Heliyon 10 (2024) e32897

Contents lists available at ScienceDirect

Heliyon

journal homepage: www.cell.com/heliyon

Research article

Selecting optimal celestial object for space observation in the realm of complex spherical fuzzy systems

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ARTICLE INFO

Keywords:
Complex spherical fuzzy sets
Yager weighted averaging operator
Yager weighted geometric operator
Decision making
Optimization
Celestial object selection

ABSTRACT

The sensible selection of celestial objects for observation by the James Web Space Telescope (JWST) is pivotal for the precise decision-making (DM) process, aligning with scientific priorities and instrument capabilities to maximize valuable data acquisition to address key astronomical questions within the constraints of limited observation time. Aggregation operators are valuable models for condensing and summarizing a finite set of data of imprecise nature. Utilization of these operators is critical when addressing multi-attribute decision-making (MCDM) challenges. The complex spherical fuzzy (CSF) framework effectively captures and represents the uncertainty that arises in a DM problem with more precision. This paper presents two novel aggregation operators, namely the complex spherical fuzzy Yager weighted averaging (CSFYWA) operator and the complex spherical fuzzy Yager weighted geometric (CSFYWG) operator. Many fundamental structural properties of these operators are delineated, and thereby an improved score function is suggested that addresses the limitations of the existing score function within the CSF system. The newly defined operators are applied to formulate an algorithm for MADM problems to tackle the challenges of ambiguous data in the selection process. Moreover, these strategies are effectively applied to handle the MADM problem of selecting the optimal astronomical object for space observation within the CSF context. Additionally, a comparative analysis is also performed to demonstrate the validity and superiority of the proposed techniques compared to the existing strategies.

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https://doi.org/10.1016/j.heliyon.2024.e32897
Received 24 March 2024; Received in revised form 6 June 2024; Accepted 11 June 2024
Available online 14 June 2024

1. Introduction

1.1. Background

In recent decades, decision-making (DM) has become crucial for navigating life's challenges. MADM holds significant importance in practical decision-making scenarios, as it facilitates an analytical evaluation of multiple criteria to rank results for a finite set of alternatives to determine the best optimal choice. It ensures that the selected option aligns with the utmost significance and requirements. In traditional decision-making approaches, the rigid boundaries of crisp set theory faced limitations in handling real-world ambiguity and imprecision. Numerous researchers and scientists have developed various mathematical theories to tackle this ambiguity in different real-life domains such as sciences, engineering, medical diagnosis, technology, etc. In order to capture the ambiguity and unpredictability of real-world issues, Zadeh [1] first put forward the remarkable theory of a fuzzy set (FS) in 1965. In this approach, human judgment may be conveyed by a membership function. Yager [2] strengthened Zadeh's seminal research by introducing the aggregation operators in fuzzy system modeling, thereby enhancing the framework's theoretical robustness and practical adaptability. Nevertheless, in many real-life scenarios, this concept is only sometimes valid. For example, the FS theory is insufficient in addressing scenarios that involve varying degrees of truth and falsity in information presentation. For this, Atanassov [3] developed an innovative framework known as intuitionistic fuzzy sets (IFSs). This framework is designed to integrate membership, non-membership, and hesitation functions effectively. With this approach, Atanassov ensures that the aggregated value of the membership degree (MD) and non-membership degree (NMD) lies inside the range of 0 and 1. This framework enables a more thorough depiction of uncertainty in decision-making. However, IFSs cannot deal with information whenever the total value of MD and NMD exceeds one. Yager [4,5] critically examined these flaws and came up with a way to extend the concept of IFSs, namely, Pythagorean fuzzy sets (PyFSs), which meet the criterion that the outcome of the square summation of its MD and NMD remains within [0,1]. The PyF settings exhibit more flexibility than both FS and IFS. Researchers across various disciplines have employed PyFS theory in numerous applications [6–8]. However, Yager restricted the endowment of MD ($\zeta(x)$) and NMD ($\kappa(x)$) pairs, requiring that the square summation of these degrees fall within the interval [0, 1], significantly reducing the adaptability in selecting the MDs and NMDs. This means that, PyFSs cannot manage the data of these membership degrees outside this quadratic inequality: $\zeta^2(x) + \kappa^2(x) > 1$. Senapati and Yager [9] presented the theory of Fermatean fuzzy sets (FFSs), a modification of PyFSs to cope with this complexity. FFSs contain the membership and non-membership functions and fulfill the requirement that the cubic summation of the MD and NMD is less than or equal to one. These sets have played a vital role in dealing with such data types. Garg et al. [10] defined the Yager aggregation operators in the FF environment to improve the COVID-19 Testing Facility. Moreover, Yager [11] extended the idea of the FFSs by introducing q-rung Orthopair fuzzy sets (q-ROFSs), characterized by the summation of q-powers of membership and non-membership degrees within the [0, 1] range, showcasing versatile applications across different fields in Refs. [12–17].

The consideration of the neutral condition has been overlooked in the above-discussed theories, which is, in fact, crucial for accurately representing human cognition. In this scenario, Cuong [18] introduced the theory of picture fuzzy set (PFS), that defines four elements (yes, abstain, no, and refusal) of a vague incident. In decision-making, human opinion can be categorized into four membership functions of PFSs: favor, abstain, disfavor, and rejection. These membership functions indicate different degrees of preference towards a particular conclusion. Furthermore, Qiyas et al. [19] presented the concept of Yager AOs within the PF environment. However, a restriction to this approach arises when a sum of three degrees exceeds one. To illustrate this claim, suppose the voting scenario where the positive grade is 0.75, the neutral grade is 0.53, and the negative grade is 0.41. Thus, PFSs cannot handle such a limitation. In 2019, Kahraman and Gundogdu [20] successfully resolved this hurdle by introducing the unique concept of a spherical FS (SFS). They modified the PFSs principle to include the sum of the squares of the membership, neutral membership (NeM), and non-membership degrees, ensuring that it remains in the unit interval. One can view the successful applications of this concept to many disciplines in Refs. [21–25].

Periodic systems or two-dimensional situations are beyond the capabilities of the aforementioned models. In order to tackle this concern, Ramot et al. introduced the complex fuzzy set (CFS) notion in Refs. [26,27]. Ramot's idea was to broaden the scope of membership from the interval [0, 1] to the unit disk of the complex plane. A CFS is represented by the amplitude term $\zeta(x)$ within the range [0,1] and the phase term $\varpi(x)$ within the range $[0, 2\pi]$. The phase term is a vital component in the formation of the CF structure. Due to this particular aspect, the CF sets are more successful than classical FS theories. However, CFS cannot articulate the intricate non-membership degree in complex-valued terms. Alkouri and Salleh [28] refined the idea of CFSs by presenting a new strategy for investigating complex IFSs (CIFSs). This extension incorporates the MD and NMD, utilizing complex numbers (CNs) within a unit disc. This framework deals with the information in which the aggregate of the real parts (and likewise imaginary parts) of MD and NMD lie in a unit disk. For more details on the development of CIFSs, we refer to Refs. [29,30]. The complex Pythagorean fuzzy set (CPyFS) was defined by Ullah et al. [31], in which the sum of the squares of the real and imaginary parts of these complex numbers should be less than or equal to one. The authors [32,33] have successfully implemented the CPyFSs in several domains of everyday life. Nevertheless, this particular environment fails to incorporate or acknowledge situations when we use four key characteristics to describe a situation. To tackle this deficiency, Akram et al. [34] established the remarkable notion of complex PFSs (CPFSs), which is defined by the membership degree $\left(\beth(x) = \zeta(x).\ e^{i2\pi\varpi(x)}\right)$, neutral degree $\left(\aleph(x) = \eta(x).\ e^{i2\pi\vartheta(x)}\right)$, and non-membership degree $\left(\Psi(x) = \kappa(x).\ e^{i2\pi\Omega(x)}\right)$, with the constraint that the sum of both the real and imaginary components of these three degrees does not surpass the unit interval. However, CPFSs encounter challenges when the summation of these three degrees exceeds 1. To tackle such problems, Ali et al. [35] developed the theory of the complex SFSs (CSFSs) to improve the limitations of CPFSs, characterized by membership, neutral, and non-membership functions in a unit disk. The amplitude $(\zeta(x), \eta(x), \kappa(x))$ and phase $(\varpi(x), \vartheta(x), \Omega(x))$ terms of MD, NeD, and NMD are restricted by the conditions $0 \leq \zeta^2(x) + \eta^2(x) + \kappa^2(x) \leq 1$ and $0 \leq \varpi^2(x) + \vartheta^2(x) + \Omega^2(x) \leq 1$. Undoubtedly, this is the most advanced approach for addressing ambiguity and uncertainty in many world problems. Numerous articles have been devoted to this remarkable concept, which has garnered considerable interest from researchers. The advancement of this framework across various fields has made it a crucial point in modern research. Akram et al. [36] introduced a hybrid DM method incorporating CSF information prioritized weighted averaging operator. The CSFS VIKOR method was utilized in a group DM approach designed by the same author in Ref. [37]. By utilizing entropy evaluation and a power operator, Naeem et al. [38] examined a CSF decision support system. Hussain et al. [39] discussed the Aczel-Alsina operators under the complex spherical fuzzy environment and their implementation in the evaluation of power-battery cars. Akram et al. [40] established the Dombi AOs in CSF structure to accomplish the MADM challenge. More developments on CSF theories can be viewed in Refs. [41,42].

The JWST, with its state-of-the-art infrared capabilities, presents a promising opportunity for astronomers to investigate new areas of the cosmos. However, the process of selecting which objects can be observed requires a comprehensive assessment of scientific objectives and technological constraints while also optimizing the telescope's performance to enable groundbreaking discoveries. The JWST operates under a limited timeframe. Although mostly powered by solar energy, the JWST requires a modest quantity of finite fuel to sustain its orbit and sensors. By considering these limitations, astronomers need to employ the decision-making process to meticulously select the most suitable target for Webb space observation, which helps to optimize scientific outputs, maximize the performance of the telescope, save a considerable amount of time, and almost guarantee that the generated data is suitable for the planned analysis. Greenhouse [43] presented the overview and status of the JWST mission. After that, Maillard [44] put forth the moon perspective, which gives us the idea of infrared astronomy beyond JWST.

1.2. Research gaps, motivations, and contributions of the current study

Information processing and decision-making are essential to a well-functioning, intelligent system. It is important to note that there are various DM problems that cannot be handled by means of different fuzzy environments like complex FF, Cq-ROFF, and CPF sets where the sum of the MD, NeD, and NMD exceeds the closed unit interval [0, 1]. The significance of these research gaps motivates us to study the theory of CSFSs, wherein the square sum of these degrees falls within [0, 1]. The utilization of aggregation operators is a strategic technique for the synthesis and consolidation of different pieces of data to distill them down into a manageable and useful resolution. Aggregation operators are a critical DM tool that allows ambiguous information from multiple sources to be distilled into a form that can be easily worked with, so the underlying trends and patterns can be more easily teased out. However, the existing models based on certain t-norms (TNs) and t-conorms (TCNs) within the context of CSF knowledge lack dynamic parameter adjustment in response to the risk tolerance of decision-makers. As a consequence, these existing strategies cannot provide accurate estimations of the algebraic product and sum in a seamless manner in their aggregation capabilities. This insensitivity may lead to premature conclusions, rendering them less adaptable and unable to account for individual preferences. Yager aggregation operators (YAOs) are extraordinarily adaptable, operating successfully in a decision-support system in a broad array of conditions, even for ambiguous tasks. Yager operators exhibit remarkable versatility in handling operational settings and demonstrate exceptional efficacy in resolving decision-making challenges. The Yager product and sum are effective substitutes for the Einstein, Dombi, Hamacher, and other algebraic products regarding intersection and union operations. One significant aspect of studying Yager aggregation operators is their ability to dynamically adjust the parametric values more effectively, thereby providing precise estimations in a seamless manner as compared to the other existing techniques. Motivated by these insights, we present the integration of YAOs into the CSF environment to mitigate all research gaps present in the existing literature.

The main aim of this study is to pinpoint the benefits of this fusion and the significance of aggregation operators. In contrast to the existing methods, our proposed methodologies significantly diminish the uncertainty in real-time applications as well. The novelty of this article is to introduce the two innovative Yager aggregation operators in the framework of the CSF environment to solve crucial MADM problems like the selection of the optimal astronomical object for space observation by JWST and to perform an in-depth comparison analysis that highlights the viability of these approaches compared to the existing methods. This important challenge is handled by designing an effective step-by-step mathematical mechanism using the proposed methodologies. The utilization of recently proposed CSFYAOs in this study helps to enhance the scientific outcomes, maximize the telescope's performance, save significant time, and ensure that the collected data is suitable for the intended analysis in comparison to other techniques.

The main objectives of this research work are outlined as follows:

1. To investigate the deficiency of the current score function by defining an enhanced score function within the context of CSF knowledge and to elucidate its validity to solve MADM problems.
1. To propose the two innovative aggregation operators for CSFNs, specifically the CSFYWA operator and the CSFYWG operator.
2. To establish the fundamental Yager operational laws that are relevant to CSFNs and to examine the inherent structural properties of the newly proposed operators.
3. To develop a sequential mathematical algorithm for the MADM problem by implementing the recently defined techniques within the CSF settings.
4. To enhance the significance and validity of the recently defined methodologies by presenting the solution to the MADM problem of selecting the optimal astronomical object for space observation by JWST.
5. To perform an in-depth comparison analysis that highlights the viability of these approaches compared to the existing methods.

The main contributions of this study are determined as follows:

1. The CSFS theory serves as a comprehensive framework that integrates and generalizes the ideas of CFS, CIFS, CPyFS, and CPFS. We examine the properties of specific operating laws between CSFNs while defining them.
2. We formulate an improved score function that addresses the limitations observed with current score functions inside the CSF environment.
3. Yager operational principles are essential tools for modeling MADM issues, although, at present, there is no published research on CSF Yager operators.
4. Considering the Yager aggregation operators' benefits, we propose some novel AOs according to the Yager t-norm and s-norm. These operators are designed for application in analyzing a problem involving two-dimensional data.
5. We explore fundamental properties such as idempotency, monotonicity, and boundedness of the newly defined operators.
6. We present a step-by-step mathematical mechanism to solve MADM problems under these strategies in CSF settings. Additionally, we conduct a comparative study to showcase the validity and practicability of the suggested approaches with the existing techniques.

The above-stated objectives are addressed in the framework of the designed methodologies as follows: In Section 2, we review some fundamental concepts and operational laws of CSFSs. In Section 3, we address the existing score function's limitations and improve it by developing a modified score function for CSFNs to solve MADM problems. In Section 4, we introduce two novel aggregation operators (AOs) in the framework of CSFSs based on Yager aggregation tools and prove their fundamental properties. In Section 5, we formulate a systematic mathematical framework to tackle the MADM problems related to CSF information using newly defined techniques and the updated score function. In addition, we highlight the significance and efficiency of the newly defined techniques by presenting the solution to the MADM problem of selecting the most suitable astronomical object for space observation by JWST. Moreover, we establish a comprehensive comparative analysis to showcase the applicability and superiority of these strategies with existing knowledge. In Section 6, we conclude this study by analyzing potential implications and summarizing the principal findings.

2. Basic concepts

In this section, we discuss the preliminary aspects of the topic at hand. These concepts and methodologies are critical for the effective comprehension of the research.

Definition 1. ([20]). Suppose that E denotes the universe of discourse. A SFS Ξ is referred to as an object of the form:

$$\Xi = \{(x, \daleth(x), \aleph(x), \Psi(x)) \mid x \,\epsilon\, E\}.$$

In the above mathematical expression, $\daleth_{\Xi} : E \to [0,1]$, $\aleph_{\Xi} : E \to [0,1]$, and Ψ_{Ξ}: $E \to [0, 1]$, are the membership, neutral, and non-membership functions, respectively, satisfying $0 \leq \daleth_{\Xi}^2(\mathrm{x}) + \aleph_{\Xi}^2(\mathrm{x}) + \Psi_{\Xi}^2(x) \leq 1$. Additionally, $\varkappa = \sqrt{1 - \left(\daleth_{\Xi}^2(\mathrm{x}) + \aleph_{\Xi}^2(\mathrm{x}) + \Psi_{\Xi}^2(\mathrm{x})\right)}$ is known as refusal degree.

Definition 2. ([26]). A CFS $\mathscr{F}$ on the universe of discourse E, is defined as:

$$\mathscr{F} = \{ x, \daleth_{\mathscr{F}}(x) \mid x \,\epsilon\, E\}.$$

In the above mathematical expression, the complex-valued membership function $\daleth_{\mathscr{F}}$ assigns each value $x \,\epsilon\, E$ to a closed unit disk in the complex plane. The membership degree $\daleth_{\mathscr{F}}\ (x)$ is expressed as $\daleth_{\mathscr{F}}\ (x) = \zeta_{\mathscr{F}}\ (x)\ e^{2\pi i \varpi_{\mathscr{F}}(x)}$, such that $\zeta_{\mathscr{F}}\ (x)\ , \varpi_{\mathscr{F}}\ (x) \in\ [0, 1]$, respectively, represent amplitude and phase terms of the membership function of CFS.

Definition 3. ([34]). A CPFS $\mathcal{P}$ on the universe of discourse E, is expressed as:

$$\mathcal{P} = \{(\mathrm{x}, \daleth_{\mathcal{P}}(x), \aleph_{\mathcal{P}}(x), \Psi_{\mathcal{P}}(x)) \mid x \,\epsilon\, E\}.$$

In the above mathematical expression, $\daleth_{\mathcal{P}}, \aleph_{\mathcal{P}}$, and $\Psi_{\mathcal{P}}$ represent the complex-valued membership, neutral, and non-membership functions, respectively. These functions map each element $x \,\epsilon\, E$ to the closed unit disk in a complex plane. Membership degree (MD) $\daleth_{\mathcal{P}}$ (x), neutral degree (NeD) $\aleph_{\mathcal{P}}(x)$, and non-membership degree $\Psi_{\mathcal{P}}\ (x)$ are expressed as follows: $\daleth_{\mathcal{P}}\ (x) = \zeta_{\mathcal{P}}\ (x)\ e^{2\pi i \varpi_{\mathcal{P}}(x)}$, $\aleph_{\mathcal{P}}(x) = \eta_{\mathcal{P}}\ (x)$ $e^{2\pi i \vartheta_{\mathcal{P}}(x)}$, and $\Psi_{\mathcal{P}}\ (x) = \kappa_{\mathcal{P}}\ (x)\ e^{2\pi i \Omega_{\mathcal{P}}(x)}$, such that $0 \leq \zeta_{\mathcal{P}}\ (x), \eta_{\mathcal{P}}\ (x), \kappa_{\mathcal{P}}\ (x), \zeta_{\mathcal{P}}\ (x) + \eta_{\mathcal{P}}\ (x) + \kappa_{\mathcal{P}}\ (x) \leq 1$ and $0 \leq \varpi_{\mathcal{P}}\ (x), \vartheta_{\mathcal{P}}\ (x), \Omega_{\mathcal{P}}\ (x)$, $\varpi_{\mathcal{P}}\ (x) + \vartheta_{\mathcal{P}}\ (x) + \Omega_{\mathcal{P}}\ (x) \leq 1$. In addition, the refusal degree is indicated by

$$\varkappa = \Gamma_{\mathcal{P}}\ (x)\ e^{2\pi i \tau_{\mathcal{P}}(x)} = (1 - (\zeta_{\mathcal{P}}(x) + \eta_{\mathcal{P}}(x) + \kappa_{\mathcal{P}}\ (x)))\ e^{2\pi i (1 - (\varpi_{\mathcal{P}}\ (x) + \vartheta_{\mathcal{P}}\ (x) + \Omega_{\mathcal{P}}\ (x)))}.$$

Definition 4. ([35]). The complex spherical fuzzy set (CSFS) · on the universe of discourse E is defined in the following manner:

$$\mathscr{Q} = \{(x, \daleth_{\mathscr{Q}}(x), \aleph_{\mathscr{Q}}(x), \Psi_{\mathscr{Q}}(x)) \mid x \,\epsilon\, E\}.$$

In the above mathematical expression, $\daleth_{\mathscr{Q}}(x), \aleph_{\mathscr{Q}}(x)$, and $\Psi_{\mathscr{Q}}(x)$ denote the complex-valued membership, neutral, and non-membership functions, respectively, that assign each element $x \in E$ to the closed unit disk in a complex plan. The membership degree (MD) $\beth_{\mathscr{C}}(x)$, neutral degree (NeD) $\aleph_{\mathscr{C}}(x)$, and non-membership degree $\Psi_{\mathscr{C}}(x)$ are expressed as follows: $\beth_{\mathscr{C}}(x) = \zeta_{\mathscr{C}}(x)\, e^{2\pi i \varpi_{\mathscr{C}}(x)}$, $\aleph_{\mathscr{C}}(x) = \eta_{\mathscr{C}}(x)\, e^{2\pi i \vartheta_{\mathscr{C}}(x)}$, and $\Psi_{\mathscr{C}}(x) = \kappa_{\mathscr{C}}(x)\, e^{2\pi i \Omega_{\mathscr{C}}(x)}$, such that $0 \le \zeta_{\mathscr{C}}(x), \eta_{\mathscr{C}}(x), \kappa_{\mathscr{C}}(x), \zeta^2_{\mathscr{C}}(x) + \eta^2_{\mathscr{C}}(x) + \kappa^2_{\mathscr{C}}(x) \le 1$ *and* $0 \le \varpi_{\mathscr{C}}(x), \vartheta_{\mathscr{C}}(x), \Omega_{\mathscr{C}}(x)$, $\varpi^2_{\mathscr{C}}(x) + \vartheta^2_{\mathscr{C}}(x) + \Omega^2_{\mathscr{C}}(x) \le 1$. Moreover, the term "refusal degree" of $\cdot$ is represented by $\mathfrak{r}$ and is expressed as:

$$\mathfrak{r} = r_{\mathscr{C}}(x)\, e^{2\pi i \tau_{\mathscr{C}}(x)} = \sqrt{1 - \left(\zeta^2_{\mathscr{C}}(x) + \eta^2_{\mathscr{C}}(x) + \kappa^2_{\mathscr{C}}(x)\right)}\; e^{2\pi i \sqrt{1 - \left(\varpi^2_{\mathscr{C}}(x) + \vartheta^2_{\mathscr{C}}(x) + \Omega^2_{\mathscr{C}}(x)\right)}}.$$

For the remaining portion of the article, membership, impartial, and non-membership degrees of $x \in E$ are denoted by the notation $x = ((\zeta, \varpi), (\eta, \vartheta), (\kappa, \Omega))$ and this particular depiction of the element x is called a CSFN, where $0 \le \zeta, \eta, \kappa, \zeta^2 + \eta^2 + \kappa^2 \le 1$ and $0 \le \varpi, \vartheta, \Omega, \varpi^2 + \vartheta^2 + \Omega^2 \le 1$.

Definition 5. ([38]). Consider $\tilde{\mho}_1 = ((\zeta_1, \varpi_1), (\eta_1, \vartheta_1), (\kappa_1, \Omega_1))$ and $\tilde{\mho}_2 = ((\zeta_2, \varpi_2), (\eta_2, \vartheta_2), (\kappa_2, \Omega_2))$ be two CSFNs. The basic operations of $\tilde{\mho}_1$ and $\tilde{\mho}_2$ are described as follows:

1. $\tilde{\mho}_1 < \tilde{\mho}_2$ iff $\zeta_1 < \zeta_2, \eta_1 < \eta_2, \kappa_1 > \kappa_2$ *and* $\varpi_1 < \varpi_2, \vartheta_1 < \vartheta_2, \Omega_1 > \Omega_2$,
2. $\tilde{\mho}_1 = \tilde{\mho}_2$ iff $\tilde{\mho}_1 \le \tilde{\mho}_2$ and $\tilde{\mho}_1 \ge \tilde{\mho}_2$,
3. $\tilde{\mho}_1^c = ((\kappa_1, \Omega_1), (\eta_1, \vartheta_1), (\zeta_1, \varpi_1))$.

Definition 6. ([37]). Let $\tilde{\mho}_i = ((\zeta_i, \varpi_i), (\eta_i, \vartheta_i), (\kappa_i, \Omega_i))$ represent the m number of CSFNs and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)^T$ denotes the associated weight vector of $\tilde{\mho}_i$ with $0 \le \beta_i \le 1$ and $\sum_{i=1}^m \beta_i = 1$. Then the aggregated value of complex spherical fuzzy weighted averaging (CSFWA) operator is stated as:

$$\text{CSFWA}(\tilde{\mho}_1, \tilde{\mho}_2, \tilde{\mho}_3, \ldots, \tilde{\mho}_m) = \left(\begin{array}{c} \left(\sqrt{1 - \prod_{i=1}^m (1 - \zeta_i^2)^{\beta_i}}, \sqrt{1 - \prod_{i=1}^m (1 - \varpi_i^2)^{\beta_i}} \right), \\ \left(\prod_{i=1}^m (\eta_i)^{\beta_i}, \prod_{i=1}^m (\vartheta_i)^{\beta_i} \right), \left(\left(\prod_{i=1}^m (\kappa_i)^{\beta_i}, \prod_{i=1}^m (\Omega_i)^{\beta_i} \right) \right) \end{array} \right).$$

Definition 7. ([37]). Let $\tilde{\mho}_i = ((\zeta_i, \varpi_i), (\eta_i, \vartheta_i), (\kappa_i, \Omega_i))$ represent the m number of CSFNs and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)^T$ denotes the associated weight vector of $\tilde{\mho}_i$ with $0 \le \beta_i \le 1$ and $\sum_{i=1}^m \beta_i = 1$. Then the aggregated value of complex spherical fuzzy weighted geometric (CSFWG) operator is formulized as:

$$\text{CSFWG}(\tilde{\mho}_1, \tilde{\mho}_2, \tilde{\mho}_3, \ldots, \tilde{\mho}_i) = \left(\begin{array}{c} \left(\left(\prod_{i=1}^m (\zeta_i)^{\beta_i}, \prod_{i=1}^m (\varpi_i)^{\beta_i} \right) \right), \left(\prod_{i=1}^m (\eta_i)^{\beta_i}, \prod_{i=1}^m (\vartheta_i)^{\beta_i} \right) \\ \left(\sqrt{1 - \prod_{i=1}^m (1 - \kappa_i^2)^{\beta_i}}, \sqrt{1 - \prod_{i=1}^m (1 - \Omega_i^2)^{\beta_i}} \right) \end{array} \right).$$

Definition 8. ([2]). Yager t-norm and s-norm on any $(c, d) \in [0, 1]^2$ and for any $\varepsilon \in (0, \infty)$ are interpreted as follows:

1. T (c, d) = 1- min (1, $((1-c)^{\varepsilon} + (1-d)^{\varepsilon})^{1/\varepsilon}$),
2. S (c, d) = min (1, $(c^{\varepsilon} + d^{\varepsilon})^{1/\varepsilon}$).

Definition 9. ([35]). Consider a CSFN $\tilde{\mho} = ((\zeta, \varpi), (\eta, \vartheta), (\kappa, \Omega))$. The score function of $\tilde{\mho}$ is expressed as follows:

$$\mathfrak{D}(\tilde{\mho}) = \frac{1}{3}\left[\left(\zeta^2 - \eta^2 - \kappa^2\right) + \left(\varpi^2 - \vartheta^2 - \Omega^2\right)\right].$$

In the above mathematical expression, $\mathfrak{D}(\tilde{\mho}) \in$ [-0.67, 0.67]. Moreover, any two CSFNs $\tilde{\mho}_1$ and $\tilde{\mho}_2$ are compared by the subsequent comparison laws:

1. If $\mathfrak{D}(\tilde{\mho}_1) < \mathfrak{D}(\tilde{\mho}_2)$, then $\tilde{\mho}_1 < \tilde{\mho}_2$,
2. If $\mathfrak{D}(\tilde{\mho}_1) > \mathfrak{D}(\tilde{\mho}_2)$, then $\tilde{\mho}_1 > \tilde{\mho}_2$,
3. If $\mathfrak{D}(\tilde{\mho}_1) = \mathfrak{D}(\tilde{\mho}_2)$, then $\tilde{\mho}_1 \sim \tilde{\mho}_2$.

3. Modified existing score function of CSFN

In this section, a specific instance is shown to highlight the limitations of the score function employed in the CSFNs formulated in

Ref. [35].

Example 1. Consider $\tilde{\mho}_1 = ((0.5, 0.3), (\frac{\sqrt{35}}{10}, 0.2), (0.3, 0.5))$ and $\tilde{\mho}_2 = ((0.1, 0.5), (0.4, 0.3), (0.2, 0.4))$ be two CSFNs. The application of definition 9 on $\tilde{\mho}_1$ and $\tilde{\mho}_2$, yields the following outcomes: $\mathfrak{H}(\tilde{\mho}_1) = \mathfrak{H}(\tilde{\mho}_2) = -0.39$. Based on comparison law 3 outlined in definition 9, it is quite evident that the CSFNs $\tilde{\mho}_1$ and $\tilde{\mho}_2$ are incomparable.

This observation highlights the drawback of the score function defined in Ref. [35]. This limitation motivates us to modify this function in the following discussion.

Definition 10. Suppose $\tilde{\mho} = ((\zeta, \varpi), (\eta, \vartheta), (\kappa, \Omega))$ is a CSFN. Then the score function of CSFN is given as:

$$\mathfrak{H}(\tilde{\mho}) = \frac{1 + 2(\zeta^2 + \varpi^2) - 3(\eta^2 + \vartheta^2) - (\kappa^2 + \Omega^2)}{3}.$$

In the above mathematical expression, the range of $\mathfrak{H}(\tilde{\mho}) \in [-\frac{1}{3}, \frac{5}{3}]$ or $[-0.33, 1.67]$. Moreover, any two CSFSs $\tilde{\mho}_1$ and $\tilde{\mho}_2$ are compared by the subsequent comparison laws:

1 If $\mathfrak{H}(\tilde{\mho}_1) < \mathfrak{H}(\tilde{\mho}_2)$, then $\tilde{\mho}_1 < \tilde{\mho}_2$,
2. If $\mathfrak{H}(\tilde{\mho}_1) > \mathfrak{H}(\tilde{\mho}_2)$, then $\tilde{\mho}_1 > \tilde{\mho}_2$,
3 If $\mathfrak{H}(\tilde{\mho}_1) = \mathfrak{H}(\tilde{\mho}_2)$, then $\tilde{\mho}_1 \sim \tilde{\mho}_2$.

The subsequent example illustrates the validity of the newly defined score function for CSFN.

Example 2. Consider $\tilde{\mho}_1 = ((0.5, 0.3), (\frac{\sqrt{35}}{10}, 0.2), (0.3, 0.5))$ and $\tilde{\mho}_2 = ((0.1, 0.5), (0.4, 0.3), (0.2, 0.4))$ be two CSFNs. The application of definition 10 on $\tilde{\mho}_1$ and $\tilde{\mho}_2$, yields the following outcomes: $\mathfrak{H}(\tilde{\mho}_1) = 0.057$ and $\mathfrak{H}(\tilde{\mho}_2) = 0.19$. Based on comparison law 1 outlined in definition 10, it is quite evident that the CSFNs $\tilde{\mho}_1 < \tilde{\mho}_2$.

The preceding discussion illustrates that the suggested score function is better and yields more precise outcomes for decision assessment.

4. Novel fuzzy aggregation operators for CSFNs and their properties

In this section, we develop Yager's operations for CSFNs. These operations are formulated by the principles of the Yager t-norm and s-norm within the context of the CSF environment. In addition, we introduce CSF Yager's aggregation operators and establish their key aspects.

Definition 11. Let ≥ 1, $\varepsilon > 0$ and $\tilde{\mho}_1 = ((\zeta_1, \varpi_1), (\eta_1, \vartheta_1), (\kappa_1, \Omega_1))$ and $\tilde{\mho}_2 = ((\zeta_2, \varpi_2), (\eta_2, \vartheta_2), (\kappa_2, \Omega_2))$ be any two CSFNs. Then Yager's operations on the $\tilde{\mho}_1$ and $\tilde{\mho}_2$ are characterized as follows:

$$1\ \tilde{\mho}_1 \oplus \tilde{\mho}_2 = \begin{pmatrix} \left(\sqrt{\min\left(1, \left(\zeta_1^{2\varepsilon} + \zeta_2^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{\min\left(1, \left(\varpi_1^{2\varepsilon} + \varpi_2^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1 - \min\left(1, \left(\left(1-\eta_1^2\right)^{\varepsilon} + \left(1-\eta_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - \min\left(1, \left(\left(1-\vartheta_1^2\right)^{\varepsilon} + \left(1-\vartheta_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1 - \min\left(1, \left(\left(1-\kappa_1^2\right)^{\varepsilon} + \left(1-\kappa_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - \min\left(1, \left(\left(1-\Omega_1^2\right)^{\varepsilon} + \left(1-\Omega_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right) \end{pmatrix}.$$

$$2.\ \tilde{\mho}_1 \otimes \tilde{\mho}_2 = \begin{pmatrix} \left(\sqrt{1 - \min\left(1, \left(\left(1-\zeta_1^2\right)^{\varepsilon} + \left(1-\zeta_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - \min\left(1, \left(\left(1-\varpi_1^2\right)^{\varepsilon} + \left(1-\varpi_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1 - \min\left(1, \left(\left(1-\eta_1^2\right)^{\varepsilon} + \left(1-\eta_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - \min\left(1, \left(\left(1-\vartheta_1^2\right)^{\varepsilon} + \left(1-\vartheta_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{\min\left(1, \left(\kappa_1^{2\varepsilon} + \kappa_2^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{\min\left(1, \left(\Omega_1^{2\varepsilon} + \Omega_2^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right) \end{pmatrix}.$$

$$3.\ \cdot\ \tilde{\text{U}}_1 = \begin{pmatrix} \left(\sqrt{min\left(1,\left(\text{ll}.\zeta_1^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{min\left(1,\left(\text{ll}.\varpi_1^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1-min\left(1,\left(\text{ll}.\left(1-\eta_1^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1-min\left(\left(1,\left(\text{ll}.\left(1-\vartheta_1^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)\right)}\right), \\ \left(\sqrt{1-min\left(1,\left(\text{ll}.\left(1-\kappa_1^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1-min\left(1,\left(\text{ll}.\left(1-\Omega_1^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right) \end{pmatrix}.$$

$$4\ \tilde{\text{U}}_1{}^{\text{ll}} = \begin{pmatrix} \left(\sqrt{1-min\left(1,\left(\text{ll}.\left(1-\zeta_1^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1-min\left(\left(1,\left(\text{ll}.\left(1-\varpi_1^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)\right)}\right), \\ \left(\sqrt{1-min\left(1,\left(\text{ll}.\left(1-\eta_1^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1-min\left(\left(1,\left(\text{ll}.\left(1-\vartheta_1^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)\right)}\right), \\ \left(\sqrt{min\left(1,\left(\text{ll}.\kappa_1^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{min\left(1,\left(\text{ll}.\Omega_1^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right) \end{pmatrix}.$$

4.1. Fundamental characteristics of complex spherical fuzzy Yager weighted averaging operator

Here, we express the complex spherical fuzzy Yager weighted averaging (CSFYWA) operator and prove its fundamental characteristics.

Definition 12. Let $\mathscr{L}$ be a collection of CSFNs, $\tilde{\text{U}}_i = ((\zeta_i, \varpi_i), (\eta_i, \vartheta_i), (\kappa_i, \Omega_i))$, i = 1, 2, …., m and $\beta = (\beta_1, \beta_2, \beta_3, \ldots, \beta_m)^T$ be an associated weight vector of these CSFNs $\tilde{\text{U}}_i$ with $0 \le \beta_i \le 1$ and $\sum_{i=1}^m \beta_i = 1$. The complex spherical fuzzy Yager weighted averaging (CSFYWA) operator is a function CSFYWA: $\mathscr{L}^m \to \mathscr{L}$ and is defined by the following rule:

$$\text{CSFYWA}\ (\tilde{\text{U}}_1, \tilde{\text{U}}_2, \tilde{\text{U}}_3, \ldots, \tilde{\text{U}}_m) = \oplus_{i=1}^m (\beta_i \tilde{\text{U}}_i)$$

$$= \begin{pmatrix} \left(\sqrt{min\left(1,\left(\sum_{i=1}^m \beta_i.\zeta_i^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{min\left(1,\left(\sum_{i=1}^m \beta_i.\varpi_i^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1-min\left(1,\left(\sum_{i=1}^m \beta_i.\left(1-\eta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1-min\left(1,\left(\sum_{i=1}^m \beta_i.\left(1-\vartheta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1-min\left(1,\left(\sum_{i=1}^m \beta_i.\left(1-\kappa_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1-min\left(1,\left(\sum_{i=1}^m \beta_i.\left(1-\Omega_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right) \end{pmatrix}. \tag{1}$$

Theorem 1. *Consider m number of CSFNs,* $\tilde{\text{U}}_i = ((\zeta_i, \varpi_i), (\eta_i, \vartheta_i), (\kappa_i, \Omega_i))$ *and associated weight vector of these CSFNs is* $\beta = (\beta_1, \beta_2, \beta_3, \ldots, \beta_m)^T$ *where* $0 \le \beta_i \le 1$ *such that* $\sum_{i=1}^m \beta_i = 1$ *and* $\varepsilon > 0$. *Then, their aggregated value in the framework of the CSFYWA operator is a CSFN and is formulated as follows*:

$$\text{CSFYWA}\left(\tilde{\text{U}}_1,\tilde{\text{U}}_2,\tilde{\text{U}}_3,\ldots,\tilde{\text{U}}_m\right)=\oplus_{i=1}^{m}\left(\beta_i\tilde{\text{U}}_i\right)$$

$$=\left(\begin{array}{c}\left(\sqrt{min\left(1,\left(\sum_{i=1}^{m}\beta_i.\zeta_i^{2\varepsilon}\right)^{1/\varepsilon}\right)},\sqrt{min\left(1,\left(\sum_{i=1}^{m}\beta_i.\varpi_i^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right),\\ \left(\sqrt{1-min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\eta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)},\sqrt{1-min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\vartheta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right),\\ \left(\sqrt{1-min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\kappa_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)},\sqrt{1-min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\Omega_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right)\end{array}\right)$$

Proof. This theorem is exemplified through the utilization of the mathematical induction process.

Consider the initial case where $m=2$. We encounter two CSFNs denoted as $\tilde{\text{U}}_1=((\zeta_1,\varpi_1),(\eta_1,\vartheta_1),(\kappa_1,\Omega_1))$ and $\tilde{\text{U}}_2=((\zeta_2,\varpi_2),(\eta_2,\vartheta_2),(\kappa_2,\Omega_2))$. Applying the operations designed for CSFNs, we acquire the following expressions:

$$\beta_1\tilde{\text{U}}_1=\left(\begin{array}{c}\left(\sqrt{min\left(1,\left(\beta_1.\zeta_1^{2\varepsilon}\right)^{1/\varepsilon}\right)},\sqrt{min\left(1,\left(\beta_1.\varpi_1^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right),\\ \left(\sqrt{1-min\left(1,\left(\beta_1.\left(1-\eta_1^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)},\sqrt{1-min\left(1,\left(\beta_1.\left(1-\vartheta_1^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right),\\ \left(\sqrt{1-min\left(1,\left(\beta_1.\left(1-\kappa_1^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)},\sqrt{1-min\left(1,\left(\beta_1.\left(1-\Omega_1^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right)\end{array}\right)$$

and

$$\beta_2\tilde{\text{U}}_2=\left(\begin{array}{c}\left(\sqrt{\min\left(1,\left(\beta_2.\zeta_2^{2\varepsilon}\right)^{1/\varepsilon}\right)},\sqrt{\min\left(1,\left(\beta_2.\varpi_2^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right),\\ \left(\sqrt{1-\min\left(1,\left(\beta_2.\left(1-\eta_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)},\sqrt{1-\min\left(1,\left(\beta_2.\left(1-\vartheta_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right),\\ \left(\sqrt{1-\min\left(1,\left(\beta_2.\left(1-\kappa_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)},\sqrt{1-\min\left(1,\left(\beta_2.\left(1-\Omega_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right)\end{array}\right)$$

In view of definition 12, the aggregated value of $\tilde{\text{U}}_1$ and $\tilde{\text{U}}_2$ is obtained as follows:

$$\text{CSFYWA}\left(\tilde{\text{U}}_1,\tilde{\text{U}}_2\right)=\beta_1\tilde{\text{U}}_1\oplus\beta_2\tilde{\text{U}}_2$$

$$= \begin{pmatrix} \left(\sqrt{min\left(1, \left(\beta_1.\zeta_1^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{min\left(1, \left(\beta_1.\varpi_1^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1 - min\left(1, \left(\beta_1.\left(1 - \eta_1^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - min\left(1, \left(\beta_1.\left(1 - \vartheta_1^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1 - min\left(1, \left(\beta_1.\left(1 - \kappa_1^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - min\left(1, \left(\beta_1.\left(1 - \Omega_1^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right) \end{pmatrix}$$

$$\oplus \begin{pmatrix} \left(\sqrt{min\left(1, \left(\beta_2.\zeta_2^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{min\left(1, \left(\beta_2.\varpi_2^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1 - min\left(1, \left(\beta_2.\left(1 - \eta_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - min\left(1, \left(\beta_2.\left(1 - \vartheta_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1 - min\left(1, \left(\beta_2.\left(1 - \kappa_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - min\left(1, \left(\beta_2.\left(1 - \Omega_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right) \end{pmatrix}$$

It concludes that

$$\text{CSFYWA}\left(\tilde{U}_1, \tilde{U}_2\right) = \begin{pmatrix} \left(\sqrt{min\left(1, \left(\sum_{i=1}^{2}\beta_i.\zeta_i^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{min\left(1, \left(\sum_{i=1}^{2}\beta_i.\varpi_i^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1 - min\left(1, \left(\sum_{i=1}^{2}\beta_i.\left(1 - \eta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - min\left(1, \left(\sum_{i=1}^{2}\beta_i.\left(1 - \vartheta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1 - min\left(1, \left(\sum_{i=1}^{2}\beta_i.\left(1 - \kappa_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - min\left(1, \left(\sum_{i=1}^{2}\beta_i.\left(1 - \Omega_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right) \end{pmatrix}.$$

Thus, the result is valid for $m = 2$.

Now, moving forward to the induction process, we assume that the result holds for m = k. Then we have

$$\text{CSFYWA}\left(\tilde{U}_1, \tilde{U}_2, \tilde{U}_3, \ldots, \tilde{U}_k\right) = \oplus_{i=1}^{k}\left(\beta_i \tilde{U}_i\right)$$

$$= \begin{pmatrix} \left(\sqrt{min\left(1, \left(\sum_{i=1}^{k}\beta_i.\zeta_i^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{min\left(1, \left(\sum_{i=1}^{k}\beta_i.\varpi_i^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1 - min\left(1, \left(\sum_{i=1}^{k}\beta_i.\left(1 - \eta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - min\left(1, \left(\sum_{i=1}^{k}\beta_i.\left(1 - \vartheta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1 - min\left(1, \left(\sum_{i=1}^{k}\beta_i.\left(1 - \kappa_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - min\left(1, \left(\sum_{i=1}^{k}\beta_i.\left(1 - \Omega_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right) \end{pmatrix}.$$

Now, we show that the result is true for $m = k + 1$.

$$\text{CSFYWA}\left(\tilde{U}_1, \tilde{U}_2, \tilde{U}_3, \ldots, \tilde{U}_k, \tilde{U}_{k+1}\right) = \left(\beta_1 \tilde{U}_1\right) \oplus \left(\beta_2 \tilde{U}_2\right) \oplus \left(\beta_3 \tilde{U}_3\right) \oplus \ldots \oplus \left(\beta_k \tilde{U}_k\right) \oplus \left(\beta_{k+1} \tilde{U}_{k+1}\right)$$

$$= \oplus_{i=1}^{k}\left(\beta_i \tilde{U}_i\right) \oplus \left(\beta_{k+1} \tilde{U}_{k+1}\right)$$

$$= \begin{pmatrix} \left(\sqrt{min\left(1, \left(\sum_{i=1}^{k} \beta_i.\zeta_i^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{min\left(1, \left(\sum_{i=1}^{k} \beta_i.\varpi_i^{2\varepsilon}\right)^{1/\varepsilon}\right)} \right), \\ \left(\sqrt{1 - min\left(1, \left(\sum_{i=1}^{k} \beta_i.\left(1 - \eta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - min\left(1, \left(\sum_{i=1}^{k} \beta_i.\left(1 - \vartheta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)} \right), \\ \left(\sqrt{1 - min\left(1, \left(\sum_{i=1}^{k} \beta_i.\left(1 - \kappa_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - min\left(1, \left(\sum_{i=1}^{k} \beta_i.\left(1 - \Omega_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)} \right) \end{pmatrix}$$

$$\oplus \begin{pmatrix} \left(\sqrt{min\left(1, \left(\beta_{k+1}.\zeta_{k+1}^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{min\left(1, \left(\beta_{k+1}.\varpi_{k+1}^{2\varepsilon}\right)^{1/\varepsilon}\right)} \right), \\ \left(\sqrt{1 - min\left(1, \left(\beta_{k+1}.\left(1 - \eta_{k+1}^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - min\left(1, \left(\beta_{k+1}.\left(1 - \vartheta_{k+1}^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)} \right), \\ \left(\sqrt{1 - min\left(1, \left(\beta_{k+1}.\left(1 - \kappa_{k+1}^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - min\left(1, \left(\beta_{k+1}.\left(1 - \Omega_{k+1}^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)} \right) \end{pmatrix}$$

Consequently,

$$\text{CSFYWA}(\tilde{\text{u}}_1, \tilde{\text{u}}_2, \tilde{\text{u}}_3, \ldots, \tilde{\text{u}}_{k+1}) = \begin{pmatrix} \left(\sqrt{min\left(1, \left(\sum_{i=1}^{k+1} \beta_i.\zeta_i^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{min\left(1, \left(\sum_{i=1}^{k+1} \beta_i.\varpi_i^{2\varepsilon}\right)^{1/\varepsilon}\right)} \right), \\ \left(\sqrt{1 - min\left(1, \left(\sum_{i=1}^{k+1} \beta_i.\left(1 - \eta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - min\left(1, \left(\sum_{i=1}^{k+1} \beta_i.\left(1 - \vartheta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)} \right), \\ \left(\sqrt{1 - min\left(1, \left(\sum_{i=1}^{k+1} \beta_i.\left(1 - \kappa_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - min\left(1, \left(\sum_{i=1}^{k+1} \beta_i.\left(1 - \Omega_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)} \right) \end{pmatrix}$$

Hence, the above shows that the result holds for each integer m.
The subsequent illustrated example verifies the aforementioned fact.

Example 3. Consider $\tilde{\text{u}}_1 = ((0.2, 0.47), (0.54, 0.36), (0.43, 0.15))$, $\tilde{\text{u}}_2 = ((0.45, 0.62), (0.28, 0.33), (0.52, 0.13))$, $\tilde{\text{u}}_3 = ((0.5, 0.3), (0.4, 0.17), (0.24, 0.34))$, and $\tilde{\text{u}}_4 = ((0.35, 0.33), (0.51, 0.45), (0.6, 0.21))$ be four CSFNs and its associated weight vector is $\beta = (0.21, 0.29, 0.34, 0.16)^T$ and $\varepsilon = 4$. Then

$$\left(\sum_{i=1}^{4} \beta_i.\zeta_i^{2\varepsilon}\right)^{1/\varepsilon} = \left(0.21(0.2)^{2(4)} + 0.29\ (0.45)^{2(4)} + 0.34\ (0.5)^{2(4)} + 0.16(0.35)^{2(4)}\right)^{1/4}$$

$= 0.2075$.
Similarly,

$$\left(\sum_{i=1}^{4} \beta_i.\varpi_i^{2\varepsilon}\right)^{1/\varepsilon} = 0.2879, \left(\sum_{i=1}^{4} \beta_i.\left(1 - \eta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon} = 0.8321,$$

$$\left(\sum_{i=1}^{4} \beta_i.\left(1 - \vartheta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon} = 0.9049, \left(\sum_{i=1}^{4} \beta_i.\left(1 - \kappa_i^2\right)^{\varepsilon}\right)^{1/\varepsilon} = 0.8279,$$

$$\left(\sum_{i=1}^{4}\beta_i.\left(1-\ \Omega_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}=0.9469.$$

This implies that CSFYWA $(\tilde{\mho}_1,\tilde{\mho}_2,\tilde{\mho}_3,\tilde{\mho}_4)=\oplus_{i=1}^{4}(\beta_i\tilde{\mho}_i)=((0.455,0.537),(0.409,0.308),(0.415,0.230))$.

It can be inferred that the aforementioned discourse yields yet again a CSFN.

Theorem 2. *(Idempotency): Consider* $\tilde{\mho}_i=((\zeta_i,\varpi_i),(\eta_i,\vartheta_i),(\kappa_i,\Omega_i))$ *and let* $\tilde{\mho}_i=\tilde{\mho}_o\ \forall\, i$, *where* $\tilde{\mho}_o=((\zeta_o,\varpi_o),(\eta_o,\vartheta_o),(\kappa_o,\Omega_o))$. *The associated weight vector of these CSFNs is* $\beta=(\beta_1,\beta_2,\beta_3,\ldots,\beta_m)^T$ *where* $0\le\beta_i\le 1$ *such that* $\sum_{i=1}^{m}\beta_i=1$ *and* $\varepsilon>0$. *Then*

CSFYWA $(\tilde{\mho}_1,\tilde{\mho}_2,\tilde{\mho}_3,\ldots,\tilde{\mho}_m)=\tilde{\mho}_o$.

Proof. Given that $\tilde{\mho}_i=\tilde{\mho}_o=((\zeta_o,\varpi_o),(\eta_o,\vartheta_o),(\kappa_o,\Omega_o))$, for all i. In the light of definition 5, $\zeta_i=\zeta_o,\varpi_i=\varpi_o,\eta_i=\eta_o,\vartheta_i=\vartheta_o,\kappa_i=\kappa_o$, and $\Omega_i=\Omega_o$. By substituting the above values $\zeta_i,\varpi_i,\eta_i,\vartheta_i,\kappa_i$, and Ω_i in Equation (1), give that

CSFYWA $(\tilde{\mho}_1,\tilde{\mho}_2,\tilde{\mho}_3,\ldots,\tilde{\mho}_m)$.

$$=\begin{pmatrix}\left(\sqrt{\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\zeta_o^{2\varepsilon}\right)^{1/\varepsilon}\right)},\sqrt{\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\varpi_o^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right),\\ \left(\sqrt{1-\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\ \eta_o^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)},\sqrt{1-\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\ \vartheta_o^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right),\\ \left(\sqrt{1-\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\ \kappa_o^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)},\sqrt{1-\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\ \Omega_o^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right)\end{pmatrix}$$

$$=\begin{pmatrix}\left(\sqrt{\min\left(1,(\zeta_o^2)\left(\sum_{i=1}^{m}\beta_i\right)^{1/\varepsilon}\right)},\sqrt{\min\left(1,(\varpi_o^2)\left(\sum_{i=1}^{m}\beta_i\right)^{1/\varepsilon}\right)}\right),\\ \left(\sqrt{1-\min\left(1,(1-\ \eta_o^2)\left(\sum_{i=1}^{m}\beta_i\right)^{1/\varepsilon}\right)},\sqrt{1-\min\left(1,(1-\ \vartheta_o^2)\left(\sum_{i=1}^{m}\beta_i\right)^{1/\varepsilon}\right)}\right),\\ \left(\sqrt{1-\min\left(1,(1-\ \kappa_o^2)\left(\sum_{i=1}^{m}\beta_i\right)^{1/\varepsilon}\right)},\sqrt{1-\min\left(1,(1-\ \Omega_o^2)\left(\sum_{i=1}^{m}\beta_i\right)^{1/\varepsilon}\right)}\right)\end{pmatrix}$$

$$=\begin{pmatrix}\left(\sqrt{\min(1,\zeta_o^2)},\sqrt{\min(1,\ \varpi_o^2)}\right),\\ \left(\sqrt{1-\min(1,(1-\ \eta_o^2))},\sqrt{1-\min(1,(1-\ \vartheta_o^2))}\right),\\ \left(\sqrt{1-\min(1,(1-\ \kappa_o^2))},\sqrt{1-\min(1,(1-\ \Omega_o^2))}\right)\end{pmatrix}$$

$$=((\zeta_o,\varpi_o),(\eta_o,\vartheta_o),(\kappa_o,\Omega_o)).$$

It follows that

CSFYWA $(\tilde{\mho}_1,\tilde{\mho}_2,\tilde{\mho}_3,\ldots,\tilde{\mho}_m)=\tilde{\mho}_o$.

Theorem 3. *(Monotonicity): Consider* $\tilde{\mho}_i=((\zeta_i,\varpi_i),(\eta_i,\vartheta_i),(\kappa_i,\Omega_i))$ *and* $\tilde{\mho}_i^*=\left((\zeta_i^*,\varpi_i^*),(\eta_i^*,\vartheta_i^*),(\kappa_i^*,\Omega_i^*)\right)$ *be any two collections of m number of CSFNs and* $\beta=(\beta_1,\beta_2,\beta_3,\ldots,\beta_m)^T$ *is an associated weight vector of these CSFNs* $\tilde{\mho}_i$, *where* $0\le\beta_i\le 1$ *such that* $\sum_{i=1}^{m}\beta_i=1$ *and* $\varepsilon>0$. *If* $\zeta_i\le\zeta_i^*$, $\varpi_i\le\varpi_i^*$, $\eta_i\le\eta_i^*$, $\vartheta_i\le\vartheta_i^*$, $\kappa_i\ge\kappa_i^*$, *and* $\Omega_i\ge\Omega_i^*$. *Then*

CSFYWA $(\tilde{\mho}_1,\tilde{\mho}_2,\tilde{\mho}_3,\ldots,\tilde{\mho}_m)\le$ CSFYWA $(\tilde{\mho}_1^*,\tilde{\mho}_2^*,\tilde{\mho}_3^*,\ldots,\tilde{\mho}_m^*)$.

Proof. Consider $\zeta_i\le\zeta_i^*\ \Rightarrow\zeta_i^{2\varepsilon}\le\left(\zeta_i^*\right)^{2\varepsilon}$.

$$\Rightarrow\ \min\left(1,\left(\sum_{i=1}^{m}\beta_i.\zeta_i^{2\varepsilon}\right)^{1/\varepsilon}\right)\le\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(\zeta_i^*\right)^{2\varepsilon}\right)^{1/\varepsilon}\right)$$

It follows that

$$\sqrt{\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\zeta_i^{2\varepsilon}\right)^{1/\varepsilon}\right)} \le \sqrt{\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(\zeta_i^*\right)^{2\varepsilon}\right)^{1/\varepsilon}\right)} \tag{2}$$

Similarly,

$$\sqrt{\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\varpi_i^{2\varepsilon}\right)^{1/\varepsilon}\right)} \le \sqrt{\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(\varpi_i^*\right)^{2\varepsilon}\right)^{1/\varepsilon}\right)} \tag{3}$$

can be established by adopting the above mathematical steps in the relation $\varpi_i \le \varpi_i^*$.
Moreover, we obtain

$$\sqrt{1-\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\eta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)} \le \sqrt{1-\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\left(\eta_i^*\right)^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)} \tag{4}$$

by following the same mathematical procedures in the relation $\eta_i \le \eta_i^*$. Additionally, employing identical mathematical steps allows us to determine

$$\sqrt{1-\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\vartheta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)} \le \sqrt{1-\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\left(\vartheta_i^*\right)^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)} \tag{5}$$

within the context of $\vartheta_i \le \vartheta_i^*$.

Moreover, consider $\kappa_i \ge \kappa_i^* \Rightarrow 1-\kappa_i^2 \le 1-\left(\kappa_i^*\right)^2$.

$$\Rightarrow \min\left(1,\left(\sum\nolimits_{i=1}^{m}\beta_i.\left(1-\kappa_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right) \le \min\left(1,\left(\sum\nolimits_{i=1}^{m}\beta_i.\left(1-\left(\kappa_i^*\right)^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right).$$

It follows that

$$\sqrt{1-\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\kappa_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)} \ge \sqrt{1-\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\left(\kappa_i^*\right)^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)} \tag{6}$$

Similarly,

$$\sqrt{1-\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\Omega_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)} \ge \sqrt{1-\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\left(\Omega_i^*\right)^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)} \tag{7}$$

can be established by utilizing the above mathematical steps in the relation $\Omega_i \ge \Omega_i^*$.

By comparing the relations from 2 to 7 and using Definition 5, we obtain that

$$\text{CSFYWA}\left(\tilde{\mho}_1,\tilde{\mho}_2,\tilde{\mho}_3,\ldots,\tilde{\mho}_m\right) \le \text{CSFYWA}\left(\tilde{\mho}_1^*,\tilde{\mho}_2^*,\tilde{\mho}_3^*,\ldots,\tilde{\mho}_m^*\right).$$

Theorem 4. *(Boundedness): Consider m number of CSFNs, $\tilde{\mho}_i = ((\zeta_i,\varpi_i),(\eta_i,\vartheta_i),(\kappa_i,\Omega_i))$ and associated weight vector of these CSFNs is $\beta = (\beta_1,\beta_2,\beta_3,\ldots,\beta_m)^T$ where $0 \le \beta_i \le 1$ such that $\sum_{i=1}^{m}\beta_i = 1$ and $\varepsilon > 0$. If $\tilde{\mho}^- = \left\{\left(\min_i\{\zeta_i\},\min_i\{\varpi_i\}\right),\left(\max_i\{\eta_i\},\max_i\{\vartheta_i\}\right),\left(\max_i\{\kappa_i\},\max_i\{\Omega_i\}\right)\right\}$ and $\tilde{\mho}^+ = \left\{\left(\max_i\{\zeta_i\},\max_i\{\varpi_i\}\right),\left(\min_i\{\eta_i\},\min_i\{\vartheta_i\}\right),\left(\min_i\{\kappa_{\rho(i)}\},\min_i\{\Omega_i\}\right)\right\}$. Then*

$$\tilde{\mho}^- \le CSFYWA\left(\tilde{\mho}_1,\tilde{\mho}_2,\tilde{\mho}_3,\ldots,\tilde{\mho}_m\right) \le \tilde{\mho}^+.$$

Proof. Consider the result obtained by using the CSFYWA operator to the collection of CSFNs, represented as $\text{CSFYWA}(\tilde{\mho}_1,\tilde{\mho}_2,\tilde{\mho}_3,\ldots,\tilde{\mho}_m) = ((\zeta,\varpi),(\eta,\vartheta),(\kappa,\Omega))$.

For each CSFN $\tilde{\mho}_i$, $\min_i\{\zeta_i\} \le \zeta_i \le \max_i\{\zeta_i\}$.

$$\Longrightarrow\left(\min_i\{\zeta_i\}\right)^{2\varepsilon} \le \zeta_i^{2\varepsilon} \le \left(\max_i\{\zeta_i\}\right)^{2\varepsilon} \Longrightarrow \left(\sum_{i=1}^{m}\beta_i.\left(\min_i\{\zeta_i\}\right)^{2\varepsilon}\right)^{1/\varepsilon} \le \left(\sum_{i=1}^{m}\beta_i.\zeta_i^{2\varepsilon}\right)^{1/\varepsilon} \le \left(\sum_{i=1}^{m}\beta_i.\left(\max_i\{\zeta_i\}\right)^{2\varepsilon}\right)^{1/\varepsilon}$$

$$\Longrightarrow\sqrt{\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(\min_i\{\zeta_i\}\right)^{2\varepsilon}\right)^{1/\varepsilon}\right)}\leq\sqrt{\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\zeta_i^{2\varepsilon}\right)^{1/\varepsilon}\right)}\leq\sqrt{\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(\max_i\{\zeta_i\}\right)^{2\varepsilon}\right)^{1/\varepsilon}\right)}$$

Because, $\sum_{i=1}^{m}\beta_i=1,$ we deduced that

$$\sqrt{\min\left(1,\left(\left(\min_i\{\zeta_i\}\right)^{2\varepsilon}\right)^{1/\varepsilon}\right)}\leq\sqrt{\min\left(1,\left(\zeta^{2\varepsilon}\right)^{1/\varepsilon}\right)}\leq\sqrt{\min\left(1,\left(\left(\max_i\{\zeta_i\}\right)^{2\varepsilon}\right)^{1/\varepsilon}\right)}$$

$$\Longrightarrow\min\left(1,\left(\left(\min_i\{\zeta_i\}\right)^{2\varepsilon}\right)^{1/\varepsilon}\right)\leq\min\left(1,\left(\zeta^{2\varepsilon}\right)^{1/\varepsilon}\right)\leq\min\left(1,\left(\left(\max_i\{\zeta_i\}\right)^{2\varepsilon}\right)^{1/\varepsilon}\right)\Longrightarrow\left(\min_i\{\zeta_i\}\right)^{2}\leq\zeta^2\leq\left(\max_i\{\zeta_i\}\right)^{2}.$$

Hence,

$$\min_i\{\zeta_i\}\leq\zeta\leq\max_i\{\zeta_i\} \tag{8}$$

Similarly, the following expression can be established by adopting the above mathematical steps.

$$\min_i\{\varpi_i\}\leq\varpi\leq\max_i\{\varpi_i\} \tag{9}$$

Moreover, consider $\max_i\{\eta_i\}\leq\eta_i\leq\min_i\{\eta_i\}\Longrightarrow\left(\max_i\{\eta_i\}\right)^2\leq\eta_i^2\leq\left(\min_i\{\eta_i\}\right)^2.$

$$\Longrightarrow 1-\left(\min_i\{\eta_i\}\right)^2\leq 1-\eta_i^2\leq 1-\left(\max_i\{\eta_i\}\right)^2$$

$$\Longrightarrow\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\left(\min_i\{\eta_i\}\right)^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)\leq\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\eta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)\leq\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\left(\max_i\{\eta_i\}\right)^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)$$

$$\Longrightarrow\sqrt{1-\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\left(\max_i\{\eta_i\}\right)^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\leq\sqrt{1-\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\eta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}$$
$$\leq\sqrt{1-\min\left(1,\left(\sum_{i=1}^{m}\beta_i.\left(1-\left(\min_i\{\eta_i\}\right)^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}$$

Because $\sum_{i=1}^{m}\beta_i=1$, we can deduce that

$$\Longrightarrow\sqrt{1-\min\left(1,\left(\left(1-\left(\max_i\{\eta_i\}\right)^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\leq\sqrt{1-\min\left(1,\left(\left(1-\eta^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\leq\sqrt{1-\min\left(1,\left(\left(1-\left(\min_i\{\eta_i\}\right)^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}$$

$$\Longrightarrow\min\left(1,\left(\left(1-\left(\min_i\{\eta_i\}\right)^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)\leq\min\left(1,\left(\left(1-\eta^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)\leq\min\left(1,\left(\left(1-\left(\max_i\{\eta_i\}\right)^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)$$

$$\Longrightarrow\left(\left(1-\left(\min_i\{\eta_i\}\right)^2\right)^{\varepsilon}\right)^{1/\varepsilon}\leq\left(\left(1-\eta^2\right)^{\varepsilon}\right)^{1/\varepsilon}\leq\left(\left(1-\left(\max_i\{\eta_i\}\right)^2\right)^{\varepsilon}\right)^{1/\varepsilon}$$

This implies that

$$\left(\max_i\{\eta_i\}\right)^2\leq\eta^2\leq\left(\min_i\{\eta_i\}\right)^2$$

Hence,

$$\max_i\{\eta_i\}\leq\eta\leq\min_i\{\eta_i\} \tag{10}$$

Similarly, we obtain

$$\max_i\{\vartheta_i\}\leq\vartheta\leq\min_i\{\vartheta_i\} \tag{11}$$

By adopting the above mathematical steps. Furthermore, we can establish

$$\max_i\{\kappa_i\} \leq \kappa \leq \min_i\{\kappa_i\} \tag{12}$$

By following the same mathematical procedures. Finally, employing identical mathematical steps allows us to determine the subsequent relation:

$$\max_i\{\Omega_i\} \leq \Omega \leq \min_i\{\Omega_i\} \tag{13}$$

Consequently, by comparing the relations from 8 to 13, we get

$$\tilde{\mho}^- \leq \text{CSFYWA}(\tilde{\mho}_1, \tilde{\mho}_2, \tilde{\mho}_3, \ldots, \tilde{\mho}_m) \leq \tilde{\mho}^+.$$

4.2. Fundamental characteristics of complex spherical fuzzy Yager weighted geometric operator

In this section, we define complex spherical fuzzy Yager weighted geometric (CSFYWG) operator and prove its fundamental characteristics.

Definition 13. Let $\mathscr{L}$ be a collection of CSFNs, $\tilde{\mho}_i = ((\zeta_i, \varpi_i), (\eta_i, \vartheta_i), (\kappa_i, \Omega_i))$, i = 1, 2, …., m and the associated weight vector of these CSFNs is $\beta_i = (\beta_1, \beta_2, \beta_3, \ldots, \beta_m)^T$ where $0 \leq \beta_i \leq 1$ such that $\sum_{i=1}^m \beta_i = 1$ and $\varepsilon > 0$.The CSF Yager weighted geometric (CSFYWG) operator is a function CSFYWG: $\mathscr{L}^m \rightarrow \mathscr{L}$ is defined by the following rule:

$$\text{CSFYWG}(\tilde{\mho}_1, \tilde{\mho}_2, \tilde{\mho}_3, \ldots, \tilde{\mho}_m) = \bigotimes_{i=1}^m (\tilde{\mho}_i)^{\beta_i} = \begin{pmatrix} \left(\sqrt{1 - \min\left(1, \left(\sum_{i=1}^m \beta_i \cdot \left(1-\zeta_i^2\right)^\varepsilon\right)^{1/\varepsilon}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{i=1}^m \beta_i \cdot \left(1-\varpi_i^2\right)^\varepsilon\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1 - \min\left(1, \left(\sum_{i=1}^m \beta_i \cdot \left(1-\eta_i^2\right)^\varepsilon\right)^{1/\varepsilon}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{i=1}^m \beta_i \cdot \left(1-\vartheta_i^2\right)^\varepsilon\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{\min\left(1, \left(\sum_{i=1}^m \beta_i . \kappa_i^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{\min\left(1, \left(\sum_{i=1}^m \beta_i . \Omega_i^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right) \end{pmatrix}. \tag{14}$$

Theorem 5. *Consider a class of m number of CSFNs,* $\tilde{\mho}_i = ((\zeta_i, \varpi_i), (\eta_i, \vartheta_i), (\kappa_i, \Omega_i))$ *and the associated weight vector of these CSFNs* $\tilde{\mho}_i$ *is* $\beta = (\beta_1, \beta_2, \beta_3, \ldots, \beta_m)^T$, *where* $0 \leq \beta_i \leq 1$ *such that* $\sum_{i=1}^m \beta_i = 1$ *and* $\varepsilon > 0$. *Then, their aggregated value in the framework of CSFYWG operator is a CSFN and is formulated as follows*:

$$\text{CSFYWG}(\tilde{\mho}_1, \tilde{\mho}_2, \tilde{\mho}_3, \ldots, \tilde{\mho}_m) = \otimes_{i=1}^m (\tilde{\mho}_i)^{\beta_i}$$

$$= \begin{pmatrix} \left(\sqrt{1 - \min\left(1, \left(\sum_{i=1}^m \beta_i \cdot \left(1-\zeta_i^2\right)^\varepsilon\right)^{1/\varepsilon}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{i=1}^m \beta_i \cdot \left(1-\varpi_i^2\right)^\varepsilon\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1 - \min\left(1, \left(\sum_{i=1}^m \beta_i \cdot \left(1-\eta_i^2\right)^\varepsilon\right)^{1/\varepsilon}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{i=1}^m \beta_i \cdot \left(1-\vartheta_i^2\right)^\varepsilon\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{\min\left(1, \left(\sum_{i=1}^m \beta_i . \kappa_i^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{\min\left(1, \left(\sum_{i=1}^m \beta_i . \Omega_i^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right) \end{pmatrix}.$$

Proof**.** This theorem is exemplified through the utilization of the mathematical induction process.

Consider the initial case where m = 2. We encounter two CSFNs denoted as $\tilde{\mho}_1 = ((\zeta_1, \varpi_1), (\eta_1, \vartheta_1), (\kappa_1, \Omega_1))$ and $\tilde{\mho}_2 = ((\zeta_2, \varpi_2), (\eta_2, \vartheta_2), (\kappa_2, \Omega_2))$. Applying the operations designed for CSFNs, we acquire the following expressions:

$$(\tilde{\mho}_1)^{\beta_1} = \begin{pmatrix} \left(\sqrt{1 - \min\left(1, \left(\beta_1 \cdot \left(1-\zeta_1^2\right)^\varepsilon\right)^{1/\varepsilon}\right)}, \sqrt{1 - \min\left(1, \left(\beta_1 \cdot \left(1-\varpi_1^2\right)^\varepsilon\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1 - \min\left(1, \left(\beta_1 \cdot \left(1-\eta_1^2\right)^\varepsilon\right)^{1/\varepsilon}\right)}, \sqrt{1 - \min\left(1, \left(\beta_1 \cdot \left(1-\vartheta_1^2\right)^\varepsilon\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{\min\left(1, \left(\beta_1 . \kappa_1^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{\min\left(1, \left(\beta_1 . \Omega_1^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right) \end{pmatrix} \text{ and}$$

$$(\tilde{\mho}_2)^{\beta_2} = \begin{pmatrix} \left(\sqrt{1-\min\left(1,\left(\beta_2.\left(1-\zeta_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1-\min\left(1,\left(\beta_2.\left(1-\varpi_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1-\min\left(1,\left(\beta_2.\left(1-\eta_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1-\min\left(1,\left(\beta_2.\left(1-\vartheta_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{\min\left(1,\left(\beta_2.\kappa_2^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{\min\left(1,\left(\beta_2.\Omega_2^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right) \end{pmatrix}.$$

In the view of definition 13, the aggregated value of $\tilde{\mho}_1$ and $\tilde{\mho}_2$ is obtained as follows:

CSFYWG $(\tilde{\mho}_1,\tilde{\mho}_2) = (\tilde{\mho}_1)^{\beta_1} \otimes (\tilde{\mho}_2)^{\beta_2}$.

$$= \begin{pmatrix} \left(\sqrt{1-min\left(1,\left(\beta_1.\left(1-\zeta_1^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1-min\left(1,\left(\beta_1.\left(1-\varpi_1^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1-min\left(1,\left(\beta_1.\left(1-\eta_1^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1-min\left(1,\left(\beta_1.\left(1-\vartheta_1^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{min\left(1,\left(\beta_1.\kappa_1^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{min\left(1,\left(\beta_1.\Omega_1^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right) \end{pmatrix}$$

$$\otimes \begin{pmatrix} \left(\sqrt{1-min\left(1,\left(\beta_2.\left(1-\zeta_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1-min\left(1,\left(\beta_2.\left(1-\varpi_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1-min\left(1,\left(\beta_2.\left(1-\eta_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1-min\left(1,\left(\beta_2.\left(1-\vartheta_2^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{min\left(1,\left(\beta_2.\kappa_2^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{min\left(1,\left(\beta_2.\Omega_2^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right) \end{pmatrix}.$$

It concludes that

$$\text{CSFYWG }(\tilde{\mho}_1,\tilde{\mho}_2) = \begin{pmatrix} \left(\sqrt{1-\min\left(1,\left(\sum_{i=1}^{2}\beta_i.\left(1-\zeta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1-\min\left(1,\left(\sum_{i=1}^{2}\beta_i.\left(1-\varpi_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1-\min\left(1,\left(\sum_{i=1}^{2}\beta_i.\left(1-\eta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1-\min\left(1,\left(\sum_{i=1}^{2}\beta_i.\left(1-\vartheta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{\min\left(1,\left(\sum_{i=1}^{2}\beta_i.\kappa_i^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{\min\left(1,\left(\sum_{i=1}^{2}\beta_i.\Omega_i^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right) \end{pmatrix}.$$

Thus, the result is valid for $m = 2$.
Next, we assume that the result holds for $m = k$. Then

$$\text{CSFYWG }(\tilde{\mho}_1,\tilde{\mho}_2,\tilde{\mho}_3,\ldots,\tilde{\mho}_k) = \otimes_{i=1}^{k}(\tilde{\mho}_i)^{\beta_i}$$

$$= \begin{pmatrix} \left(\sqrt{1-min\left(1,\left(\sum_{i=1}^{k}\beta_i.\left(1-\zeta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1-min\left(1,\left(\sum_{i=1}^{k}\beta_i.\left(1-\varpi_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{1-min\left(1,\left(\sum_{i=1}^{k}\beta_i.\left(1-\eta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1-min\left(1,\left(\sum_{i=1}^{k}\beta_i.\left(1-\vartheta_i^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right), \\ \left(\sqrt{min\left(1,\left(\sum_{i=1}^{k}\beta_i.\kappa_i^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{min\left(1,\left(\sum_{i=1}^{k}\beta_i.\Omega_i^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right) \end{pmatrix}.$$

Now, we show that result is true for $m = k+1$.

$$\text{CSFYWG }(\tilde{\mho}_1,\tilde{\mho}_2,\tilde{\mho}_3,\ldots,\tilde{\mho}_k,\tilde{\mho}_{k+1}) = (\tilde{\mho}_1)^{\beta_1}\otimes(\tilde{\mho}_2)^{\beta_2}\otimes(\tilde{\mho}_3)^{\beta_3}\otimes\ldots\otimes(\tilde{\mho}_k)^{\beta_k}\otimes(\tilde{\mho}_{k+1})^{\beta_{k+1}}$$

$$= \otimes_{i=1}^{k}(\tilde{\mho}_i)^{\beta_i}\otimes(\tilde{\mho}_{k+1})^{\beta_{k+1}}$$

$$= \left(\begin{array}{c} \left(\sqrt{1 - min\left(1, \left(\sum_{i=1}^{k} \beta_i.(1-\zeta_i^2)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - min\left(1, \left(\sum_{i=1}^{k} \beta_i.(1-\varpi_i^2)^{\varepsilon}\right)^{1/\varepsilon}\right)} \right), \\ \left(\sqrt{1 - min\left(1, \left(\sum_{i=1}^{k} \beta_i.(1-\eta_i^2)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - min\left(1, \left(\sum_{i=1}^{k} \beta_i.(1-\vartheta_i^2)^{\varepsilon}\right)^{1/\varepsilon}\right)} \right), \\ \left(\sqrt{min\left(1, \left(\sum_{i=1}^{k} \beta_i.\kappa_i^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{min\left(1, \left(\sum_{i=1}^{k} \beta_i.\Omega_i^{2\varepsilon}\right)^{1/\varepsilon}\right)} \right) \end{array} \right)$$

$$\otimes \left(\begin{array}{c} \left(\sqrt{1 - min\left(1, \left(\beta_{k+1}.(1-\zeta_{k+1}^2)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - min\left(1, \left(\beta_{k+1}.(1-\varpi_{k+1}^2)^{\varepsilon}\right)^{1/\varepsilon}\right)} \right), \\ \left(\sqrt{1 - min\left(1, \left(\beta_{k+1}.(1-\eta_{k+1}^2)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - min\left(1, \left(\beta_{k+1}.(1-\vartheta_{k+1}^2)^{\varepsilon}\right)^{1/\varepsilon}\right)} \right), \\ \left(\sqrt{min\left(1, \left(\beta_{k+1}.\kappa_{k+1}^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{min\left(1, \left(\beta_{k+1}.\Omega_{k+1}^{2\varepsilon}\right)^{1/\varepsilon}\right)} \right) \end{array} \right).$$

Consequently,

$$\text{CSFYWG}\,(\tilde{\mathrm{U}}_1, \tilde{\mathrm{U}}_2, \tilde{\mathrm{U}}_3, \ldots, \tilde{\mathrm{U}}_{k+1}) = \left(\begin{array}{c} \left(\sqrt{1 - min\left(1, \left(\sum_{i=1}^{k+1} \beta_i.(1-\zeta_i^2)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - min\left(1, \left(\sum_{i=1}^{k+1} \beta_i.(1-\varpi_i^2)^{\varepsilon}\right)^{1/\varepsilon}\right)} \right), \\ \left(\sqrt{1 - min\left(1, \left(\sum_{i=1}^{k+1} \beta_i.(1-\eta_i^2)^{\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{1 - min\left(1, \left(\sum_{i=1}^{k+1} \beta_i.(1-\vartheta_i^2)^{\varepsilon}\right)^{1/\varepsilon}\right)} \right), \\ \left(\sqrt{min\left(1, \left(\sum_{i=1}^{k+1} \beta_i.\kappa_i^{2\varepsilon}\right)^{1/\varepsilon}\right)}, \sqrt{min\left(1, \left(\sum_{i=1}^{k+1} \beta_i.\Omega_i^{2\varepsilon}\right)^{1/\varepsilon}\right)} \right) \end{array} \right)$$

Thus, the above concludes that the outcome is valid for all integer values of m.
The subsequent illustrated example verifies the aforementioned fact.

Example 4. Consider $\tilde{\mathrm{U}}_1$ = ((0.6, 0.36), (0.45, 0.43), (0.21, 0.2)), $\tilde{\mathrm{U}}_2$ = ((0.48, 0.22), (0.18, 0.33), (0.44, 0.23)), $\tilde{\mathrm{U}}_3$ = ((0.52, 0.15), (0.35, 0.57), (0.25, 0.34)), and $\tilde{\mathrm{U}}_4$ = ((0.15, 0.23), (0.52, 0.35), (0.53, 0.41)) be four CSFNs and its associated weighted vector is $\beta = (0.21, 0.29, 0.34, 0.16)^T$ and $\varepsilon = 5$. Then

$$\left(\sum_{i=1}^{4} \beta_i.(1-\zeta_i^2)^{\varepsilon}\right)^{1/\varepsilon} = \left(0.21\left(1-(0.6)^2\right)^5 + 0.29\left(1-(0.48)^2\right)^5 + 0.34\left(1- \quad (0.52)^2\right)^5 + 0.16\left(1-(0.15)^2\right)^5\right)^{1/5}$$

$= 0.7932$,
Similarly,

$$\left(\sum_{i=1}^{4} \beta_i.(1-\varpi_i^2)^{\varepsilon}\right)^{1/\varepsilon} = 0.9457,$$

$$\left(\sum_{i=1}^{4} \beta_i.(1-\eta_i^2)^{\varepsilon}\right)^{1/\varepsilon} = 0.8786, \left(\sum_{i=1}^{4} \beta_i.(1-\vartheta_i^2)^{\varepsilon}\right)^{1/\varepsilon} = 0.8194,$$

$$\left(\sum_{i=1}^{4} \beta_i.\kappa_i^{2\varepsilon}\right)^{1/\varepsilon} = 0.2047, \left(\sum_{i=1}^{4} \beta_i.\Omega_i^{2\varepsilon}\right)^{1/\varepsilon} = 0.1234.$$

This implies that, CSFYWG $(\tilde{\mathrm{U}}_1, \tilde{\mathrm{U}}_2, \tilde{\mathrm{U}}_3, \tilde{\mathrm{U}}_4) = \otimes_{i=1}^{4}(\tilde{\mathrm{U}}_i)^{\beta_i} = ((0.455, 0.233), (0.348, 0.425), (0.452, 0.351))$.
Subsequently, it can be inferred that the aforementioned discourse yields yet again a CSFN.

Theorem 6. *(Idempotency): Consider* $\tilde{\mathrm{U}}_i = ((\zeta_i, \varpi_i), (\eta_i, \vartheta_i), (\kappa_i, \Omega_i))$, $i = 1, 2, \ldots, m$ *and let* $\tilde{\mathrm{U}}_i = \tilde{\mathrm{U}}_o \ \forall\, i$, *where* $\tilde{\mathrm{U}}_o = ((\zeta_o, \varpi_o), (\eta_o, \vartheta_o), (\kappa_o, \Omega_o))$. *The associated weight vector of these CSFNs is* $\beta = (\beta_1, \beta_2, \beta_3, \ldots, \beta_m)^T$ *where* $0 \le \beta_i \le 1$ *such that* $\sum_{i=1}^{m} \beta_i = 1$ *and* $\varepsilon > 0$. *Then*

CSFYWG $(\text{Ŭ}_1, \text{Ŭ}_2, \text{Ŭ}_3, \ldots, \text{Ŭ}_m) = \text{Ŭ}_o$.

Proof. The proof of this theorem is easy to demonstrate, as stated in Theorem 2.

Theorem 7. *(Monotonicity): Consider* $\text{Ŭ}_i = ((\zeta_i, \varpi_i), (\eta_i, \vartheta_i), (\kappa_i, \Omega_i))$ *and* $\text{Ŭ}_i^* = ((\zeta_i^*, \varpi_i^*), (\eta_i^*, \vartheta_i^*), (\kappa_i^*, \Omega_i^*))$ *be any two collections of m number of CSFNs for* $i = 1, 2, \ldots, m$ *and* $\beta = (\beta_1, \beta_2, \beta_3, \ldots, \beta_m)^T$ *is an associated weight vector of these CSFNs* Ŭ_i *where* $0 \le \beta_i \le 1$ *such that* $\sum_{i=1}^{m} \beta_i = 1$ *and* $\varepsilon > 0$. *If* $\zeta_i \le \zeta_i^*, \varpi_i \le \varpi_i^*, \eta_i \le \eta_i^*, \vartheta_i \le \vartheta_i^*, \kappa_i \ge \kappa_i^*$, *and* $\Omega_i \ge \Omega_i^*$ $(i = 1, 2, \ldots, m)$. *Then*

CSFYWG $(\text{Ŭ}_1, \text{Ŭ}_2, \text{Ŭ}_3, \ldots, \text{Ŭ}_m) \le$ CSFYWG $(\text{Ŭ}_1^*, \text{Ŭ}_2^*, \text{Ŭ}_3^*, \ldots, \text{Ŭ}_m^*)$.

Proof. The proof of this theorem is analogous to that of theorem 3.

Theorem 8. *(Boundedness): Consider m number of CSFNs,* $\text{Ŭ}_i = ((\zeta_i, \varpi_i), (\eta_i, \vartheta_i), (\kappa_i, \Omega_i))$, $i = 1, 2, \ldots, m$ *and associated weight vector of these CSFNs is* $\beta = (\beta_1, \beta_2, \beta_3, \ldots, \beta_m)^T$ *where* $0 \le \beta_i \le 1$ *such that* $\sum_{i=1}^{m} \beta_i = 1$ *and* $\varepsilon > 0$. *If* $\text{Ŭ}^- = \left\{ \left(\min_i\{\zeta_i\}, \min_i\{\varpi_i\} \right), \left(\min_i\{\eta_i\}, \min_i\{\vartheta_i\} \right), \left(\max_i\{\kappa_i\}, \max_i\{\Omega_i\} \right) \right\}$ *and* $\text{Ŭ}^+ = \left\{ \left(\max_i\{\zeta_i\}, \max_i\{\varpi_i\} \right), \left(\max_i\{\eta_i\}, \max_i\{\vartheta_i\} \right), \left(\min_i\{\kappa_i\}, \min_i\{\Omega_i\} \right) \right\}$. *Then*

$$\text{Ŭ}^- \le CSFYWG(\text{Ŭ}_1, \text{Ŭ}_2, \text{Ŭ}_3, \ldots, \text{Ŭ}_m) \le \text{Ŭ}^+.$$

proof: The proof of this theorem is analogous to that of theorem 4.

5. Application of proposed strategies in MADM problem

This section focuses on the analysis of MADM problems that are associated with CSF information. We perform this by employing the CSF Yager aggregation operators (YAOs) discussed earlier. Now, we establish a systematic mathematical approach to address the MADM problems associated with CSF information, utilizing the powerful CSF Yager aggregation operators (YAOs). Additionally, we structure the following process that effectively navigates the MADM problem.

- For this, suppose $\mathscr{Q} = (\mathscr{Q}_1, \mathscr{Q}_2, \mathscr{Q}_3, \ldots, \mathscr{Q}_m)$ is a collection of alternatives.
- Consider $\mathscr{C} = (\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \ldots, \mathscr{C}_n)$ is a set of attributes, and $\beta = (\beta_1, \beta_2, \beta_3, \ldots, \beta_n)^T$ is an associated weight vector of these attributes with $\beta_j \in [0, 1]$ and $\sum_{j=1}^{n} \beta_j = 1$.
- Let the decision matrix $\mathscr{D} = (\mathscr{G}_{ij})_{m \times n} = ((\zeta_{ij}, \varpi_{ij}), (\eta_{ij}, \vartheta_{ij}), (\kappa_{ij}, \Omega_{ij}))_{m \times n}$, be the collection of complex spherical fuzzy information from the expert evaluation about the finite number of alternatives $\mathscr{Q}_i$ adhering to the criteria $\mathscr{C}_j$.

The key steps to solve MADM problem in the framework of newly defined CSFYWA and CSFYWG operators are outlined as follows:

Step 1. Formation of the decision matrix

Formulate the CSF decision matrix in the framework of the information obtained from the decision-makers in the following way:

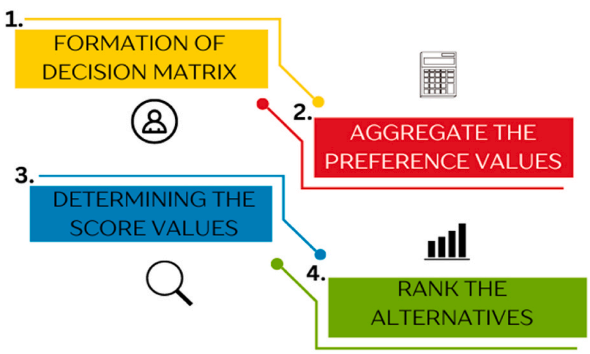

Fig. 1. Flow chart of the proposed technique.

$$\mathscr{D}=\begin{pmatrix}\begin{pmatrix}(\zeta_{11},\varpi_{11}),(\eta_{11},\vartheta_{11}),\\(\kappa_{11},\Omega_{11})\end{pmatrix} & \begin{pmatrix}(\zeta_{12},\varpi_{12}),(\eta_{12},\vartheta_{12}),\\(\kappa_{12},\Omega_{12})\end{pmatrix} & \cdots & \begin{pmatrix}(\zeta_{1n},\varpi_{1n}),(\eta_{1n},\vartheta_{1n}),\\(\kappa_{1n},\Omega_{1n})\end{pmatrix}\\ \begin{pmatrix}(\zeta_{21},\varpi_{21}),(\eta_{21},\vartheta_{21}),\\(\kappa_{21},\Omega_{21})\end{pmatrix} & \begin{pmatrix}(\zeta_{22},\varpi_{22}),(\eta_{22},\vartheta_{22}),\\(\kappa_{22},\Omega_{22})\end{pmatrix} & \cdots & \begin{pmatrix}(\zeta_{2n},\varpi_{2n}),(\eta_{2n},\vartheta_{2n}),\\(\kappa_{2n},\Omega_{2n})\end{pmatrix}\\ \vdots & \vdots & \ddots & \vdots\\ \begin{pmatrix}(\zeta_{m1},\varpi_{m1}),(\eta_{m1},\vartheta_{m1}),\\(\kappa_{m1},\Omega_{m1})\end{pmatrix} & \begin{pmatrix}(\zeta_{m2},\varpi_{m2},),(\eta_{m2},\vartheta_{m2}),\\(\kappa_{m2},\Omega_{m2})\end{pmatrix} & \cdots & \begin{pmatrix}(\zeta_{mn},\varpi_{mn}),(\eta_{mn},\vartheta_{mn}),\\(\kappa_{mn},\Omega_{mn})\end{pmatrix}\end{pmatrix}.$$

Step 2. Evaluation of the aggregated preference values:

(a) Obtain the aggregated preference values $\mathscr{G}_i$ of all alternatives $\mathscr{Q}_i$ by means of CSFYWA operator in the following way:

$\mathscr{G}_i = \text{CSFYWA}\ (\mathscr{G}_{i1}, \mathscr{G}_{i2}, \ldots, \mathscr{G}_{in})$

$$=\begin{pmatrix}\left(\sqrt{min\left(1,\left(\sum_{j=1}^{n}\beta_j.\zeta_{ij}^{2\varepsilon}\right)^{1/\varepsilon}\right)},\sqrt{min\left(1,\left(\sum_{j=1}^{n}\beta_j.\varpi_{ij}^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right),\\ \left(\sqrt{1-min\left(1,\left(\sum_{j=1}^{n}\beta_j.\left(1-\eta_{ij}^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)},\sqrt{1-min\left(1,\left(\sum_{j=1}^{n}\beta_j.\left(1-\vartheta_{ij}^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right),\\ \left(\sqrt{1-min\left(1,\left(\sum_{j=1}^{n}\beta_j.\left(1-\kappa_{ij}^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)},\sqrt{1-min\left(1,\left(\sum_{j=1}^{n}\beta_j.\left(1-\Omega_{ij}^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right)\end{pmatrix}$$

(b) Compute the aggregated preference values $\mathscr{G}_i$ of all alternatives $\mathscr{Q}_i$ using CSFYWG operator in the following manner:

$\mathscr{G}_i = \text{CSFYWG}\ (\mathscr{G}_{i1}, \mathscr{G}_{i2}, \ldots, \mathscr{G}_{in})$

$$=\begin{pmatrix}\left(\sqrt{1-min\left(1,\left(\sum_{j=1}^{n}\beta_j.\left(1-\zeta_{ij}^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)},\sqrt{1-min\left(1,\left(\sum_{j=1}^{n}\beta_j.\left(1-\varpi_{ij}^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right),\\ \left(\sqrt{1-min\left(1,\left(\sum_{j=1}^{n}\beta_j.\left(1-\eta_{ij}^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)},\sqrt{1-min\left(1,\left(\sum_{j=1}^{n}\beta_j.\left(1-\vartheta_{ij}^2\right)^{\varepsilon}\right)^{1/\varepsilon}\right)}\right),\\ \left(\sqrt{min\left(1,\left(\sum_{j=1}^{n}\beta_j.\kappa_{ij}^{2\varepsilon}\right)^{1/\varepsilon}\right)},\sqrt{min\left(1,\left(\sum_{j=1}^{n}\beta_j.\Omega_{ij}^{2\varepsilon}\right)^{1/\varepsilon}\right)}\right)\end{pmatrix}$$

Step 3. Determining the score values:

Calculate the score values of $\mathscr{G}_i$ corresponding to each alternative $\mathscr{Q}_i$ by using Definition 10.

Step 4. Rank the alternatives

Arrange the alternatives $\mathscr{Q}_i$ and identify the optimal one using $\mathfrak{H}\ (\mathscr{G}_i)$.

The flowchart presented in Fig. 1 provides a graphical representation of the aforementioned methodologies. The pseudocode for the developed mechanisms is presented in Table 1.

5.1. Practical relevance of proposed strategies within the domain of space observation

- Space observation is the study of distinct galaxies, planets, and other celestial objects. It is important for the understanding of the universe and its practical applications on Earth. Astronomers employ satellites, telescopes, space probes, and spectroscopes to conduct observations and gather data on celestial bodies inside and beyond the solar system. The utilization of these instruments and the corresponding technologies enables astronomers to scrutinize and analyze the data, facilitating scientific exploration of the solar system and the universe. The practical relevance of the proposed strategies to solve decision-making problems related to space

Table 1
Pseudocode for the developed mechanisms.

```
                    Algorithm MADM_CSFYWA_CSFYWG
Input: Decision matrix D, CSFYWA operator, CSFYWG operator,
Definition 10
Output: Ranked alternatives
Step 1: Formation of the decision matrix
1. Formulate the CSF decision matrix D based on the information from
decision-makers
Step 2: Evaluation of the aggregated preference values
2. Initialize an empty list G = []
3. For each alternative Q_i in D:
    a. Compute G_i using the CSFYWA operator
    b. Compute G_i using the CSFYWG operator
    c. Append G_i to G
Step 3: Determining the score values
4. Initialize an empty list S = []
5. For each G_i in G:
    a. Calculate the score value S_i of G_i using Definition 10
    b. Append S_i to S
Step 4: Rank the alternatives
6. Sort the alternatives Q_i based on their corresponding S_i values
in descending order
7. Identify the optimal alternative Q_opt as the one with the highest
S_i value
Return: Ranked list of alternatives and Q_opt
```

observations may be outlined as follows: (i) selection of the best celestial object for space observation by the James Webb Space Telescope; (ii) detection of anomalies on the Mars Rover Power System; and (iii) selection of a most suitable space probe target for an asteroid mining mission.

- We employ the newly defined methods to resolve the aforementioned DM issues within the space observation context.
- For instance, the following example describes the solution of a certain decision-making problem within the framework of space observation using the proposed techniques.

Problem formulation: Machine learning techniques are highly efficient instruments in astronomical jobs for categorizing objects based on their distinct characteristics. The morphological categorization of galaxies at various redshifts constitutes one of the potentially useful applications. Our aim is to determine the most effective of five supervised machine learning techniques (multi-photometry diagrams, naive Bayes, logistic regression, support vector machines, and random forests) for the automated classification of galaxy morphology by employing newly defined CSFY aggregation operators for the SDSS data.

Conclusion: For the morphological classification of binary galaxies, the support-vector machine and random forest methods implemented with the Scikit-learn machine learning library in Python yield the maximum degree of precision. In particular, the support-vector machine achieves a success rate of 96.4 %, while the random forest model achieves 95.5 %. Thus, the support-vector machine is the optimal method for the automated classification of galaxy morphology.

5.2. *Case study: selection of the best celestial object for space observation by James Webb Space Telescope*

The James Webb Space Telescope (JWST) is a powerful revolutionary space observatory that is a collaboration between NASA, ESA, and CSA. The Webb mission expanded upon the legacy of the Hubble Space Telescope's impressive imaging capability and the scientific capabilities of the Spitzer Space Telescope to detect light in the mid-infrared range and beyond the visible spectrum. This telescope is not our average stargazer; it is designed to see things in a completely different light, the infrared (IR) spectrum. Because IR light traverses through the dense gas clouds that hide the visible light, Webb will unveil the initially hidden areas of the cosmos. This means it can study celestial objects invisible to regular telescopes, like the earliest galaxies, brown dwarfs, exoplanets, and the birth of stars. However, it is set to change how we understand the universe now that it's up there. The JWST comes with some incredible tools that help us unlock the secrets of the cosmos.

- One of its most important missions is studying exoplanet atmospheres, especially those that resemble Earth and revolve within the habitable zones of neighboring stars. By utilizing its exceptional infrared technology, the James Webb can dissect these atmospheres and find signs of life. In short, it's helping answer the big question: "Are we alone in the universe?"
- The JWST has another big and crucial task: studying galaxies born in the universe's early days. Producing images of these galaxies is no mean feat, by relying on their infrared vision, we'll be able to look out and analyze these ancient galaxies to learn not just where we came from but where we're all going—literally. The JWST can tell us where the very first galaxies came from, how galaxies change and die, and how stars and even black holes are born.

In a nutshell, the JWST is like a space detector with an incredibly fancy infrared magnifying glass. Its primary mission is to discover the universe's origins and structure. However, it may also make some truly groundbreaking breakthroughs by pursuing everything from Earth-like exoplanets to ancient galaxies. One of its breakthroughs was in September, when NASA created a significant impact by revealing that its James Webb Space Telescope had detected "dimethyl sulfide" (DMS), a gas exclusively produced by lifeforms on Earth, which could potentially indicate the presence of life on the exoplanet K2-18b. The JWST really is a disruptor of science, and this is one step closer to its wild missions.

The challenges for object selection involve the following:

i. Limited observation time: Compared to other telescopes, the JWST has a finite amount of time to make observations. Determining which objectives have the most incredible scientific value is challenging, and prioritizing and scheduling observations for selected targets is difficult.
ii. Competitive Demand: Given the autonomy of the observatory, we can only imagine that astronomers inside and outside of the space agency are going to compete fiercely to get their observation targets picked. After all, it's no mean feat to use an observatory like this to fulfill a wide variety of research interests and captivating scientific objectives.
iii. Technical limitations of other telescopes: The wavelength ranges and observational capabilities vary among telescopes, such as the Hubble Space Telescope. Celestial objects detected by the JWST in the infrared spectrum may have lacked extensive research available or may not have been observable by telescopes that operate in visible or ultraviolet wavelengths. Therefore, there is a lack of comprehensive data on these objects.

Table 2
CSF decision matrix.

	$\mathscr{C}_1$	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_4$
$\mathscr{Q}_1$	((0.35, 0.59), (0.21, 0.16), (0.31, 0.24))	((0.65, 0.23), (0.21, 0.31), (0.14, 0.13))	((0.55, 0.37), (0.10, 0.12), (0.25, 0.23))	((0.45, 0.51), (0.18, 0.10), (0.30, 0.21))
$\mathscr{Q}_2$	((0.53, 0.30) ,(0.22, 0.41), (0.10, 0.25))	((0.40, 0.50), (0.32, 0.22), (0.20, 0.16))	((0.5, 0.35), (0.24, 0.20), (0.19, 0.15))	((0.38, 0.45), (0.29, 0.20), (0.20, 0.23))
$\mathscr{Q}_3$	((0.36, 0.57), (0.31, 0.44), (0.22, 0.27))	((0.44, 0.39), (0.16, 0.33), (0.46, 0.15))	((0.51, 0.45), (0.29, 0.53), (0.51, 0.65))	((0.25, 0.79), (0.36, 0.25), (0.49, 0.41))
$\mathscr{Q}_4$	((0.62, 0.49), (0.36, 0.12), (0.21, 0.35))	((0.73, 0.39), (0.24, 0.36), (0.46, 0.5))	((0.29, 0.40), (0.43, 0.47), (0.55, 0.32))	((0.81, 0.39), (0.26, 0.65), (0.31, 0.24))
$\mathscr{Q}_5$	((0.85, 0.65), (0.32, 0.27), (0.27, 0.41))	((0.52, 0.30), (0.20, 0.15), (0.47, 0.32))	((0.67, 0.52), (0.38, 0.25), (0.23, 0.42))	((0.56, 0.35), (019, 0.27), (0.42, 0.33))

Table 3
Aggregated values by using CSFYWA operator.

Alternatives	$\mathscr{G}_i$
$\mathscr{Q}_1$	$((0.566, 0.481), (0.174, 0.367), (0.318, 0.307))$
$\mathscr{Q}_2$	$((0.476, 0.434), (0.269, 0.258), (0.179, 0.191))$
$\mathscr{Q}_3$	$((0.453, 0.607), (0.275, 0.411), (0.436, 0.398))$
$\mathscr{Q}_4$	$((0.683, 0.422), (0.338, 0.405), (0.419, 0.369))$
$\mathscr{Q}_5$	$((0.703, 0.534), (0.293, 0.233), (0.346, 0.422))$

iv. Data Transmission and Processing: It is critical to the success of the James Webb Space Telescope (JWST) that the vast volumes of data it produces are not only transmitted but are also processed in an extremely efficient manner. Bottlenecks in the administration of data through the pipelines are to be avoided, as are unnecessary delays in the extraction of the scientific insights they have been sent to deliver.

5.2.1. *Empirical validation*

The telescope is equipped with a deployable Optical Telescope Element (OTE) that is loaded with four science instruments (SIs): the Mid Infrared Instrument (MIRI), which is jointly developed by the EC and JPL, the Near Infrared Camera (NIRCam) from the University of Arizona, and the Fine Guidance Sensor (FGS)/Near Infrared Imaging Slitless Spectrometer (NIRISS) from the CSA, assembled into an Integrated Science Instrument Module (ISIM) that provides imagery and spectroscopy in the near-infrared band between 0.6 and 5 µm and in the mid-infrared band between 5 and 28.1 µm.

Moreover, the studies suggest that the criteria on the basis of which a suitable object is selected for space observation by the JWST are: ($\mathscr{C}_1$) Resource constraints: The time-constrained observation, ($\mathscr{C}_2$) Feasibility: Webb has certain limitations on when it can see certain targets due to a combination of safety considerations for the observatory and the position of the target in elliptic coordinates, ($\mathscr{C}_3$) Scientific Importance: the target should have a potential contribution to the advancement of scientific knowledge, ($\mathscr{C}_4$) Observational History: A demonstration that the unique capabilities of JWST are required to achieve the science goals of the program that combines JWST information with the data from other space missions or ground-based observatories.

The subsequent discourse provides the solution to the MADM problem as a numerical illustration to demonstrate the effectiveness of the proposed novel technique.

Consider the set of five astronomical objects $\{\mathscr{Q}_1$ = Distant Galaxies, $\mathscr{Q}_2$ = Staller Nurseries, $\mathscr{Q}_3$ = Black holes & Active Galactic Nuclei, $\mathscr{Q}_4$ = Comets, $\mathscr{Q}_5$ = Exoplanets$\}$, which need to be selected and prioritized for observation in space by JWST. JWST has limited observing time and resources, and there are multiple criteria according to the weight vector $(0.21, 0.29, 0.34, 0.16)^T$ for target selection, including

$\mathscr{C}_1$: Resources Constraints
$\mathscr{C}_2$: Feasibility
$\mathscr{C}_3$: Scientific Importance
$\mathscr{C}_4$: Observational History and Redundancy.

To make a CSFN, these factors can be subdivided into two distinct characteristics, as outlined below:

- Scientific importance consists of the potential for a paradigm shift and alignment with mission goals.
- Feasibility consists of target visibility and technical capabilities.
- Observational history and redundancy consist of the frequency of previous observations and temporal considerations.
- Resource constraints consist of time allocation and instrument availability.

The key phases to solve MADM problem in the framework of CSFYW aggregation operators are outlined as follows:

Step 1. Table 2 summarizes the decision-maker's opinions about each alternative $\mathscr{Q}_i$ for each criterion $\mathscr{C}_j$ in the form of CSFN.

The following discussion aggregates different expert assessments into a collective one using CSFYWA operator.

Step 2. (a) The application of the CSFYWA operator on the values presented in Table 2 for the particular value $\varepsilon = 3$, yields the following Table 3.

Table 4
Aggregated values by using CSFYWG operator.

Alternatives	$\mathscr{G}_i$
$\mathscr{Q}_1$	$((0.516, 0.403), (0.174, 0.367), (0.472, 0.402))$
$\mathscr{Q}_2$	$((0.458, 0.401), (0.269, 0.258), (0.189, 0.209))$
$\mathscr{Q}_3$	$((0.419, 0.483), (0.275, 0.411), (0.471, 0.546))$
$\mathscr{Q}_4$	$((0.546, 0.415), (0.338, 0.405), (0.481, 0.418))$
$\mathscr{Q}_5$	$((0.625, 0.455), (0.293, 0.233), (0.401, 0.436))$

Table 5
The aggregated values of existing operators.

	SFYWA [24]	SFYWG [24]	CSFWA [37]	CSFWG [37]
$\mathscr{O}_1$	$\begin{pmatrix} 0.566, 0.174, \\ 0.318 \end{pmatrix}$	$\begin{pmatrix} 0.516, 0.174, \\ 0.472 \end{pmatrix}$	$\begin{pmatrix} (0.540, 0.429), (0.159, 0.163), \\ (0.228, 0.194) \end{pmatrix}$	$\begin{pmatrix} (0.508, 0.374), (0.159, 0.163), \\ (0.249, 0.205) \end{pmatrix}$
$\mathscr{O}_2$	$\begin{pmatrix} 0.476, 0.269, \\ 0.179 \end{pmatrix}$	$\begin{pmatrix} 0.458, 0.269, \\ 0.189 \end{pmatrix}$	$\begin{pmatrix} (0.464, 0.409), (0.264, 0.239), \\ (0.169, 0.182) \end{pmatrix}$	$\begin{pmatrix} (0.454, 0.391), (0.264, 0.239), \\ (0.180, 0.192) \end{pmatrix}$
$\mathscr{O}_3$	$\begin{pmatrix} 0.453, 0.275, \\ 0.436 \end{pmatrix}$	$\begin{pmatrix} 0.419, 0.275, \\ 0.471 \end{pmatrix}$	$\begin{pmatrix} (0.429, 0.552), (0.256, 0.395), \\ (0.412, 0.328) \end{pmatrix}$	$\begin{pmatrix} (0.405, 0.496), (0.256, 0.395), \\ (0.449, 0.461) \end{pmatrix}$
$\mathscr{O}_4$	$\begin{pmatrix} 0.683, 0.338, \\ 0.419 \end{pmatrix}$	$\begin{pmatrix} 0.546, 0.338, \\ 0.481 \end{pmatrix}$	$\begin{pmatrix} (0.628, 0.417), (0.323, 0.344), \\ (0.389, 0.354) \end{pmatrix}$	$\begin{pmatrix} (0.524, 0.413), (0.323, 0.344), \\ (0.442, 0.381) \end{pmatrix}$
$\mathscr{O}_5$	$\begin{pmatrix} 0.703, 0.293, \\ 0.346 \end{pmatrix}$	$\begin{pmatrix} 0.625, 0.293, \\ 0.401 \end{pmatrix}$	$\begin{pmatrix} (0.681, 0.487), (0.272, 0.222), \\ (0.322, 0.372) \end{pmatrix}$	$\begin{pmatrix} (0.636, 0.436), (0.272, 0.222), \\ (0.359, 0.378) \end{pmatrix}$

Table 6
Ranking of alternatives.

Method	$\mathscr{O}_1$	$\mathscr{O}_2$	$\mathscr{O}_3$	$\mathscr{O}_4$	$\mathscr{O}_5$	Ranking
SFYWA [24]	0.4829	0.4013	0.3309	0.4709	0.5373	$\mathscr{O}_5 > \mathscr{O}_1 > \mathscr{O}_4 > \mathscr{O}_2 > \mathscr{O}_3$
SFYWG [24]	0.4065	0.3889	0.3009	0.3407	0.4550	$\mathscr{O}_5 > \mathscr{O}_1 > \mathscr{O}_2 > \mathscr{O}_4 > \mathscr{O}_3$
CSFWA [37]	0.5693	0.4414	0.3454	0.3974	0.5967	$\mathscr{O}_5 > \mathscr{O}_1 > \mathscr{O}_2 > \mathscr{O}_4 > \mathscr{O}_3$
CSFWG [37]	0.5124	0.4229	0.2470	0.2939	0.5157	$\mathscr{O}_5 > \mathscr{O}_1 > \mathscr{O}_2 > \mathscr{O}_4 > \mathscr{O}_3$
Proposed CSFYWA	0.4711	0.4481	0.3550	0.3809	0.6135	$\mathscr{O}_5 > \mathscr{O}_1 > \mathscr{O}_2 > \mathscr{O}_4 > \mathscr{O}_3$
Proposed CSFYWG	0.4224	0.4147	0.1604	0.2329	0.4747	$\mathscr{O}_5 > \mathscr{O}_1 > \mathscr{O}_2 > \mathscr{O}_4 > \mathscr{O}_3$

Table 7
Ranking the alternatives of previous observatory.

Methods	Ranking
SFYWA [24]	$\mathscr{O}_5 > \mathscr{O}_2 > \mathscr{O}_1 > \mathscr{O}_4 > \mathscr{O}_3$
SFYWG [24]	$\mathscr{O}_5 > \mathscr{O}_2 > \mathscr{O}_1 > \mathscr{O}_4 > \mathscr{O}_3$
CSFWA [37]	$\mathscr{O}_5 > \mathscr{O}_1 > \mathscr{O}_2 > \mathscr{O}_4 > \mathscr{O}_3$
CSFWG [37]	$\mathscr{O}_5 > \mathscr{O}_1 > \mathscr{O}_2 > \mathscr{O}_4 > \mathscr{O}_3$
Proposed CSFYWA	$\mathscr{O}_5 > \mathscr{O}_2 > \mathscr{O}_1 > \mathscr{O}_4 > \mathscr{O}_3$
Proposed CSFYWG	$\mathscr{O}_5 > \mathscr{O}_1 > \mathscr{O}_2 > \mathscr{O}_4 > \mathscr{O}_3$

Step 3. We determine the score values assigned to all CSF numbers obtained in Table 3 in view of Definition 10 as follows: $\mathfrak{D}(\mathscr{O}_5) = 0.6135$, $\mathfrak{D}(\mathscr{O}_1) = 0.4711$, $\mathfrak{D}(\mathscr{O}_2) = 0.4481$, $\mathfrak{D}(\mathscr{O}_4) = 0.3809$, and $\mathfrak{D}(\mathscr{O}_3) = 0.3550$.

Step 4. Since $\mathfrak{D}(\mathscr{O}_5) > \mathfrak{D}(\mathscr{O}_1) > \mathfrak{D}(\mathscr{O}_2) > \mathfrak{D}(\mathscr{O}_4) > \mathfrak{D}(\mathscr{O}_3)$, therefore the ranking order of alternatives is $\mathscr{O}_5 > \mathscr{O}_1 > \mathscr{O}_2 > \mathscr{O}_4 > \mathscr{O}_3$. Thus, exoplanet is the most suitable astronomical object for the space observation by the JWST.

Similarly, within the context of the CSFYWG operator, the solution to the above MADM problem is as follows:

Step 1. Summarize all the data collected from experts' evaluation of each alternative $\mathscr{O}_i$ for each attribute $\mathscr{C}_j$ in the form of a CSFN, displayed in Table 2.

Step 2. (b) The application of CSFYWG operator on the values provided in Table 2, specifically for the given value $\varepsilon = 3$, yields the following Table 4:

Step 3. We determine the score values assigned to each CSF number obtained in Table 4 by using the proposed score function as follows:

$\mathfrak{D}\ (\mathscr{O}_5) = 0.4747$, $\mathfrak{D}\ (\mathscr{O}_1) = 0.4224$, $\mathfrak{D}\ (\mathscr{O}_2) = 4147$, $\mathfrak{D}\ (\mathscr{O}_4) = 0.2329$, and $\mathfrak{D}\ (\mathscr{O}_3) = 0.1604$.

Step 4. Since $\mathfrak{D}(\mathscr{O}_5) > \mathfrak{D}(\mathscr{O}_1) > \mathfrak{D}(\mathscr{O}_2) > \mathfrak{D}(\mathscr{O}_4) > \mathfrak{D}(\mathscr{O}_3)$, therefore the ranking order of alternatives is $\mathscr{O}_5 > \mathscr{O}_1 > \mathscr{O}_2 > \mathscr{O}_4 > \mathscr{O}_3$. Hence, exoplanet is the optimal astronomical object for observational purposes in space through the JWST.

5.3. Comparative analysis

This sub-section aims to provide a comprehensive assessment to demonstrate the significance and validity of the suggested techniques. This comparative analysis utilizes proven methodologies to assess different alternatives carefully, as described in Refs. [24, 37]. In this discussion, we compare the CSFYWA and CSFYWG operators to established SFYWA, SFYWG, CSFWA, and CSFWG operators. The following Table 5 summarizes the outcomes obtained by these aggregation operators, and Table 6 illustrates the ranking of alternatives.

Moreover, the following Table 7 describes the validation of the proposed strategies on the real data of previous space mission like Hubble telescope.

By comparing Tables 5 and 7, it is apparent that the newly proposed methodologies effectively address the MADM problem associated with space observation using the JWST, as well as the previous space missions such as the Hubble telescope.

The preceding discussion demonstrates that the newly defined methodologies provide more comprehensive methods than existing approaches due to their superior preferences. A lot of information is lost in the framework of techniques developed in Ref. [37] because these techniques cannot be customized to fit specific needs, making them less adaptable and unable to consider individual preferences or changes in context. However, the newly introduced operators exhibit greater flexibility due to their use of parametric values. In addition, notice that the operators suggested by Chinaram et al. [24] are specific instances of these proposed operators, as these techniques can only deal with one dimension during the aggregation process. These particular limitations of SF Yager aggregation operators can cause a significant loss of information, whereas CSF Yager aggregation operators effectively address this situation without losing important information. As a result, the methodologies described here are more generic and successful in dealing with MADM problems.

6. Conclusion

Decision making is an essential scientific methodology that provides methods and instruments for effectively navigating such intricacies and uncertainties when confronted with unsuitable data and challenging-to-solve issues. A systematic DM presented in this article is utilized to choose the most appropriate target for JWST space observation. The utilization of recently proposed CSF Yager aggregation operators in this study helps to enhance the scientific outcomes, maximize the telescope's performance, save significant time, and ensure that the collected data is suitable for the intended analysis in comparison to other techniques. In this research article, the notions of CSFYWA and CSFYWG operators have been established. We have formulated a novel score function for CSFNs that surpasses current methods, pinpointing optimal alternatives more effectively. We have also explored operational laws within the CSF environment and established the structural properties of the newly defined CSFY aggregation operators.

Expanding our contributions, we have designed robust MADM methodologies grounded in our proposed AOs and score function. We have successfully applied these techniques to identify the most suitable astronomical object for observation in space by JSWT, thereby showcasing the validity and superior performance of our proposed approaches compared to the existing methods.

6.1. Limitations of the current study

Although the methodologies designed in this article have many advantages, they still have a few limitations:

1. These techniques cannot deal with problems where the square sum of membership, neutral, and non-membership degrees exceeds 1.
2. The absence of an intrinsic dynamic adjustment mechanism to mitigate MCDM issues renders this research potentially inadequate for scenarios involving data collection at different time intervals.

6.2. Potential future research directions of the current study

We will expand the scope of the current study to more generalized environments, specifically, complex interval-valued SF, CS dynamic fuzzy, and Ct-SF environments, to rectify the above-mentioned limitations in the provided scheme. In addition, we will intend to enhance the versatility and applicability of our model in diverse decision-making scenarios, such as decision support systems in health care, cybersecurity, IOT network optimization, natural disaster response planning, and environmental impact assessment.

Data availability

No data were used to support this study.

CRediT authorship contribution statement

Asima Razzaque: Writing – review & editing, Funding acquisition, Data curation. **Masfa Nasrullah Ansari:** Writing – original draft, Investigation, Conceptualization. **Dilshad Alghazzawi:** Writing – review & editing, Validation, Formal analysis. **Hamiden Abd El-Wahed Khalifa:** Validation, Methodology, Formal analysis. **Alhanouf Alburaikan:** Investigation, Formal analysis, Data curation. **Abdul Razaq:** Validation, Supervision, Methodology.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [KFU241221].

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