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Synchronization in output-coupled temporal Boolean networks

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This paper presents an analytical study of synchronization in an array of output-coupled temporal Boolean networks. A temporal Boolean network (TBN) is a logical dynamic system developed to model Boolean networks with regulatory delays. Both state delay and output delay are considered, and these two delays are assumed to be different. By referring to the algebraic representations of logical dynamics and using the semi-tensor product of matrices, the output-coupled TBNs are firstly converted into a discrete-time algebraic evolution system, and then the relationship between the states of coupled TBNs and the initial state sequence is obtained. Then, some necessary and sufficient conditions are derived for the synchronization of an array of TBNs with an arbitrary given initial state sequence. Two numerical examples including one epigenetic model are finally given to illustrate the obtained results.

A central focus of modern biology concerns understanding the function of a cell, the manner how a cell performs its function and the ways in which cellular systems fail in disease. With the rapid development of systems biology[1,2], genetic regulatory networks have recently become a newly developing area and have attracted much attention due to its close relationship with neural networks, protein webs and other biological systems[3–7]. A genetic regulatory network contains a group of DNA elements in a cell which interact with others directly or indirectly through their mRNA or protein products. Thus, a genetic regulatory network governs the expression levels of the mRNA and the protein products. With time going, various types of genetic regulatory networks have been proposed, such as Markov-type genetic networks[8] and Boolean networks[1]. Each of these models has its own advantages and limitations in terms of the following considerations: the accuracy of the approach relate to the level of genetic, biological phenomenon being modelled; the experimental feasibility of model construction; and the computational complexity of the model inference from the available data.

The concept of Boolean network (BN) was firstly proposed by Kauffman[1] as a simplified model of the behavior of large networks of randomly interconnected binary (on-off) genes. One of the main advantages of Boolean networks is the comprehensibility owing to the simplicity of the representation. A BN can effectively model a genetic regulatory network with its gene products (the outputs) and the substances from outside or the environment that can affect the gene (the inputs). In a BN, the state of each node has only two unique states: 1 (on) or 0 (off). For a gene, "on" corresponds to the gene being expressed; For inputs and outputs, "on" corresponds to the relative substances being present. Moreover, each node evolves its state according to a Boolean function, which is a logical function, at each discrete time point.

Since the BNs can provide a general description of the behavior of many living organisms, they have attracted great attention from researchers in different fields. There are various examples of genetic regulatory networks modeled using BNs, such as the yeast cell-cycle networks[9]; the control of the mammalian cell cycle[10]; the yeast transcriptional networks[11]; the network predicting the expression of the segment polarity genes in the fly Drosophila melanogaster[12], and so on. Recently, a new method called semi-tensor product (STP) of matrices has been firstly proposed by Cheng and his colleagues[13,14]. By using the method of STP, a Boolean function can be uniquely expressed as an algebraic expression, and hence a BN can be expressed as a standard discrete-time algebraic system. This method has been proved to be quite useful to study BNs concerning the topological structure, such as the fixed points, cycles, basin of attractors and the transient time. Based on this approach, many interesting results have been obtained in the last few years, such as stabilization of BNs with impulsive effects[15], optimal control of logical control networks[16], etc.

A curious ability of some real-world systems is that they can evolve in perfect synchronization. In the past decades, the phenomenon of collective behavior has inspired a large amount of research, such as synchronization analysis and control of complex networks[17–21], consensus in multi-agent systems[22,23], synchronization of

Kauffman networks[24], and cooperation of networks[25–27], etc. It was demonstrated that the nodes in a second-order Kuramoto model perform a cascade of transitions toward a synchronous macroscopic state[28]. A time delay, which is a source of instability, exists in various engineering systems such as chemical processes, or long transmission lines in pneumatic systems. Some interesting results have also been obtained concerning the effect of time-delay on network synchronization[29–31]. A processing coupling-delay is effectively used to annihilate the quenching of oscillation in a network of coupled oscillators by switching the stability of amplitude death[32]. Recently, interests have also been extended to synchronization of BNs, due to their potential applications in biology, chemistry, as well as engineering[33,34].

Synchronization and controllability are structural properties of a system, and they are fundamental concepts in control theory. Recently, synchronization of two delay-free deterministic BNs and synchronization in an array of coupled BNs have been respectively studied in[33,34] and the synchronization of multi-valued logical networks have been studied in[35]. In[36], we investigated the synchronization in an array of BNs with time delay. It is well known that time delay is very common in real world and also is inevitable in biological systems and engineering problems. Moreover, in many situations, time delay can not be ignored in practice, since it may heavily affect the dynamics of the networks[37,38]. One interesting kind of BNs, called temporal Boolean networks (TBNs) were developed to model regulatory delays in[39]. TBNs own a more complex structure when the time delays in Boolean variables are allowed to be different. TBNs can approximate real cellular regulatory networks or genetic networks better than BNs. Hence, it is desirable to study synchronization of TBNs. In the past few years, many results about TBNs have been obtained, such as output controllability of temporal Boolean control networks (TBCNs) and complete synchronization of TBNs[40]. However, there has been no results investigating synchronization in an array of output-coupled TBNs, to the best of our knowledge. In[40], the authors studied the synchronization of two coupled TBNs in a drive-response configuration. Sometimes, in the real world, we need to judge whether an array of networks can be synchronized or not. The output coupling phenomenon is ubiquitous in many biological networks and physical systems. For instance, a cellular system contains many different components to synthesis macromolecular in cells. In an idealized model of a metabolic feed-back control cycle, each unit component consists of three parts: a genetic locus, a ribosome and a cellular locus. Each of these three parts interacts with each other by its corresponding products, such as mRNA, protein and metabolic species. Furthermore, different components interacts with each other by its corresponding metabolic products, which is a type of output coupling. Thus, investigating synchronization in an array of output-coupled TBNs is meaningful and challenging, and can be helpful to disclose the control mechanisms for macromolecular synthesis and the behavior of a cell.

Motivated by the above discussions, in this paper, we focus on synchronization in an array of output-coupled TBNs. Our main objective is to obtain a general analytical approach to study synchronization in output-coupled TBNs. Both state delay and output delay are considered in the output-coupled TBNs. The main results are derived by using the method of semi-tensor product of matrices which can convert a deterministic BN into a unique algebraic representation. By using the algebraic representations of TBNs, we derive some necessary and sufficient conditions for the synchronization in an array of TBNs.

Some basic notations: The standard notations will be used in this paper. Throughout this paper, the notation $\mathbf{1}_k$ denotes the k-dimensional column vector with all entries equalling to 1. Denote the set of two logical variables by $\mathcal{D} = \{1,0\}$, where $1 \sim T$ means "true" and $0 \sim F$ means "false". A logical variable u will take a value from $\mathcal{D}$, which is expressed as $u \in \mathcal{D}$. Define a delta set as

$\Delta_n := \{\delta_n^i | i = 1,2,\ldots,n\}$, where δ_n^i is the ith column of the identity matrix I_n. We denote the ith column of matrix A by $\text{Col}_i(A)$ and denote the set of columns of matrix A by $\text{Col}(A)$. We also denote the ith row of matrix A by $\text{Row}_i(A)$ and denote the set of rows of matrix A by $\text{Row}(A)$. An $n \times m$ matrix L is called a logical matrix if the columns of L, denoted by $\text{Col}(L)$, are in the form of δ_n^k. That is to say, $\text{Col}(L) \subseteq \Delta_n$. Denote the set of $n \times s$ logical matrices by $\mathcal{L}_{n \times s}$. If $L \in L_{n \times s}$, it can be expressed as $L = [\delta_n^{i_1}, \delta_n^{i_2}, \ldots, \delta_n^{i_s}]$. For simplicity, it can be denoted as $L = \delta_n[i_1, \ldots, i_s]$. We use $\otimes_{i=1}^M X_i(t)$ to denote $X_1(t) \otimes X_2(t) \otimes \cdots \otimes X_M(t)$, and use $\ltimes_{j=1}^N y_i(t)$ to denote $y_1(t) \ltimes y_2(t) \ltimes \cdots \ltimes y_N(t)$.

Results and Methods

In this paper, we study the following array of M output-coupled BNs, with each one being an N-node system:

$$
\begin{cases}
X_j^i(t+1) = f_j^i\Big(X_j^1(t),\ldots,X_j^N(t),X_j^1(t-1),\ldots,X_j^N(t-1),\ldots, \\
\qquad\quad X_j^1(t-\tau_1),\ldots,X_j^N(t-\tau_1),y_1(t),\ldots,y_M(t)\Big), \\
\qquad\quad i = 1,2,\ldots,N \\
y_j(t) = g_j\Big(X_j^1(t),\ldots,X_j^N(t),X_j^1(t-1),\ldots,X_j^N(t-1),\ldots, \\
\qquad\quad X_j^1(t-\tau_2+1),\ldots,X_j^N(t-\tau_2+1),X_j^1(t-\tau_2),\ldots, \\
\qquad\quad X_j^N(t-\tau_2)\Big), j = 1,2,\ldots,M
\end{cases}
\tag{1}
$$

where X_j^i is the ith node of the jth TBN, y_j is the binary output, $f_j^i : \{1,0\}^{(\tau_1+1)N+M} \to \{1,0\}$, $g_j : \{1,0\}^{(\tau_2+1)N} \to \{1,0\}$ are Boolean functions, and τ_1, τ_2 are positive integers describing the state delay and the output delay. In many biological networks and genetic regulatory networks, time delay is ubiquitous due to finite signal propagation time, finite chemical reaction times, memory effects, and so on. The time delay τ_1 describes delay occurring between states, while time delay τ_2 describes delay resulting from output. For example, in a genetic regulatory networks, the level of a gene maybe affected by its past μ gene levels. Different genes interact by its corresponding gene products, which may cost time to be transmitted from one gene to another one. Thus, model (1) can be an efficient model to describe some biological networks or genetic regulatory networks with state delay and output delay. We simply denote $X_j(t) = \Big(X_j^1(t), X_j^2(t), \ldots, X_j^N(t)\Big)^T$ the states of the TBNs (1) at time instant t. We observe that the state evolution of the output-coupled TBN (1) depends on the initial state sequences, and there are two cases as follows. Case (I): $\tau_1 \geq \tau_2$, the initial state sequence is $X_j(0)$, $X_j(-1), \ldots, X_j(-\tau_1)$ for $j = 1, 2, \ldots, M$; case (II): $\tau_1 < \tau_2$, the initial state sequence is $X_j(0), X_j(-1), \ldots, X_j(-\tau_2)$ for $j = 1, 2, \ldots, M$. Now, we define complete synchronization in the array of TBNs (1) as follows:

Definition 1 *The array of TBNs are said to be synchronized if for every initial state sequence, there exists a positive integer k, such that $t \geq k$ implies $X_i(t) = X_j(t)$ for every distinct $1 \leq i, j \leq M$.*

Remark 1 *In this definition, the value of k depends on the initial state sequences. Nevertheless, since the $\{1,0\}^{(\tau_1+1)N}$ (or $\{1,0\}^{(\tau_2+1)N}$) is a finite set, we can always choose a constant k large enough which is independent of the initial states sequence.*

In the following, we establish some necessary and sufficient criteria for synchronization in an array of output-coupled TBNs. To acquire a strict analysis, we ought to apply algebraic representations of the TBNs according to the STP theory.

Let $X_j(t) = X_j^1(t) \ltimes X_j^2(t) \ltimes \cdots \ltimes X_j^N(t)$ for $j = 1, \ldots, M$ and $y(t) = y_1(t) \ltimes y_2(t) \ltimes \cdots \ltimes y_M(t)$. Let F_j^i and G_j be the structure matrix

of f_j^i and g_j. Then we can make $A_j(t,\tau_2) = X_j(t)X_j(t-1)\cdots X_j(t-\tau_2)$ for $j = 1, 2, \ldots, M$ to obtain the algebraic representations of $y(t)$. By equation (1) and the algebraic representations of Boolean functions, we have

$$y_j(t) = G_j X_j(t) X_j(t-1) \cdots X_j(t-\tau_2) = G_j A_j(t,\tau_2). \quad (2)$$

Hence, we get $y(t) = \left(\otimes_{j=1}^M G_j\right) \ltimes_{j=1}^M A_j(t,\tau_2) \triangleq G \ltimes_{j=1}^M A_j(t,\tau_2)$, where $G = \otimes_{j=1}^M G_j$. To obtain the accurate algebraic expressions of $\ltimes_{j=1}^M A_j(t,\tau_2)$, we also make $M_j(t,\tau_2) = X_j(t-1)\cdots X_j(t-\tau_2)$ to facilitate the expression. Then we have

$$\begin{aligned}
\ltimes_{j=1}^M A_j(t) &= X_1(t)X_1(t-1)\cdots X_1(t-\tau_2)X_2(t)\cdots X_2(t-\tau_2)\\
&\quad \cdots X_M(t)\cdots X_M(t-\tau_2)\\
&= X_1(t)M_1(t,\tau_2)X_2(t)M_2(t,\tau_2)X_3(t)M_3(t,\tau_2)\\
&\quad \cdots X_M(t)M_M(t,\tau_2)\\
&= X_1(t)W_{[2^N,2^{\tau_2 N}]}X_2(t)M_1(t,\tau_2)M_2(t,\tau_2)X_3(t)\\
&\quad M_3(t,\tau_2)\cdots X_M(t)M_M(t,\tau_2)\\
&= \left(I_{2^N}\otimes W_{[2^N,2^{\tau_2 N}]}\right)X_1(t)X_2(t)M_1(t,\tau_2)M_2(t,\tau_2)\\
&\quad X_3(t)M_3(t,\tau_2)\cdots X_M(t)M_M(t,\tau_2)\\
&\quad \vdots\\
&= \ltimes_{i=1}^{M-1}\left(I_{2^{iN}}\otimes W_{[2^N,2^{i\tau_2 N}]}\right)\left(\ltimes_{j=1}^M X_j(t)\right)M_1(t,\tau_2)\\
&\quad M_2(t,\tau_2)\cdots M_M(t,\tau_2)\\
&\triangleq W_0\left(\ltimes_{j=1}^M X_j(t)\right)M_1(t,\tau_2)M_2(t,\tau_2)\cdots M_M(t,\tau_2)\\
&= W_0\left(\ltimes_{j=1}^M X_j(t)\right)X_1(t-1)\cdots X_1(t-\tau_2)\cdots X_M(t-1)\\
&\quad \cdots X_M(t-\tau_2),
\end{aligned} \quad (3)$$

where $W_0 = \ltimes_{i=1}^{M-1}\left(I_{2^{iN}}\otimes W_{[2^N,2^{i\tau_2 N}]}\right)$. By induction, we yield

$$\ltimes_{j=1}^M A_j(t,\tau_2) \triangleq \widehat{W}_1 \ltimes_{j=1}^M X_j(t)\ltimes_{j=1}^M X_j(t-1)\cdots \ltimes_{j=1}^M X_j(t-\tau_2), \quad (4)$$

where $\widehat{W}_1 = W_0 \ltimes_{j=1}^{\tau_2-1}\left(I_{2^{jNM}}\otimes W_j\right)$, $W_j = \ltimes_{i=1}^{M-1}\left(I_{2^{iN}}\otimes W_{[2^N,2^{iN(\tau_2-j)}]}\right)$, $j = 1, \ldots, \tau_2 - 1$. Hence, we have $y(t) = G\widehat{W}_1 \ltimes_{j=1}^M X_j(t)\ltimes_{j=1}^M X_j(t-1)\cdots \ltimes_{j=1}^M X_j(t-\tau_2)$.

Now, we try to obtain the algebraic representations of the states $X_j(t+1)$. By equation (1) and algebraic representations of Boolean functions, we have $X_j^i(t+1) = F_j^i X_j(t)X_j(t-1)\cdots X_j(t-\tau_1)y(t)$. By denoting $C(t,\tau_1) = X_j(t)X_j(t-1)\cdots X_j(t-\tau_1)y(t)$ for simplicity, we then get

$$\begin{aligned}
X_j(t+1) &= F_j^1 C(t,\tau_1)F_j^2 C(t,\tau_1)\cdots F_j^N C(t,\tau_1)\\
&= F_j^1\left(I_{2^{(\tau_1+1)N+M}}\otimes F_j^2\right)\Phi_{(\tau_1+1)N+M}C(t,\tau_1)\cdots F_j^N C(t,\tau_1) (5)\\
&\triangleq F_j C(t,\tau_1) = F_j X_j(t)X_j(t-1)\cdots X_j(t-\tau_1)y(t).
\end{aligned}$$

where $F_j = F_j^1 \ltimes_{j=2}^N\left\{\left(I_{2^{(\tau_1+1)N+M}}\otimes F_j^i\right)\Phi_{(\tau_1+1)N+M}\right\}$.

That is to say, we have obtained the following equivalent algebraic representations of output-coupled TBNs (1):

$$\begin{cases}
X_j(t+1) = F_j X_j(t)X_j(t-1)\cdots X_j(t-\tau_1)y(t)\\
y(t) = G\widehat{W}_1 \ltimes_{j=1}^M X_j(t)\ltimes_{j=1}^M X_j(t-1)\cdots \ltimes_{j=1}^M X_j(t-\tau_2)
\end{cases} \quad (6)$$

where F_j is an $2^N \times 2^{(\tau_1+1)N+M}$ matrix, G is an $2^M \times 2^{(\tau_2+1)NM}$ matrix and $\widehat{W}_1$ is an $2^{(\tau_2+1)NM} \times 2^{(\tau_2+1)NM}$ matrix.

From equation (6), it is easy to see that the state of each TBN is determined by its delayed states and the output $y(t)$, which means these TBNs are coupled through the output $y(t)$. Let $\ltimes_{j=1}^M X_j(t) = X_1(t)\ltimes X_2(t)\ltimes\cdots\ltimes X_M(t)$, using equation (6) and mathematical iteration, we can derive the relationship between $\ltimes_{j=1}^M X_j(t)$ and corresponding initial state sequence.

Before we derive the necessary and sufficient conditions for synchronization, we need the following proposition. From this proposition, we can acquire the relationship between the state $X_j(t)$ at time t and the initial state sequence which will be employed to obtain the criteria for synchronization of TBNs.

Proposition 1 Let $F = \otimes_{j=1}^M F_j$, $W = W_{\left[2^M, 2^{(\tau_1+1)N}\right]}\ltimes_{i=2}^M$ $\left\{\left(I_{2^M}\otimes W_{\left[2^M,2^{i(\tau_1+1)N}\right]}\right)\Phi_M\right\}$ and $\widehat{W}_2 = W'_0\ltimes_{j=1}^{\tau_1-1}\left(I_{2^{jMN}}\otimes W'_j\right)$, where $W'_j = \ltimes_{i=1}^{M-1}\left(I_{2^{iN}}\otimes W_{\left[2^N,2^{i(\tau_1-j)N}\right]}\right)$, $j = 0, 1, \ldots, \tau_1 - 1$. Then we have

$$\begin{aligned}
\ltimes_{j=1}^M X_j(t+1) &= FWG\widehat{W}_1\left(I_{2^{(\tau_2+1)NM}}\otimes \widehat{W}_2\right)\ltimes_{j=1}^M X_j(t)\ltimes_{j=1}^M X_j(t-1)\\
&\quad \cdots \ltimes_{j=1}^M X_j(t-\tau_2)\ltimes_{j=1}^M X_j(t)\ltimes_{j=1}^M X_j(t-1) \quad (7)\\
&\quad \cdots \ltimes_{j=1}^M X_j(t-\tau_1).
\end{aligned}$$

Remark 2 *According to equation (7), we need to know whether τ_1 is larger than τ_2 or not by considering these two cases: Case (I): $\tau_1 \geq \tau_2$, which means that the initial state sequence is $X_j(0), \ldots, X_j(-\tau_1)$ for $j = 1, 2, \ldots, M$; Case (II): $\tau_1 < \tau_2$, which implies the initial state sequence is $X_j(0), \ldots, X_j(-\tau_2)$ for $j = 1, 2, \ldots, M$. In the following sequel, according to different case, we derive some necessary and sufficient conditions for synchronization in an array of output-coupled TBNs. We firstly consider Case (I), and then Case (II).*

Case I: $\tau_1 \geq \tau_2$. Now let us first consider Case (I): $\tau_1 \geq \tau_2$, with the initial state $X_j(0), \ldots, X_j(-\tau_1)$ for $j = 1, 2, \ldots, M$. Since τ_1 and τ_2 describe the state delay and the output delay, in this case, it means the state delay is not smaller than the output delay. In a cellular system, it means the delay caused during the processes of the transcription of a gene is not smaller than that caused during the processes of the translation of a gene. Based on Proposition 1, we obtain the following proposition, which is derived to show the explicit relationship between the state $X_j(t)$ and the initial states $X_j(0), \ldots, X_j(-\tau_1)$.

Proposition 2 Let $\Theta_1 = FWG\widehat{W}_1\left(I_{2^{(\tau_2+1)NM}}\otimes \widehat{W}_2\right)\Phi_{(\tau_2+1)NM}$ and $\Xi_1 = \Theta_1 W_{\left[2^{\tau_1 NM},2^{(\tau_1+1)NM}\right]}\Phi_{\tau_1 NM}$, then we have

$$\ltimes_{j=1}^M X_j(t+1) = \Theta_1 \ltimes_{j=1}^M X_j(t)\ltimes_{j=1}^M X_j(t-1)\cdots \ltimes_{j=1}^M X_j(t-\tau_1) \quad (8)$$

and

$$\ltimes_{j=1}^M X_j(t) = \left(I_{2^{NM}}\otimes 1_{2^{\tau_1 NM}}^T\right)\Xi_1^t\left\{\ltimes_{j=1}^M X_j(0)\ltimes_{j=1}^M X_j(-1)\cdots \ltimes_{j=1}^M X_j(-\tau_1)\right\}. \quad (9)$$

In the first case, i.e. $\tau_1 \geq \tau_2$, which means the state delay is larger than the output delay, we can observe the value of states at time $t+1$ depends on its past $\tau_1 + 1$ states. Then, using mathematical analysis and STP, we can obtain that the value of states at time t has a specific connection with the initial sequence, $\ltimes_{j=1}^M X_j(0), \ltimes_{j=1}^M X_j(-1), \ldots, \ltimes_{j=1}^M X_j(-\tau_1)$. The left side of equation (9) is the STP of state of each TBN at the discrete time point t, while the right side is product of several matrices and the initial state sequences. Since $\left(I_{2^{MN}}\otimes 1_{2^{\tau_1 NM}}^T\right)$ and Ξ_1 are constant matrices, we can present our necessary and

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sufficient criterion for synchronization in the array of output-coupled TBNs as following.

Theorem 1 *Let equation (6) be the algebraic representations of the array of TBNs (1). Then synchronization occurs iff there exists a positive integer k satisfying $1 \leq k \leq k_0$ such that*

$$Col\left[\left(I_{2^{NM}} \otimes 1_{2^{\tau_1 NM}}^T\right)\Xi_1^k\right] \subseteq \left\{\delta_{2^{MN}}^{\lambda_i} : \lambda_i = 1 + \frac{(i-1)(2^{MN}-1)}{2^N-1}, i=1,2,\ldots,2^N\right\} \quad (10)$$

where

$$k_0 = min\left\{i \geq 1 : \left(I_{2^{NM}} \otimes 1_{2^{\tau_1 NM}}^T\right)\Xi_1^i = \left(I_{2^{NM}} \otimes 1_{2^{\tau_1 NM}}^T\right)\Xi_1^j, j > i\right\}. \quad (11)$$

Remark 3 *In equation (11) in Theorem 1, we can use $k_0' = min\left\{i \geq 1 : \Xi_1^i = \Xi_1^j, j > i\right\}$ to simplify it but meanwhile the restriction is strengthened. By $k_0' = min\left\{i > 1 : \Xi_1^i = \Xi_1^j, j > i\right\}$, there must exist an integer $j > k_0'$ such that $\Xi_1^{k_0'} = \Xi_1^j$. Then we have $\left(I_{2^{NM}} \otimes 1_{2^{\tau_1 NM}}^T\right)\Xi_1^{k_0'} = \left(I_{2^{NM}} \otimes 1_{2^{\tau_1 NM}}^T\right)\Xi_1^j$. Hence, we can conclude that $k_0 \leq k_0'$. Equation (10) implies that after finite time steps, i.e. k_0 steps, we can judge whether the array of TBNs can be synchronized or not. If this array of TBNs can not be synchronized before the k_0-th steps, then it can not be synchronized at any time. Thus, Theorem 1 provides an efficient criterion for synchronization in an array of output-coupled TBNs.*

Remark 4 *If the coupled TBNs can be synchronized after certain times k, then we have $X_j^i(t) = X_{j'}^{i'}(t)$ for $1 \leq j \neq j' \leq M, 1 \leq i \neq i' \leq N$ and $t \geq k$. Since the set of state values is a finite set, i.e. $\left\{\delta_{2^N}^1, \delta_{2^N}^2, \ldots, \delta_{2^N}^{2^N}\right\}$, we can suppose each synchronous state to be $X_j^i = \delta_{2^N}^i, 1 \leq j \leq M, 1 \leq i \leq N$. Because of the definition of $\bowtie_{j=1}^M X_j(t) = X_1(t) \bowtie \cdots \bowtie X_M(t)$, we can obtain the synchronous set of $\bowtie_{j=1}^M X_j(t)$, which is $\left\{\delta_{2^{MN}}^{\lambda_i} : \lambda_i = 1 + \frac{(i-1)(2^{MN}-1)}{2^N-1}, i=1,2,\ldots,2^N\right\}$. According to equation (9), if we have an initial sequence, i.e. $\bowtie_{j=1}^M X_j(0)\bowtie_{j=1}^M X_j(-1) \cdots \bowtie_{j=1}^M X_j(-\tau_1) = \delta_{2^{NM(\tau_1+1)}}^i$, we can obtain $\bowtie_{j=1}^M X_j(t) = Col_i\left[\left(I_{2^{NM}} \otimes 1_{2^{\tau_1 NM}}^T\right)\Xi_1^t\right]$. Thus, $Col\left[\left(I_{2^{NM}} \otimes 1_{2^{\tau_1 NM}}^T\right)\Xi_1^t\right]$ is the total possible state value of $\bowtie_{j=1}^M X_j(t)$. In order to judge whether the coupled TBNs can be synchronized or not, we just need to calculate whether there exists a positive integer k such that all possible state value of $\bowtie_{j=1}^M X_j(k)$ are in the synchronous set $\left\{\delta_{2^{MN}}^{\lambda_i} : \lambda_i = 1 + \frac{(i-1)(2^{MN}-1)}{2^N-1}, i=1,2,\ldots,2^N\right\}$.*

As we know, Ξ_1 is a logical matrix and so is the matrix of $I_{2^{MN}} \otimes 1_{2^{\tau_1 NM}}^T$. Moreover, if $(\Xi_1)_{ij} = 1$, then we have $\left(I_{2^{NM}} \otimes 1_{2^{\tau_1 NM}}^T\right)Col_j(\Xi_1) = Col_i\left(I_{2^{NM}} \otimes 1_{2^{\tau_1 NM}}^T\right)$ which means taking the ith column of matrix $I_{2^{NM}} \otimes 1_{2^{\tau_1 NM}}^T$. Then, we can easily derive the following simplified criterion for synchronization in an array of output-coupled TBNs.

Corollary 1 *Let $\Omega_1 = \left\{(\lambda_i - 1)^{2^{\tau_1 NM}} + j : \lambda_i = 1 + \frac{(i-1)(2^{NM}-1)}{2^N-1}, i=1,2,\ldots,2^N, j=1,\ldots,2^{\tau_1 NM}\right\}$. Then the array of output-coupled TBNs (1) can be synchronized iff there exists a positive integer k such that $Row_i\left(\Xi_1^k\right) = \underbrace{(0,0,\ldots,0)}_{2^{(\tau_1+1)NM}}, i \in \left\{1,2,\ldots,2^{(\tau_1+1)NM}\right\}/\Omega_1$.*

In Proposition 2, we use the variable substitution $Z_1(t) = \bowtie_{j=1}^M X_j(t) \cdots \bowtie_{j=1}^M X_j(t-\tau_1)$ to analyze the problem of synchronization in an array of output-coupled TBNs. Based on the variable substitution, we derive the necessary and sufficient criterion for synchronization, i.e. Theorem 1. In the following sequel, we use the mathematical induction without using the variable substitution

$Z_1(t) = \bowtie_{j=1}^M X_j(t) \cdots \bowtie_{j=1}^M X_j(t-\tau_1)$ to analyze the synchronization problem. According to equation (8), we obtain another form of relationship between the state $X_j(t)$ and the initial state sequence $X_j(0)$, $X_j(-1)$, …, $X_j(-\tau_1)$ by mathematical induction.

Proposition 3 *Let Θ_1 defined as the same as in Lemma 2, then we get that*

$$\bowtie_{j=1}^M X_j(t) = \Theta_1^t L_t^{\Theta_1} \bowtie_{j=1}^M X_j(0)\bowtie_{j=1}^M X_j(-1) \cdots \bowtie_{j=1}^M X_j(-\tau_1) \quad (12)$$

where

$$L_t^{\Theta_1} = \begin{cases} I_{2^{(\tau_1+1)NM}}, & t=1, \\ L_1^{\Theta_1} W_{\left[2^{\tau_1 NM},2^{(\tau_1+1)NM}\right]}\Phi_{\tau_1 NM}, & t=2, \\ L_{t-1}^{\Theta_1}\bowtie_{j=t-2}^1\left\{\left[I_{2^{(\tau_1+1)NM}} \otimes \left(\Theta_1^j L_j^{\Theta_1}\right)\right]\Phi_{(\tau_1+1)NM}\right\} & \\ \quad W_{\left[2^{(\tau_1+2-t)NM},2^{(\tau_1+1)NM}\right]}\Phi_{(\tau_1+2-t)NM}, & 3 \leq t \leq \tau_1+1, \\ L_{t-1}^{\Theta_1}\bowtie_{j=t-2}^{t-1-\tau_1}\left\{\left[I_{2^{(\tau_1+1)NM}} \otimes \left(\Theta_1^j L_j^{\Theta_1}\right)\right]\Phi_{(\tau_1+1)NM}\right\}, & t \geq \tau_1+2. \end{cases} \quad (13)$$

By Proposition 3, we then present another necessary and sufficient synchronization criterion for the array of TBNs.

Theorem 2 *The array of TBNs can be synchronized iff there exists a positive integer k such that*

$$Col\left(\Theta_1^k L_k^{\Theta_1}\right) \subseteq \left\{\delta_{2^{MN}}^{\lambda_i} : \lambda_i = 1 + \frac{(i-1)(2^{MN}-1)}{2^N-1}, i=1,2,\ldots,2^N\right\} \quad (14)$$

where $L_k^{\Theta_1}$ is defined in Proposition 3.

Remark 5 *According to equation (12), if we have an initial sequence, i.e. $\bowtie_{j=1}^M X_j(0)\bowtie_{j=1}^M X_j(-1) \cdots \bowtie_{j=1}^M X_j(-\tau_1) = \delta_{2^{NM(\tau_1+1)}}^i$, we can obtain that the value of states at time k is $Col_i\left[\Theta_1^k L_k^{\Theta_1}\right]$. Therefore, $Col\left[\Theta_1^k L_k^{\Theta_1}\right]$ is the total possible state values at time t. In order to judge whether the coupled TBNs can be synchronized or not, we just need to know whether there exists a positive integer k such that the possible state values are in the synchronous set $\left\{\delta_{2^{MN}}^{\lambda_i} : \lambda_i = 1 + \frac{(i-1)(2^{MN}-1)}{2^N-1}, i=1,2,\ldots,2^N\right\}$.*

Case II: $\tau_1 < \tau_2$. Now we turn to case (II): $\tau_1 < \tau_2$, where the initial state sequence is $X_j(0), X_j(-1), \ldots, X_j(-\tau_2)$ for $j = 1, 2, \ldots, M$. In this case, it means the state delay is smaller than the output delay. The following proposition is firstly presented to explicitly express the relationship between the current state and the initial state sequence.

Proposition 4 *Let $\Theta_2 = FW G \widehat{W_1}\left(I_{2^{(\tau_2+1)NM}} \otimes \widehat{W_2}\right)W_{\left[2^{(\tau_1+1)NM},2^{(\tau_2+1)NM}\right]}\Phi_{(\tau_1+1)NM}$ and $\Xi_2 = \Theta_2 W_{\left[2^{\tau_2 NM},2^{(\tau_2+1)NM}\right]}$, then we have*

$$\bowtie_{j=1}^M X_j(t+1) = \Theta_2\bowtie_{j=1}^M X_j(t)\bowtie_{j=1}^M X_j(t-1) \cdots \bowtie_{j=1}^M X_j(t-\tau_2) \quad (15)$$

and

$$\bowtie_{j=1}^M X_j(t) = \left(I_{2^{NM}} \otimes 1_{2^{\tau_2 NM}}^T\right)\Xi_2^t\left\{\bowtie_{j=1}^M X_j(0)\bowtie_{j=1}^M X_j(-1) \cdots \bowtie_{j=1}^M X_j(-\tau_2)\right\}. \quad (16)$$

In this case ($\tau_1 < \tau_2$), the initial state sequence is $X_j(0), \ldots, X_j(-\tau_2)$ for $j = 1, 2, \ldots, M$. Thus, the STP of state of each TBN is determined by the initial state sequence and a transition matrix $\left(I_{2^{MN}} \otimes 1_{2^{\tau_2 NM}}^T\right)\Xi_2^t$. Now we can present the following synchronization criterion which is similar with Theorem 1.

Theorem 3 *Let equation (6) be the algebraic representations of the array of TBNs (1). Then synchronization occurs iff there exists a positive integer k satisfying $1 \leq k \leq k_0$ such that*

$$Col\left[\left(I_{2^{NM}} \otimes 1_{2^{\tau_2 NM}}^T\right)\Xi_2^k\right] \subseteq \left\{\delta_{2^{MN}}^{\lambda_i} : \lambda_i = 1 + \frac{(i-1)(2^{MN}-1)}{2^N-1}, i=1,2,\ldots,2^N\right\} \quad (17)$$
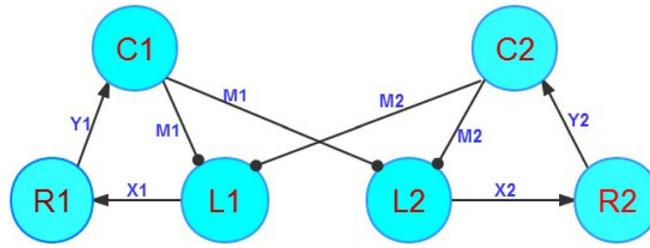
where

Figure 1 | Boolean model of a coupled macromolecular synthesis in a cell with each unit consisting of a genetic locus (L_1 or L_2), a cellular structure (R_1 or R_2) and a cellular locus (C_1 or C_2). Arrows correspond to positive interactions, circles to negative ones.

$$k_0 = min\left\{i : i \geq 1, \left(I_{2^{NM}} \otimes 1^T_{2^{\tau_2 NM}}\right)\Xi_2^i = \left(I_{2^{NM}} \otimes 1^T_{2^{\tau_2 NM}}\right)\Xi_2^j \text{ for some } j > i\right\}. \quad (18)$$

Similar to Theorem 1, we can also obtain the same synchronous set,

which is $\left\{\delta_{2^{MN}}^{\lambda_i} : \lambda_i = 1 + \dfrac{(i-1)(2^{MN}-1)}{2^N-1}, i = 1, 2, \ldots, 2^N\right\}$.

According to equation (16), we can obtain the total possible state values as $Col\left[\left(I_{2^{NM}} \otimes 1^T_{2^{\tau_2 NM}}\right)\Xi_2^k\right]$. As long as there exists a positive integer k such that the possible state values at time k are in the synchronous set, then we can draw a conclusion that the coupled TBNs can be synchronized.

Remark 6 *In equation (18) in Theorem 3, we can also use* $k'_0 = min\left\{i \geq 1 : \Xi_2^i = \Xi_2^j, j > i\right\}$ *to simplify it but meanwhile the restriction is strengthened. And we can also get $k_0 \leq k'_0$. Equation (17) implies that after finite time steps, i.e. k_0 steps, we can judge whether the array of TBNs can be synchronized or not. If after k_0 steps, this array of TBNs still can not be synchronized, then it can not be synchronized at any time. Thus, Theorem 3 also provides an efficient criterion for synchronization in an array of output-coupled TBNs.*

Since $\left(I_{2^{NM}} \otimes 1^T_{2^{\tau_2 NM}}\right)$ and Ξ_2 are logical matrices, we can derive the following simplified criterion for synchronization in an array of output-coupled TBNs.

Corollary 2 *Let* $\Omega_2 = \left\{(\lambda_i - 1)2^{\tau_2 NM} + j : \lambda_i = 1 + \dfrac{(i-1)(2^{NM}-1)}{2^N-1},\right.$
$\left. i = 1, 2, \ldots, 2^N, j = 1, \ldots, 2^{\tau_1 NM}\right\}$. *Then the array of temporal Boolean networks can be synchronized iff there exists a positive integer k such that* $Row_i\left(\Xi_2^k\right) = \underbrace{(0, 0, \ldots, 0)}_{2^{(\tau_2+1)NM}}, i \in \left\{1, 2, \ldots, 2^{(\tau_2+1)NM}\right\}/\Omega_2$.

Similar to Proposition 3, we can also use mathematical induction to obtain following proposition which implies the relationship between the current state and the initial state sequence by using Proposition 4.

Proposition 5 *Let Θ_2 be defined the same as in Proposition 4, then we can obtain that*

$$\ltimes_{j=1}^M X_j(t) = \Theta_2^t L_t^{\Theta_2} \ltimes_{j=1}^M X_j(0) \ltimes_{j=1}^M X_j(-1) \ldots \ltimes_{j=1}^M X_j(-\tau_2) \quad (19)$$

where

$$L_t^{\Theta_2} = \begin{cases} I_{2^{(\tau_2+1)NM}}, & t = 1, \\ L_1^{\Theta_2} W_{\left[2^{\tau_2 NM}, 2^{(\tau_2+1)NM}\right]} \Phi_{\tau_2 NM}, & t = 2, \\ L_{t-1}^{\Theta_2} \ltimes_{j=t-2}^1 \left\{\left[I_{2^{(\tau_2+1)NM}} \otimes \left(\Theta_2^j L_j^{\Theta_2}\right)\right] \Phi_{(\tau_2+1)NM}\right\} & \\ \quad W_{\left[2^{(\tau_2+2-t)NM}, 2^{(\tau_2+1)NM}\right]} \Phi_{(\tau_2+2-t)NM}, & 3 \leq t \leq \tau_2 + 1, \\ L_{t-1}^{\Theta_2} \ltimes_{j=t-2}^{t-1-\tau_2} \left\{\left[I_{2^{(\tau_2+1)NM}} \otimes \left(\Theta_2^j L_j^{\Theta_2}\right)\right] \Phi_{(\tau_2+1)NM}\right\}, & t \geq \tau_2 + 2. \end{cases} \quad (20)$$

According to Proposition 5, we can obtain following synchronization criterion to judge the array of output-coupled TBNs can be synchronized or not, which is similar to Theorem 2.

Theorem 4 *The array of TBNs can be synchronized iff there exists a positive integer k such that*

$$Col\left(\Theta_2^k L_k^{\Theta_2}\right) \subseteq \left\{\delta_{2^{MN}}^{\lambda_i} : \lambda_i = 1 + \frac{(i-1)(2^{MN}-1)}{2^N-1}, i = 1, 2, \ldots, 2^N\right\} \quad (21)$$

where $L_k^{\Theta_2}$ is defined in Proposition 5.

According to equation (19), we obtain the total possible state values are $Col\left(\Theta_2^k L_k^{\Theta_2}\right)$. Following similar discussions in Remark 5, the coupled TBNs can be synchronized if and only if there exists an integer k such that the possible state values are in the synchronization set.

Numerical Simulation

In this section, we present two numerical examples to illustrate our main theoretical results, which include a practical epigenetic example.

Example 1 *Let us first consider the BN which is used to model a molecular biological system[41]. It is an epigenetic model proposed by Goodwin[42] of control mechanisms for the macromolecular synthesis in cells including: genetic locus L_i, cellular structure R_i (ribosome) and cellular locus C_i, $i = 1, 2$. As is shown in Fig. 1, L_i synthesises mRNA (X_i) to the cellular structure R_i, where its activity results in the synthesis of a protein, denoted by Y_i. The protein then travels to some cellular locus C_i. A fraction of a metabolic species M_i generated by the protein returns to the genetic locus L_i. Moreover, a reciprocal interaction occurs from L_2 to L_1. Based on the analysis in[41], the variables and Boolean functions of the model are given in Table I.*

Let $f_1 = (f_1^1, f_1^2, f_1^3) = (\neg x_1^3 \wedge y_2, x_1^1, x_1^2)$, $f_2 = (f_2^1, f_2^2, f_2^3) = (\neg x_2^3 \wedge y_1, x_2^1, x_2^2)$, $g_1 = \neg x_1^3$, $g_2 = \neg x_2^3$. During the process of synthesis in cells, several environmental conditions such as temperature, growth rate or concentration of nutrition, may cause a time delay of state and output. Thus, it is reasonable to use TBNs in form (1) to model the network. Meanwhile, we can use the following case to illustrate the impacts of time delay occurred in the process of synthesis in cells towards the synchronization problem. Put $x_j(t) = x_j^1(t) \ltimes x_j^2(t) \ltimes x_j^3(t)$ for $j = 1, 2$, and $y(t) = y_1(t) \ltimes y_2(t)$. In the TBNs, we let the initial state sequences be $x_1(-1) = (1, 1, 1)$, $x_2(-1) = (0, 0, 0)$, $x_1(0) = (0, 0, 0)$, $x_2(0) = (1, 1, 1)$.

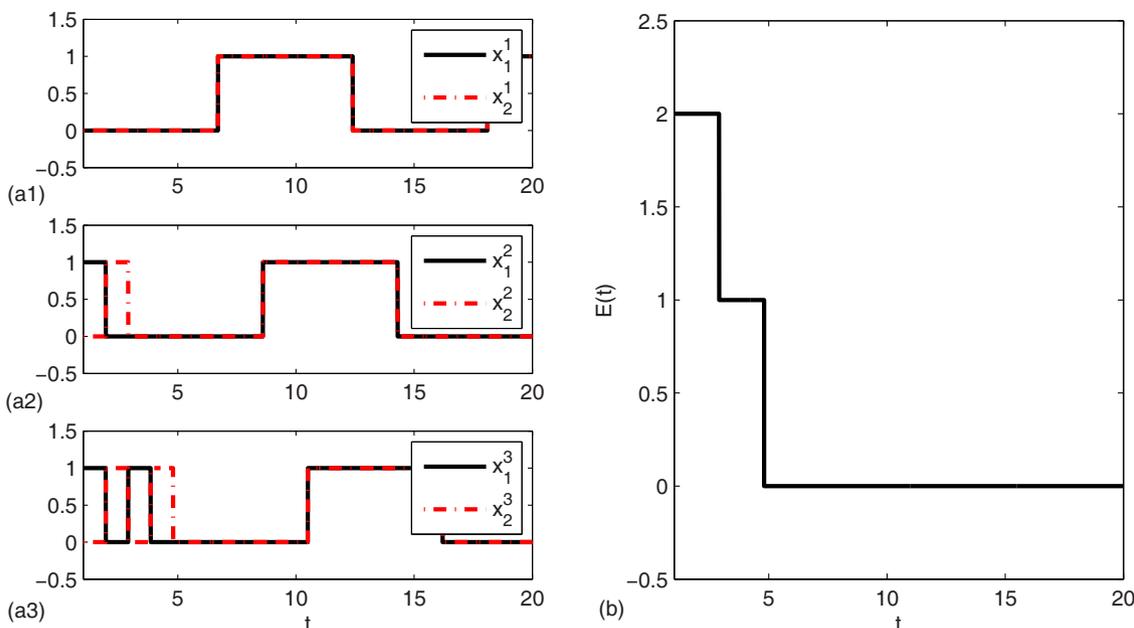*Consider: $\tau_1 = \tau_2 = 1$ which means that the state time delay is equal to the output time delay.*

Table I | Boolean functions of the Boolean model of coupled macromolecular synthesis in cells

variable	Boolean function
$x_1^1 = $ genetic locus L_1	$\neg x_1^3 \wedge y_2$
$x_1^2 = $ ribosome R_1	x_1^1
$x_1^3 = $ cellular locus C_1	x_1^2
$x_2^1 = $ genetic locus L_2	$\neg x_2^3 \wedge y_1$
$x_2^2 = $ ribosome R_2	x_2^1
$x_2^3 = $ cellular locus C_2	x_2^1

Figure 2 | (a): Time evolution of TBNs (22) with the given initial state sequences. The black solid line presents the state evolution of x_1, while the red dash line presents the state evolution of x_2. (b): Total synchronization error $E(t)$ of the coupled BNs (22) with the given initial state sequences.

By using the STP of matrices, we also obtain the following algebraic representation:

$$\begin{cases} x_1(t+1) = Fx_1(t-1)y_2(t) \\ x_2(t+1) = Fx_2(t-1)y_1(t) \\ y_1(t) = Gx_1(t-1) \\ y_2(t) = Gx_2(t-1) \end{cases} \tag{22}$$

where $F = \delta_8[5, 5, 1, 5, 6, 6, 2, 6, 7, 7, 3, 7, 8, 8, 4, 8]$ and $G = \delta_8[2, 1, 2, 1, 2, 1, 2, 1]$.

In order to see whether these two TBNs can be synchronized or not, we should calculate Ξ_1^t according to Theorem 1. Direct computation gives that $\Xi_1^5 = \Xi_1^{17}$, then we have $k'_0 = 5$. According to Remark 3, we have $k_0 \leq k'_0 = 5$. Moreover, we obtain that $\left(I_{2^6} \otimes 1_{2^6}^T\right)\Xi_1^5 \subseteq \left\{\delta_{64}^i | i = 1, 10, 19, 28, 37, 46, 55, 64\right\}$. Hence, the TBNs can be synchro-

nized by Theorem 1. Define the total synchronization error: $E(t) = \sum_{i=1}^3 \left|x_1^i(t) - x_2^i(t)\right|$. Fig. 2(a) shows the time evolution of the TBNs and Fig. 2(b) shows the total synchronization error $E(t)$. By Proposition 2, we also get Ξ_1. Due to the complexity of this matrix, we just present the indexes of the first 50 columns as follows:

$$\begin{aligned} \Xi_1 = \delta_{2^{12}}[&2305, 2305, 2369, 2369, 2433, 2433, 2497, 2497, \\ &65, 2433, 129, 2497, 193, 2817, 2817, 2881, 2881, \\ &2945, 2945, 3009, 3009, 2817, 513, 2881, 577, 2945, \\ &641, 3009, 705, 3329, 3329, 3393, 3393, 3457, 3457, \\ &3521, 3521, 3329, 1025, 3393, 1089, 3457, 1153, \\ &3521, 1217, 3841, 3841, \ldots] \end{aligned} \tag{23}$$
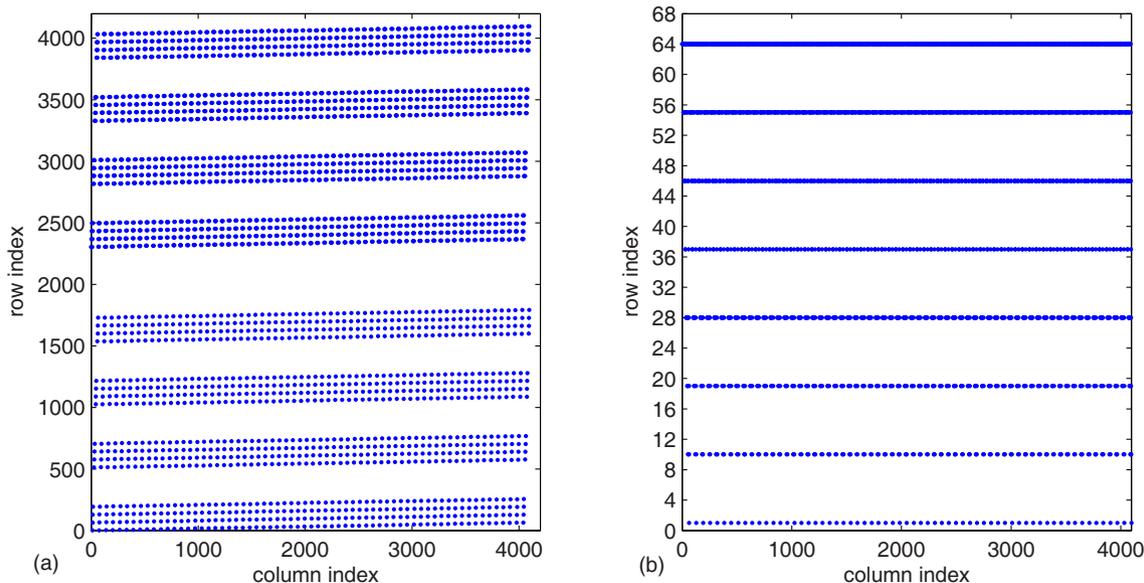

Figure 3 | (a): The whole row indexes of each column of matrix Ξ_1. (b): The whole row indexes of each column of matrix $\left(I_{2^6} \otimes 1_{2^6}^T\right)\Xi_1^5$. Each point corresponds to the row index of each column, which implies the position of 1.
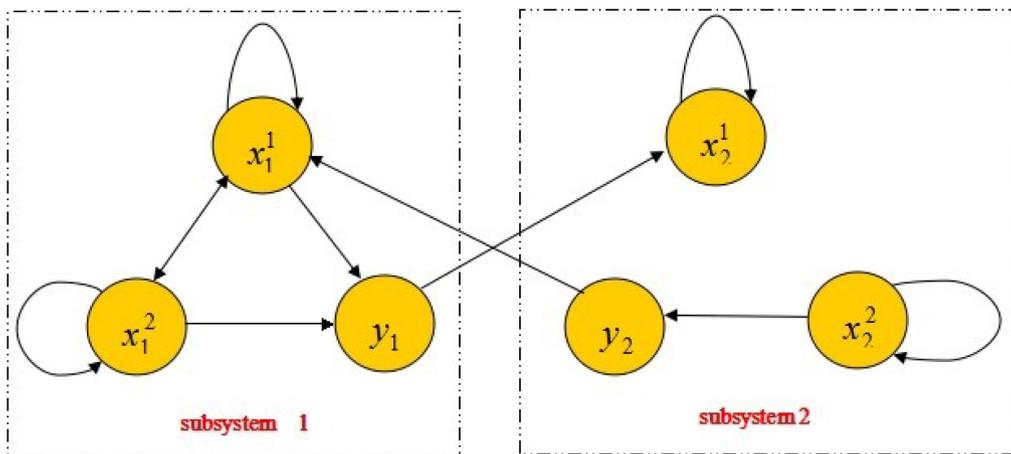
Figure 4 | Two coupled BNs with each network consisting of two nodes and one output. Arrows correspond to interactions between nodes and outputs.

and Fig. 3(a) shows the whole row indexes of each column of matrix Ξ_1, Fig. 3(b) shows the row indexes of each column of matrix $\left(I_{2^6} \otimes 1_{2^6}^T\right)\Xi_1^5$.

Example 2 In Fig. 4, we have a system consisting of two coupled TBNs with delay $\tau_1 = \tau_2 = 1$, each TBN with two nodes. And its dynamics is described as follows:

$$\begin{cases} x_1^1(t+1) = \left[x_1^2(t) \wedge x_1^1(t)\right] \to y_2(t), \\ x_1^2(t+1) = \left[x_1^1(t) \leftrightarrow x_1^2(t)\right], \\ y_1(t) = x_1^1(t) \vee x_1^2(t-1). \end{cases}$$

$$\begin{cases} x_2^1(t+1) = x_2^1(t-1) \leftrightarrow y_1(t), \\ x_2^2(t+1) = x_2^2(t) \wedge x_2^2(t-1), \\ y_2(t) = x_2^2(t-1). \end{cases} \tag{24}$$

Denote $x_j(t) = x_j^1(t) \ltimes x_j^2(t)$ for $j = 1, 2$ and $y(t) = y_1(t) \ltimes y_2(t)$, then we can express the system into its algebraic form by using the STP of matrices as follows:

$$\begin{cases} x_1^1(t+1) = F_1^1 x_1(t) x_1(t-1) y(t) \\ x_1^2(t+1) = F_1^2 x_1(t) x_1(t-1) y(t) \\ y_1(t) = G_1 x_1(t) x_1(t-1) \end{cases} \tag{25}$$

and

$$\begin{cases} x_2^1(t+1) = F_2^1 x_2(t) x_2(t-1) y(t) \\ x_2^2(t+1) = F_2^2 x_2(t) x_2(t-1) y(t) \\ y_2(t) = G_2 x_2(t) x_2(t-1) \end{cases} \tag{26}$$

We obtain the explicit expression of the matrices $F_1^1, F_1^2, F_2^1, F_2^2, G_1$ and G_2, by using STP. Here, we omit the procedure of computation for simplicity.

As we can see this system belongs to case (I). By Proposition 2, we obtain the matrix Ξ_1 and Fig. 5 shows the row indexes of each columns of matrix Ξ_1. In order to judge whether these two coupled TBNs can be synchronized or not, we should calculate Ξ_1^t according to Theorem 1. Direct computation shows that $\left(I_{2^4} \otimes 1_{2^4}^T\right)\Xi_1^8 = \left(I_{2^4} \otimes I_{2^4}^T\right)\Xi_1^{13}$ by using
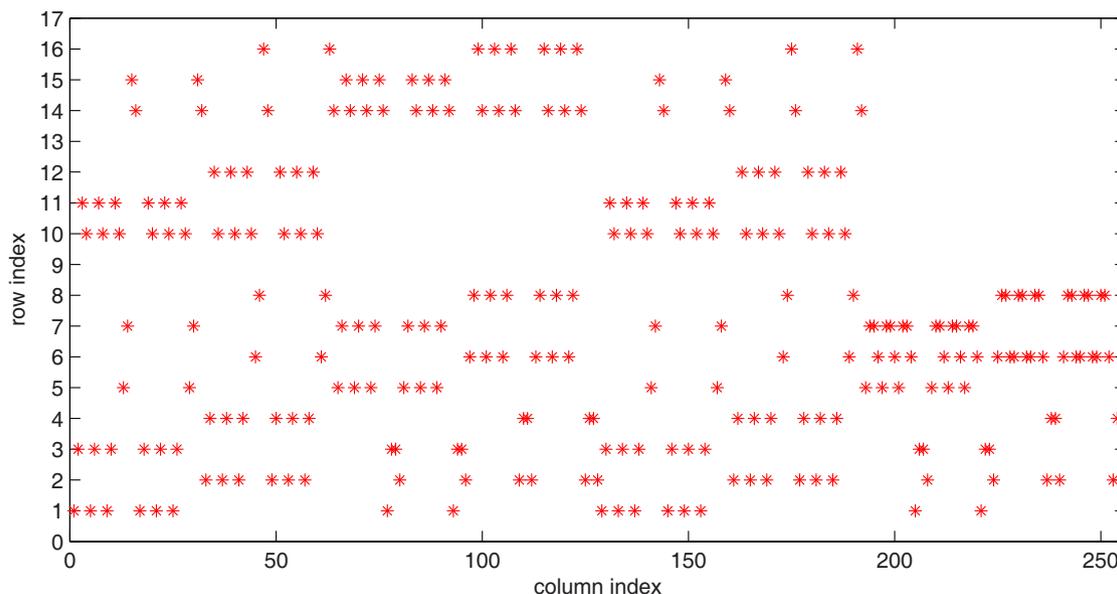

Figure 5 | The whole row indexes of each column of matrix Ξ_1 obtained in system (24). Each point corresponds to the row index of each column, which implies the position of 1.
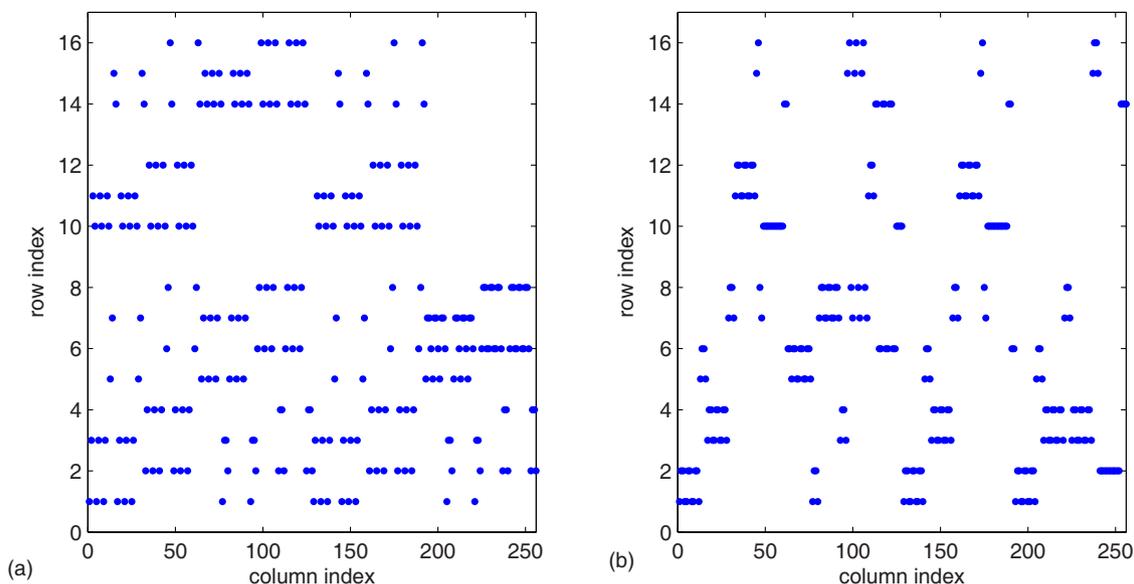
Figure 6 | (a): The whole row index of each column of matrix $\left(I_{2^4} \otimes \mathbf{1}_{2^4}^T\right)\Xi_1$ obtained in system (24). (b): The whole row index of each column of matrix $\left(I_{2^4} \otimes \mathbf{1}_{2^4}^T\right)\Xi_1^2$. Each point corresponds to the row index of each column, which implies the position of 1.

the equation (11), then we have $k_0 = 8$. However, we get that $\Xi_1^9 = \Xi_1^{14}$, while $\Xi_1^8 \neq \Xi_1^{13}$. By using $k'_0 = min\left\{i \geq 1 : \Xi_1^i = \Xi_1^j, j > i\right\}$, we have $k'_0 = 9$ which verifies Remark 3. It is easy to verify that $Col\left[\left(I_{2^4} \otimes \mathbf{1}_{2^4}^T\right)\Xi_1^t\right] \nsubseteq \left\{\delta_{2^4}^i : i = 1,6,11,16\right\}$ for $1 \leq t \leq 8$. In Fig. 6 to Fig. 9, we plot the whole row index of columns of each matrix $\left(I_{2^4} \otimes \mathbf{1}_{2^4}^T\right)\Xi_1^i$, $i = 1, 2, \ldots, 8$. From these figures, we can also draw the conclusion that $Col\left[\left(I_{2^4} \otimes \mathbf{1}_{2^4}^T\right)\Xi_1^t\right] \nsubseteq \left\{\delta_{2^4}^i : i = 1,6,11,16\right\}$ for $1 \leq t \leq 8$. Hence, this output-coupled TBNs can not be synchronized according to Theorem 1.

In the following, we can also define the total synchronization error of the output-coupled TBNs: $E(t) = \left|x_1^1(t) - x_2^1(t)\right| + \left|x_1^2(t) - x_2^2(t)\right|$. Fig. 10(a) shows the time evolution of the state $x_1(t)$ and $x_2(t)$ of the array of TBNs (24), and Fig. 10(b) shows the totaly synchronization error $E(t)$ of the TBNs (24). From the state evolution and the total synchronization error of the TBNs (24), we can draw the conclusion

that the TBNs can not synchronized. Our analytical results are well illustrated by this example.

Remark 7 *Due to the complexity of computation, Fig. 10(a) and Fig. 10(b) just plot time evolution and the total synchronization error of the TBNs with time up to $t = 100$. To some extent, Fig. 10(a) and Fig. 10(b) can illustrate that the coupled TBNs can not be synchronized. In fact, we have already made the computation of these TBNs with time up to $t = 10000$, observing that these TBNs can not be synchronized. Since the simulation result can not be well expressed in the figures with time $t = 10000$, we just plot Fig. 10(a) and Fig. 10(b) till time $t = 100$ only.*

Conclusions

In this paper, we have studied synchronization in an array of output-coupled TBNs. Both a state delay τ_1 and an output delay τ_2 are considered in the model. In a genetic regulatory network, state corre-
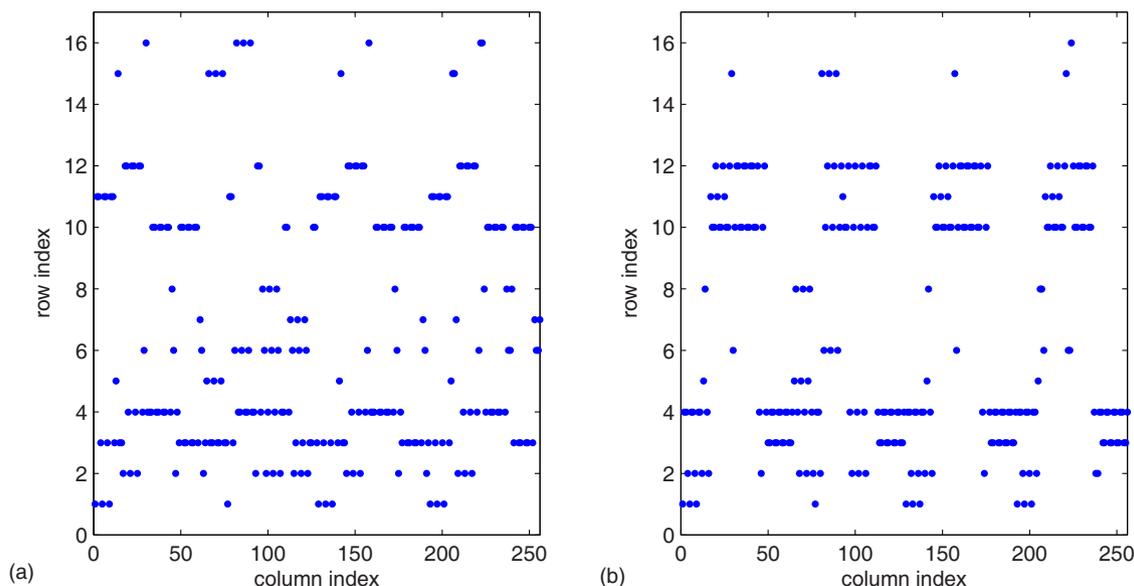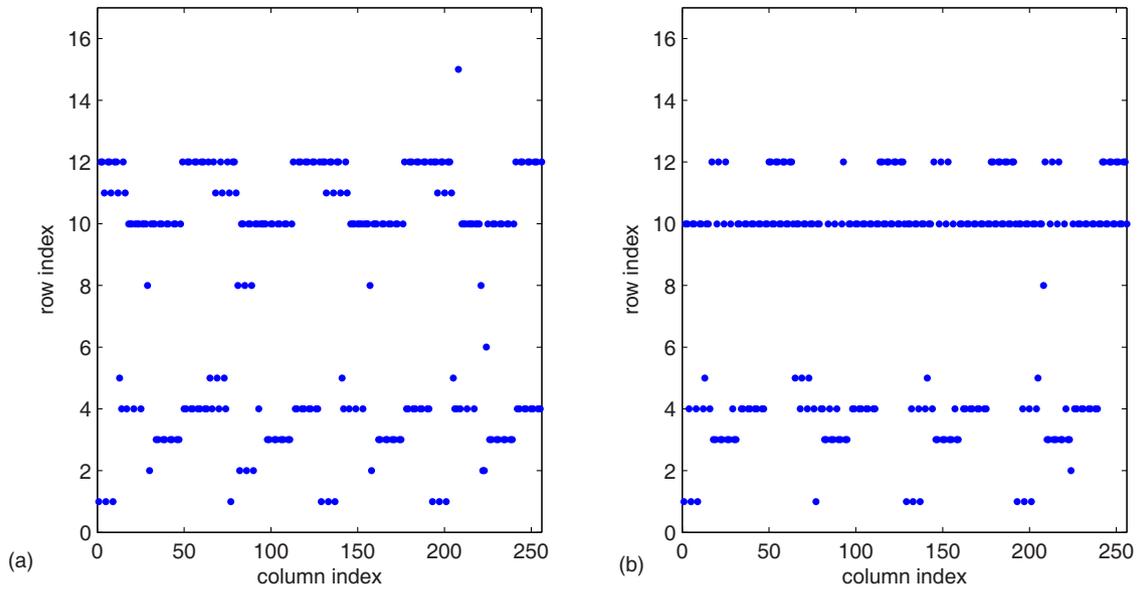

Figure 7 | (a): The whole row index of each column of matrix $\left(I_{2^4} \otimes \mathbf{1}_{2^4}^T\right)\Xi_1^3$ obtained in system (24). (b): The whole row index of each column of matrix $\left(I_{2^4} \otimes \mathbf{1}_{2^4}^T\right)\Xi_1^4$. Each point corresponds to the row index of each column, which implies the position of 1.

Figure 8 | (a): The whole row index of each column of matrix $\left(I_{2^4}\otimes \mathbf{1}_{2^4}^T\right)\Xi_1^5$ obtained in system (24). (b): The whole row index of each column of matrix $\left(I_{2^4}\otimes \mathbf{1}_{2^4}^T\right)\Xi_1^6$. Each point corresponds to the row index of each column, which implies the position of 1.

sponds to the level of gene, while the output corresponds to gene products. In real world, many evolutionary processes may experience external or internal changes. These changes maybe due to the environmental disturbances or the interconnections of other genes, which can heavily affect the evolutional processes of genes, such as the transcription process or the translation process. By using the semi-tensor product of matrices, we first convert the TBNs into an unique discrete time algebraic system. Based on the equivalent algebraic representations of TBNs, we then study the coupled TBNs by considering the two cases: $\tau_1 \geq \tau_2$ and $\tau_1 < \tau_2$. For each case, we derive necessary and sufficient synchronization criteria for the array of output-coupled TBNs. Different types of criteria are derived from different aspects. These criteria provide efficient ways to judge whether an array of output-coupled TBNs can be synchronized or not. And

they also implies that if the array of TBNs can not be synchronized among the upper bound time, then they can not be synchronized at any time, which means we only need to judge finite time steps. Finally, two illustrative examples including a practical epigenetic example are given to show the validity of our theoretical results. One interesting topic in the near future is to design logical controllers to make coupled TBNs synchronized.

This mechanism is expected to be applicable to a wide range of biological systems, because synchronization phenomenon is very commonly in biological systems. Moreover, it is expected to help to understand the nature of cellular function, the behavior of genes and the manner in which cells execute and control a large number of operations which are required for normal function and the ways in which cellular systems fail in disease.
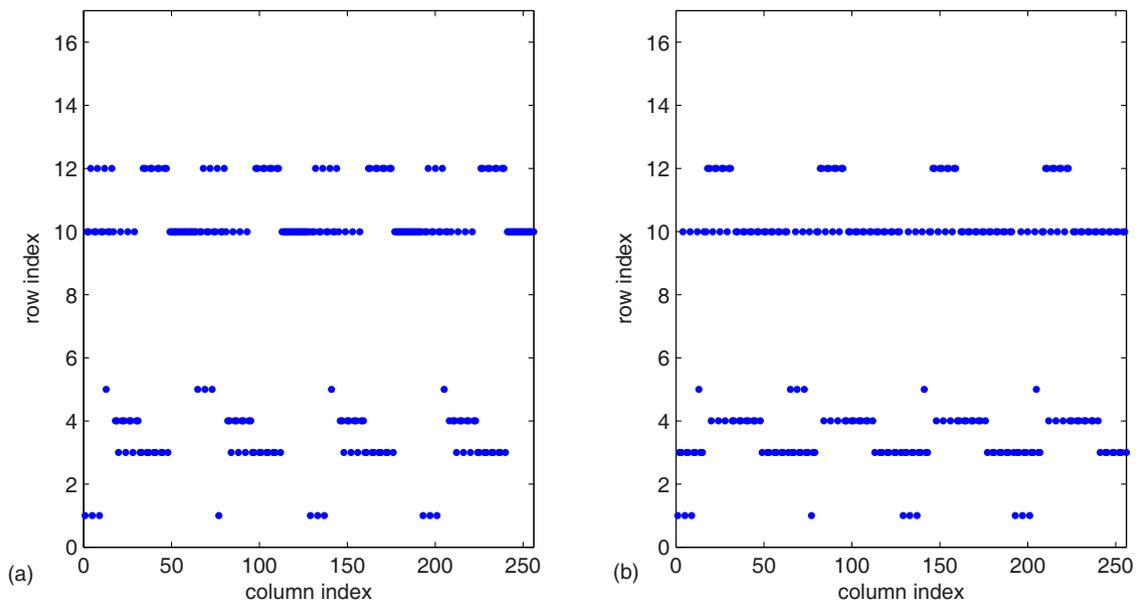

Figure 9 | (a): The whole row index of each column of matrix $\left(I_{2^4}\otimes \mathbf{1}_{2^4}^T\right)\Xi_1^5$ obtained in system (24). (b): The whole row index of each column of matrix $\left(I_{2^4}\otimes \mathbf{1}_{2^4}^T\right)\Xi_1^6$. Each point corresponds to the row index of each column, which implies the position of 1.
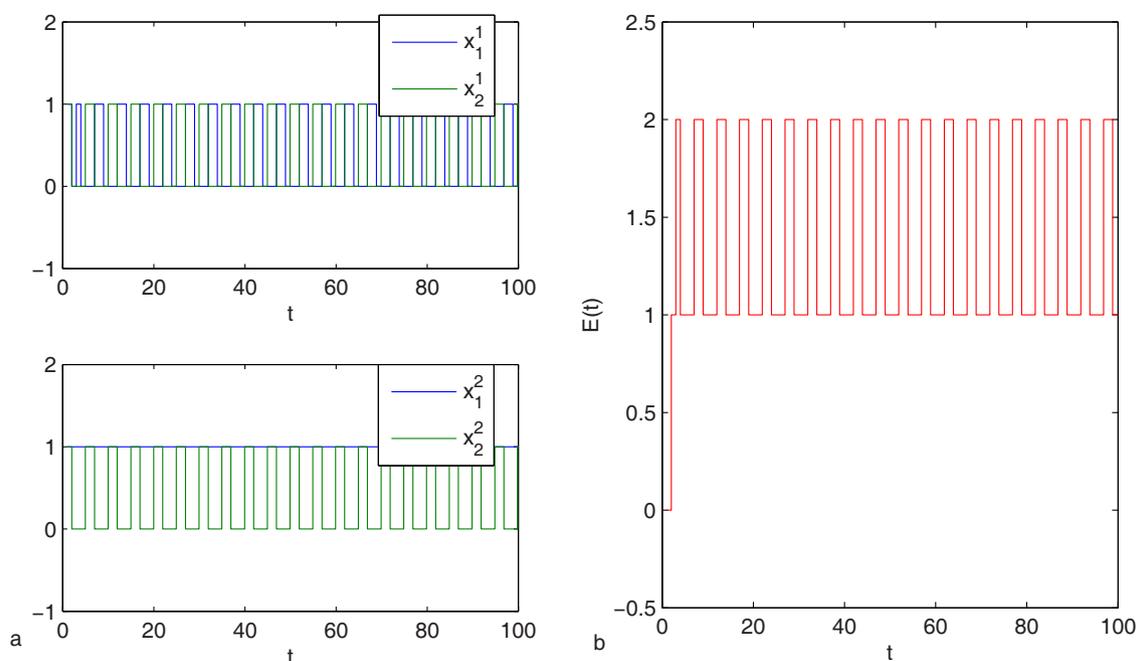
Figure 10 | (a): Time evolution of the TBNs (24) with initial state sequences $x_1(-1) = (0,0)$, $x_2(-1) = (1,1)$, $x_1(0) = (1,0)$, $x_2(0) = (0,1)$. The red solid line presents the state evolution of x_1, while the green solid line presents the state evolution of x_2. (b): Total synchronization error $E(t)$ of the TBNs (24). Fig. (a) shows states x_1^1 and x_2^1, states x_1^2 and x_2^2 can not reach synchronization till time to 100, while Fig. (b) shows the total synchronization error fluctuates between value 2 and value 0. Both Fig. (a) and Fig. (b) imply that the coupled TBNs can not be synchronized.

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Acknowledgments

Author contributions

J.L., J.Z., Y.T., T.H., J.C. and J.K. designed and performed the research, analyzed the results and wrote the paper.

Additional information

Supplementary information accompanies this paper at http://www.nature.com/scientificreports

Competing financial interests: The authors declare no competing financial interests.

How to cite this article: Lu, J. *et al.* Synchronization in output-coupled temporal Boolean networks. *Sci. Rep.* **4**, 6292; DOI:10.1038/srep06292 (2014).