$$\left[u^* = \frac{u}{U_0},\ \xi^* = \frac{U_0}{\nu}\xi,\ t^* = \frac{U_0^{\ 2}}{\nu}t,\ \theta = \frac{\Theta - \Theta_\infty}{\Theta_w - \Theta_\infty}\right], \tag{11}$$

Into Eqs (7–9), we get

$$u_t = c_2u_{\xi\xi} - Hu + Gr_0\theta \quad t,\ \xi > 0, \tag{12}$$

$$\theta_t = c_4\theta_{\xi\xi} + c_5\theta; \quad \xi,\ t > 0 \tag{13}$$

$$\left.\begin{array}{l} u = 0,\ \theta = 0;\ \xi \geq 0,\ t < 0 \\ u = H(t)\cos(\omega t) \text{ or } u = \sin(\omega t),\ \theta_\xi = -\lambda(1 + \theta);\ t \geq 0,\ \xi = 0 \\ u \to 0,\ \theta \to 0 \text{ as } \xi \to \infty \end{array}\right\} \tag{14}$$

where

$$\begin{aligned}
\varphi_1 &= (1-\phi)^{2.5}\left[1-\phi+\phi\left(\frac{\rho_s}{\rho_f}\right)\right],\ c_1 = 1+\frac{3(\sigma-1)\phi}{(\sigma+2)-(\sigma-1)\phi},\ c_2 = \left(1+\frac{1}{\gamma}\right)\frac{1}{\varphi_1},\\
c_3 &= 1-\phi+\phi\frac{(\rho c_p)_s}{(\rho c_p)_f},\ c_4 = \frac{\lambda_{nf}(1+Nr_0)}{\nu\ \mathrm{Pr}\ c_3},\ c_5 = \frac{\nu Q_0}{U_0^2 k_f},\ \lambda_{nf} = \frac{k_{nf}}{k_f},\\
M &= \frac{c_1\sigma_f \nu B_0^2}{\rho_f U_0^2},\ Gr = \frac{\nu g(\rho\beta)_f}{U_0^3\rho_f}(\Theta_w-\Theta_\infty),\ \frac{1}{K} = \frac{\nu_f^2\psi}{kU_0^2},\ H = \frac{M}{\varphi_2}+\frac{c_2}{\varphi_1 K},\\
\varphi_2 &= 1-\phi+\phi\frac{\rho_s}{\rho_f},\ Nr = \frac{16\sigma^*\Theta_\infty^3}{3k_f k^*},\ \mathrm{Pr}\frac{(\rho c_p)_f}{k_f},\ \lambda = \frac{h_s\nu}{U_0},\ Gr_0 = \frac{\varphi_3}{\varphi_2}\frac{\nu g(\rho\beta)_f}{U_0^3\rho_f}(\Theta_w-\Theta_\infty)\\
\varphi_3 &= 1-\phi+\phi\frac{(\rho\beta)_s}{(\rho\beta)_f},\ Nr_0 = \frac{\lambda_{nf}}{k_{nf}}\frac{16\sigma^*\Theta_\infty^3}{3k_f k^*}
\end{aligned}$$

where $\frac{1}{K}$ is permeability of pours medium, M is the magnetic parameter, Gr is thermal Grashof number, Pr is Prandtl number, Nr is radiation parameter, and λ is Newtonian heating parameter.

Laplace Transform Solution

Laplace transforms of Eqs (12, 13) gives:

$$c_2\overline{u_{\xi\xi}} - (q+H)\bar{u} = -Gr_0\bar{\theta}, \tag{15}$$

$$c_4\overline{\theta_{\xi\xi}} - (q-c_5)\bar{\theta} = 0, \tag{16}$$

$$\left|\begin{array}{l}\bar{u} = 0,\ \bar{\theta} = 0;\ \xi \ge 0,\ q < 0\\ \bar{u} = \dfrac{\omega}{q^2+\omega^2},\ \bar{\theta}_\xi = -\lambda\left(\dfrac{1}{q}+\bar{\theta}\right);\ q \ge 0,\ \xi = 0\\ \bar{u} \to 0,\ \bar{\theta} \to 0 \text{ as } \xi \to \infty\end{array}\right| \tag{17}$$

Eq. (16) using Eq. (17) gives:

$$\bar{\theta}(\xi, q) = \frac{1}{q}\left(\frac{\lambda\sqrt{c_4}}{\sqrt{q-c_5}-\lambda\sqrt{c_4}}\right)e^{-\xi\frac{\sqrt{q-c_5}}{\sqrt{c_4}}}. \tag{18}$$

After taking the inverse Laplace of Eq. (18):

$$\theta(\xi, t) = \frac{1}{2}\lambda\sqrt{c_4}e^{-\xi\sqrt{\frac{-c_5}{c_4}}}\int_0^t\left[\begin{array}{l}e^{c_5(t-\tau)}\left\{\dfrac{1}{\sqrt{\pi}\sqrt{t-\tau}} + \lambda\sqrt{c_4}e^{c_6{}^2(t-\tau)}erfc(-\lambda\sqrt{c_4}\sqrt{t-\tau})\right\}*\\ \left\{2 - erfc\left(\dfrac{2\sqrt{-c_5c_4}\tau-\xi}{2\sqrt{c_4\tau}}\right) + e^{2y\sqrt{\frac{-c5}{c4}}}erfc\left(\dfrac{2\sqrt{-c_5c_4}\tau+\xi}{2\sqrt{c_4\tau}}\right)\right\}\end{array}\right]d\tau. \tag{19}$$

Solution of Eq. (15) is:

$$\bar{u}(\xi, q) = [\bar{u}_a(\xi, q) + \bar{u}_b(\xi, q) + \bar{u}_c(\xi, q) + \bar{u}_d(\xi, q)]. \tag{20}$$

Arranging Eq. (20) as:

$$\begin{aligned}
\bar{u}_a(\xi, q) &= \frac{\omega}{q^2+\omega^2}e^{-\xi\frac{\sqrt{q+H}}{\sqrt{c_2}}},\\
\bar{u}_b(\xi, q) &= \frac{A}{q}\left(e^{-\xi\frac{\sqrt{q-c_5}}{\sqrt{c_4}}} - e^{-\xi\frac{\sqrt{q+H}}{\sqrt{c_2}}}\right),\\
\bar{u}_c(\xi, q) &= \frac{B_1}{q+\frac{c_7}{c_6}}\left(e^{-\xi\frac{\sqrt{q-c_5}}{\sqrt{c_4}}} - e^{-\xi\frac{\sqrt{q+H}}{\sqrt{c_2}}}\right),\\
\bar{u}_d(\xi, q) &= \frac{C}{\sqrt{q-c_5}-\lambda\sqrt{c_4}}\left(e^{-\xi\frac{\sqrt{q-c_5}}{\sqrt{c_4}}} - e^{-\xi\frac{\sqrt{q+H}}{\sqrt{c_2}}}\right),
\end{aligned} \tag{21}$$

where

$$A = \frac{c_8}{c_7(\sqrt{c_5} - \lambda\sqrt{c_4})},\ B = \frac{c_8 c_6}{c_7\left(\lambda\sqrt{c_4} - \sqrt{\frac{c_7}{c_6} - c_5}\right)},\ C = \frac{c_8}{\lambda^2 c_4 + c_5\{c_6(\lambda^2 c_4 + c_5) + c_7\}},$$

$$B_1 = \frac{B}{c_6},\ c_6 = c_4 - c_2,\ c_7 = c_4 H + c_2 c_5,\ c_8 = c_4 Gr_0 \lambda\sqrt{c_4},$$

Upon inversion:

$$u(\xi, t) = [u_a(\xi, t) + u_b(\xi, t) + u_c(\xi, t) + u_d(\xi, t)], \tag{22}$$

where

$$\begin{aligned} u_a(\xi, t) = {} & \frac{1}{4\mathrm{i}} \mathrm{e}^{\mathrm{i}tw}\left[\mathrm{e}^{-\xi\sqrt{\frac{H+\mathrm{i}w}{c_2}}} erfc\left\{\frac{\xi}{2\sqrt{c_2 t}} - \sqrt{t(H+\mathrm{i}w)}\right\}\right. \\ & \left. + \mathrm{e}^{\xi\sqrt{\frac{H+\mathrm{i}w}{c_2}}} erfc\left\{\frac{\xi}{2\sqrt{c_2 t}} + \sqrt{t(H+\mathrm{i}w)}\right\}\right], \\ & - \frac{1}{4\mathrm{i}} \mathrm{e}^{-itw}\left[\mathrm{e}^{-\xi\sqrt{\frac{H-\mathrm{i}w}{c_2}}} erfc\left\{\frac{\xi}{2\sqrt{c_2 t}} - \sqrt{t(H-\mathrm{i}w)}\right\}\right. \\ & \left. + \mathrm{e}^{\xi\sqrt{\frac{H-\mathrm{i}w}{c_2}}} erfc\left\{\frac{\xi}{2\sqrt{c_2 t}} + \sqrt{t(H-\mathrm{i}w)}\right\}\right] \end{aligned} \tag{23}$$

$$u_b(\xi, t) = \frac{1}{2} A \left\{ \begin{array}{l} \mathrm{e}^{-\sqrt{-\frac{c_5}{c_4}}\xi}\left(2 - erfc\left[\frac{2\sqrt{-c_4 c_5}\,t - \xi}{2\sqrt{c_4 t}}\right] + \mathrm{e}^{2\sqrt{-\frac{c_5}{c_4}}\xi} erfc\left[\frac{2\sqrt{-c_4 c_5}\,t + \xi}{2\sqrt{c_4 t}}\right]\right) \\ -\mathrm{e}^{-\sqrt{\frac{H}{c_2}}\xi}\left(2 - erfc\left[\frac{2\sqrt{c_2 H}\,t - \xi}{2\sqrt{c_2 t}}\right] + \mathrm{e}^{2\sqrt{\frac{H}{c_2}}\xi} erfc\left[\frac{2\sqrt{c_2 H}\,t + \xi}{2\sqrt{c_2 t}}\right]\right) \end{array} \right\}, \tag{24}$$

$$u_c(\xi, t) = \frac{B_1 e^{\frac{c_7}{c_6}}}{2} \left\{ \begin{array}{l} e^{-\xi i \frac{\sqrt{c_5 + \frac{c_7}{c_6}}}{\sqrt{c_4}}} erfc\left(\frac{\xi}{2\sqrt{t}\sqrt{c_4}} - i\sqrt{\left(c_5 + \frac{c_7}{c_6}\right)t}\right) + e^{\xi i \frac{\sqrt{c_5 + \frac{c_7}{c_6}}}{\sqrt{c_4}}} erfc\left(\frac{\xi}{2\sqrt{t}\sqrt{c_4}} + i\sqrt{\left(c_5 + \frac{c_7}{c_6}\right)t}\right) \\ -e^{-\xi \frac{\sqrt{H - \frac{c_7}{c_6}}}{\sqrt{c_2}}} erfc\left(\frac{\xi}{2\sqrt{t}\sqrt{c_2}} - \sqrt{\left(H - \frac{c_7}{c_6}\right)t}\right) - e^{\xi \frac{\sqrt{H - \frac{c_7}{c_6}}}{\sqrt{c_2}}} erfc\left(\frac{\xi}{2\sqrt{t}\sqrt{c_2}} + \sqrt{\left(H - \frac{c_7}{c_6}\right)t}\right) \end{array} \right\}, \tag{25}$$

$$\begin{aligned} u_d(\xi, t) = {} & \frac{\xi C}{2\sqrt{c_4}\sqrt{\pi}} \int_0^t \left\{ \mathrm{e}^{c_5(t-\tau)}\left(\frac{1}{\sqrt{\pi}\sqrt{t-\tau}} + c_6 \mathrm{e}^{c_6{}^2(t-\tau)} erfc[-c_6\sqrt{t-\tau}]\right) * \frac{\mathrm{e}^{c_5\tau - \frac{\xi^2}{4c_4\tau}}}{\tau^{3/2}} \right\} d\tau \\ & - \frac{\xi C}{2\sqrt{c_2}\sqrt{\pi}} \int_0^t \left\{ \mathrm{e}^{c_5(t-\tau)}\left(\frac{1}{\sqrt{\pi}\sqrt{t-\tau}} + c_6 \mathrm{e}^{c_6{}^2(t-\tau)} erfc[-c_6\sqrt{t-\tau}]\right) * \frac{\mathrm{e}^{-H\tau - \frac{\xi^2}{4c_2\tau}}}{\tau^{3/2}} \right\} d\tau. \end{aligned} \tag{26}$$

Particular Cases

In order to link our found solutions with published literature, the following particular cases are examined by taking some parameters absent.

Making $Gr = \gamma = 0$ and $\mathrm{Re} = 1$ in Eq. (22), reduces to:

$$\begin{aligned} u(\xi, t) = {} & \frac{1}{4\mathrm{i}} \mathrm{e}^{\mathrm{i}tw}\left[\mathrm{e}^{-\xi\sqrt{H+\mathrm{i}w}} erfc\left\{\frac{\xi}{2\sqrt{t}} - \sqrt{t(H+\mathrm{i}w)}\right\}\right. \\ & \left. + \mathrm{e}^{\xi\sqrt{H+\mathrm{i}w}} erfc\left\{\frac{\xi}{2\sqrt{t}} + \sqrt{t(H+\mathrm{i}w)}\right\}\right] \\ & - \frac{1}{4\mathrm{i}} \mathrm{e}^{-itw}\left[\mathrm{e}^{-\xi\sqrt{H-\mathrm{i}w}} erfc\left\{\frac{\xi}{2\sqrt{t}} - \sqrt{t(H-\mathrm{i}w)}\right\}\right. \\ & \left. + \mathrm{e}^{\xi\sqrt{H-\mathrm{i}w}} erfc\left\{\frac{\xi}{2\sqrt{t}} + \sqrt{t(H-\mathrm{i}w)}\right\}\right], \end{aligned} \tag{27}$$

which is identical to results of[61], Eq. (24).

Taking $M = \frac{1}{k} = 0$ in the above relation, we get:

$$u(\xi,\,t) = \frac{1}{4\mathrm{i}}\mathrm{e}^{\mathrm{i}tw}\left[\mathrm{e}^{-\xi\sqrt{\mathrm{i}w}}\mathit{erfc}\left\{\frac{\xi}{2\sqrt{t}} - \sqrt{\mathrm{i}wt}\right\} + e^{\xi\sqrt{\mathrm{i}w}}\mathit{erfc}\left\{\frac{\xi}{2\sqrt{t}} + \sqrt{\mathrm{i}wt}\right\}\right] - \frac{1}{4\mathrm{i}}\mathrm{e}^{-\mathrm{i}tw}\left[\mathrm{e}^{-\xi\sqrt{\mathrm{i}w}}\mathit{erfc}\left\{\frac{\xi}{2\sqrt{t}} - \sqrt{-\mathrm{i}wt}\right\} + e^{\xi\sqrt{\mathrm{i}w}}\mathit{erfc}\left\{\frac{\xi}{2\sqrt{t}} + \sqrt{-\mathrm{i}wt}\right\}\right], \tag{28}$$

Which is in accordance with[61], Eq. (25).

Taking $Gr = M = \frac{1}{k} = \frac{1}{\gamma} = 0$, in Eq. (22), it moderates to:

$$u(\xi,\,t) = \frac{1}{4\mathrm{i}}\mathrm{e}^{\mathrm{i}tw}\left[\mathrm{e}^{-\xi\sqrt{\mathrm{Re}\mathrm{i}w}}\mathit{erfc}\left\{\frac{\xi\sqrt{\mathrm{Re}}}{2\sqrt{t}} - \sqrt{\mathrm{i}wt}\right\} + e^{\xi\sqrt{\mathrm{Re}\mathrm{i}w}}\mathit{erfc}\left\{\frac{\xi\sqrt{\mathrm{Re}}}{2\sqrt{t}} + \sqrt{\mathrm{i}wt}\right\}\right] - \frac{1}{4\mathrm{i}}\mathrm{e}^{-\mathrm{i}tw}\left[\mathrm{e}^{-\xi\sqrt{-\mathrm{Re}\mathrm{i}w}}\mathit{erfc}\left\{\frac{\xi\sqrt{\mathrm{Re}}}{2\sqrt{t}} - \sqrt{-\mathrm{i}wt}\right\} + e^{\xi\sqrt{-\mathrm{Re}\mathrm{i}w}}\mathit{erfc}\left\{\frac{\xi\sqrt{\mathrm{Re}}}{2\sqrt{t}} + \sqrt{-\mathrm{i}wt}\right\}\right], \tag{29}$$

Identical to[58], Eq. (35).

Skin friction and Nusselt Number

$$C_f = \frac{1}{(1-\phi)^{2.5}}\left(1 + \frac{1}{\gamma}\right)\frac{\partial u(\xi,\,t)}{\partial\xi}\bigg|_{\xi=0}, \tag{30}$$

$$\begin{aligned} C_f = {} & \frac{1}{2}\mathrm{i}\mathrm{e}^{-\mathrm{i}tw}\frac{1}{(1-\phi)^{2.5}}\left(1+\frac{1}{\gamma}\right) \\ & \times \left[\begin{matrix} 1 - e^{2\mathrm{i}tw} - \dfrac{\mathrm{e}^{-t(H-\mathrm{i}w)}}{\sqrt{c_2 t}\sqrt{\pi}} - \sqrt{\dfrac{H-\mathrm{i}w}{c_2}}\mathit{erf}[\sqrt{t(H-\mathrm{i}w)}] \\ + \mathrm{e}^{2\mathrm{i}tw}\left\{\dfrac{\mathrm{e}^{-t(H+\mathrm{i}w)}}{\sqrt{c_2 t}\sqrt{\pi}} + \sqrt{\dfrac{H+\mathrm{i}w}{c_2}}\mathit{erf}[\sqrt{t(H-\mathrm{i}w)}]\right\} \end{matrix}\right] \\ & - A\left\{\frac{\mathrm{e}^{-Ht}}{\sqrt{c_2 t}\sqrt{\pi}} + \sqrt{\frac{H}{c_2}}\mathit{erfc}(\sqrt{Ht})\right\} \\ & - B_1 e^{\frac{c_7}{c_6}}\left\{\frac{\mathrm{e}^{\left(c_5+\frac{c_7}{c_6}\right)t}}{\sqrt{c_4 t}\sqrt{\pi}} + i\frac{\sqrt{c_5+\frac{c_7}{c_6}}}{\sqrt{c_4}} - \frac{\mathrm{e}^{-\left(H-\frac{c_7}{c_6}\right)t}}{\sqrt{c_2 t}\sqrt{\pi}} - \frac{\sqrt{H-\frac{c_7}{c_6}}}{\sqrt{c_2}}\right\} \\ & + \left(\frac{C}{2\sqrt{c_4}\sqrt{\pi}} - \frac{C}{2\sqrt{c_2}\sqrt{\pi}}\right) \\ & \times \int_0^t\left\{\frac{\mathrm{e}^{c_5(t-\tau)}}{\tau^{3/2}}\left(\frac{1}{\sqrt{\pi}\sqrt{t-\tau}} + c_6 e^{c_6^2(t-\tau)}\mathit{erfc}[-c_6\sqrt{t-\tau}]\right)\right\}d\tau. \end{aligned} \tag{31}$$

$$Nu = -\lambda_{nf}\frac{\partial\theta(\xi,\,t)}{\partial\xi}\bigg|_{\xi=0}, \tag{32}$$

$$Nu = -\frac{1}{2}\lambda_{nf}\lambda\sqrt{c_4}\int_0^t\left[\begin{matrix} e^{c_5(t-\tau)}\left\{\dfrac{1}{\sqrt{\pi}\sqrt{t-\tau}} + \lambda\sqrt{c_4}e^{c_6^2(t-\tau)}\mathit{erfc}(-\lambda\sqrt{c_4}\sqrt{t-\tau})\right\} \\ *\{2 - \mathit{erfc}(\sqrt{-c_5\tau})\} \end{matrix}\right]d\tau. \tag{33}$$

Discussion

In this section different parameters including γ, ϕ, Gr, M, K, Pr, Nr Figs 2–11 are plotted. Geometry of problem is shown in Fig. 1. The influence of γ on $u(y, t)$ which shows oscillatory behavior increasing first then decreasing is highlighted in Fig. 2.

Figures 3 and 4 show effects of ϕ on $u(\xi, t)$ and $\theta(\xi, t)$.φ is take in between $0 \le \phi \le 0.04$ due to sedimentation when the range goes above from 0.08. It is observed in both cases if the nano particles volume fraction ϕ is increased it leads to the decreasing of temperature and velocity profile.

Figure 5 highlights the effect of Gr for Sodium alginate -based, Casson nanofluids on velocity profile. It is found that with increasing Gr, velocity increases. Because increasing effect in Gr, due to increase of buoyancy forces and decrease of viscous forces. Figure 6 the effect of $M = 0, 1, 2$ on the velocity profile. $u(\xi, t)$ decreases due to increasing dragging force. $M = 0$, shows absence of MHD. Figure 7 shows K effect of on $u(\xi, t)$. Velocity decrease due to decreasing friction. Figure 8 highlights that profile of velocity is increased with increasing radiation parameter Nr. The effect is studied for TiO_2 nano particle.

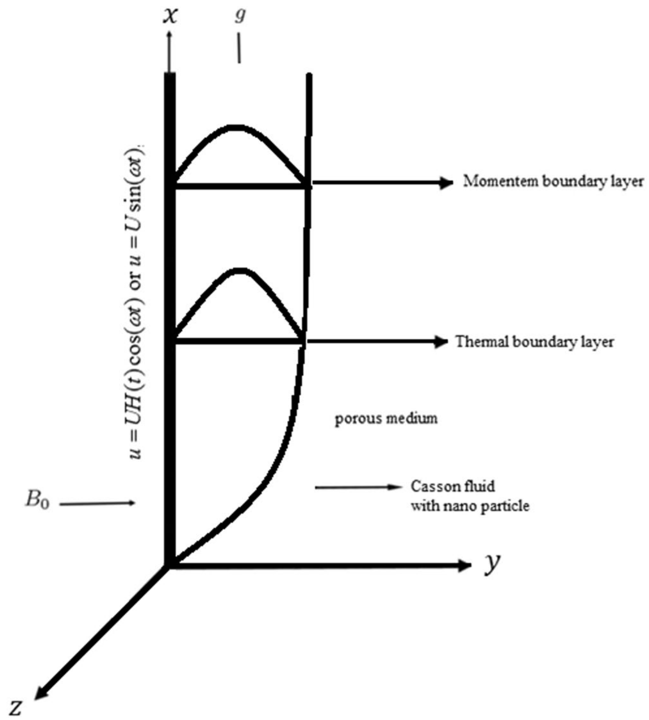

Figure 1. Geometry of the flow.

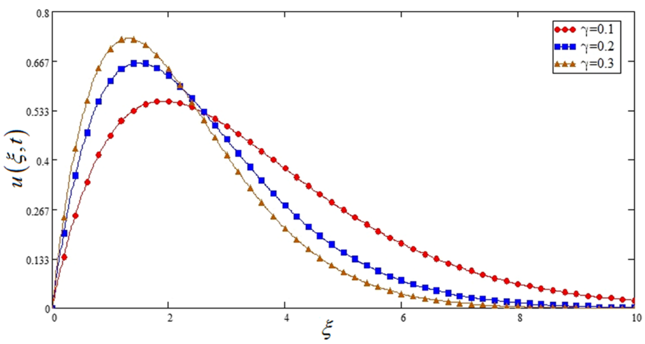

Figure 2. Effects of Casson fluid parameter γ on the velocity profile of Sodium alginate based Casson nanofluid when $\mathrm{Pr} = 0.7$, $Gr = 2$ and $\varphi = 0.04$.

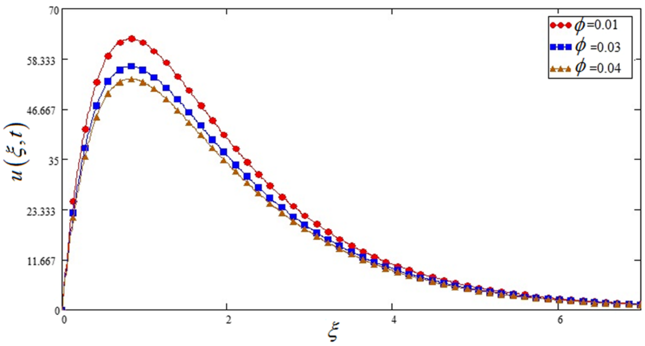

Figure 3. Effects of nano particles volume fraction parameter φ on the velocity profile of Sodium alginate based nano fluid when $Gr = 0.2$, $Nr = 0.2$ and $t = 1$.

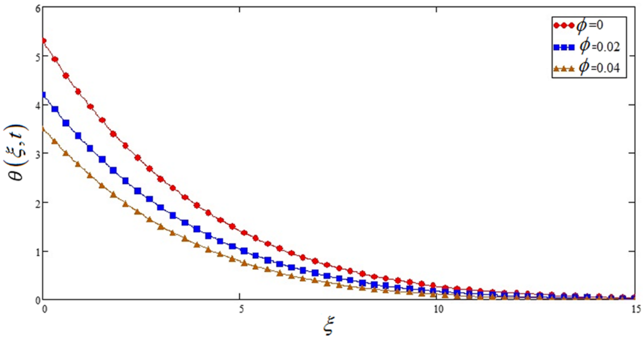

Figure 4. Effects of nano particles volume fraction parameter φ on the temperature profile of Sodium alginate based nano fluid when $\mathrm{Pr}=5$ and $t=1$.

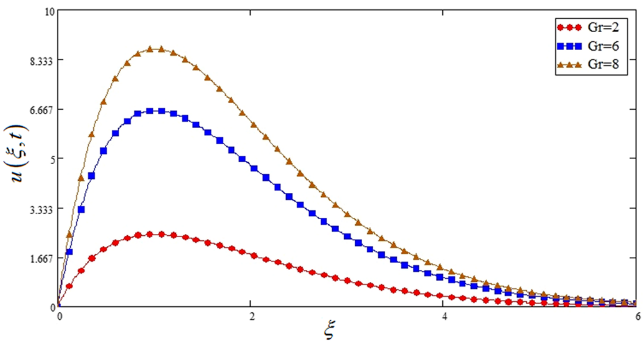

Figure 5. Effects of thermal Grashof number Gr on the velocity profile of Sodium alginate based Casson nano fluid when $\mathrm{Pr}=0.7$, $Nr=2$, $\varphi=0.04$ and $t=1$.

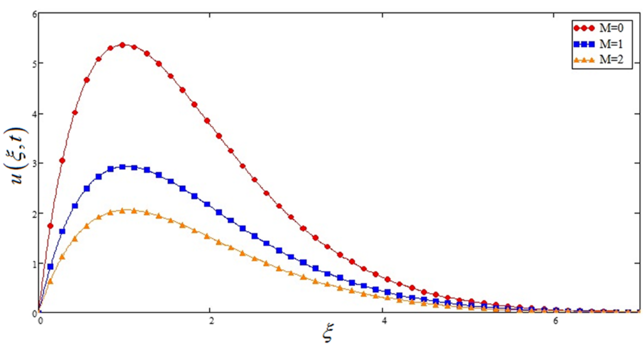

Figure 6. Effects of magnetic parameter M on the velocity profile of Sodium alginate based Casson nano fluid when $\mathrm{Pr}=0.7$, $Nr=2$, $Gr=10$, $k=2$ and $t=1$.

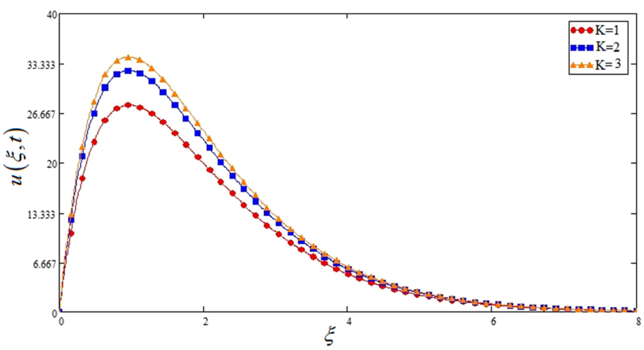

Figure 7. Effects of permeability of porous medium k on the velocity profile of Sodium alginate based nano fluid when $\mathrm{Pr}=10$, $Gr=10$, $Nr=8$, $\varphi=0.04$ and $t=1$.

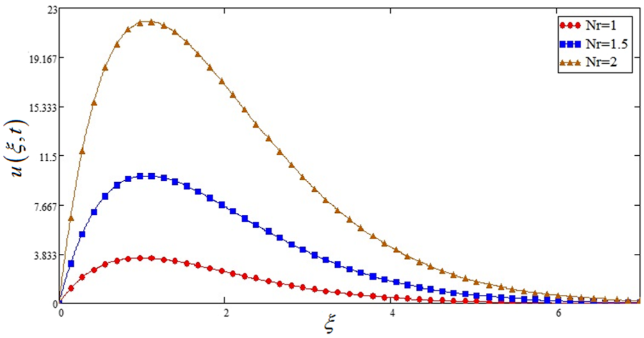

Figure 8. Effects of radiation parameter Nr for TiO_2 on the velocity profile of Sodium alginate based nano fluid when $\Pr = 0.7$, $Gr = 8$, $\varphi = 0.04$ and $t = 1$.

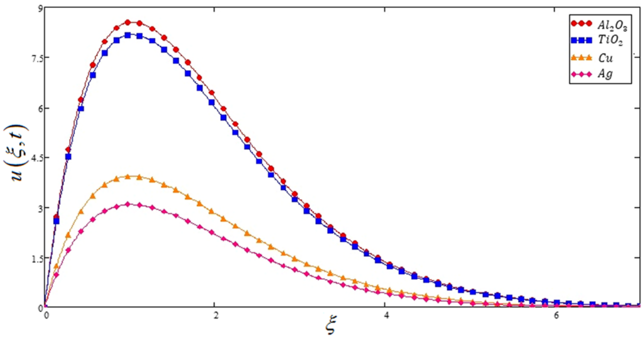

Figure 9. Comparison of velocities profiles for different types of nano particles for Casson nanofluids when $\Pr = 0.71$, $Gr = 10$, $Nr = 2$, $\varphi = 0.04$ and $t = 1$.

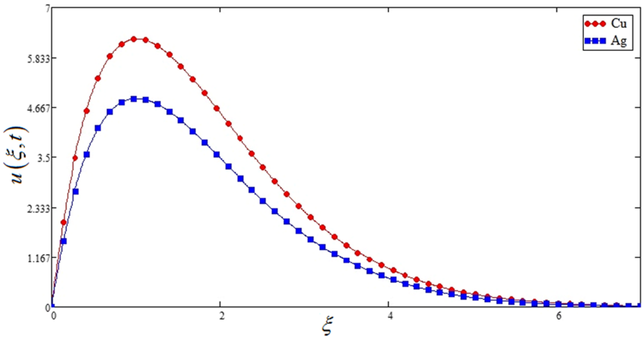

Figure 10. Comparison of velocities profiles of Cu and Ag Casson nanofluids when $\Pr = 0.71$, $Gr = 10$, $Nr = 2$, $\varphi = 0.04$ and $t = 1$.

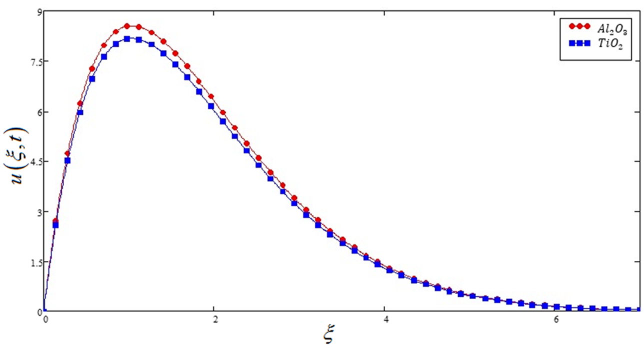

Figure 11. Comparison of velocities profiles of Al_2O_3 and TiO_3 for Casson nanofluids when $\Pr = 0.71$, $Gr = 10$, $Nr = 2$, $\varphi = 0.04$ and $t = 1$.

The impact of two different types of nano particles (Al_2O_3 Sodium alginate -based Casson nanofluid and *Ag*-Sodium alginate -based nanofluid) on profile of velocity is studied in Fig. 9. The profile of velocity is greater for Al_2O_3 Sodium alginate -based Casson nanofluid and lower profile velocity for *Ag*-Sodium alginate -based nanofluid is observed.

Figure 10 highlights the comparison of both (*Cu* Sodium alginate -based Casson nanofluid and *Ag*-Sodium alginate -based nanofluid) on $u(\xi, t)$. Velocity of *Ag*-Sodium alginate -based nanofluid is lower than copper Sodium alginate -based nanofluid. This shows that *Cu* nano particles have more thermal diffusivity compare to *Ag* which is physically true. Furthermore, the same comparison is study for Al_2O_3 and TiO_2 models in Fig. 11, which shows that Aluminum oxide Al_2O_3 nano particles have high thermal diffusivity as compare to Titanium oxide TiO_2.

Conclusion

The following remarks are concluded from this work:

- $u(\xi, t)$ decreases as γ increases
- Temperature and velocity profile are decreasing with increasing nano particles volume Fraction ϕ.
- Al_2O_3 nanofluid has higher velocity from TiO_2 nanofluid and *Cu* nanofluid has higher velocity from *Ag* nanofluid.
- The pours medium *K* and MHD M show opposite behavior.

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Author Contributions

A.K. and I.K. designed the study; D.K. and F.K. conducted the experiments with technical assistance from F.A. and M.I.D.K. analyzed the data and wrote the paper; A.K., I.K. and M.I. provided general assistance. All authors have read and approved the final submission.

Additional Information

Competing Interests: The authors declare no competing interests.

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