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The effect of the fear factor on the dynamics of an eco-epidemiological system with standard incidence rate

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A R T I C L E I N F O

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A B S T R A C T

In order to protect endangered prey, ecologists suggest introducing parasites into predators which have achieved the expected goal in practice. Then how to explain the inherent mechanism and validate the effectiveness of this approach theoretically? In response to this question, we propose an eco-epidemiological system with the standard incidence rate and the anti-predator behavior in this paper, where the predator population is infected by parasites. We show the existence and local stability of equilibria for the system, and verify the occurrence of Hopf bifurcation. Theoretical and numerical results suggest that the fear effect reduces the density of the predator population but has no effect on the density of prey population. In addition, the cost of fear may not only break the stability of the equilibrium of the system, but also induce the equilibrium to change from unstable to stable. Based on the theoretical analysis, we confirm that introducing parasites into the predator population is an effective method to protect endangered prey.

1. Introduction

The prey-predator interaction system is one of the dominant themes in theoretical ecology (Haque & Greenhalgh, 2010; Kuang & Beretta, 1998). Without predators, the prey population would give rise to over grazing and directly affect the natural plant life cycle; without prey, the predators would starve from hunger and die; moreover, the existence of predators is indispensable to the long-term existence of prey to a certain extent (Sen et al., 2022; Wang et al., 2018). In biological control, considerable evidence substantiates the way that introducing parasites into invasive predators can effectively control predator population and protect endangered prey (Bulai & Hilker, 2019; Courchamp et al., 1999). The introduced parasites resemble in the mode of infectious diseases transmitted in predators. Once parasites or infectious diseases transmit in prey or predator population, the inherent prey-predator cycles will be broken and the interactions will be more complex (Bate & Hilker, 2013a; Gao et al., 2013). Eco-epidemiological system, devoting to investigate the mechanism of parasite and disease transmission in predator-prey interaction, has gained widely concern in recent years (Auger et al., 2009; Kooi et al., 2011). These results suggested that parasites and infectious diseases can greatly influence the dynamics of the host population and other related species (Cojocaru et al., 2020; Hadeler & Freedman, 1989; Bate & Hilker, 2013b; Zhang et al., 2022; pada Das, 2011).

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Fear factor has profound effects on preys, and one of the most influential effect is the caused anti-predator behaviors (Das & Samanta, 2018; Dubey et al., 2021; Wang et al., 2019, 2020). These behaviors may directly affect the growth rate of prey population, and change the physiological characteristics of the preys (Zanette et al., 2011). For the representative research work on the prey-predator dynamics with the cost of anti-predator behaviors, we recommend the articles written by Wang et al. (Wang et al., 2016; Wang & Zou, 2017). The term accounts for the cost of anti-predator behaviors may take different forms, such as $\frac{1}{kY}$ (Wang et al., 2016; Zhang et al., 2023), e^{-kY} (Das & Samanta, 2018; Li & Tian, 2023), $\eta+\frac{\alpha(1-\eta)}{\alpha+Y}$ (Sarkara & Khajanchi, 2020). Generally, the fear effect can stabilize or destabilize the considered predator-prey systems, and may cause a significant reduction in the biomass of the population, which are demonstrated in (Hossain et al., 2020; Pal, 2020; Sarkara & Khajanchi, 2020; Zhang et al., 2019). These references inevitably lead us to further explore the impact of fear effect on the dynamics of eco-epidemiological systems, especially (Liu et al., 2021; Sarkar & Khajanchi, 2022). Barman et al. (Barman et al., 2021) investigated the impact of fear effect on a predator-prey model with disease in predator. They found that the fear effect can reduce the size of species including predator and prey, and may trigger a phenomenon where diseases in the predator population tend to become extinct. But in most of these works, the growth of prey population is unrestricted, which often goes against reality (Haque & Venturino, 2007; Li & Tian, 2023). Hence, it is more meaningful to study the dynamics of eco-epidemiological systems with fear effect under the assumption that the prey grow logistically.

In my earlier work, we established the following predator-prey system that incorporates infectious disease in predator population and the cost of anti-predator behaviors (Zhang et al., 2023):

$$\begin{cases}\frac{dX}{dt}=\frac{rX}{1+k(S+I)}-\frac{rX^2}{K}-\frac{aXS}{1+bX},\\ \frac{dS}{dt}=\frac{eaXS}{1+bX}-d_1S-\beta SI,\\ \frac{dI}{dt}=\beta SI-d_2I,\end{cases} \tag{1.1}$$

where, X, S, I represent the densities of prey, susceptible predator and infected predator at time t, respectively. Here, r is the intrinsic growth rate of the logistically growing prey and K is the carrying capacity of the prey population. The parameter k represents the level of fear which drives anti-predator behaviours of the prey. a denotes the predation coefficient, e is the biomass conversion constant, and b is the predators handling time of a prey. β represents the effective contact rate between healthy predator and infected predator. d_1 and d_2 are the mortality rate of the susceptible predator and infected predator, respectively.

We summarize the equilibria for system (1.1) as follows:

(1) Trivial equilibrium: $E_0=(0,0,0)$;
(2) Axial equilibrium: $E_1=(K,0,0)$;
(3) Planar equilibrium: $E_2=(X_2,S_2,0)$ exists if $ea-bd_1>0$ and $K>\frac{d_1}{ea-bd_1}$ hold, where X_2 and S_2 are defined in Eq (3.1) of (Zhang et al., 2023).
(4) If $ea-bd_1>0$ and $r>\frac{ad_2(\beta+k(d_2-d_1))}{\beta^2}$ hold, then system (1.1) has at least one positive equilibrium $E_3=(X_3,S_3,I_3)$, where X_3, S_3 and I_3 are provided in [23, Theorem 3.1].

We found that high level of the fear effect leads to complex dynamics and the infected predator can go to extinction.

Disease incidence, the rate at which new infections occur, is a critical term in modelling the transmission mechanism of infectious diseases (Naji & Mustafa, 2012; Upadhyay & Roy, 2014). The commonly used incidence terms mainly include bilinear type (or mass action type) βSI, saturated type $\frac{\beta SI}{1+mI}$, standard type $\frac{\beta SI}{S+I}$ (Haque et al., 2009; Wang & Feng, 2017). In system (1.1), bilinear incidence term is used to model the early stage of disease transmission ideally assumed that the susceptible and infected predators are homogeneous mixing. However as the infected transmission progressing, the mixing is no longer homogeneous, and the standard incidence term could better catch the transmission feature and lead to unexpected dynamics (Adak & Bairagi, 2015; Dutta et al., 2022; Liu, 2011; Saha et al., 2018; Tan et al., 2022; Xu et al., 2020; Yang & Wang, 2019).

Motivated by above analyses and based on system (1.1), in this paper, we investigate the following prey parasites-infected-predator system with standard incidence rate and the fear factor:

$$\begin{cases} \dfrac{dX}{dt} = \dfrac{rX}{1+k(S+I)} - \dfrac{rX^2}{K} - \dfrac{aXS}{1+bX}, \\ \dfrac{dS}{dt} = \dfrac{eaXS}{1+bX} - d_1 S - \dfrac{\beta SI}{S+I}, \\ \dfrac{dI}{dt} = \dfrac{\beta SI}{S+I} - d_2 I. \end{cases} \tag{1.2}$$

Model (1.2) can be applied to the wildlife-livestock interacting with the infectious diseases. It is known that the livestock constitutes on average 37% of the agricultural gross domestic product, and is one of the mainly important and rapidly expanding commercial agricultural sectors worldwide (Wiethoelter et al., 2015). However, most of infectious diseases will induce the direct losses to this sector through increased mortality and reduced livestock productivity, as well as the indirect losses associated with cost of control, loss of trade, decreased market values, and food insecurity. Moreover, the infectious diseases that are shared between species also represent a potential burden to the whole ecosystem, affecting biodiversity, changing behavior or composition of animal populations, and even relegating species to the fringe of extinction (Wiethoelter et al., 2015). Therefore, it is mostly important and reasonable to construct a mathematical model to describe the wildlife-livestock interacting with the infectious diseases.

Pathogen maintenance within wildlife populations and spillover to livestock has been reported as a precursor to disease emergence in humans.

The rest of this paper is arranged as follows. In Section 2, we prove the positivity and boundedness of the solutions for system (1.2), define the basic reproduction number and present the existence of equilibria. In Section 3, we analyze the stability of equilibria and show the existence of Hopf bifurcation. In Section 4, numerical simulations are performed to substantiate our analytical results. Finally, Section 5 is concerned with a discussion of main results.

2. Basic reproduction number and equilibria

From the perspective of biology, we consider the solution $(X(t), S(t), I(t))$ of system (1.2) on

$$\mathbb{R}^3_+ = \{(X(t), S(t), I(t)) \in \mathbb{R}^3 : X(t) \geq 0, S(t) \geq 0, I(t) \geq 0\}.$$

Similar with the proof of Theorem 2.1 in (Zhang et al., 2023), we get the following results.

Theorem 2.1. *Each solution of system (1.2) with initial value $(X(0), S(0), I(0)) \in \mathbb{R}^3_+$ is positive and ultimately bounded, and all these positive solutions are defined in the following positive bounded invariant:*

$$\Gamma := \left\{(X(t), S(t), I(t)) \in \mathbb{R}^3_+ : 0 \leq X(t) < K, 0 \leq eX(t) + S(t) + I(t) \leq \frac{eK(r+d_1)^2}{4rd_1}\right\}.$$

Following (Tan et al., 2022), we define the basic reproduction number $\mathcal{R}_0$ for system (1.2) as follows:

$$\mathcal{R}_0 = \frac{\beta}{d_2},$$

which is one of the most important quantities in epidemiology. Here, $\mathcal{R}_0$ represents the average number of secondary infections during an mean infectious period.

One can easily obtain the existence theorem of the planar equilibrium E_1 listed in Theorem 2.2, the proof is standard, hence we omit it here.

Theorem 2.2. *If $ea - bd_1 > 0$ holds, then system (1.2) has a planar equilibrium $E_1 = (X_1, S_1, 0)$, where $X_1 = \frac{d_1}{ea - bd_1}$ and S_1 is the positive root of the equation:*

$$H(S) = akKS^2 + (Ka + krX_1 + bkrX_1^2)S - r(1+bX_1)(K - X_1) = 0.$$

On the other hand, we mainly focus on the existence of positive equilibrium $E_2 = (X_2, S_2, I_2)$ of system (1.2). Any positive equilibrium (X_2, S_2, I_2) of system (1.2) satisfies

$$\begin{cases} \dfrac{r}{1+k(S_2+I_2)} - \dfrac{rX_2}{K} - \dfrac{aS_2}{1+bX_2} = 0, \\ e\dfrac{aX_2}{1+bX_2} - d_1 - \dfrac{\beta I_2}{S_2+I_2} = 0, \\ \dfrac{\beta S_2}{S_2+I_2} - d_2 = 0, \end{cases}$$

which yields

$$X_2 = \frac{d_1 + d_2(\mathcal{R}_0 - 1)}{ea - bd_1 - bd_2(\mathcal{R}_0 - 1)}, \quad S_2 = \frac{I_2}{\mathcal{R}_0 - 1}$$

and I_2 is the positive root of (2.1) in $\left(\frac{d_1}{ea-bd_1}, +\infty\right)$:

$$Q(I) = m_2 I^2 + m_1 I + m_0 = 0, \tag{2.1}$$

where

$$m_2 := akK\beta d_2,$$
$$m_1 := (aK + \mathcal{R}_0 X_2 kr(1 + bX_2))(\mathcal{R}_0 - 1),$$
$$m_0 := r(\mathcal{R}_0 - 1)^2(1 + bX_2)(X_2 - K).$$

Therefore, if $\mathcal{R}_0 > 1$ and $\frac{ea-bd_1}{b(\mathcal{R}_0-1)} > d_2$ hold, there exists one positive equilibrium E_2 for system (1.2).

Theorem 2.3. If $\mathcal{R}_0 > 1$ and $\frac{ea-bd_1}{b(\mathcal{R}_0-1)} > d_2$ hold, then system (1.2) has a positive equilibrium $E_2 = (X_2, S_2, I_2)$, where $X_2 = \frac{d_1+d_2(\mathcal{R}_0-1)}{ea-bd_1-bd_2(\mathcal{R}_0-1)}$, $S_2 = \frac{I_2}{\mathcal{R}_0-1}$ and I_2 is the positive root of (2.1).

Remark 2.1. *Compared with the existence of the equilibria for system (1.1) shown in the Introduction, the system (1.2) no longer has the trivial equilibrium and the axial equilibrium. This indicates that predator population and prey population for system (1.2) always coexist in nature, so the balance of the ecosystem will not be disrupted. Significantly, the dynamic results of the eco-epidemiological model (1.2), which conforms to the dynamical characteristics of infectious diseases (the incidence function of the disease adopts the standard form instead of the bilinear form), reveal an important conclusion that introducing parasites into the predator population can achieve the goal of protecting endangered prey.*

3. Stability analysis

3.1. Local stability of equilibria

Firstly, we will show the stability of the planar equilibrium E_1 of system (1.2) by setting

$$K_1 := \frac{ea + bd_1}{b(ea - bd_1)}, \quad k_1 := \frac{Kb(ea - bd_1)^2(Kb(ea - bd_1) - (ea + bd_1))}{ae^2 r(ea + bd_1)}. \tag{3.1}$$

Theorem 3.1. *For system (1.2), assume that* $ea - bd_1 > 0$ *and* $\mathcal{R}_0 < 1$.

(P1) If one of the following inequalitiesholds, then the planar equilibrium E_1 is locally asymptotic stable.

(I) $K \le K_1$;
(II) $K > K_1$ and $k > k_1$

(P2) If $K > K_1$ and $k < k_1$ hold, then the planar equilibrium E_1 is unstable.
Proof**.** The Jacobian matrix of system (1.2) at E_1 is given by

$$J(E_1)=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & 0 & -\mathcal{R}_0 d_2 \\ 0 & 0 & d_2(\mathcal{R}_0-1) \end{pmatrix}$$

where

$$a_{11}=X_1\left(-\frac{r}{K}+\frac{abS_1}{(1+bX_1)^2}\right),$$
$$a_{12}=\frac{-krX_1}{(1+kS_1)^2}-\frac{aX_1}{1+bX_1}<0,$$
$$a_{13}=\frac{-krX_1}{(1+kS_1)^2}<0,$$
$$a_{21}=\frac{eaS_1}{(1+bX_1)^2}>0.$$

Hence, the characteristic equation of $J(E_1)$ is given as

$$(\lambda^2-a_{11}\lambda-a_{12}a_{21})(\lambda-d_2(\mathcal{R}_0-1))=0. \tag{3.2}$$

Clearly, $J(E_1)$ has three eigenvalues, where λ_1, λ_2 are the roots of

$$\lambda^2-a_{11}\lambda-a_{12}a_{21}=0$$

and $\lambda_3=d_2(\mathcal{R}_0-1)$. Here, when $\mathcal{R}_0<1$, we have $\lambda_3<0$.

Clearly, if $a_{11}<0$, then λ_1, $\lambda_2<0$. By calculating, we have

$$a_{11}=\frac{X_1}{aKe^2}(Kb(ea-bd_1)^2S_1-rae^2):=\Phi(S_1).$$

Note $\widehat{S}=\frac{rae^2}{Kb(ea-bd_1)^2}$ is the root of $\Phi(S_1)=0$. Then, we have

$$H(\widehat{S})=\frac{er(ae^2r(ea+bd_1)k-Kb(ea-bd_1)^2(Kb(ea-bd_1)-(ea+bd_1)))}{K(ea-bd_1)^3b^2}.$$

Hence, we can determine the sign of a_{11}.

(1) If $K<K_1$, or $K>K_1$ and $k>k_1$, then we can obtain $H(\widehat{S})>0$, which yields $a_{11}<0$, then λ_1, $\lambda_2<0$.
(2) If $K>K_1$ and $k<k_1$ hold, then we can obtain that $H(\widehat{S})<0$, which yields $a_{11}>0$, then $\max\{\lambda_1, \lambda_2\}>0$.

This completes the proof.

Remark 3.1. Theorem 3.1 illustrates an interesting point that the high level of fear is beneficial to stabilizing planar equilibrium E_1. That is to say, the higher the level of fear which drives anti-predator behavior of prey, the more likely it is to help the infected predator become extinct, and thus the disease existing in the predator population can be controlled.

Next, we will show the stability of the positive equilibrium E_2 of system (1.2).

The Jacobian matrix of system (1.2) at E_2 is given as

$$J(E_2)=\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & \frac{d_2(\mathcal{R}_0-1)}{\mathcal{R}_0} & -\frac{d_2}{\mathcal{R}_0} \\ 0 & \frac{d_2(\mathcal{R}_0-1)^2}{\mathcal{R}_0} & -\frac{d_2(\mathcal{R}_0-1)}{\mathcal{R}_0} \end{pmatrix}$$

where

$$\begin{aligned} b_{11} &= X_2\left(-\frac{r}{K}+\frac{abS_2}{(1+bX_2)^2}\right),\\ b_{12} &= \frac{-krX_2}{(1+k(S_2+I_2))^2}-\frac{aX_2}{1+bX_2}<0,\\ b_{13} &= \frac{-krX_2}{(1+k(S_2+I_2))^2}<0,\\ b_{21} &= \frac{eaS_2}{(1+bX_2)^2}>0. \end{aligned} \tag{3.3}$$

The characteristic equation of $J(E_2)$ is given as

$$\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0, \tag{3.4}$$

where

$$\begin{aligned} A_1 &= -b_{11},\\ A_2 &= \frac{eaS_2X_2(ak^2(S_2+I_2)^2+kr(1+bX_2)+2ak(S_2+I_2)+a)}{(1+bX_2)^3(1+k(S_2+I_2))^2}>0,\\ A_3 &= \frac{d_2aeS_2X_2(\mathcal{R}_0-1)(ak^2(S_2+I_2)^2+\mathcal{R}_0kr(1+bX_2)+2ak(S_2+I_2)+a)}{\mathcal{R}_0(1+bX_2)^3(1+k(S_2+I_2))^2}. \end{aligned} \tag{3.5}$$

According to Routh-Hurwitz criterion, the positive equilibrium E_2 is locally asymptotically stable when $A_1>0$, $A_1A_2-A_3>0$ and $\mathcal{R}_0>1$.

Therefore, we can establish the following statement.

Theorem 3.2. *Assuming that $\mathcal{R}_0>1$ and $\frac{ea-bd_1}{b(\mathcal{R}_0-1)}>d_2$ hold, then the positive equilibrium E_2 of system (1.2) is locally asymptotically stable if $A_1>0$ and $A_1A_2-A_3>0$ hold, where $A_i(i=1, 2, 3)$ is defined as in Eq (3.5). Otherwise, it is unstable.*

Remark 3.2. *It is very regrettable that the result of theorem 3.2 cannot clearly show the impact of fear factor on the stability of positive equilibrium E_2. Considering that it is indispensable for us to grasp the influence of fear effect on the dynamics of population system, we will use numerical simulation to further discuss later.*

For the sake of facilitating the follow-up research on how the cost of fear affects the dynamics of the population system, it is necessary to give the following results here.

Remark 3.3. For the case without the fear factor, i.e., $k=0$, we get the following results:

(1) If $\mathcal{R}_0<1$, $ea-bd_1>0$ and $K<K_1$, then there exists a planar equilibrium $\widetilde{E}_1=\left(\frac{d_1}{ea-bd_1},\frac{er(K(ea-bd_1)-d_1)}{K(ea-bd_1)^2},0\right)$, which is locally asymptotic stable.
(2) If $\mathcal{R}_0>1$, $\frac{ea-bd_1}{b(\mathcal{R}_0-1)}>d_2$ and $K>\frac{d_2(\mathcal{R}_0-1)+d_1}{ea-bd_1-bd_2(\mathcal{R}_0-1)}$ hold, then there exists a positive equilibrium $\widetilde{E}_2=(\widetilde{X}_2,\widetilde{S}_2,\widetilde{I}_2)$, where

$$\widetilde{X}_2:=\frac{d_2(\mathcal{R}_0-1)+d_1}{ea-bd_1-bd_2(\mathcal{R}_0-1)},\widetilde{S}_2:=\frac{er(((ea-bd_1)-bd_2(\mathcal{R}_0-1))K-(d_2(\mathcal{R}_0-1)+d_1))}{K(ea-bd_1-bd_2(\mathcal{R}_0-1))^2},$$
$$\widetilde{I}_2:=\frac{er(\mathcal{R}_0-1)(((ea-bd_1)-bd_2(\mathcal{R}_0-1))K-(d_2(\mathcal{R}_0-1)+d_1))}{K(ea-bd_1-bd_2(\mathcal{R}_0-1))^2}.$$

And when $K<\frac{\mathcal{R}_0r(d_1+d_2(\mathcal{R}_0-1))(bd_2(\mathcal{R}_0-1)+ea+bd_1)}{(ea-bd_1-bd_2(\mathcal{R}_0-1))(d_2(\mathcal{R}_0-1)(\mathcal{R}_0br+ea)+\mathcal{R}_0bd_1r)}$, $\widetilde{E}_2$ is locally asymptotic stable.

3.2. Hopf bifurcation

In this subsection, we take k as the bifurcation parameter. The characteristic equation of system (1.2) at E_2 is given by Eq (3.4), and $A_i(k)(k=1, 2, 3)$ are defined as Eq (3.5).

Theorem 3.3. *Hopf bifurcation near the positive equilibrium E_2 for system* (1.2) *occurs whenever the critical parameter k attains the value $k = k_h$ in the domain*:

$$\Omega = \left\{k_h \in \mathbb{R}^+ | \Delta(k_h) := [A_1(k)A_2(k) - A_3(k)]|_{k=k_h} = 0 \text{ with } A_2(k_h) > 0, \left[\frac{\mathrm{d}\Delta(k)}{\mathrm{d}k}\right]|_{k=k_h} \neq 0\right\}.$$

Proof. If $k = k_h$, the characteristic equation (3.4) is

$$\lambda^3 + A_1(k_h)\lambda^2 + A_2(k_h)\lambda + A_3(k_h) = 0, \tag{3.6}$$

which can be factorized as following

$$(\lambda^2 + A_2(k_h))(\lambda + A_1(k_h)) = 0. \tag{3.7}$$

Eq (3.7) has three roots: $\lambda_1 = i\sqrt{A_2(k_h)}$, $\lambda_2 = -i\sqrt{A_2(k_h)}$ and $\lambda_3 = -A_1(k_h)$. These roots are of the form $\lambda_1 = p_1(k) + ip_2(k)$, $\lambda_2 = p_1(k) - ip_2(k)$ and $\lambda_3 = -p_3(k)$, where $p_i(k)(i = 1, 2, 3)$ are real numbers.

By the characteristic equation (3.4), we have

$$\frac{\mathrm{d}\lambda}{\mathrm{d}k} = -\frac{\lambda^2 A_1' + \lambda A_2' + A_3'}{3\lambda^2 + 2A_1\lambda + A_2}, \tag{3.8}$$

where $' = \frac{\mathrm{d}}{\mathrm{d}k}$. Substituting $\lambda = i\sqrt{A_2}$ into Eq (3.8), we get

$$\frac{A_3' - A_2A_1' + iA_2'\sqrt{A_2}}{2(A_2 - iA_1\sqrt{A_2})} = -\frac{\frac{\mathrm{d}\Delta(k)}{\mathrm{d}k}}{2(A_1^2 + A_2)} + i\left[\frac{\sqrt{A_2}A_2'}{2A_2} - \frac{A_1\sqrt{A_2}\ \frac{\mathrm{d}\Delta(k)}{\mathrm{d}k}}{2A_2(A_1^2 + A_2)}\right],$$

which means that

$$\left[\frac{\mathrm{dRe}(\lambda)}{\mathrm{d}k}\right]|_{k=k_h} = -\frac{\frac{\mathrm{d}\Delta(k)}{\mathrm{d}k}}{2(A_1^2 + A_2)}|_{k=k_h}.$$

Using monotonicity condition $\frac{\mathrm{dRe}(\lambda)}{\mathrm{d}k}|_{k=k_h} \neq 0$, the condition $\frac{\mathrm{d}\Delta(k)}{\mathrm{d}k}|_{k=k_h} \neq 0$ guarantees the existence of Hopf bifurcation.

4. Numerical simulations

In this section, we fully discuss the intricate impact of fear factor on the dynamics of system (1.2) in which the predator is infected by parasites, contributing to the ecological balance. We apply numerical simulations to focus on the stability of equilibria E_1 and E_2, with parameter values shown in Table 1.

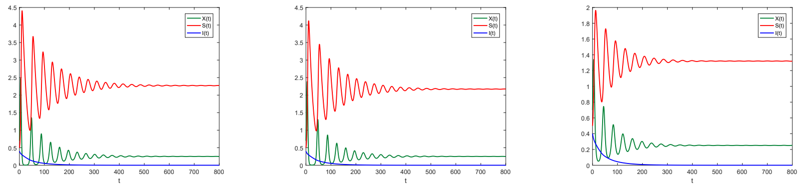
4.1. The stability of E_1

Unless otherwise statement, we use the following parameters in this subsection

$$a = 0.3, b = 0.1, e = 0.68, d_1 = 0.05, r = 0.7, d_2 = 0.08, \beta = 0.065,$$

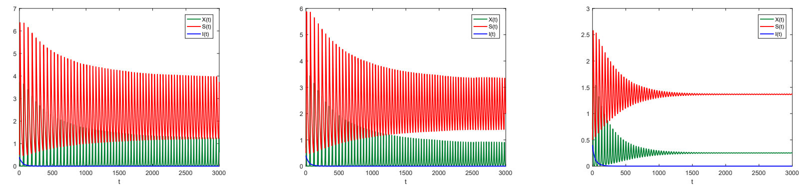
Table 1
The values of parameters for model (1.2).

Paras	Meanings	Range	References
r	Intrinsic growth rate of prey	0.59 – 1	[(Sarkara & Khajanchi, 2020), (Barman et al., 2021), (Hethcote et al., 2004)]
k	Fear effect level of prey	0 – 60	[(Wang et al., 2016), (Wang & Zou, 2017), (Zhang et al., 2019)]
K	Carrying capacity of the prey	2.6 – 100	[(Hossain et al., 2020), (Hethcote et al., 2004)]
a	Predation coefficients	0.2 – 0.7	[(Zhang et al., 2023), (Dutta et al., 2022)]
b	Predators handling time of a prey	0.1 – 3	[(Kooi et al., 2011), (Zhang et al., 2023)]
e	Biomass conversion constant	0.02 – 0.9	[(Zhang et al., 2023), (Dutta et al., 2022)]
β	Transmissibility coefficient	0.05 – 2	[(Zhang et al., 2022), (Zhang et al., 2023), (Dutta et al., 2022)]
d_1	Mortality rates of the susceptible predator	0.04 – 0.1	[(Zhang et al., 2022), (Barman et al., 2021)]
d_2	Mortality rates of the infected predator	0.08 – 0.22	[(Zhang et al., 2022), (Barman et al., 2021)]



(a) $k = 0$ (b) $k = 0.02$ (c) $k = 0.5$

Fig. 1. The paths of the solution $(X(t), S(t), I(t))$ for system (1.2) with $K = 5$.



(a) $k = 0$ (b) $k = 0.02$ (c) $k = 0.5$

Fig. 2. The paths of the solution $(X(t), S(t), I(t))$ for system (1.2) with $K = 11$.

then we have

$$ea - bd_1 = 0.199 > 0, \;\; \mathcal{R}_0 = 0.81 < 1, \;\; K_1 = \frac{ea + bd_1}{b(ea - bd_1)} = 10.5,$$

which means that system (1.2) has only one planar equilibrium E_1. And the diseases existing in the predator population are eliminated in this case.

Example 5.1.1: The impact of *k* under small environmental carrying capacity

We start our numerical simulation with the environmental carrying capacity $K = 5 < K_1$, then according to Theorem 3.1 (P1−I), we conclude that E_1 is locally asymptotic stable. The numerical simulations showed in Fig. 1 provide the graphs of $(X(t), S(t), I(t))$ for system (1.2) when $k = 0, 0.02, 0.5$ respectively. For $k = 0$, $E_1 = (0.25, 2.27, 0)$ is stable; for $k = 0.02$, $E_1 = (0.25, 2.17, 0)$ is stable; for $k = 0.5$, $E_1 = (0.25, 1.32, 0)$ is stable. Hence, with the increases of the fear level, the number of susceptible predator population decreases, but the number of prey population does not change.

Example 5.1.2: The impact of *k* under large environmental carrying capacity

For the large environmental carrying capacity $K = 11 > K_1$, there exists abundant phenomena about the influence of the fear effect on population dynamics. Through calculation, we have $k_1 = 0.0213$, which guides us to take different values of k to study the dynamics of system (1.2). As an example, we adopt $k = 0$, then system (1.2) without the fear factor has the equilibrium $E_1 = (0.25, 2.34, 0)$, which is unstable (Fig. 2(a)). Next, one take $k = 0.02 < k_1$, then system (1.2) has a planar equilibrium $E_1 = (0.25, 2.24, 0)$, which is unstable (Fig. 2(b)). When we increase the cost of the fear to $k = 0.5$, then $k > k_1$, as a consequence result of Theorem 3.1 (P1-II), there exists a stable equilibrium $E_1 = (0.25, 1.37, 0)$ for system (1.2) (Fig. 2(c)). These numerical investigations under the assumption of relatively large environmental carrying capacity show that the fear effect not only reduces the number of susceptible predators which is similar to the case of small environmental carrying capacity showed in Fig. 1, but also may be conducive to the changing of the planar equilibrium E_1 from unstable to stable.

4.2. The stability of E_2

In this subsection, we firstly fix the parameters of system (1.2) as follows:

$$r = 0.7, b = 0.1, e = 0.68, \beta = 0.26, d_1 = 0.05.$$

Then we analyze the influence of the level of fear k on the positive equilibrium E_2 of system (1.2).

Example 5.2.1: The impact of *k* under the assumption that E_2 for system (1.2) without fear factors is stable

After taking

$$a = 0.3, \;\; d_2 = 0.08, \;\; K = 6.9$$

we have

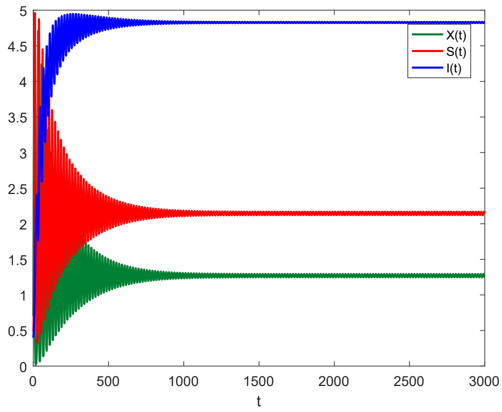


Fig. 3. The paths of the solution $(X(t), S(t), I(t))$ for system (1.2) with $a = 0.3$, $d_2 = 0.08$, $K = 6.9$ when $k = 0$.

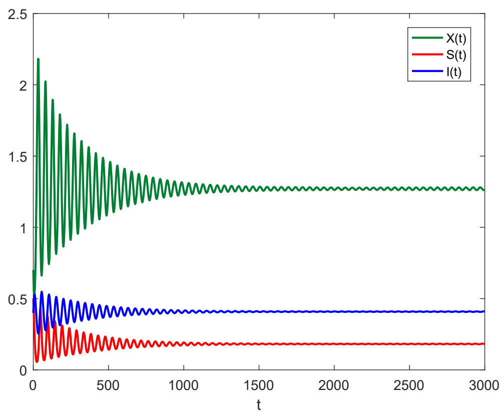


Fig. 4. The paths of the solution $(X(t), S(t), I(t))$ for system (1.2) with $a = 0.3$, $d_2 = 0.08$, $K = 6.9$ when $k = 5$.

$$k_h = 9.635051, \quad \mathcal{R}_0 = 3.25 > 1, \quad \frac{ea - bd_1}{b(\mathcal{R}_0 - 1)} = 0.88 > d_2,$$

$$K > \frac{d_2(\mathcal{R}_0 - 1) + d_1}{ea - bd_1 - bd_2(\mathcal{R}_0 - 1)} = 1.27,$$

$$K < \frac{\mathcal{R}_0 r(d_1 + d_2(\mathcal{R}_0 - 1))(bd_2(\mathcal{R}_0 - 1) + ea + bd_1)}{(ea - bd_1 - bd_2(\mathcal{R}_0 - 1))(d_2(\mathcal{R}_0 - 1)(\mathcal{R}_0 br + ea) + \mathcal{R}_0 bd_1 r)} = 7.37,$$

which means that system (1.2) without fear factors, i.e., $k = 0$, has a planar equilibrium $E_1 = (0.25, 2.31, 0)$ and a locally asymptotic stable $E_2 = (1.27, 2.15, 4.83)$. The numerical results are presented in Fig. 3.

When the growth rate of prey decreases due to the change of prey behavior caused by the presence of predators, we find that system (1.2) shows complex dynamics. Here, we will conduct a series of detailed analysis. Assume that the level of fear is small, i.e. $k = 5$, then we have $A_1 = 0.124 > 0$, $A_1A_2 - A_3 = 0.0002 > 0$. This reveals that the positive equilibrium $E_2 = (1.27, 0.18, 0.41)$ is stable, which is confirmed by the numerical example shown in Fig. 4. What these tell us is that the fear factor reduces the number of susceptible predator and infected predator, but it does not break the stability of E_2. Furthermore, results for the level of fear $k = k_h$, which shows $A_1 = 0.126 > 0$, and $A_1A_2 - A_3 = 0$ by calculations, provide further insight. We can see that system (1.2) undergoes a Hopf bifurcation and there is a limit cycle around $E_2 = (1.27, 0.11, 0.25)$ (Fig. 5). If one increase the cost of fear k to $k = 15 > k_h$ such that $A_1 = 0.127 > 0$ and $A_1A_2 - A_3 = -0.0001 < 0$, the profile in Fig. 6 demonstrates that the equilibrium $E_2 = (1.27, 0.08, 0.17)$ is unstable. It should come as no surprise that in general, the increase of the level of fear not

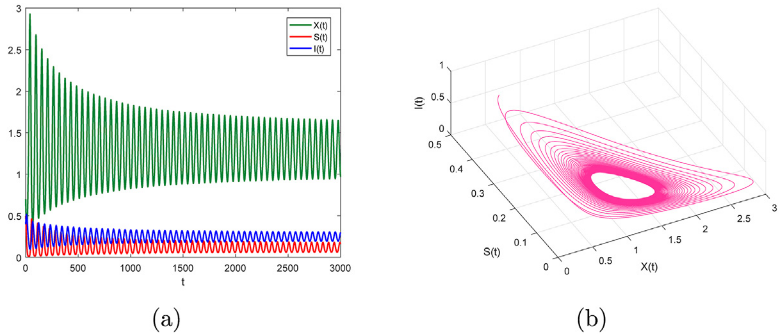


Fig. 5. The paths of the solution ($X(t)$, $S(t)$, $I(t)$) for system (1.2) with $a = 0.3$, $d_2 = 0.08$, $K = 6.9$ when $k = k_h$.

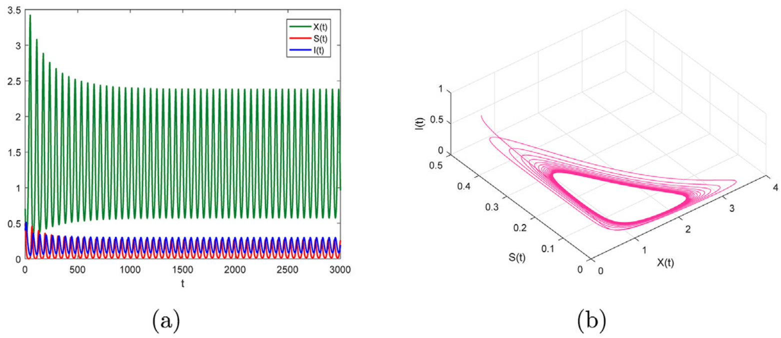


Fig. 6. The paths of the solution ($X(t)$, $S(t)$, $I(t)$) for system (1.2) with $a = 0.3$, $d_2 = 0.08$, $K = 6.9$ when $k = 15$.

only results in the decrease of the number of predators including susceptible and infected predators, but also induces the positive equilibrium E_2 to change from stable to unstable.

Example 5.2.2: The impact of k under the assumption that E_2 for system (1.2) without fear factors is unstable

Through setting the following parameter values:

$$a = 0.2,\ d_2 = 0.2,\ K = 8,$$

the numerical experiments starting from the case that the positive equilibrium E_2 for system (1.2) without fear effect is unstable are now initialized. In this case, we have

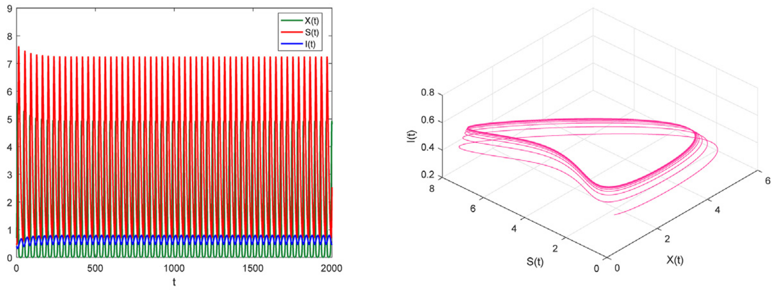
$$k_h = 0.299949,\ \ \mathcal{R}_0 = 1.3 > 1,\ \ \frac{ea - bd_1}{b(\mathcal{R}_0 - 1)} = 4.37 > d_2,$$

$$K > \frac{d_2(\mathcal{R}_0 - 1) + d_1}{ea - bd_1 - bd_2(\mathcal{R}_0 - 1)} = 0.88,$$

$$K > \frac{\mathcal{R}_0 r(d_1 + d_2(\mathcal{R}_0 - 1))(bd_2(\mathcal{R}_0 - 1) + ea + bd_1)}{(ea - bd_1 - bd_2(\mathcal{R}_0 - 1))(d_2(\mathcal{R}_0 - 1)(\mathcal{R}_0 br + ea) + \mathcal{R}_0 bd_1 r)} = 6.48.$$

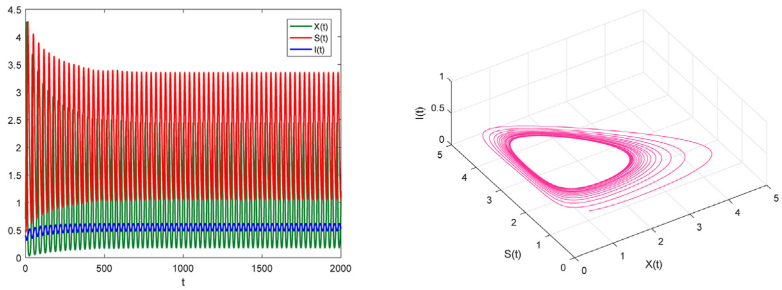
Thus, system (1.2) without fear factors, i.e., $k = 0$, has a planar equilibrium $E_1 = (0.38, 3.46, 0)$ and an unstable equilibrium $E_2 = (0.88, 3.39, 1.02)$.

In order to determine whether the impact of the fear effect on the stability of E_2 is different from the conclusion revealed in Example 5.2.1 for this case, we similarly adopt three different choices of k to analyze the population dynamics of system (1.2). When we take $k = 0.2 < k_h$, system (1.2) has a unique positive equilibrium $E_2 = (0.88, 2.06, 0.62)$ and one can obtain $A_1 = 0.05 > 0$ and $A_1A_2 - A_3 = -0.0002 < 0$, which means that E_2 is unstable (Fig. 8). When we take $k = k_h$ and obtain $A_1 = 0.050 > 0$ and $A_1A_2 - A_3 = 0$, Fig. 9 shows that system (1.2) exhibits limit cycle around $E_2 = (0.88, 1.81, 0.54)$. When we take $k = 1 > k_h$, the simple calculation shows that $A_1 = 0.060 > 0$ and $A_1A_2 - A_3 = 0.0003 > 0$, and hence there is a stable equilibrium $E_2 = (0.88, 1.13, 0.34)$ for system (1.2), which are supported by the numerical simulations displayed in Fig. 10. Hence, comparing the results given in Example 5.2.1, we see that the different implication induced by the fear factor k is that the dynamics of E_2 may change from unstable to stable with the increase of the value of k.



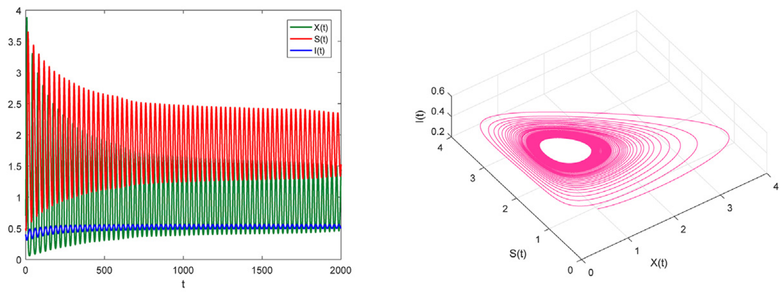
(a) (b)

Fig. 7. The paths of the solution ($X(t)$, $S(t)$, $I(t)$) for system (1.2) with $a = 0.2$, $d_2 = 0.2$, $K = 8$ when $k = 0$.



(a) (b)

Fig. 8. The paths of the solution ($X(t)$, $S(t)$, $I(t)$) for system (1.2) with $a = 0.2$, $d_2 = 0.2$, $K = 8$ when $k = 0.2$.



(a) (b)

Fig. 9. The paths of the solution ($X(t)$, $S(t)$, $I(t)$) for system (1.2) with $a = 0.2$, $d_2 = 0.2$, $K = 8$ when $k = k_h$.

5. Conclusion and discussion

In this paper, we proposed an eco-epidemiological system (1.2) involving standard incidence rate to investigate the effect of the anti-predator behaviors caused by the fear factor on the population dynamics of prey and predator. Compared with the system (1.1) with bilinear incidence rate, our analysis reveals that for system (1.2), the predator population and the prey population do not tend to extinction and always coexist in nature, which means that the ecosystem is in a balanced state.

The impacts of the fear factor on the dynamics for predator-prey system (1.2) are summarized as follows.

(1) **Fear effect leads to a reduction of predator population.**

As far as our observation is concerned, fear effect on the population density is only to reduce the number of the predator population, but does not affect the number of the prey population (for example, see Fig. 1).

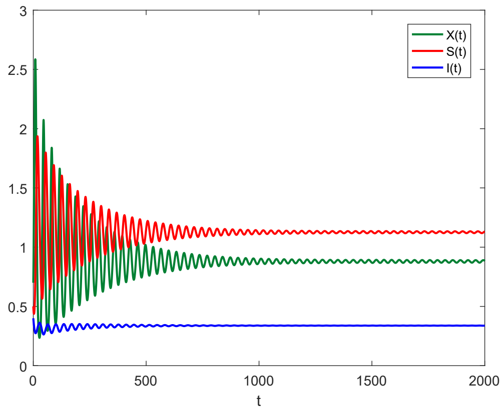


Fig. 10. The paths of the solution ($X(t)$, $S(t)$, $I(t)$) for system (1.2) with $a=0.2$, $d_2=0.2$, $K=8$ when $k=1$.

(2) **Fear effect can change the stability of E_1 (disease free when $\mathcal{R}_0<1$).**

If the environmental carrying capacity is relatively small, the planar equilibrium E_1 of system (1.2) without fear effect is stable and the introduction of fear effect does not change its stability (see Fig. 1), while if the environmental carrying capacity is large, the planar equilibrium E_1 of the system (1.2) without fear effect is unstable and the increase of the level of fear make it from unstable to stable (see Fig. 2).

(3) **Fear effect can change the stability of E_2 (disease persistence when $\mathcal{R}_0>1$).**

The small level of fear, i.e., $k<k_h$ can not cause any change in the stability of the positive equilibrium E_2 for system (1.2) (see Figs. 3–4 and Figs. 7–8). When $k=k_h$, the fear factor can destabilize the stability of E_2 and be beneficial to the occurrence of periodic oscillation (see Figs. 5 and 9). The high level of fear factor, i.e. $k>k_h$, may change E_2 from stable to unstable or from unstable to stable (see Figs. 6 and 10).

Although this manuscript seem simple and conventional, numerical methods, stability analysis are all conventional approaches, the mathematical modelling research on these phenomena presented in this manuscript is lack since mathematical modelling for some natural phenomena is always challenging and difficult. In fact, this manuscript is progressive for quantitative studies of the qualitative issues, and obtains some theoretical evidences about these natural phenomena, so the research of this manuscript is still necessary. In addition, The data in this paper are given by the authors according to the scope of references, the real data associated with this manuscript are lacking, and the real data fitting is out of my fields, hence, the real data fitting could not be realized at present. However, I will continue this work in the future.

A natural question, what's the corresponding dynamics if the parasite was not introduced into the predator population? For the convenience of readers, we summarize the main results as follows. The system without parasite is a special of system (1.2):

$$\begin{cases} \dfrac{dX}{dt}=\dfrac{rX}{1+kY}-\dfrac{rX^2}{K}-\dfrac{aXY}{1+bX}, \\ \dfrac{dY}{dt}=\dfrac{eaXY}{1+bX}-d_1Y, \end{cases} \tag{5.1}$$

where X, Y represent the densities of prey and predator at time t, respectively. Similarly, we have the following results.

(i) The trivial equilibrium $\widehat{E}_0=(0,0)$ is unstable.
(ii) If one of the following inequalities holds:
 (ii-1) $ea-bd_1<0$;
 (ii-2) $ea-bd_1>0$ and $K<\frac{d_1}{ea-bd_1}$,

then the axial equilibrium $\widehat{E}_1=(K,0)$ is stable; while $\widehat{E}_1=(K,0)$ is unstable if $ea-bd_1>0$ and $K>\frac{d_1}{ea-bd_1}$.

(iii) Assume that $ea - bd_1 > 0$. Setting

$$\widehat{K} := \frac{e^2a^2 - b^2d_1^2 + \sqrt{(e^2a^2 - b^2d_1^2)(4ae^2kr + e^2a^2 - b^2d_1^2)}}{2b(ea - bd_1)^2},$$

if $\frac{d_1}{ea-bd_1} < K < \widehat{K}$ holds, then the positive equilibrium $E_2 = (X_2, Y_2)$ is stable, which if $K > \widehat{K}$ holds, then the positive equilibrium $E_2 = (X_2, Y_2)$ is unstable, where $X_2 = \frac{d_1}{ea-bd_1}$ and Y_2 is the positive root of the following equation:

$$P(Y) = Kk(ea - bd_1)^2Y^2 + (K(ea - bd_1)^2 + d_1ekr)Y - er(K(ea - bd_1) - d_1).$$

Hence, for system (5.1), the predator population may become extinct, and even both the prey and predator may become extinct. Combined with the results of system (1.2), we theoretically assert that the measures to prevent the extinction of endangered prey by introducing parasites into the predator population are feasible in the biological control.

CRediT authorship contribution statement

Chunmei Zhang: Writing – original draft, Writing – review & editing.

Declaration of competing interest

The author declare there is no conflicts of interest.

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