Heliyon 9 (2023) e18323

Contents lists available at ScienceDirect

Heliyon

journal homepage: www.cell.com/heliyon

A new three way decision making technique for supplier selection in logistics service value Cocreation under intuitionistic double hierarchy linguistic term set

Abbas Qadir [a], Saleem Abdullah [a,*], Tarik Lamoudan [b], Faisal Khan [c], Saifullah Khan [a]

[a] Department of Mathematics, Abdul Wali Khan University Mardan, Pakistan
[b] Department of Mathematics, College of Science and Arts, Muhayil, King Khalid University, 61413 Abha, Saudi Arabia
[c] Department of Electrical and Electronic Engineering, College of Science and Engineering National University of Ireland Galway, Ireland

ARTICLE INFO

Keywords:
Double hierarchy linguistic term set
Intuitionistic fuzzy sets to double hierarchy linguistic term set
Supplier selection in logistics service value Cocreation
Three way decision making

ABSTRACT

In today's business world, choosing a logistics supplier is a critical factor for companies to improve operational efficiency and reduce business costs. With the development of market economy, it is very difficult for companies to choose a suitable logistics provider according to specific rules. Therefore, this study proposes a new three-way decision making (TWD) technique for supplier selection in logistics service value creation. For this, we first develop a new concept called intuitionistic double hierarchy linguistic term set (IDHLTSs) that can describe uncertainty and ambiguity in a more flexible way. Some Hamacher aggregation operators for collecting IDHLTSs information and its basic aspect are proposed. The unknown weight vector for decision experts and criteria is determined by using entropy measures. In addition, the conditional probability is determined using TOPSIS which makes the decision making process more rational. And the decision result is conducted according to minimum loss principle. Finally, an example of 3 PL supplier selection in the logistics service value co-creation environment and comparison is given to validate and demonstrate the effectiveness of the developed method.

1. Introduction

The global third-party logistics (3 PL) sector has expanded in recent years and is becoming increasingly important as a means of dealing with rapid changes in the global competitive environment. As a result of a rising tendency towards outsourcing logistics activities, shippers have been forced to choose the best acceptable 3 PL provider. The usage of 3 PL providers can result in significant benefits such as lower logistical costs and fixed logistics assets, higher order fill rates, and shorter average order-cycle lengths and cash-to-cash cycles. If an appropriate 3 PL provider is not selected, serious problems can occur, such as low-quality logistics services and contract nonfulfillment. As a result, the shipper's reputation, image, and trust may suffer. Therefore, in today's highly competitive business environment, more and more companies outsource their logistics services to 3 PL provider to reduce costs and improve business efficiency. The importance of supplier selection for effective logistics and supply chain management has received a lot of attention [1]. The 3 PL supplier selection issue is a common multi-criteria group decision-making problem (MCGDM) in complex

* Corresponding author.
E-mail addresses: abbasqadir@awkum.edu.pk (A. Qadir), saleemabdullah@awkum.edu.pk (S. Abdullah), lamoudan@kku.edu.sa (T. Lamoudan), f.khan4@nuigalway.ie (F. Khan), saifullah.wazir@awkum.edu.pk (S. Khan).

https://doi.org/10.1016/j.heliyon.2023.e18323
Received 21 February 2023; Received in revised form 8 July 2023; Accepted 13 July 2023
Available online 16 July 2023

business environments because it focuses on 3 PL supplier selection criteria [2–5] and the selection of models approaches of 3 PL supplier [6,7]. Gergin et al. [8], integrated a new method to select the most suitable supplier for a company engage in activities in the automotive supply industry. Riaz and Farid [9] proposed a new method for 3 PL selection under a picture fuzzy set. Most decision data given by the decision maker in many scenarios is often inexact or uncertain due to the complex decision-making (DM) environment and a lack of data. To handle such complex DM problems, Zadeh [10], discovered a fuzzy set (FS) by implying that each element of a fixed set has a degree of membership (*MD*) belonging to [0,1]. FS creates a powerful notion of applicability compared to traditional mathematical binary representations. Thus, researchers studied the fuzzy theory to introduce similarity measures and fuzzy DM approaches. But this theory is limited by the lack of assigning a degree of non-membership to each element of a fixed set. Therefore, Atanassov [11] introduced an intuitionistic fuzzy set (IFS) by adding the non-membership (*NMD*) degree to the FS theory satisfying the condition $MD + NMD \leq 1$. IFS is an important FS extension and is considered a good way to handle DM problems. Since its inception, the concept and results of IFS, as well as its application to DM problems, have been extensively studied. While FS and IFS are becoming popular techniques, peoples are more familiar with using linguistic term sets (LTSs) in fact to convey their assessment data, like "very bad", "somewhat bad", "excellent", etc. Therefore, LTSs can effectively link with difficult circumstances. Since Zadeh [12] introduced the structure computing with word (CWW), also explained other extensions of LTSs [13–16]. Further Gou et al. [17,18], extended LTS and defined a double hierarchy linguistic term set (DHLTS) by considering the membership degree in the form of a linguistic variable for strong modeling of expert expressions. DHLTS is the combination of two sets namely first hierarchy and second hierarchy linguistic term sets, allowing more flexibility to describe uncertainty and ambiguity. In real-world problems mostly it is difficult for decision-makers (DMs) to provide an accurate assessment of the attributes. DHLTS solves these difficulties and conveys appropriate data more conveniently in complex expressions than single LTSs. Many researchers have used this concept successfully. Xang et al. [19], introduced the Hamacher aggregation operator and applied it to the three-way decision (TWD) problem. DHLTS is a powerful tool but there is no way to assign a degree of non-membership to an element individually and the degree of membership in DHLTS does not cover the uncertainty of an element in most DM problems. To fill this gap it is necessary to define a new theory by studying IFS and DHLTSs to properly handle DM problems. Aggregation operators (AOs) are tools for combining n-tuples of information into a single useful form. AOs find extensive applications in decision-making. So to express the operations in fuzzy set theory as a generalization of Boolean logical connectives, the researcher used the terms triangular norm (t-norm) and triangular co-norm (t-conorm). Menger [20] pioneered the concepts of t-norm and t-conorm in the context of probabilistic metric spaces in 1942. Later, the development of the t-norm and t-conorms was successfully carried out by Schweizer and Sklar [21]. The concepts of t-norm and t-conorm attracted the interest of scholars. Numerous scholars have made contributions to this topic and have proposed the multipurpose t-norms, such as the radical product, algebraic product, Lukasiewicz t-norm, Yager t-norm, Schweizer and Sklar t-norm, and Frank t-norm, as well as the corresponding t-conorms. Zimmermann and Zysno [22] later discovered that t-norm and t-conorm exhibit major behavior. This led to the invention of the compensation and averaging operators [23], which produce results that lie within an interval. In 1978, Hamacher [24] introduced a parameterized t-norm and its dual t-conorm as a generalization of Einstein product and Einstein sum, which are more broad and flexible than other existing norms. To the best of our knowledge and the above analysis up-till now no application with the hybrid study of IFSs and DHLTSs by applying Hamacher aggregation operators is reported. Therefore, this study motivates and fills the existing research gap to investigate a new concept namely intuitionistic double hierarchy linguistic term set (IDHLTSs) by studying IFSs and DHLTSs.

The main objectives of this work are as follows:

(i) To define a hybrid notion intuitionistic double hierarchy linguistic term set (IDHLTSs) by extending DHLTSs to an intuitionistic fuzzy set, which has greater application flexibility in real-world DM problems.
(ii) To compare the intuitionistic double hierarchy linguistic numbers (DHLTNs), we define a new score and accuracy function
(iii) Hamacher t-norm and t-conorm with flexible operational parameters have great significance as they incorporate the properties of several other widely used operators. The objective of the proposed operators is to present Hamacher weighted averaging operators in IDHLTSs circumstances.
(iv) It is very essential to form operational laws during the aggregation procedure. Therefore, we develop some basic operations on IDHLTSs utilizing Hamacher operators by keeping the advantages of IDHLTSs.
(v) Finding unknown weight vectors for decision-makers or criteria is a critical issue. To address this issue, the entropy measure of the IDHLTS-based process is used to obtain the weights of the DM and criteria to avoid adverse effects of the weights.
(vi) We further investigate the theoretical and practical interpretation of the proposed tool by solving numerical examples.

1.1. Related work

In real life, traditional MCGDM [25,26] only provides a ranking of alternatives without providing specific recommendations to decision-makers. The three-way decisions (TWDs) overcome this constraint since they are a decision-making technique that is consistent with people's thought processes. Thus Yao [27–29], proposed the three-way decision-making (TWDs) process and showed an important decision-making principle. Intuitively, the Bayesian method [30,31] classifies objects into three distinct regions. Whenever an object is assigned to a positive region, negative region, or boundary region, this is an indication that the DM should accept the object, reject the object, or delay the decision respectively. Although TWDs are relevant to human decision-making patterns, they have been applied in many domains, including health care [32,33], investment [34] and activity rehabilitation [35] decisions. To better express the loss function (LFs) in TWDs, more extended structures of fuzzy sets have been introduced in the process of TWDs, including dual hesitant fuzzy sets [36–38], fuzzy set [39] and triangular fuzzy number [40]. Herbert and Yao [41] investigated game theory related to LFs determination techniques for constructing loss function matrices. Jia et al. [42], suggested a correction problem based on the link between loss functions and limit values and solved the optimization problem to yield limit values. Jia et al. [43], suggested some new approaches for computing LFs based on multiple-criteria environments. However, in practice, the LFs are evaluated by decision-makers according to their own historical experience and knowledge, and this study adopts the same method. Many scholars have studied the determination of conditional probability, which is another critical component of TWDs. Ye et al. [39], initially used the entropy weight method to calculate attribute weights and then used weighted aggregation to calculate conditional probability. While Liang et al. [44], applied the maximizing deviation approach [45] to first determine attribute weights and then use a technique called order performance by similarity to Ideal Solutions (TOPSIS) to achieve conditional probability. Wang et al. [46], calculated the conditional probability using DM methods based on third-generation prospect theory and then used the method for grey relational analysis (GRA) [47] to achieve the conditional probability. Liu and Yang [48], developed a decision-theoretic rough set (DTRS) model and applied it to TWDs. From the literature review, there are no such tools and implementations of the TWDs technique with the hybrid notion of DHLTs and IFSs where the weights of criteria and experts are completely unknown. Therefore, the motivation of this work is to investigate the above-mentioned specific goals.

From the above-mentioned goals the major contributions and factors of this work are as follows:

(i) The considerable contribution of this study is to define a new theory called intuitionistic double hierarchy linguistic term sets (IDHLTSs) because Gou et al. [17,18], developed DHLTSs by considering only the membership degree, but this idea has some limitations due to the lack of non-membership degrees. So we generalized this concept by adding the non-membership degree and applying the DHLTSs on IFS and defined IDHLTSs to study attributes and LFs values. They are adaptable tools that allow decision-makers to provide assessments in the form of IDHLTSs.
(ii) To make capital of the parametric and flexible framework of Hamacher operations under the competitive and innovative model of IDHLTSs to accumulate the decision-making.
(iii) The novelty of the proposed operators is due to their flexible structure and authentic outputs as they compile the IDHLTSs data deploying the brilliance of Hamacher operations, whereas the existing operators, developed based on Hamacher norms, are not applicable for IDHLTSs data due to nonavailability or strick condition of IDHLTSs.
(iv) The entropy and distance measures were established for finding the unknown weight vector of experts and criteria.
(v) Discuss the TOPSIS and GRA methods for calculating conditional probability using Hamacher aggregation operations and their further expected losses and score functions.
(vi) To make an accurate decision a novel TWDs technique is given to select the best result based on three main principles.

The summary of this paper is as follows: Section 2, includes the basic concepts related to IFSS, LTSs, and DHLTSs. Section 3, includes a novel notion of intuitionistic double hierarchy linguistic term set (IDHLTSs), score function accuracy function, and distance measure. Section 4, proposed some Hamacher aggregation operators such as the intuitionistic double hierarchy linguistic Hamacher weighted averaging (IDHLHWA) operator, intuitionistic double hierarchy linguistic Hamacher ordered weighted averaging (IDHLHOWA) operator, and also fundamental properties such as Idempotency and Boundedness are discussed. Section 5, proposed the algorithm for determining the conditional probability based on the TOPSIS methods. Section 6, introduces the Novel DHLDTRS model for the selection of the best optimal result. Section 7, describes the application of the proposed method by solving numerical examples to make an accurate decision. Section 8, we compare the proposed method with other MADM methods to demonstrate the applicability of our proposed method and to show the limitations and advantages. In section 9, explain the conclusion of the article.

Following Table A represented the detail description of acronyms used in this work. Similarly, Table B summarized all the symbols used in this paper.

Table A
Description of acronyms used in this work.

Acronym	Description
MCGDM	Multi criteria group decision making
MADM	Multi attribute decision making
DMs/EM	Decision-maker(s)/Expert matrix
DHLDTRS	Double hierarchy linguistic decision-theoretic rough set
MD	Membership degree
NMD	Non-membership degree
LTSs	Linguistic term set
DHLTSs	Double hierarchy linguistic term set
GRA	Grey relational analysis
PIS^{+}	Positive ideal solution
NIS^{-}	Negative ideal solution
AOs	Aggregation operators
IO, RIO, LIO	Ideal opinion matrix, Right ideal matrix, Left ideal matrix
TWDs	Three way decisions
IDHLTSs	Intuitionistic double hierarchy linguistic term set

(continued on next page)

Table A (*continued*)

Acronym	Description
LF	Loss function
DTRS	Decision-theoretic rough set

Table B
Representation of Symbols used in this work.

Symbols	Representation
U	Non empty universal set
x	$x \in U$
$u_{\underline{A}}(x), v_{\underline{A}}(x)$	Membership degree, Non-membership degree
$S_\alpha(x)$	First hierarchy linguistic terms (Membership degree)
$S_\beta(x)$	First hierarchy linguistic terms (Non-membership degree)
Q_k	Second hierarchy linguistic terms (Membership degree)
Q_l	Second hierarchy linguistic terms (Non-membership degree)
$\overline{A}_{\overline{B}}$	Intuitionistic double hierarchy linguistic term set
$\overline{S}_{\overline{\mathscr{Q}}}$	Double hierarchy linguistic term set
Sc, Ac	Score function, Accuracy function
τ, δ	Any even number
t, α, β, k, l	$t, \alpha, \beta \in [0, \tau], k, l \in [0, \delta]$
a_P, a_B, a_N	Positive, boundary, and negative regions
$T_\gamma^H(u, v), S_\gamma^H(u, v)$	Family of Hamacher t-norm (Product) and t-conorm (Sum)
λ, γ	$\lambda, \gamma > 0 (\in \mathbb{R})$
$(M)^{\prime} = [\overline{A}_{\overline{B}ij}]_{m\times n}$	Expert evaluation matrix
$dIO_i, dRIO_i, dLIO_i$	Distance from *IO*, *RIO*, *LIO* to $(M)^{\prime}$
$\bigoplus_H, \bigodot_H, \vee, \wedge$	Hamacher Addition, Hamacher Multiplication, Union, Intersection
ω, ϖ	Expert weight, Criteria weight
$U_i, \mathscr{L}_j$	Alternatives, Criteria
$CI, E_i(\Psi)$	closeness indices, Entropy measure
g_{ij}^+, g_{ij}^-	Grey relational coefficient from PIS^+ and NIS^-

2. Preliminaries

Here we will put forward the notions of IFS, linguistic term set (LTSs), intuitionistic fuzzy linguistic term set (IFLTSs) and double hierarchy linguistic term sets (DHLTSs). These concepts will connect our study with upcoming sections.

Definition 1. [11] Let U is a non empty set, then intuitionistic fuzzy set (IFS) is mathematically defined in equation (1) as follows.

$$\underline{A} = \{x, \langle u_{\underline{A}}(x), v_{\underline{A}}(x)\rangle | x \in U\} \tag{1}$$

Where $u_{\underline{A}}(x) \in [0,1]$ and $v_{\underline{A}}(x) \in [0,1]$ are represents the membership and non-membership degree respectively, belong to $[0,1]$ with conditions $\left(u_{\underline{A}}(x), v_{\underline{A}}(x)\right) \leq 1$..

Definition 2. [13] A non-empty set S with odd cardinality is known as a linguistic term set (LTSs), i.e. $\overline{S} = \{S_t | t \in [0, \tau]\}$, whereas S_t is the possible linguistic term with linguistic variable.

Definition 3. [14] Let U be a universal set and $\overline{S} = \{S_t | S_0 \leq S_t \leq S_\tau, t \in [0, \tau]\}$ be a continuous linguistic term set. Then intuitionistic fuzzy linguistic term set (IFLTSs), is defined in the finite universe of discourse U mathematically with the form given in equation (2) as

$$\underline{A} = \{x, \langle S_\alpha(x), S_\beta(x)\rangle | x \in U\} \tag{2}$$

Where $S_\alpha(x)$ and $S_\beta(x)$ denotes the membership and non-membership degree in the from linguistic term such that $\alpha + \beta \leq \tau$ or $((\alpha/\tau) + (\beta/\tau)) \leq 1$. For simplicity it is denoted by $\underline{A} = \langle S_\alpha(x), S_\beta(x)\rangle$..

Definition 4. [17] Let $\overline{S} = \{S_\alpha | \alpha = -\tau, ..., -1, 0, 1, ..., \tau\}$ and $\overline{Q} = \{Q_k | k = -\delta, ..., -1, 0, 1, ..., \delta\}$ be the first hierarchy and second hierarchy LTSs, then the structure given in equation (3)

$$\overline{S}_{\overline{Q}} = \{S_{\alpha\langle Q_k\rangle} | \alpha = -\tau, ..., -1, 0, 1, ..., \tau; k = -\delta, ..., -1, 0, 1, ..., \delta\}, \tag{3}$$

is said to be double hierarchy linguistic term sets (DHLTSs), denoted by $\overline{S}_{\overline{Q}}$, where S_α is the first hierarchy and Q_k represents the second hierarchy linguistic terms respectively.

3. Intuitionistic double hierarchy linguistic term sets

Here we will develop the hybrid notion of IFS and DHLTSs to obtained the notion of intuitionistic double hierarchy linguistic term set (IDHLTSs) and initiate the new score and accuracy functions and also put forward its basic operations in detailed.

Definition 5. Let $\overline{A} = \{\langle S_\alpha, S_\beta\rangle | \alpha, \beta = 0, 1, ..., \tau\}$ be the first and $\overline{B} = \{\langle Q_k, Q_l\rangle | k, l = 0, 1, ..., \delta\}$ be the second hierarchy linguistic term sets, then the mathematical structure as given in equation (4)

$$\overline{A}_{\overline{B}} = \left\{\langle S_{\alpha\langle Q_k\rangle}, S_{\beta\langle Q_l\rangle}\rangle \middle| \alpha, \beta = 0, 1, ..., \tau; k, l = 0, 1, ..., \delta\right\} \tag{4}$$

is said to be intuitionistic double hierarchy linguistic term sets (IDHLTSs). Where S_α, S_β represents membership and non-membership degree of first hierarchy linguistic term sets and Q_k, Q_l is the membership and non-membership degree of second hierarchy linguistic term sets, such that $\alpha + \beta \leq \tau$ and $k + l \leq \delta$ or $((\alpha/\tau) + (\beta/\tau)) \leq 1$ and $((k/\delta) + (l/\delta)) \leq 1$.

Definition 6. Let $\overline{A}_{\overline{B}i} = \langle S_{\alpha_i\langle Q_{k_i}\rangle}, S_{\beta_i\langle Q_{l_i}\rangle}\rangle$ be the intuitionistic double hierarchy linguistic term set. Then mathematically the score and accuracy function are denoted and defined in equations (5) and (6) respectively as follows.

$$Sc = \frac{\left(\left(\frac{\alpha_i}{\tau}\right) + \left(\frac{k_i}{\delta}\right) - \left(\frac{\beta_i}{\tau}\right) - \left(\frac{l_i}{\delta}\right)\right)}{2} \in [-1, 1] \tag{5}$$

$$Ac = \frac{\left(\left(\frac{\alpha_i}{\tau}\right) + \left(\frac{k_i}{\delta}\right) + \left(\frac{\beta_i}{\tau}\right) + \left(\frac{l_i}{\delta}\right)\right)}{2} \in [0, 1] \tag{6}$$

Definition 7. Consider $\overline{A}_{\overline{B}1} = \langle S_{\alpha_1\langle Q_{k_1}\rangle}, S_{\beta_1\langle Q_{l_1}\rangle}\rangle$ and $\overline{A}_{\overline{B}2} = \langle S_{\alpha_2\langle Q_{k_2}\rangle}, S_{\beta_2\langle Q_{l_2}\rangle}\rangle$ be two intuitionistic double hierarchy linguistic numbers (IDHLNs). Then the distance measure between any two IDHLTSs for any $\lambda > 0 (\in \mathbb{R})$ is defined in equation (7) follows as

$$d(\overline{A}_{\overline{B}1}, \overline{A}_{\overline{B}2}) = \left[\frac{1}{4n}\sum_{i=1}^{2}\left(\begin{array}{c}\left|S_{\left(\frac{\alpha_1}{\tau}\right)} - S_{\left(\frac{\alpha_2}{\tau}\right)}\right|^\lambda + \left|Q_{\left(\frac{k_1}{\delta}\right)} - Q_{\left(\frac{k_2}{\delta}\right)}\right|^\lambda + \\ \left|S_{\left(\frac{\beta_1}{\tau}\right)} - S_{\left(\frac{\beta_2}{\tau}\right)}\right|^\lambda + \left|Q_{\left(\frac{l_1}{\delta}\right)} - Q_{\left(\frac{l_2}{\delta}\right)}\right|^\lambda\end{array}\right)\right]^{\frac{1}{\lambda}} \tag{7}$$

4. Hamacher t-norm and t-conorm

This section provides Hamacher t-norm and t-conorm for IDHLTSs. The Hamacher t-norm and t-conorm [24], are more flexible and generalized from algebraic and Einstein triangular norms [49,50] which are mathematically given below.

Definition 8. Let $u, v \in \mathbb{R}$ and $\gamma > 0$, then Hamacher t-norm and t-conorm are defined in equations (8) and (9) as.

$$T^H_\gamma(u, v) = \frac{uv}{\gamma + (1-\gamma)(u + v - uv)} \tag{8}$$

$$S^H_\gamma(u, v) = \frac{u + v - uv - (1-\gamma)uv}{1 - (1-\gamma)uv} \tag{9}$$

The Hamacher norm reduces to the algebraic norm at $\gamma = 1$ and to the Einstein norm at $\gamma = 2$..

4.1. Hamacher operational laws of intuitionistic double hierarchy linguistic term set

According to the Hamacher t-norm and t-conorm, a few hesitant Hamacher operators given that to aggregate [51] hesitant fuzzy information. Because the Hamacher t-norm and t-conorm operator laws are logical and closed, a few closed operational laws are defined as follows.

Definition 9. Let $\overline{A}_{\overline{B}1} = \langle S_{\alpha_1\langle Q_{k_1}\rangle}, S_{\beta_1\langle Q_{l_1}\rangle}\rangle$ and $\overline{A}_{\overline{B}2} = \langle S_{\alpha_2\langle Q_{k_2}\rangle}, S_{\beta_2\langle Q_{l_2}\rangle}\rangle$ be two intuitionistic double hierarchy linguistic term set and $\gamma \geq 0, 0 \leq \lambda \leq 1$, then the algebraic Hamacher operational laws for intuitionistic double hierarchy linguistic term set are in equations (10) and (11) as follows.

$$\overline{A}_{\overline{B}1}\bigoplus\overline{A}_{\overline{B}2}=\left\{\begin{array}{l}\left(\begin{array}{l}S_{\tau\left(\frac{\left(\frac{\alpha_1}{\tau}\right)+\left(\frac{\alpha_2}{\tau}\right)-\left(\frac{\alpha_1}{\tau}\right).\left(\frac{\alpha_2}{\tau}\right)-(1-\gamma)\left(\frac{\alpha_1}{\tau}\right).\left(\frac{\alpha_2}{\tau}\right)}{1-(1-\gamma)\left(\frac{\alpha_1}{\tau}\right).\left(\frac{\alpha_2}{\tau}\right)}\right)}\\ \langle Q_{\delta\left(\frac{\left(\frac{k_1}{\delta}\right)+\left(\frac{k_2}{\delta}\right)-\left(\frac{k_1}{\delta}\right).\left(\frac{k_2}{\delta}\right)-(1-\gamma)\left(\frac{k_1}{\delta}\right).\left(\frac{k_2}{\delta}\right)}{1-(1-\gamma)\left(\frac{k_1}{\delta}\right).\left(\frac{k_2}{\delta}\right)}\right)}\rangle\end{array}\right)\\ S_{\tau\left(\frac{\left(\frac{\beta_1}{\tau}\right).\left(\frac{\beta_2}{\tau}\right)}{\gamma+(1-\gamma)\left(\frac{\beta_1}{\tau}\right)+\left(\frac{\beta_2}{\tau}\right)-\left(\frac{\beta_1}{\tau}\right).\left(\frac{\beta_2}{\tau}\right)}\right)}\langle Q_{\delta\left(\frac{\left(\frac{l_1}{\delta}\right).\left(\frac{l_2}{\delta}\right)}{\gamma+(1-\gamma)\left(\frac{l_1}{\delta}\right)+\left(\frac{l_2}{\delta}\right)-\left(\frac{l_1}{\delta}\right).\left(\frac{l_2}{\delta}\right)}\right)}\rangle\end{array}\right\} \tag{10}$$

$$\lambda\bigodot_H\overline{A}_{\overline{B}1}=\left\{\begin{array}{l}\left(\begin{array}{l}S_{\tau\left(\frac{\left(1+(\gamma-1).\left(\frac{\alpha_1}{\tau}\right)\right)^{\lambda}-\left(1-\left(\frac{\alpha_1}{\tau}\right)\right)^{\lambda}}{\left(1+(\gamma-1).\left(\frac{\alpha_1}{\tau}\right)\right)^{\lambda}+(\gamma-1).\left(1-\left(\frac{\alpha_1}{\tau}\right)\right)^{\lambda}}\right)}\\ \langle Q_{\delta\left(\frac{\left(1+(\gamma-1).\left(\frac{k_1}{\delta}\right)\right)^{\lambda}-\left(1-\left(\frac{k_1}{\delta}\right)\right)^{\lambda}}{\left(1+(\gamma-1).\left(\frac{k_1}{\delta}\right)\right)^{\lambda}+(\gamma-1).\left(1-\left(\frac{k_1}{\delta}\right)\right)^{\lambda}}\right)}\rangle\end{array}\right)\\ S_{\tau\left(\frac{\gamma.\left(\left(\frac{\beta_1}{\tau}\right)\right)^{\lambda}}{\left(1+(\gamma-1).\left(\frac{\beta_1}{\tau}\right)\right)^{\lambda}+(\gamma-1).\left(1-\left(\frac{\beta_1}{\tau}\right)\right)^{\lambda}}\right)}\langle Q_{\delta\left(\frac{\gamma.\left(\left(\frac{l_1}{\tau}\right)\right)^{\lambda}}{\left(1+(\gamma-1).\left(\frac{l_1}{\tau}\right)\right)^{\lambda}+(\gamma-1).\left(1-\left(\frac{l_1}{\tau}\right)\right)^{\lambda}}\right)}\rangle\end{array}\right\} \tag{11}$$

Example 1. Let $(\tau,\delta=6)$ and $\overline{A}_{\overline{B}1}=\langle S_{1\langle Q_3\rangle},S_{4\langle Q_3\rangle}\rangle$, $\overline{A}_{\overline{B}2}=\langle S_{3\langle Q_2\rangle},S_{1\langle Q_4\rangle}\rangle$ be two IDHLNs, and $\gamma=2,\lambda=0.6$, then

$$(1)\ \overline{A}_{\overline{B}1}\bigoplus\overline{A}_{\overline{B}2}=\left\{\begin{array}{l}\left(\begin{array}{l}S_{6\left(\frac{\left(\frac{1}{6}\right)+\left(\frac{3}{6}\right)-\left(\frac{1}{6}\right).\left(\frac{3}{6}\right)-(1-2)\left(\frac{1}{6}\right).\left(\frac{3}{6}\right)}{1-(1-2)\left(\frac{1}{6}\right).\left(\frac{3}{6}\right)}\right)}\\ \langle Q_{6\left(\frac{\left(\frac{3}{6}\right)+\left(\frac{2}{6}\right)-\left(\frac{3}{6}\right).\left(\frac{2}{6}\right)-(1-2)\left(\frac{3}{6}\right).\left(\frac{2}{6}\right)}{1-(1-2)\left(\frac{3}{6}\right).\left(\frac{2}{6}\right)}\right)}\rangle\end{array}\right)\\ S_{6\left(\frac{\left(\frac{4}{6}\right)\left(\frac{1}{6}\right)}{2+(1-2)\left(\frac{4}{6}\right)+\left(\frac{1}{6}\right)-\left(\frac{4}{6}\right).\left(\frac{1}{6}\right)}\right)}\langle Q_{6\left(\frac{\left(\frac{3}{6}\right)\left(\frac{4}{6}\right)}{2+(1-2)\left(\frac{3}{6}\right)+\left(\frac{4}{6}\right)-\left(\frac{3}{6}\right).\left(\frac{4}{6}\right)}\right)}\rangle\end{array}\right\}.$$

$$=\langle S_{3.69\langle Q_{4.28}\rangle},S_{0.48\langle Q_{1.09}\rangle}\rangle$$

$$(2).\ 0.6\bigodot_H\overline{A}_{\overline{B}1}=\left\{\begin{array}{l}\left(\begin{array}{l}S_{6\left(\frac{\left(1+(2-1).\left(\frac{1}{6}\right)\right)^{0.6}-\left(1-\left(\frac{1}{6}\right)\right)^{0.6}}{\left(1+(2-1).\left(\frac{1}{6}\right)\right)^{0.6}+(2-1).\left(1-\left(\frac{1}{6}\right)\right)^{0.6}}\right)}\\ \langle Q_{6\left(\frac{\left(1+(2-1).\left(\frac{3}{6}\right)\right)^{0.6}-\left(1-\left(\frac{3}{6}\right)\right)^{0.6}}{\left(1+(2-1).\left(\frac{3}{6}\right)\right)^{0.6}+(2-1).\left(1-\left(\frac{3}{6}\right)\right)^{0.6}}\right)}\rangle\end{array}\right)\\ S_{6\left(\frac{2.\left(\left(\frac{4}{6}\right)\right)^{0.6}}{\left(1+(2-1).\left(\frac{4}{6}\right)\right)^{0.6}+(2-1).\left(1-\left(\frac{4}{6}\right)\right)^{0.6}}\right)}\langle Q_{6\left(\frac{2.\left(\left(\frac{3}{6}\right)\right)^{0.6}}{\left(1+(2-1).\left(\frac{3}{6}\right)\right)^{0.6}+(2-1).\left(1-\left(\frac{3}{6}\right)\right)^{0.6}}\right)}\rangle\end{array}\right\}.$$

$$=\langle S_{0.60\langle Q_{1.90}\rangle},S_{0.83\langle Q_{0.68}\rangle}\rangle$$

Definition 10. Let $\overline{A}_{\overline{B}i}=\langle S_{\alpha_i\langle Q_{k_i}\rangle},S_{\beta_i\langle Q_{l_i}\rangle}\rangle$ $(i=1,2,...,n)$ be the collection of IDHLNs and $\omega=(\omega_1,\omega_2,...,\omega_n)^T$ represent weight vectors of given collection restricted to $\omega_i>0$, $\sum_{i=1}^{n}\omega_i=1$. Then based on the above operational laws intuitionistic double hierarchy

Hamacher weighted averaging operator is mapping $IDHLHWA : \Omega^n \rightarrow \Omega$ is defined in equation (12) as.

$$IDHLHWA(\overline{A}_{\overline{B}1}, \overline{A}_{\overline{B}2}, ..., \overline{A}_{\overline{B}n}) = \bigoplus_{i=1}^{n} (\omega_i \bigodot_H \overline{A}_{\overline{B}i}) \tag{12}$$

According to the above definition the aggregated result of *IDHLHWA* is given in Theorem 1.

Theorem 1. *Let* $\overline{A}_{\overline{B}i} = \langle S_{\alpha_i \langle Q_{k_i} \rangle}, S_{\beta_i \langle Q_{l_i} \rangle} \rangle$ $(i = 1, 2, ..., n)$ *be the collection of IDHLNs and* $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ *represent weight vectors of given collection restricted to* $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i = 1$. *Then intuitionistic double hierarchy Hamacher weighted averaging IDHLHWA operator is defined in* equation (13) *as.*

$$IDHLHWA(\overline{A}_{\overline{B}1}, \overline{A}_{\overline{B}2}, ..., \overline{A}_{\overline{B}n}) = \bigoplus_{i=1}^{n} (\omega_i \bigodot_H \overline{A}_{\overline{B}i})$$

$$= \left[\begin{array}{l} \left(S_{\tau\left(\frac{\prod_{i=1}^{n}\left(1+(\gamma-1).\left(\frac{\alpha_i}{\tau}\right)\right)^{\omega_i} - \prod_{i=1}^{n}\left(1-\left(\frac{\alpha_i}{\tau}\right)\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\gamma-1).\left(\frac{\alpha_i}{\tau}\right)\right)^{\omega_i} + (\gamma-1).\prod_{i=1}^{n}\left(1-\left(\frac{\alpha_i}{\tau}\right)\right)^{\omega_i}} \right)} \left\langle Q_{\delta\left(\frac{\prod_{i=1}^{n}\left(1+(\gamma-1).\left(\frac{k_i}{\delta}\right)\right)^{\omega_i} - \prod_{i=1}^{n}\left(1-\left(\frac{k_i}{\delta}\right)\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\gamma-1).\left(\frac{k_i}{\delta}\right)\right)^{\omega_i} + (\gamma-1)\prod_{i=1}^{n}\left(1-\left(\frac{k_i}{\delta}\right)\right)^{\omega_i}} \right)} \right\rangle \right), \\ \left(S_{\tau\left(\frac{\gamma.\prod_{i=1}^{n}\left(\frac{\beta_i}{\tau}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\gamma-1).\left(\frac{\beta_i}{\tau}\right)\right)^{\omega_i} + (\gamma-1)\prod_{i=1}^{n}\left(1-\left(\frac{\beta_i}{\tau}\right)\right)^{\omega_i}} \right)} \left\langle Q_{\delta\left(\frac{\gamma.\prod_{i=1}^{n}\left(\frac{l_i}{\delta}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\gamma-1).\left(\frac{l_i}{\delta}\right)\right)^{\omega_i} + (\gamma-1)\prod_{i=1}^{n}\left(1-\left(\frac{l_i}{\delta}\right)\right)^{\omega_i}} \right)} \right\rangle \right) \end{array} \right] \tag{13}$$

Theorem 2. Let $\overline{A}_{\overline{B}i} = \langle S_{\alpha_i \langle Q_{k_i} \rangle}, S_{\beta_i \langle Q_{l_i} \rangle} \rangle$ $(i = 1, 2, ..., n)$ be the collection of IDHLNs and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ represent weight vectors of given collection restricted to $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i = 1$. Then its desirable properties are as follows in equations (14)–(16).

(1). (*Idempotency*): *Suppose* $\overline{A}_{\overline{B}i} = \langle S_{\alpha_i \langle Q_{k_i} \rangle}, S_{\beta_i \langle Q_{l_i} \rangle} \rangle$ $(i = 1, 2, ..., n)$ *be the collection of IDHLNs, if* $\overline{A}_{\overline{B}i} = \langle S_{\alpha \langle Q_k \rangle}, S_{\beta \langle Q_l \rangle} \rangle$ *for all* i, *then*

$$IDHLHWA(\overline{A}_{\overline{B}1}, \overline{A}_{\overline{B}2}, ..., \overline{A}_{\overline{B}n}) = \overline{A}_{\overline{B}} = \langle S_{\alpha \langle Q_k \rangle}, S_{\beta \langle Q_l \rangle} \rangle \tag{14}$$

(2). (*Monotonicity*): *Suppose* $C_{Di} = \langle S^*_{\alpha_i \langle Q^*_{k_i} \rangle}, S^*_{\beta_i \langle Q^*_{l_i} \rangle} \rangle (i \in \mathbb{N})$ *be another collection of IDHLNs such that* $S^*_{\alpha_i} \geq S_{\alpha_i}$, $S^*_{\beta_i} \leq S_{\beta_i}$ *and* $Q^*_{k_i} \geq Q_{k_i}$, $Q^*_{l_i} \leq Q_{l_i}$ *then*

$$IDHLHWA(\overline{A}_{\overline{B}1}, \overline{A}_{\overline{B}2}, ..., \overline{A}_{\overline{B}n}) \leq IDHLHWA(C_{D1}, C_{D2}, ..., C_{Dn}) \tag{15}$$

(3). (*Boundedness*): *Let* $(\overline{A}_{\overline{B}i})^- = \min\limits_{1 \leq i \leq n} \langle S_{\alpha_i \langle Q_{k_i} \rangle}, S_{\beta_i \langle Q_{l_i} \rangle} \rangle$ *and* $(\overline{A}_{\overline{B}i})^+ = \max\limits_{1 \leq i \leq n} \langle S_{\alpha_i \langle Q_{k_i} \rangle}, S_{\beta_i \langle Q_{l_i} \rangle} \rangle$, *then*

$$(\overline{A}_{\overline{B}i})^- \leq IDHLHWA(\overline{A}_{\overline{B}1}, \overline{A}_{\overline{B}2}, ..., \overline{A}_{\overline{B}n}) \leq (\overline{A}_{\overline{B}i})^+ \tag{16}$$

Definition 11. *Let* $\overline{A}_{\overline{B}i} = \langle S_{\alpha_i \langle Q_{k_i} \rangle}, S_{\beta_i \langle Q_{l_i} \rangle} \rangle$ $(i = 1, 2, ..., n)$ *be the collection of IDHLNs and* $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ *represent weight vectors of given collection restricted to* $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i = 1$. *Then based on the above operational laws intuitionistic double hierarchy Hamacher ordered weighted averaging operator is mapping* $IDHLHWA : \Omega^n \rightarrow \Omega$ *is defined in* equation (17) *as.*

$$IDHLHOWA(\overline{A}_{\overline{B}1},\overline{A}_{\overline{B}2},...,\overline{A}_{\overline{B}n})=\bigoplus_{i=1}^{n}(\omega_i\bigodot_H\overline{A}_{\wp\overline{B}_i}) \quad (17)$$

According to the above definition the aggregated result of IDHLHOWA is given in Theorem 3.

Theorem 3. *Let $\overline{A}_{\overline{B}i}=\langle S_{\alpha_i\langle Q_{k_i}\rangle},S_{\beta_i\langle Q_{l_i}\rangle}\rangle$ $(i=1,2,...,n)$ be the collection of IDHLNs and $\omega=(\omega_1,\omega_2,...,\omega_n)^T$ represent weight vectors of given collection restricted to $\omega_i>0$, $\sum_{i=1}^{n}\omega_i=1$. Then intuitionistic double hierarchy Hamacher ordered weighted averaging IDHLHWA operator is defined in* equation (18) *as.*

$$IDHLHOWA(\overline{A}_{\overline{B}1},\overline{A}_{\overline{B}2},...,\overline{A}_{\overline{B}n})=\bigoplus_{i=1}^{n}(\omega_i\bigodot_H\overline{A}_{\wp\overline{B}_i})$$

$$=\left[\begin{array}{l}\left(S_{\tau\left(\frac{\prod_{i=1}^{n}\left(1+(\gamma-1).\left(\frac{\alpha_{\wp i}}{\tau}\right)\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\left(\frac{\alpha_{\wp i}}{\tau}\right)\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\gamma-1).\left(\frac{\alpha_{\wp i}}{\tau}\right)\right)^{\omega_i}+(\gamma-1)\prod_{i=1}^{n}\left(1-\left(\frac{\alpha_{\wp i}}{\tau}\right)\right)^{\omega_i}}\right)}\langle Q_{\delta\left(\frac{\prod_{i=1}^{n}\left(1+(\gamma-1).\left(\frac{k_{\wp i}}{\delta}\right)\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\left(\frac{k_{\wp i}}{\delta}\right)\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\gamma-1).\left(\frac{k_{\wp i}}{\delta}\right)\right)^{\omega_i}+(\gamma-1).\prod_{i=1}^{n}\left(1-\left(\frac{k_{\wp i}}{\delta}\right)\right)^{\omega_i}}\right)}\rangle\right),\\ \left(S_{\tau\left(\frac{\gamma.\prod_{i=1}^{n}\left(\frac{\beta_{\wp i}}{\tau}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\gamma-1).\left(\frac{\beta_{\wp i}}{\tau}\right)\right)^{\omega_i}+(\gamma-1)\prod_{i=1}^{n}\left(1-\left(\frac{\beta_{\wp i}}{\tau}\right)\right)^{\omega_i}}\right)}\langle Q_{\delta\left(\frac{\gamma.\prod_{i=1}^{n}\left(\frac{l_{\wp i}}{\delta}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\gamma-1).\left(\frac{l_{\wp i}}{\delta}\right)\right)^{\omega_i}+(\gamma-1)\prod_{i=1}^{n}\left(1-\left(\frac{l_{\wp i}}{\delta}\right)\right)^{\omega_i}}\right)}\rangle\right)\end{array}\right] \quad (18)$$

Where $\omega_i\bigodot_H\overline{A}_{\wp\overline{B}_i}$ represent the largest value of the given collection.

Theorem 4. Let $\overline{A}_{\overline{B}i}=\langle S_{\alpha_i\langle Q_{k_i}\rangle},S_{\beta_i\langle Q_{l_i}\rangle}\rangle$ $(i=1,2,...,n)$ be the collection of IDHLNs and $\omega=(\omega_1,\omega_2,...,\omega_n)^T$ represent weight vectors of given collection restricted to $\omega_i>0$, $\sum_{i=1}^{n}\omega_i=1$. Then its desirable properties are as follows in equations (19)–(21).

(1). (*Idempotency*): *Suppose $\overline{A}_{\overline{B}i}=\langle S_{\alpha_i\langle Q_{k_i}\rangle},S_{\beta_i\langle Q_{l_i}\rangle}\rangle$ $(i=1,2,...,n)$ be the collection of IDHLNs, if $\overline{A}_{\overline{B}i}=\langle S_{\alpha\langle Q_k\rangle},S_{\beta\langle Q_l\rangle}\rangle$ for all i, then*

$$IDHLHOWA(\overline{A}_{\overline{B}1},\overline{A}_{\overline{B}2},...,\overline{A}_{\overline{B}n})=\overline{A}_{\overline{B}}=\langle S_{\alpha\langle Q_k\rangle},S_{\beta\langle Q_l\rangle}\rangle \quad (19)$$

(2). (*Monotonicity*): *Suppose $C_{Di}=\langle S^*_{\alpha_i\langle Q^*_{k_i}\rangle},S^*_{\beta_i\langle Q^*_{l_i}\rangle}\rangle(i\in\mathbb{N})$ be another collection of IDHLNs such that $S^*_{\alpha_i}\geq S_{\alpha_i}$, $S^*_{\beta_i}\leq S_{\beta_i}$ and $Q^*_{k_i}\geq Q_{k_i}$, $Q^*_{l_i}\leq Q_{l_i}$ then*

$$IDHLHOWA(\overline{A}_{\overline{B}1},\overline{A}_{\overline{B}2},...,\overline{A}_{\overline{B}n})\leq IDHLHOWA(C_{D1},C_{D2},...,C_{Dn}) \quad (20)$$

(3). (*Boundedness*): *Let $(\overline{A}_{\overline{B}i})^-=\min_{1\leq i\leq n}\langle S_{\alpha_i\langle Q_{k_i}\rangle},S_{\beta_i\langle Q_{l_i}\rangle}\rangle$ and $(\overline{A}_{\overline{B}i})^+=\max_{1\leq i\leq n}\langle S_{\alpha_i\langle Q_{k_i}\rangle},S_{\beta_i\langle Q_{l_i}\rangle}\rangle$, then*

$$(\overline{A}_{\overline{B}i})^-\leq IDHLHOWA(\overline{A}_{\overline{B}1},\overline{A}_{\overline{B}2},...,\overline{A}_{\overline{B}n})\leq(\overline{A}_{\overline{B}i})^+ \quad (21)$$

5. Conditional probability based on GRA method

The TWD approach is based on two main elements namely; LF and conditional probability. We first defined IDHLTSs to find the conditional probability.

Let $U_i = \{u_1, u_2, \ldots, u_m\}$ be the set of alternatives and $\mathscr{L}_j = \{\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_n\}$ be conditional attribute in the form of IDHLTSs with unknown weight vectors. Also, let there be e number of decision experts, $E_{\ell}(\ell = 1, 2, 3, \ldots, e)$ with unknown weights vector to provide an evaluation reports for each alternative based on conditional attribute in the form of IDHLTS. Then the decision expert matrix is represented in equation (22).

$$(M)^{\ell} = \left[\overline{A}^{\ell}_{\overline{B}_{ij}}\right]_{m\times n} (\ell = 1, 2, 3, \ldots, e) \tag{22}$$

The conditional probabilities based on the TOPSIS method with IDHLTSs are calculated in the following four steps.

Phase-I (Finding Expert weights)

In this phase, first, construct the decision expert's matrix in the form of IDHLTSs with unknown weights of each expert matrix. So when the weights of experts are unknown it is very difficult for decision expert to make an accurate decision. Hence it is important to evaluate the weights of each decision expert matrix. For this, we first construct the ideal opinion matrix, right ideal and left ideal opinion matrix, represented by (IO), (RIO) and (LIO) respectively. Then we determine the distance measure denoted by (dIO_i), $(dRIO_i)$ and $(dLIO_i)$ from decision experts matrix $(M)^{\ell}$ to IO, RIO and LIO. Further, we find the closeness index and at last calculate the weights of each decision expert matrix. The stepwise detail is as follows.

(I-a) Construct decision expert's matrix in the form of IDHLTSs as follows in equation (23).

$$(M)^{\ell} = \left[\overline{A}^{\ell}_{\overline{B}_{ij}}\right]_{m\times n} = \begin{matrix} \\ u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{matrix} \begin{bmatrix} \mathscr{L}_1 & \mathscr{L}_2 & \mathscr{L}_3 & \mathscr{L}_n \\ \overline{A}^{\ell}_{\overline{B}_{11}} & \overline{A}^{\ell}_{\overline{B}_{12}} & \cdots & \overline{A}^{\ell}_{\overline{B}_{1n}} \\ \overline{A}^{\ell}_{\overline{B}_{21}} & \overline{A}^{\ell}_{\overline{B}_{22}} & \cdots & \overline{A}^{\ell}_{\overline{B}_{2n}} \\ \vdots & \cdots & \ddots & \vdots \\ \overline{A}^{\ell}_{\overline{B}_{m1}} & \overline{A}^{\ell}_{\overline{B}_{m2}} & \cdots & \overline{A}^{\ell}_{\overline{B}_{mn}} \end{bmatrix} \tag{23}$$

Where $\overline{A}^{\ell}_{\overline{B}_{ij}} = \langle S^{\ell}_{\alpha_{ij}\langle Q_{k_{ij}}\rangle}, S^{\ell}_{\beta_{ij}\langle Q_{l_{ij}}\rangle} \rangle$ $(i = 1, 2, \ldots, m)$, $(j = 1, 2, \ldots, m)$ and $(\ell = 1, 2, \ldots, e)$..

(I-b) Construct the Ideal opinion matrix IO by using equation (25) that is closer to each decision expert matrix as follows in equation (24).

$$IO = \begin{bmatrix} IO_{11} & IO_{12} & \cdots & IO_{1n} \\ IO_{21} & IO_{22} & \cdots & IO_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ IO_{m1} & IO_{m2} & \cdots & IO_{mn} \end{bmatrix} \tag{24}$$

Where $IO_{ij} = \sum_{\ell=1}^{e} \frac{1}{e} \langle S^{\ell}_{\alpha_{ij}\langle Q_{k_{ij}}\rangle}, S^{\ell}_{\beta_{ij}\langle Q_{l_{ij}}\rangle} \rangle = .$

$$\left[\left(S_{\tau\left(\frac{\prod_{\ell=1}^{e}\left(1+(\gamma-1).\left(\frac{\alpha_i}{\tau}\right)\right)^{\frac{1}{e}} - \prod_{\ell=1}^{e}\left(1-\left(\frac{\alpha_i}{\tau}\right)\right)^{\frac{1}{e}}}{\prod_{\ell=1}^{e}\left(1+(\gamma-1).\left(\frac{\alpha_i}{\tau}\right)\right)^{\frac{1}{e}} + (\gamma-1).\prod_{\ell=1}^{e}\left(1-\left(\frac{\alpha_i}{\tau}\right)\right)^{\frac{1}{e}}}\right)} \langle Q_{\delta\left(\frac{\prod_{\ell=1}^{e}\left(1+(\gamma-1).\left(\frac{k_i}{\delta}\right)\right)^{\frac{1}{e}} - \prod_{\ell=1}^{e}\left(1-\left(\frac{k_i}{\delta}\right)\right)^{\frac{1}{e}}}{\prod_{\ell=1}^{e}\left(1+(\gamma-1).\left(\frac{k_i}{\delta}\right)\right)^{\frac{1}{e}} + (\gamma-1)\prod_{\ell=1}^{e}\left(1-\left(\frac{k_i}{\delta}\right)\right)^{\frac{1}{e}}}\right)} \rangle \right),\right.$$

$$\left.\left(S_{\tau\left(\frac{\gamma\cdot\prod_{\ell=1}^{e}\left(\frac{\beta_i}{\tau}\right)^{\frac{1}{e}}}{\prod_{\ell=1}^{e}\left(1+(\gamma-1).\left(\frac{\beta_i}{\tau}\right)\right)^{\frac{1}{e}}+(\gamma-1)\prod_{\ell=1}^{e}\left(1-\left(\frac{\beta_i}{\tau}\right)\right)^{\frac{1}{e}}}\right)}\langle Q_{\delta\left(\frac{\gamma\cdot\prod_{\ell=1}^{e}\left(\frac{l_i}{\delta}\right)^{\frac{1}{e}}}{\prod_{\ell=1}^{e}\left(1+(\gamma-1).\left(\frac{l_i}{\delta}\right)\right)^{\frac{1}{e}}+(\gamma-1)\prod_{\ell=1}^{n}\left(1-\left(\frac{l_i}{\delta}\right)\right)^{\frac{1}{e}}}\right)}\rangle\right)\right] \tag{25}$$

(I-c) Compute the right ideal opinion *RIO* and left ideal opinion *LIO* matrix's by using equations (27) and (29) as follows in equations (26) and (28):

$$RIO=\begin{bmatrix} RIO_{11} & RIO_{12} & \cdots & RIO_{1n} \\ RIO_{21} & RIO_{22} & \cdots & RIO_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ RIO_{m1} & RIO_{m2} & \cdots & RIO_{mn} \end{bmatrix} \tag{26}$$

Where

$$RIO_{ij}=\left\{\max_{\ell}\left(Sc\langle S^{\ell}_{\alpha_{ij}\langle Q_{k_{ij}}\rangle}, S^{\ell}_{\beta_{ij}\langle Q_{l_{ij}}\rangle}\rangle\right)\right\} \tag{27}$$

$$LIO=\begin{bmatrix} LIO_{11} & LIO_{12} & \cdots & LIO_{1n} \\ LIO_{21} & LIO_{22} & \cdots & LIO_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ LIO_{m1} & LIO_{m2} & \cdots & LIO_{mn} \end{bmatrix} \tag{28}$$

Where

$$LIO_{ij}=\left\{\min_{\ell}\left(Sc\langle S^{\ell}_{\alpha_{ij}\langle Q_{k_{ij}}\rangle}, S^{\ell}_{\beta_{ij}\langle Q_{l_{ij}}\rangle}\rangle\right)\right\} \tag{29}$$

(I-d) Calculate the distance measure denoted by dIO_i, $dRIO_i$ and $dLIO_i$ from each decision experts matrix $(M)^{\ell}$ to *IO*, *RIO* and *LIO* matrixs by using equation (7).

(I-e) Evaluate closeness indices (CIs) "Yue [52]" by using equation (30).

$$CI^{(\ell)}=\frac{\sum_{i=1}^{m} dRIO_i+\sum_{i=1}^{m} dLIO_i}{\sum_{i=1}^{m} dIO_i+\sum_{i=1}^{m} dRIO_i+\sum_{i=1}^{m} dLIO_i} \tag{30}$$

(I-f) **Calculate** the decision expert's weight by the following formula (31) as:

$$\omega^{(\ell)}=\frac{CI^{(\ell)}}{\sum_{i=1}^{\ell} CI^{(\ell)}} \tag{31}$$

For $\ell=1,2,\dots,e$.

Phase II (Finding criteria weights)

(II-a) Aggregate the expert matrix's M^{ℓ} to single matrix M by using intuitionistic double hierarchy linguistic weighted averaging operators' definition 10 as given in (32).

$$M=\left[\overline{A}_{\overline{B}_{ij}}\right]_{m\times n}=\begin{matrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{matrix}\begin{bmatrix} \mathscr{L}_1 & \mathscr{L}_2 & \mathscr{L}_3 & \mathscr{L}_n \\ \overline{A}_{\overline{B}_{11}} & \overline{A}_{\overline{B}_{12}} & \cdots & \overline{A}_{\overline{B}_{1n}} \\ \overline{A}_{\overline{B}_{21}} & \overline{A}_{\overline{B}_{22}} & \cdots & \overline{A}_{\overline{B}_{2n}} \\ \vdots & \cdots & \ddots & \vdots \\ \overline{A}_{\overline{B}_{m1}} & \overline{A}_{\overline{B}_{m2}} & \cdots & \overline{A}_{\overline{B}_{mn}} \end{bmatrix} \tag{32}$$

Where $\overline{A}_{\overline{B}_{ij}}=\langle S_{\alpha_{ij}\langle Q_{k_{ij}}\rangle}, S_{\beta_{ij}\langle Q_{l_{ij}}\rangle}\rangle$ for $(i=1,2,\dots,m)$ and $(j=1,2,\dots,n)$.

(II-b) By using equation (33) calculate the entropy measure [53] corresponding to each criteria of aggregated matrix.

$$E_i(\Psi)=\frac{1}{n}\sum_{j=1}^{n}\left[\left\{\sqrt{2}\ cos\ \pi\left(\frac{\left(\frac{\alpha_i}{\tau}\right)-\left(\frac{\beta_i}{\delta}\right)}{4}\right)+\sqrt{2}\ cos\ \pi\left(\frac{\left(\frac{k_i}{\tau}\right)-\left(\frac{l_i}{\delta}\right)}{4}\right)-1\right\}\times\frac{1}{\sqrt{2}-1}\right]. \tag{33}$$

The weights of the criteria are determined by the following equations (34) and (35).

$$E_j(A)=\frac{1}{m}\sum_{i=1}^{m}E_i(\Psi), j=1,2,...,n \tag{34}$$

Implies that

$$\varpi_j=\frac{E_j(A)}{\sum_{j=1}^{n}E_j(A)} \tag{35}$$

Phase-III (Conditional Probability based on TOPSIS method)

(III-a) Compute the positive ideal solution (*PIS*) and negative ideal solution (*NIS*) of IDHLTSs by equations (36) and (37). i.e. $U^+=(u_1^+,u_2^+,u_3^+,\ldots,u_n^+)$ and $U^-=(u_1^-,u_2^-,u_3^-,\ldots,u_n^-)$ as follows:

$$u_j^+=\left\{\max_{1\le i\le n}\left(Sc\langle S_{\alpha_{ij}\langle Q_{k_{ij}}\rangle},S_{\beta_{ij}\langle Q_{l_{ij}}\rangle}\rangle\right)\right\} \tag{36}$$

And

$$u_j^-=\left\{\min_{1\le i\le n}\left(Sc\langle S_{\alpha_{ij}\langle Q_{k_{ij}}\rangle},S_{\beta_{ij}\langle Q_{l_{ij}}\rangle}\rangle\right)\right\} \tag{37}$$

Where $(j=1,2,...,m)$. When the TWDs, PIS and NIS are added together, they are equal to the set of states, X and X^C..

(III-b) **Determine** the distance of each u_i object of aggregated matrix to u_j^+ and u_j^- by equations (38) and (39).

$$d\left(u_i,u_j^+\right)=\left[\frac{1}{4n}\sum_{j=1}^{n}\varpi_j\begin{pmatrix}\left|S_{\left(\frac{\alpha_{ij}}{\tau}\right)}-S_{\left(\frac{\alpha_j^+}{\tau}\right)}\right|^{\lambda}+\left|Q_{\left(\frac{k_{ij}}{\delta}\right)}-Q_{\left(\frac{k_j^+}{\delta}\right)}\right|^{\lambda}+\\ \left|S_{\left(\frac{\beta_{ij}}{\tau}\right)}-S_{\left(\frac{\beta_j^+}{\tau}\right)}\right|^{\lambda}+\left|Q_{\left(\frac{l_{ij}}{\delta}\right)}-Q_{\left(\frac{l_j^+}{\delta}\right)}\right|^{\lambda}\end{pmatrix}\right]^{\frac{1}{\lambda}} \tag{38}$$

$$d\left(u_i,u_j^-\right)=\left[\frac{1}{4n}\sum_{j=1}^{n}\varpi_j\begin{pmatrix}\left|S_{\left(\frac{\alpha_{ij}}{\tau}\right)}-S_{\left(\frac{\alpha_j^-}{\tau}\right)}\right|^{\lambda}+\left|Q_{\left(\frac{k_{ij}}{\delta}\right)}-Q_{\left(\frac{k_j^-}{\delta}\right)}\right|^{\lambda}+\\ \left|S_{\left(\frac{\beta_{ij}}{\tau}\right)}-S_{\left(\frac{\beta_j^-}{\tau}\right)}\right|^{\lambda}+\left|Q_{\left(\frac{l_{ij}}{\delta}\right)}-Q_{\left(\frac{l_j^-}{\delta}\right)}\right|^{\lambda}\end{pmatrix}\right]^{\frac{1}{\lambda}} \tag{39}$$

(III-c) Calculate the relative closeness (RC) of the object u_i by equation (40) and denoted by $\mathscr{T}_i$..

$$\mathscr{T}_i=\frac{d\left(u_i,u_j^-\right)}{d\left(u_i,u_j^+\right)+d\left(u_i,u_j^-\right)} \tag{40}$$

(III-d) Here $\mathscr{T}_i$ assume to be conditional probability represented by (*Pr*) of an object belongs to the state X as follows in equation (41).

$$Pr(X\,/\,u_i)=\mathscr{T}_i \tag{41}$$

Where $0\le Pr(X\,/u_i)\le 1$..

6. A novel DTRSs model with DHLDTRS expression of loss functions

Phase-IV: As from the definition of IDHLSs, they consists of two terms namely first and second hierarchy linguistic term sets, which easily handle uncertainty and vagueness more than a single term set. In this section, we attempt to express the loss function in TWDs using IDHLTSs and suggest a new DTRSs model based on IDHLTSs information. Here, we addressed the loss functions in TWDs using intuitionistic double hierarchy linguistic number (IDHLTNs) as well as how to build a new DTRS model for IDHLTNs. This model consist of two states such as $\psi = \{X, X^c\}$ which expressed an element belong to X or not, and regard to three actions like $\varphi = \{a_P, a_B, a_N\}$. Where a_P, a_B, a_N shows actions which is applied for determining the objects u_i such as a_P denotes $u_i \in POS(X)$ positive region, a_B denotes $u_i \in BND(X)$ boundary region and a_N denotes $u_i \in NEG(X)$ of respectively. The overall situation are represents of an object, while the judgement are represented by the action. Here we construct the LF matrix for the IDHLTS environment given in Table 1.

From Table 1, we see that the determined LFs are IDHLTNs. $h_{\rho PP}, h_{\rho BP}$ and $h_{\rho NP}$ represent loss degrees with DHLNs produced by taking actions of a_P, a_B and a_N, for u given state X, respectively. Similarly the loss degrees generated by conducting the same actions on u specific state . So, in this case, $h_\rho \neq \varphi$. Based on the definition of IDHLNs and the semantics of DTRS [38,54] the acceptable relation are given in equations 42 and 43.

$$h_{\rho PP} \preccurlyeq h_{\rho BP} \prec h_{\rho NP} \tag{42}$$

$$h_{\rho NN} \preccurlyeq h_{\rho BN} \prec h_{\rho NN} \tag{43}$$

That is, the loss degrees of incorrect judgment is more than the loss degree of delaying decision, and both of these loss degree is more than the loss degree of correct judgement. The conditional probability is one of the important part of Bayesian decision making technique [30,31].

$Pr(X|u_i), Pr(X^c|u_i)$, represents the conditional probability of an object u_i belonging to X and X^c respectively. They are all related to real numbers, such that $PPr(X|u_i), Pr(X^c|u_i) = 1$. Thus, given in object u_i the expected loss for the corresponding action $R(a_\Lambda|u_i)$ where $(\Lambda = P, B, N)$ can be determined in following equations (44)–(46).

$$R(a_P|u_i) = Pr(X|u_i) \odot_H h_{\rho PP} \oplus_H Pr(X^c|u_i) \odot_H h_{\rho PN} \tag{44}$$

$$R(a_B|u_i) = Pr(X|u_i) \odot_H h_{\rho BP} \oplus_H Pr(X^c|u_i) \odot_H h_{\rho BN} \tag{45}$$

$$R(a_N|u_i) = Pr(X|u_i) \odot_H h_{\rho NP} \oplus_H Pr(X^c|u_i) \odot_H h_{\rho NN} \tag{46}$$

Based on the minimum loss decision rules can be derived by using the result given [27,28], which are given in equations (47)–(49) as follows

(1) Decide $\mathrm{u_i} \in \mathrm{POS(X)}$ indicate that the action are acceptable, if

$$\mathrm{Sc(R(a_P|u_i)) \leq Sc(R(a_B|u_i)) \leq Sc(R(a_N|u_i))} \tag{47}$$

(2) Decide $u_i \in BND(X)$, shows the action are delayed, if

$$Sc(R(a_B|u_i)) \leq Sc(R(a_P|u_i)) \leq Sc(R(a_N|u_i)) \tag{48}$$

(3) Decide $u_i \in NEG(X)$, represents the action are rejected, if

$$Sc(R(a_N|u_i)) \leq Sc(R(a_P|u_i)) \leq Sc(R(a_B|u_i)) \tag{49}$$

Based on the results mentioned above, we propose a novel MCGDM method in the environment of IDHLTSs. As shown in Fig. 1, the description of the proposed method is given in the following short steps.

Step 1. On the basis of the practical context, we determine the elements of the IDHLTSs information system including alternatives and criteria.

Step 2. According to the distance measure of ideal opinion, left ideal opinion, and right ideal opinion, and the closeness indices and decision expert's weights are attained.

Step 3. Determine the aggregated matrix by using the weights of the decision expert weight and the proposed IDHLHWA operator.

Step (4). The weights of the criteria can be evaluated by entropy measure. The weights of criteria reflect the importance of these criteria in the evaluation system.

Step (5). Determine u_j^+ and u_j^- based on formula (36) and (37). Combining the TWDs, PIS^+ and NIS^- are equivalent to the set of

Table 1
Loss functions.

	$X(P)$	$X^c(N)$
a_P	$h_{\rho PP} = \langle S_{\alpha_{PP}\langle Q_{k_{PP}}\rangle}, S_{\beta_{PP}\langle Q_{l_{PP}}\rangle} \rangle$	$h_{\rho PN} = \langle S_{\alpha_{PN}\langle Q_{k_{PN}}\rangle}, S_{\beta_{PN}\langle Q_{l_{PN}}\rangle} \rangle$
a_B	$h_{\rho BP} = \langle S_{\alpha_{BP}\langle Q_{k_{BP}}\rangle}, S_{\beta_{BP}\langle Q_{l_{BP}}\rangle} \rangle$	$h_{\rho BN} = \langle S_{\alpha_{BN}\langle Q_{k_{BN}}\rangle}, S_{\beta_{BN}\langle Q_{l_{BN}}\rangle} \rangle$
a_N	$h_{\rho NP} = \langle S_{\alpha_{NP}\langle Q_{k_{NP}}\rangle}, S_{\beta_{NP}\langle Q_{l_{NP}}\rangle} \rangle$	$h_{\rho NN} = \langle S_{\alpha_{NN}\langle Q_{k_{NN}}\rangle}, S_{\beta_{NN}\langle Q_{l_{NN}}\rangle} \rangle$

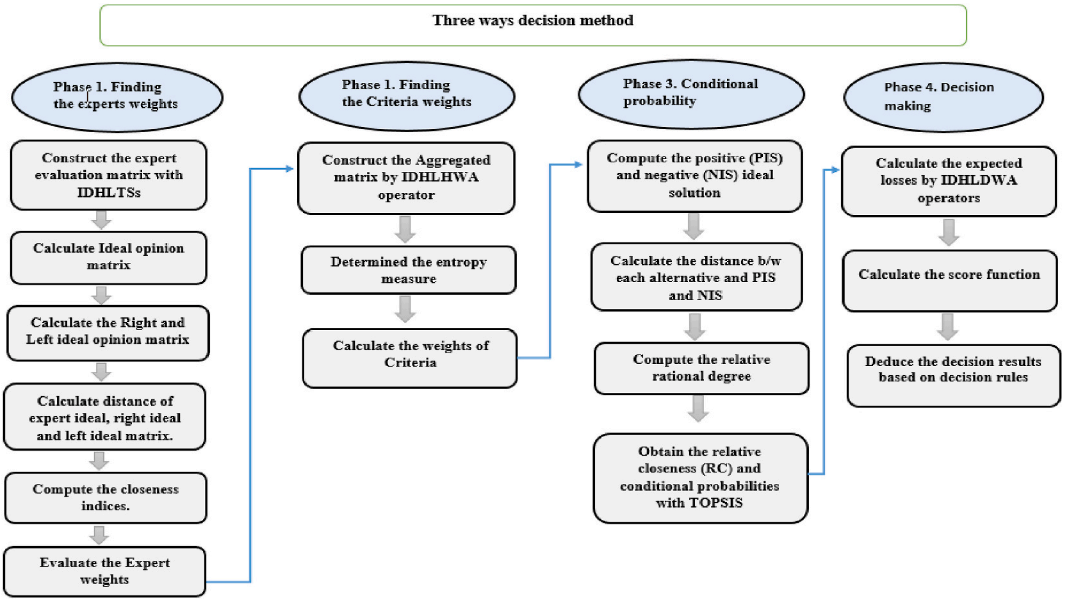

Fig. 1. Graphical representation of proposed method.

states, that is X and X^c. Then determine the distance of each u_i object of aggregated matrix to u_j^+ and the u_j^- by equations (38) and (39), and the relative closeness coefficient (RC) expressed as $\mathscr{T}_i$ can be calculated by formulas (40-41). And the conditional probability that the object belongs to the state X can be estimated based on the TOPSIS method.

Step (6). Based on the Hamacher aggregation operators, the expected loss of each action can be aggregated respectively by formulas (44-46). And then, the score functions of expected losses can be calculated.

Step (7). In light of the decision rules (47–49), we can ascertain the decision result for each object further. And the decision rules (47–49) are deduced by the minimum-loss principle.

7. Application of proposed method

A practical DM problem concerning to selection of sustainable Logistics supplier considered as an example in this section to validate the applicability and practicality of the developed methodology.

7.1. Case study

Logistics supplier selection is a multi-criteria problem that necessitates the consideration of several attributes. Thus, for logistics provider selection, Spencer et al. [2] and Govindan et al. [9] identified 23 and 35 potential attributes, respectively. Aicha identified 11 key 3 PL criteria, each with its own set of criteria [5]. Although the above selection criteria are often used in the selection of 3 PLs and they are operational in nature, supply chain strategy and service value creation elements are rarely considered in the selection of logistics providers in past studies. In the scenario of logistics service value co-creation, it is necessary to reconsider the selection criteria.

The main principle of creating and maintaining customer relationships and the main objective and process of economic exchange is the creation of value [55]. In today's supply chain management climate, greater and greater businesses recognize the potential of logistics service value co-creation with customers. Logistics service value co-creation has emerged as a new option for 3 PLs to gain a competitive edge. While also providing clients with more tailored product and service offerings [56]. One of the best critical concerns for logistics service value co-creation in a supply chain management environment is supplier selection [57]. The combination of conventional selection variables including cost, quality, fast response, and location with new aspects in service value co-creation like as unique value creation, knowledge management, and service development is an emerging trend in 3 PL supplier selection. We combine conventional operational-focused selection characteristics with value co-creation focused supply chain management strategic selection criteria to provide a broad set of selection criteria for logistics service value co-creation scenarios. The step wise procedure are given in next section.

7.1.1. Numerical steps

In this section, we will explore at a group of logistic supply chain DM challenges that include a 4 PL solution supplier searching for the best 3 PL provider for service value co-creation with its client (an international manufacturing company group). Assume there are six global 3 PL suppliers $U_i(i = 1, 2, \ldots, 6)$ and three decision experts with unknown weight $\omega^{(\ell)}$ from various professional disciplines are included in the decision-making. Let there are six criteria $\mathscr{L}_j(i = 1, 2, \ldots, 6)$ with unknown weight information ϖ_j for selecting 3 PL suppliers in a service value co-creation environment, the detail description are as follows.

$\mathscr{L}_1$: Value collaboration ability: The foundation of value co-creation is value collaboration, which should be presented from the perspectives of strategic, business aim, and market.

$\mathscr{L}_2$: Knowledge matching ability: Knowledge is essential in logistics service value co-creation, knowledge management is included in the value co-creation process.

$\mathscr{L}_3$: Service innovation ability: Service innovation demonstrates the 3 PL ability to achieve service value co-creation in logistics service solutions.

$\mathscr{L}_4$: Quality of service: The conventional assessment quality for 3 PL selection is service quality, which also shows service value co-creation.

$\mathscr{L}_5$: Resource interaction ability: Logistics resources are the material basis forco-creation service value. The logistics service is provided combination and interaction with various logistic resources from the participating 3 PLs.

$\mathscr{L}_6$: Risk analysis: Risk is an important factor in logistic service value Cocreation order delay sare the risk for all companies which is an important outsource their logistics activities.

The decision results are summarized in the following steps four Phases.

Phase-I (Expert weights):

(I-a) Construct the experts evaluation matrix E_1, E_2 and E_3 in the form of intuitionistic double hierarchy linguistic term sets, so the linguistic term set is denoted by $S = \{S_0 = \text{medium}, S_1 = \text{low}, S_2 = \text{sligtly low}, S_3 = \text{very low}, S_4 = \text{high}, S_5 = \text{slightly high}, S_6 = \text{very high}\}$ and $Q = \{Q_0 = \text{right}, Q_1 = \text{only right}, Q_2 = \text{much}, Q_3 = \text{very much}, Q_4 = \text{little}, Q_5 = \text{just little}, Q_6 = \text{extremely little}\}$ are defined based on the following set as follows in Tables 2a, 2b, 3a, 3b, 4a and 4b:

(I-b) Calculate the Ideal opinion matrix by using equation (25) as shown in Table-5a, 5b:

(I-c) Evaluated the right and left ideal matrix by using equations 27 and 29 is shown in Table-6a, 6b, 7a and 7b:

(I-d, e, f) Based on equation (7) determine the distance measure denoted by dIO_i, $dRIO_i$ and $dLIO_i$ and by using equations 30 and 31 the weights of experts are determined as follows:

$$\omega^1 = 0.353, \omega^2 = 0.324, \omega^3 = 0.323$$

Phase-II (Criteria weights)

(II-a) Aggregate all the expert matrix's M^{ℓ} to single matrix M by using the proposed aggregation (IDHLHWA) operators is given in Table-8a, 8b:

(II-b, c) Calculate the entropy measure (33) of aggregated matrix, and by equations (34) and (35) calculate the criteria weights as follows.

$$\varpi_1 = 0.184, \varpi_2 = 0.17, \varpi_3 = 0.176, \varpi_4 = 0.17, \varpi_5 = 0.124, \varpi_6 = 0.176.$$

Phase-III (Conditional probability)

(III-a) Applying equations (36) and (37) to determine the *PIS* and *NIS* as follows.

$$u_j^+ = \begin{pmatrix} \langle S_{6\langle Q_{2.94}\rangle}, S_{0\langle Q_{1.26}\rangle}\rangle, \langle S_{3.4\langle Q_{2.57}\rangle}, S_{1.62\langle Q_2\rangle}\rangle, \langle S_{2.57\langle Q_{2.19}\rangle}, S_{2.13\langle Q_{2.1}\rangle}\rangle, \\ \langle S_{2.77\langle Q_{2.04}\rangle}, S_{2.13\langle Q_{1.6}\rangle}\rangle, \langle S_{6\langle Q_{1.2}\rangle}, S_{0\langle Q_{3.33}\rangle}\rangle, \langle S_{3.42\langle Q_{2.47}\rangle}, S_{1.6\langle Q_{1.69}\rangle}\rangle \end{pmatrix}$$

$$u_j^- = \begin{pmatrix} \langle S_{1.77\langle Q_{1.98}\rangle}, S_{1.57\langle Q_{2.63}\rangle}\rangle, \langle S_{1.52\langle Q_{1.88}\rangle}, S_{2\langle Q_{1.62}\rangle}\rangle, \langle S_{1.78\langle Q_{1.79}\rangle}, S_{2.1\langle Q_0\rangle}\rangle, \\ \langle S_{2.69\langle Q_{1.82}\rangle}, S_{0\langle Q_0\rangle}\rangle, \langle S_{0.88\langle Q_{3.12}\rangle}, S_{2.3\langle Q_0\rangle}\rangle, \langle S_{1.47\langle Q_{1.52}\rangle}, S_{1.8\langle Q_{2.36}\rangle}\rangle \end{pmatrix}$$

(III-b) By equations (38) and (39) the distance of each object u_i of aggregated matrix to u_j^+ and u_j^- are calculated in Table-9.

(III-c, d). The relative closeness $\mathscr{F}_i$ and conditional probability are evaluated by equations (40) and (41) in Table-10.

Phase-IV (Decision making)

(IV-a) Construct the loss functions matrix in the form of IDHLTSs is given in Table 11 as follows:

(IV-b) On the basis of loss functions and conditional probability, the expected losses can be determined apply equations (44)–(46). Consider that $\gamma = 2$ is a parameter. To make the comparison easier, the expected loss is finally transform into score functions. The result is given in Table 12.

(IV-c) Determine the decision result for each object further using the decision rules (1), (2) and (3) based on the minimum loss

Table-2a
Expert matrix E_1.

	$\mathscr{L}_1$	$\mathscr{L}_2$	$\mathscr{L}_3$
u_1	$\langle S_{3\langle Q_2\rangle}, S_{1\langle Q_3\rangle}\rangle$	$\langle S_{2\langle Q_3\rangle}, S_{3\langle Q_2\rangle}\rangle$	$\langle S_{0\langle Q_2\rangle}, S_{3\langle Q_4\rangle}\rangle$
u_2	$\langle S_{2\langle Q_1\rangle}, S_{3\langle Q_4\rangle}\rangle$	$\langle S_{5\langle Q_1\rangle}, S_{1\langle Q_1\rangle}\rangle$	$\langle S_{3\langle Q_1\rangle}, S_{1\langle Q_2\rangle}\rangle$
u_3	$\langle S_{4\langle Q_2\rangle}, S_{1\langle Q_3\rangle}\rangle$	$\langle S_{2\langle Q_4\rangle}, S_{2\langle Q_1\rangle}\rangle$	$\langle S_{3\langle Q_2\rangle}, S_{1\langle Q_4\rangle}\rangle$
u_4	$\langle S_{6\langle Q_4\rangle}, S_{0\langle Q_1\rangle}\rangle$	$\langle S_{4\langle Q_3\rangle}, S_{1\langle Q_2\rangle}\rangle$	$\langle S_{5\langle Q_2\rangle}, S_{1\langle Q_3\rangle}\rangle$
u_5	$\langle S_{1\langle Q_2\rangle}, S_{4\langle Q_3\rangle}\rangle$	$\langle S_{4\langle Q_2\rangle}, S_{0\langle Q_3\rangle}\rangle$	$\langle S_{6\langle Q_2\rangle}, S_{0\langle Q_2\rangle}\rangle$
u_6	$\langle S_{2\langle Q_3\rangle}, S_{4\langle Q_2\rangle}\rangle$	$\langle S_{5\langle Q_3\rangle}, S_{1\langle Q_2\rangle}\rangle$	$\langle S_{2\langle Q_4\rangle}, S_{3\langle Q_1\rangle}\rangle$

Table-2b
Expert matrix E_1.

	$\mathscr{L}_4$	$\mathscr{L}_5$	$\mathscr{L}_6$
u_1	$\langle S_{1\langle Q_3\rangle}, S_{4\langle Q_3\rangle}\rangle$	$\langle S_{4\langle Q_3\rangle}, S_{1\langle Q_2\rangle}\rangle$	$\langle S_{3\langle Q_2\rangle}, S_{1\langle Q_3\rangle}\rangle$
u_2	$\langle S_{3\langle Q_5\rangle}, S_{1\langle Q_1\rangle}\rangle$	$\langle S_{6\langle Q_1\rangle}, S_{0\langle Q_4\rangle}\rangle$	$\langle S_{2\langle Q_3\rangle}, S_{2\langle Q_2\rangle}\rangle$
u_3	$\langle S_{4\langle Q_2\rangle}, S_{1\langle Q_0\rangle}\rangle$	$\langle S_{1\langle Q_4\rangle}, S_{2\langle Q_0\rangle}\rangle$	$\langle S_{4\langle Q_1\rangle}, S_{1\langle Q_3\rangle}\rangle$
u_4	$\langle S_{0\langle Q_3\rangle}, S_{4\langle Q_2\rangle}\rangle$	$\langle S_{1\langle Q_3\rangle}, S_{3\langle Q_2\rangle}\rangle$	$\langle S_{0\langle Q_0\rangle}, S_{2\langle Q_4\rangle}\rangle$
u_5	$\langle S_{1\langle Q_5\rangle}, S_{4\langle Q_0\rangle}\rangle$	$\langle S_{4\langle Q_2\rangle}, S_{1\langle Q_0\rangle}\rangle$	$\langle S_{3\langle Q_4\rangle}, S_{3\langle Q_1\rangle}\rangle$
u_6	$\langle S_{3\langle Q_1\rangle}, S_{3\langle Q_2\rangle}\rangle$	$\langle S_{2\langle Q_1\rangle}, S_{3\langle Q_1\rangle}\rangle$	$\langle S_{5\langle Q_3\rangle}, S_{0\langle Q_2\rangle}\rangle$

Table-3a
Expert matrix E_2.

	$\mathscr{L}_1$	$\mathscr{L}_2$	$\mathscr{L}_3$
u_1	$\langle S_{1\langle Q_0\rangle}, S_{2\langle Q_5\rangle}\rangle$	$\langle S_{3\langle Q_2\rangle}, S_{2\langle Q_1\rangle}\rangle$	$\langle S_{2\langle Q_0\rangle}, S_{4\langle Q_5\rangle}\rangle$
u_2	$\langle S_{4\langle Q_2\rangle}, S_{1\langle Q_3\rangle}\rangle$	$\langle S_{0\langle Q_1\rangle}, S_{4\langle Q_3\rangle}\rangle$	$\langle S_{2\langle Q_3\rangle}, S_{2\langle Q_2\rangle}\rangle$
u_3	$\langle S_{2\langle Q_2\rangle}, S_{3\langle Q_3\rangle}\rangle$	$\langle S_{3\langle Q_1\rangle}, S_{2\langle Q_4\rangle}\rangle$	$\langle S_{2\langle Q_2\rangle}, S_{3\langle Q_0\rangle}\rangle$
u_4	$\langle S_{2\langle Q_1\rangle}, S_{0\langle Q_1\rangle}\rangle$	$\langle S_{2\langle Q_1\rangle}, S_{3\langle Q_2\rangle}\rangle$	$\langle S_{2\langle Q_1\rangle}, S_{3\langle Q_1\rangle}\rangle$
u_5	$\langle S_{1\langle Q_3\rangle}, S_{4\langle Q_2\rangle}\rangle$	$\langle S_{3\langle Q_2\rangle}, S_{2\langle Q_2\rangle}\rangle$	$\langle S_{3\langle Q_2\rangle}, S_{1\langle Q_1\rangle}\rangle$
u_6	$\langle S_{1\langle Q_5\rangle}, S_{4\langle Q_0\rangle}\rangle$	$\langle S_{2\langle Q_1\rangle}, S_{4\langle Q_2\rangle}\rangle$	$\langle S_{2\langle Q_1\rangle}, S_{3\langle Q_3\rangle}\rangle$

Table-3b
Expert matrix E_2.

	$\mathscr{L}_4$	$\mathscr{L}_5$	$\mathscr{L}_6$
u_1	$\langle S_{3\langle Q_2\rangle}, S_{1\langle Q_4\rangle}\rangle$	$\langle S_{3\langle Q_2\rangle}, S_{1\langle Q_2\rangle}\rangle$	$\langle S_{2\langle Q_3\rangle}, S_{2\langle Q_1\rangle}\rangle$
u_2	$\langle S_{1\langle Q_0\rangle}, S_{3\langle Q_3\rangle}\rangle$	$\langle S_{4\langle Q_2\rangle}, S_{0\langle Q_3\rangle}\rangle$	$\langle S_{4\langle Q_1\rangle}, S_{1\langle Q_3\rangle}\rangle$
u_3	$\langle S_{4\langle Q_1\rangle}, S_{0\langle Q_3\rangle}\rangle$	$\langle S_{2\langle Q_3\rangle}, S_{3\langle Q_0\rangle}\rangle$	$\langle S_{2\langle Q_2\rangle}, S_{2\langle Q_0\rangle}\rangle$
u_4	$\langle S_{1\langle Q_2\rangle}, S_{4\langle Q_2\rangle}\rangle$	$\langle S_{4\langle Q_1\rangle}, S_{1\langle Q_3\rangle}\rangle$	$\langle S_{5\langle Q_4\rangle}, S_{1\langle Q_1\rangle}\rangle$
u_5	$\langle S_{1\langle Q_0\rangle}, S_{1\langle Q_6\rangle}\rangle$	$\langle S_{6\langle Q_4\rangle}, S_{0\langle Q_1\rangle}\rangle$	$\langle S_{0\langle Q_3\rangle}, S_{6\langle Q_2\rangle}\rangle$
u_6	$\langle S_{3\langle Q_1\rangle}, S_{1\langle Q_2\rangle}\rangle$	$\langle S_{3\langle Q_3\rangle}, S_{1\langle Q_2\rangle}\rangle$	$\langle S_{0\langle Q_0\rangle}, S_{3\langle Q_4\rangle}\rangle$

Table-4a
Expert matrix E_3.

	$\mathscr{L}_1$	$\mathscr{L}_2$	$\mathscr{L}_3$
u_1	$\langle S_{2\langle Q_4\rangle}, S_{2\langle Q_1\rangle}\rangle$	$\langle S_{4\langle Q_1\rangle}, S_{2\langle Q_3\rangle}\rangle$	$\langle S_{1\langle Q_4\rangle}, S_{5\langle Q_3\rangle}\rangle$
u_2	$\langle S_{0\langle Q_1\rangle}, S_{5\langle Q_3\rangle}\rangle$	$\langle S_{2\langle Q_2\rangle}, S_{3\langle Q_3\rangle}\rangle$	$\langle S_{0\langle Q_2\rangle}, S_{3\langle Q_1\rangle}\rangle$
u_3	$\langle S_{2\langle Q_1\rangle}, S_{2\langle Q_3\rangle}\rangle$	$\langle S_{0\langle Q_1\rangle}, S_{2\langle Q_1\rangle}\rangle$	$\langle S_{1\langle Q_2\rangle}, S_{3\langle Q_2\rangle}\rangle$
u_4	$\langle S_{3\langle Q_4\rangle}, S_{3\langle Q_2\rangle}\rangle$	$\langle S_{2\langle Q_3\rangle}, S_{1\langle Q_2\rangle}\rangle$	$\langle S_{3\langle Q_4\rangle}, S_{2\langle Q_1\rangle}\rangle$
u_5	$\langle S_{2\langle Q_3\rangle}, S_{2\langle Q_1\rangle}\rangle$	$\langle S_{3\langle Q_2\rangle}, S_{3\langle Q_2\rangle}\rangle$	$\langle S_{2\langle Q_3\rangle}, S_{2\langle Q_0\rangle}\rangle$
u_6	$\langle S_{1\langle Q_3\rangle}, S_{1\langle Q_2\rangle}\rangle$	$\langle S_{3\langle Q_4\rangle}, S_{1\langle Q_2\rangle}\rangle$	$\langle S_{4\langle Q_2\rangle}, S_{1\langle Q_1\rangle}\rangle$

Table-4b
Expert matrix E_3.

	$\mathscr{L}_4$	$\mathscr{L}_5$	$\mathscr{L}_6$
u_1	$\langle S_{4\langle Q_1\rangle}, S_{1\langle Q_3\rangle}\rangle$	$\langle S_{3\langle Q_3\rangle}, S_{1\langle Q_2\rangle}\rangle$	$\langle S_{0\langle Q_0\rangle}, S_{3\langle Q_4\rangle}\rangle$
u_2	$\langle S_{3\langle Q_3\rangle}, S_{2\langle Q_1\rangle}\rangle$	$\langle S_{4\langle Q_1\rangle}, S_{1\langle Q_3\rangle}\rangle$	$\langle S_{3\langle Q_4\rangle}, S_{1\langle Q_1\rangle}\rangle$
u_3	$\langle S_{0\langle Q_3\rangle}, S_{2\langle Q_2\rangle}\rangle$	$\langle S_{0\langle Q_3\rangle}, S_{2\langle Q_1\rangle}\rangle$	$\langle S_{2\langle Q_3\rangle}, S_{1\langle Q_2\rangle}\rangle$
u_4	$\langle S_{3\langle Q_4\rangle}, S_{1\langle Q_1\rangle}\rangle$	$\langle S_{2\langle Q_4\rangle}, S_{2\langle Q_1\rangle}\rangle$	$\langle S_{4\langle Q_3\rangle}, S_{2\langle Q_1\rangle}\rangle$
u_5	$\langle S_{2\langle Q_6\rangle}, S_{2\langle Q_0\rangle}\rangle$	$\langle S_{2\langle Q_3\rangle}, S_{3\langle Q_2\rangle}\rangle$	$\langle S_{2\langle Q_3\rangle}, S_{2\langle Q_0\rangle}\rangle$
u_6	$\langle S_{3\langle Q_4\rangle}, S_{3\langle Q_1\rangle}\rangle$	$\langle S_{4\langle Q_2\rangle}, S_{1\langle Q_3\rangle}\rangle$	$\langle S_{4\langle Q_1\rangle}, S_{1\langle Q_3\rangle}\rangle$

principle. Based on the rules (1), (2) and (3), the final result of each object's decision can be determined that as $POS(X) = \{u_2, u_4, u_6\}$, $BND(X) = \{u_1, u_3, u_5\}$ and $NEG(X) = \varphi$. The result are shown in Fig. 2. From the above result we analyze that u_2, u_4, u_6 are considered to be selected and u_1, u_3, u_5 can gather extra information and await future decisions.

Table-5a
Ideal opinion matrix.

	$\mathscr{L}_1$	$\mathscr{L}_2$	$\mathscr{L}_3$
u_1	$\langle S_{2.02\langle Q_{2.18}\rangle}, S_{1.61\langle Q_{2.64}\rangle}\rangle$	$\langle S_{3.05\langle Q_{2.02}\rangle}, S_{2.32\langle Q_{1.86}\rangle}\rangle$	$\langle S_{1.01\langle Q_{2.18}\rangle}, S_{4\langle Q_{2.91}\rangle}\rangle$
u_2	$\langle S_{2.18\langle Q_{1.33}\rangle}, S_{2.64\langle Q_{3.34}\rangle}\rangle$	$\langle S_{2.82\langle Q_{1.33}\rangle}, S_{2.38\langle Q_{2.13}\rangle}\rangle$	$\langle S_{1.72\langle Q_{2}\rangle}, S_{1.86\langle Q_{2.32}\rangle}\rangle$
u_3	$\langle S_{2.75\langle Q_{1.66}\rangle}, S_{1.86\langle Q_{3.02}\rangle}\rangle$	$\langle S_{1.72\langle Q_{2.16}\rangle}, S_{2.02\langle Q_{1.67}\rangle}\rangle$	$\langle S_{2.02\langle Q_{1.98}\rangle}, S_{2.13\langle Q_{0}\rangle}\rangle$
u_4	$\langle S_{6\langle Q_{3.16}\rangle}, S_{0\langle Q_{1.28}\rangle}\rangle$	$\langle S_{2.75\langle Q_{2.37}\rangle}, S_{1.48\langle Q_{2.02}\rangle}\rangle$	$\langle S_{3.59\langle Q_{2.46}\rangle}, S_{1.86\langle Q_{1.50}\rangle}\rangle$
u_5	$\langle S_{1.33\langle Q_{2.66}\rangle}, S_{3.25\langle Q_{1.86}\rangle}\rangle$	$\langle S_{3.34\langle Q_{1.98}\rangle}, S_{0\langle Q_{2.32}\rangle}\rangle$	$\langle S_{6\langle Q_{2.33}\rangle}, S_{0\langle Q_{0}\rangle}\rangle$
u_6	$\langle S_{1.33\langle Q_{3.84}\rangle}, S_{2.63\langle Q_{0}\rangle}\rangle$	$\langle S_{3.59\langle Q_{2.78}\rangle}, S_{1.67\langle Q_{2.02}\rangle}\rangle$	$\langle S_{2.75\langle Q_{2.45}\rangle}, S_{2.13\langle Q_{2.1}\rangle}\rangle$

Table-5b
Ideal opinion matrix.

	$\mathscr{L}_4$	$\mathscr{L}_5$	$\mathscr{L}_6$
u_1	$\langle S_{2.78\langle Q_{2.02}\rangle}, S_{1.67\langle Q_{3.34}\rangle}\rangle$	$\langle S_{3.34\langle Q_{2.66}\rangle}, S_{1.01\langle Q_{2.1}\rangle}\rangle$	$\langle S_{1.72\langle Q_{1.72}\rangle}, S_{1.86\langle Q_{2.38}\rangle}\rangle$
u_2	$\langle S_{2.37\langle Q_{3.12}\rangle}, S_{1.86\langle Q_{1.48}\rangle}\rangle$	$\langle S_{6\langle Q_{1.33}\rangle}, S_{0\langle Q_{3.34}\rangle}\rangle$	$\langle S_{3.05\langle Q_{2.78}\rangle}, S_{1.28\langle Q_{1.85}\rangle}\rangle$
u_3	$\langle S_{2.92\langle Q_{2.02}\rangle}, S_{0\langle Q_{0}\rangle}\rangle$	$\langle S_{1.01\langle Q_{3.34}\rangle}, S_{2.32\langle Q_{0}\rangle}\rangle$	$\langle S_{2.75\langle Q_{2.37}\rangle}, S_{1.28\langle Q_{0}\rangle}\rangle$
u_4	$\langle S_{1.39\langle Q_{3.05}\rangle}, S_{2.62\langle Q_{1.61}\rangle}\rangle$	$\langle S_{2.45\langle Q_{2.78}\rangle}, S_{1.86\langle Q_{1.86}\rangle}\rangle$	$\langle S_{3.47\langle Q_{2.52}\rangle}, S_{1.61\langle Q_{1.67}\rangle}\rangle$
u_5	$\langle S_{1.33\langle Q_{6}\rangle}, S_{2.08\langle Q_{0}\rangle}\rangle$	$\langle S_{6\langle Q_{3.05}\rangle}, S_{0\langle Q_{0}\rangle}\rangle$	$\langle S_{1.72\langle Q_{3.34}\rangle}, S_{4.2\langle Q_{0}\rangle}\rangle$
u_6	$\langle S_{2.97\langle Q_{2.16}\rangle}, S_{2.13\langle Q_{1.61}\rangle}\rangle$	$\langle S_{3.05\langle Q_{2}\rangle}, S_{1.48\langle Q_{1.86}\rangle}\rangle$	$\langle S_{3.48\langle Q_{1.39}\rangle}, S_{0\langle Q_{2.95}\rangle}\rangle$

Table-6a
Right ideal matrix.

	$\mathscr{L}_1$	$\mathscr{L}_2$	$\mathscr{L}_3$
u_1	$\langle S_{2\langle Q_4\rangle}, S_{2\langle Q_1\rangle}\rangle$	$\langle S_{3\langle Q_2\rangle}, S_{2\langle Q_1\rangle}\rangle$	$\langle S_{1\langle Q_4\rangle}, S_{5\langle Q_1\rangle}\rangle$
u_2	$\langle S_{4\langle Q_2\rangle}, S_{1\langle Q_3\rangle}\rangle$	$\langle S_{5\langle Q_1\rangle}, S_{1\langle Q_1\rangle}\rangle$	$\langle S_{2\langle Q_3\rangle}, S_{2\langle Q_2\rangle}\rangle$
u_3	$\langle S_{4\langle Q_2\rangle}, S_{1\langle Q_3\rangle}\rangle$	$\langle S_{2\langle Q_4\rangle}, S_{2\langle Q_1\rangle}\rangle$	$\langle S_{2\langle Q_2\rangle}, S_{3\langle Q_0\rangle}\rangle$
u_4	$\langle S_{6\langle Q_4\rangle}, S_{0\langle Q_1\rangle}\rangle$	$\langle S_{4\langle Q_3\rangle}, S_{1\langle Q_2\rangle}\rangle$	$\langle S_{3\langle Q_4\rangle}, S_{2\langle Q_1\rangle}\rangle$
u_5	$\langle S_{2\langle Q_3\rangle}, S_{2\langle Q_1\rangle}\rangle$	$\langle S_{4\langle Q_2\rangle}, S_{0\langle Q_3\rangle}\rangle$	$\langle S_{3\langle Q_2\rangle}, S_{1\langle Q_1\rangle}\rangle$
u_6	$\langle S_{1\langle Q_5\rangle}, S_{4\langle Q_0\rangle}\rangle$	$\langle S_{5\langle Q_3\rangle}, S_{1\langle Q_2\rangle}\rangle$	$\langle S_{2\langle Q_4\rangle}, S_{3\langle Q_1\rangle}\rangle$

Table-6b
Right ideal matrix.

	$\mathscr{L}_4$	$\mathscr{L}_5$	$\mathscr{L}_6$
u_1	$\langle S_{4\langle Q_1\rangle}, S_{1\langle Q_3\rangle}\rangle$	$\langle S_{3\langle Q_3\rangle}, S_{1\langle Q_2\rangle}\rangle$	$\langle S_{2\langle Q_3\rangle}, S_{2\langle Q_1\rangle}\rangle$
u_2	$\langle S_{3\langle Q_5\rangle}, S_{1\langle Q_1\rangle}\rangle$	$\langle S_{6\langle Q_1\rangle}, S_{0\langle Q_4\rangle}\rangle$	$\langle S_{3\langle Q_4\rangle}, S_{1\langle Q_1\rangle}\rangle$
u_3	$\langle S_{4\langle Q_2\rangle}, S_{1\langle Q_0\rangle}\rangle$	$\langle S_{2\langle Q_3\rangle}, S_{3\langle Q_0\rangle}\rangle$	$\langle S_{2\langle Q_3\rangle}, S_{2\langle Q_0\rangle}\rangle$
u_4	$\langle S_{3\langle Q_4\rangle}, S_{1\langle Q_1\rangle}\rangle$	$\langle S_{2\langle Q_4\rangle}, S_{2\langle Q_1\rangle}\rangle$	$\langle S_{5\langle Q_4\rangle}, S_{1\langle Q_1\rangle}\rangle$
u_5	$\langle S_{2\langle Q_6\rangle}, S_{2\langle Q_0\rangle}\rangle$	$\langle S_{6\langle Q_4\rangle}, S_{0\langle Q_1\rangle}\rangle$	$\langle S_{2\langle Q_3\rangle}, S_{2\langle Q_0\rangle}\rangle$
u_6	$\langle S_{3\langle Q_4\rangle}, S_{3\langle Q_1\rangle}\rangle$	$\langle S_{3\langle Q_3\rangle}, S_{1\langle Q_2\rangle}\rangle$	$\langle S_{5\langle Q_3\rangle}, S_{0\langle Q_2\rangle}\rangle$

Table-7a
Left ideal matrix.

	$\mathscr{L}_1$	$\mathscr{L}_2$	$\mathscr{L}_3$
u_1	$\langle S_{1\langle Q_0\rangle}, S_{2\langle Q_5\rangle}\rangle$	$\langle S_{4\langle Q_1\rangle}, S_{2\langle Q_3\rangle}\rangle$	$\langle S_{1\langle Q_4\rangle}, S_{5\langle Q_1\rangle}\rangle$
u_2	$\langle S_{2\langle Q_1\rangle}, S_{3\langle Q_4\rangle}\rangle$	$\langle S_{2\langle Q_2\rangle}, S_{3\langle Q_3\rangle}\rangle$	$\langle S_{0\langle Q_2\rangle}, S_{3\langle Q_3\rangle}\rangle$
u_3	$\langle S_{2\langle Q_1\rangle}, S_{2\langle Q_3\rangle}\rangle$	$\langle S_{0\langle Q_1\rangle}, S_{2\langle Q_1\rangle}\rangle$	$\langle S_{1\langle Q_2\rangle}, S_{3\langle Q_2\rangle}\rangle$
u_4	$\langle S_{2\langle Q_1\rangle}, S_{0\langle Q_1\rangle}\rangle$	$\langle S_{2\langle Q_1\rangle}, S_{3\langle Q_2\rangle}\rangle$	$\langle S_{2\langle Q_1\rangle}, S_{3\langle Q_1\rangle}\rangle$
u_5	$\langle S_{1\langle Q_3\rangle}, S_{4\langle Q_2\rangle}\rangle$	$\langle S_{3\langle Q_2\rangle}, S_{2\langle Q_2\rangle}\rangle$	$\langle S_{3\langle Q_2\rangle}, S_{1\langle Q_1\rangle}\rangle$
u_6	$\langle S_{2\langle Q_3\rangle}, S_{4\langle Q_2\rangle}\rangle$	$\langle S_{2\langle Q_1\rangle}, S_{4\langle Q_2\rangle}\rangle$	$\langle S_{2\langle Q_4\rangle}, S_{3\langle Q_3\rangle}\rangle$

8. Comparison section

In this section, we compare our proposed method with the GRA method and discuss the advantages and implementation of the proposed method. To determine the conditional probability based on GRA with TWDs is proposed by Liang et al., [47].

Hence this comparison is taken by considering the decision expert matrix and the same expert weights $\omega = (0.353, 0.324, 0.323)^T$ and criteria weights $\varpi = (0.184, 0.17, 0.176, 0.17, 0.124, 0.176)^T$ as we have calculated in above example. Aggregate the all the de-

Table-7b
Left ideal matrix.

	$\mathscr{L}_4$	$\mathscr{L}_5$	$\mathscr{L}_6$
u_1	$\langle S_{1\langle Q_3\rangle}, S_{4\langle Q_3\rangle}\rangle$	$\langle S_{3\langle Q_2\rangle}, S_{1\langle Q_2\rangle}\rangle$	$\langle S_{0\langle Q_0\rangle}, S_{3\langle Q_4\rangle}\rangle$
u_2	$\langle S_{1\langle Q_0\rangle}, S_{3\langle Q_3\rangle}\rangle$	$\langle S_{4\langle Q_1\rangle}, S_{1\langle Q_3\rangle}\rangle$	$\langle S_{2\langle Q_3\rangle}, S_{2\langle Q_2\rangle}\rangle$
u_3	$\langle S_{0\langle Q_3\rangle}, S_{2\langle Q_2\rangle}\rangle$	$\langle S_{0\langle Q_3\rangle}, S_{2\langle Q_1\rangle}\rangle$	$\langle S_{4\langle Q_1\rangle}, S_{1\langle Q_3\rangle}\rangle$
u_4	$\langle S_{0\langle Q_3\rangle}, S_{4\langle Q_2\rangle}\rangle$	$\langle S_{1\langle Q_3\rangle}, S_{3\langle Q_2\rangle}\rangle$	$\langle S_{0\langle Q_0\rangle}, S_{2\langle Q_4\rangle}\rangle$
u_5	$\langle S_{1\langle Q_0\rangle}, S_{1\langle Q_6\rangle}\rangle$	$\langle S_{2\langle Q_3\rangle}, S_{3\langle Q_2\rangle}\rangle$	$\langle S_{0\langle Q_3\rangle}, S_{6\langle Q_2\rangle}\rangle$
u_6	$\langle S_{3\langle Q_1\rangle}, S_{3\langle Q_2\rangle}\rangle$	$\langle S_{4\langle Q_2\rangle}, S_{1\langle Q_3\rangle}\rangle$	$\langle S_{0\langle Q_0\rangle}, S_{3\langle Q_4\rangle}\rangle$

Table-8a
Aggregated IDHLT matrix.

	$\mathscr{L}_1$	$\mathscr{L}_2$	$\mathscr{L}_3$
u_1	$\langle S_{1.77\langle Q_{1.98}\rangle}, S_{1.57\langle Q_{2.63}\rangle}\rangle$	$\langle S_{2.98\langle Q_{1.78}\rangle}, S_{2.31\langle Q_{1.84}\rangle}\rangle$	$\langle S_{0.99\langle Q_{1.98}\rangle}, S_{3.95\langle Q_{2.9}\rangle}\rangle$
u_2	$\langle S_{1.99\langle Q_{1.2}\rangle}, S_{2.62\langle Q_{3.33}\rangle}\rangle$	$\langle S_{2.58\langle Q_{1.2}\rangle}, S_{2.29\langle Q_{2.1}\rangle}\rangle$	$\langle S_{1.45\langle Q_{1.9}\rangle}, S_{1.8\langle Q_{2.29}\rangle}\rangle$
u_3	$\langle S_{2.5\langle Q_{1.46}\rangle}, S_{1.8\langle Q_3\rangle}\rangle$	$\langle S_{1.52\langle Q_{1.88}\rangle}, S_{2\langle Q_{1.62}\rangle}\rangle$	$\langle S_{1.78\langle Q_{1.79}\rangle}, S_{2.1\langle Q_0\rangle}\rangle$
u_4	$\langle S_{6\langle Q_{2.94}\rangle}, S_{0\langle Q_{1.26}\rangle}\rangle$	$\langle S_{2.5\langle Q_{2.13}\rangle}, S_{1.44\langle Q_2\rangle}\rangle$	$\langle S_{3.39\langle Q_{2.28}\rangle}, S_{1.8\langle Q_{1.49}\rangle}\rangle$
u_5	$\langle S_{1.2\langle Q_{2.49}\rangle}, S_{3.26\langle Q_{1.87}\rangle}\rangle$	$\langle S_{3.12\langle Q_{1.79}\rangle}, S_{0\langle Q_{2.31}\rangle}\rangle$	$\langle S_{6\langle Q_{2.14}\rangle}, S_{0\langle Q_0\rangle}\rangle$
u_6	$\langle S_{1.11\langle Q_{3.68}\rangle}, S_{2.64\langle Q_0\rangle}\rangle$	$\langle S_{3.4\langle Q_{2.57}\rangle}, S_{1.62\langle Q_2\rangle}\rangle$	$\langle S_{2.57\langle Q_{2.19}\rangle}, S_{2.13\langle Q_{2.1}\rangle}\rangle$

Table-8b
Aggregated IDHLT matrix.

	$\mathscr{L}_4$	$\mathscr{L}_5$	$\mathscr{L}_6$
u_1	$\langle S_{2.9\langle Q_{2.67}\rangle}, S_{1.78\langle Q_{1.69}\rangle}\rangle$	$\langle S_{3.13\langle Q_{2.44}\rangle}, S_{1\langle Q_2\rangle}\rangle$	$\langle S_{1.47\langle Q_{1.52}\rangle}, S_{1.8\langle Q_{2.36}\rangle}\rangle$
u_2	$\langle S_{2.13\langle Q_{2.89}\rangle}, S_{1.8\langle Q_{1.44}\rangle}\rangle$	$\langle S_{6\langle Q_{1.2}\rangle}, S_{0\langle Q_{3.33}\rangle}\rangle$	$\langle S_{2.89\langle Q_{2.57}\rangle}, S_{1.28\langle Q_{1.84}\rangle}\rangle$
u_3	$\langle S_{2.69\langle Q_{1.82}\rangle}, S_{0\langle Q_0\rangle}\rangle$	$\langle S_{0.88\langle Q_{3.12}\rangle}, S_{2.3\langle Q_0\rangle}\rangle$	$\langle S_{2.49\langle Q_{2.25}\rangle}, S_{1.25\langle Q_0\rangle}\rangle$
u_4	$\langle S_{1.37\langle Q_{2.84}\rangle}, S_{2.64\langle Q_{1.6}\rangle}\rangle$	$\langle S_{2.34\langle Q_{2.57}\rangle}, S_{1.87\langle Q_{1.8}\rangle}\rangle$	$\langle S_{3.42\langle Q_{2.47}\rangle}, S_{1.6\langle Q_{1.69}\rangle}\rangle$
u_5	$\langle S_{1.2\langle Q_6\rangle}, S_{2.1\langle Q_0\rangle}\rangle$	$\langle S_{6\langle Q_{2.89}\rangle}, S_{0\langle Q_0\rangle}\rangle$	$\langle S_{1.45\langle Q_{3.12}\rangle}, S_{4.15\langle Q_0\rangle}\rangle$
u_6	$\langle S_{2.77\langle Q_{2.04}\rangle}, S_{2.13\langle Q_{1.6}\rangle}\rangle$	$\langle S_{2.89\langle Q_{1.9}\rangle}, S_{1.5\langle Q_{1.8}\rangle}\rangle$	$\langle S_{3.27\langle Q_{1.11}\rangle}, S_{0\langle Q_{2.89}\rangle}\rangle$

Table-9
Distance of each u_i to u_j^+ and u_j^-.

U	u_1	u_2	u_3	u_4	u_5	u_6
$d(u_i, u_j^+)$	0.0245	0.0346	0.0081	0.0341	0.0312	0.0332
$d(u_i, u_j^-)$	0.0349	0.0217	0.0428	0.0171	0.0335	0.0253

Table-10
The RC and Conditional Probability with TOPSIS method.

U	u_1	u_2	u_3	u_4	u_5	u_6
$\mathscr{T}_i$	0.4125	0.6145	0.1591	0.6654	0.4815	0.5675
$Pr(X\|u_i)$	0.4125	0.6145	0.1591	0.6654	0.4815	0.5675

Table-11
Loss function information.

	$X(P)$	$X^c(N)$
a_P	$\langle S_{2.17\langle Q_{1.35}\rangle}, S_{2.63\langle Q_{3.3}\rangle}\rangle$	$\langle S_{2.63\langle Q_{3.3}\rangle}, S_{2.17\langle Q_{1.35}\rangle}\rangle$
a_B	$\langle S_{1.71\langle Q_{2.12}\rangle}, S_{2\langle Q_{1.65}\rangle}\rangle$	$\langle S_{2\langle Q_{1.65}\rangle}, S_{1.71\langle Q_{2.12}\rangle}\rangle$
a_N	$\langle S_{2.73\langle Q_{1.66}\rangle}, S_{1.86\langle Q_3\rangle}\rangle$	$\langle S_{1.86\langle Q_3\rangle}, S_{\beta_{2.73}\langle Q_{1.66}\rangle}\rangle$

cision matrix to a single decision matrix by using proposed aggregation operators and the *PIS* and *NIS* are calculated by equations (36) and (37) as

Table-12
Score functions of expected losses.

U	u_1	u_2	u_3	u_4	u_5	u_6
$Sc(R(a_P\|u_i))$	0.2781	0.2123	0.3571	0.1957	0.2549	0.2273
$Sc(R(a_B\|u_i))$	0.2170	0.2850	0.0951	0.2999	0.2429	0.2710
$Sc(R(a_N\|u_i))$	0.2584	0.2999	0.1738	0.3084	0.2747	0.2917

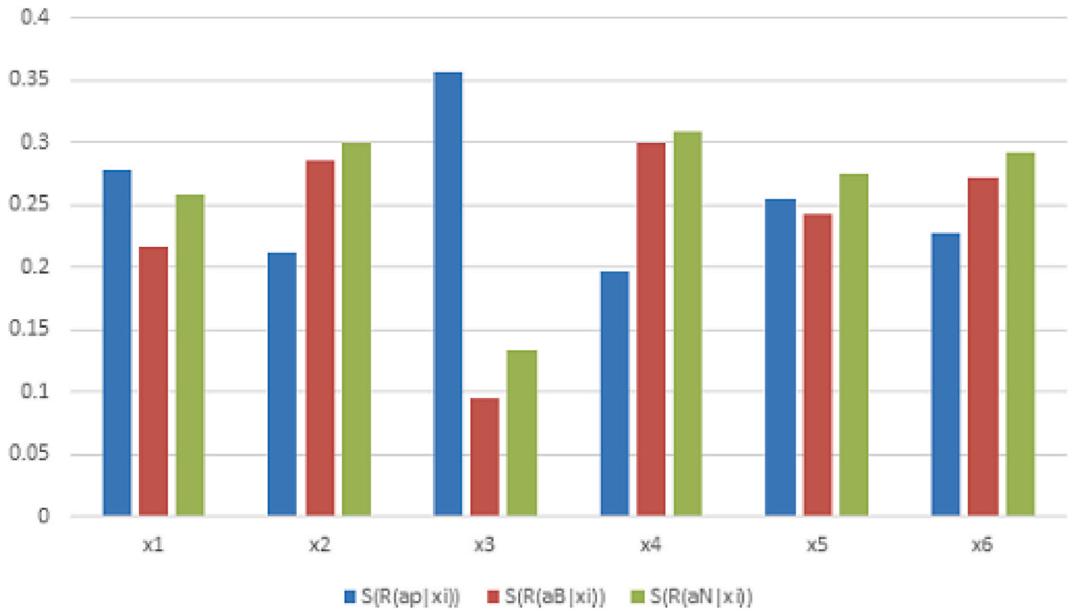

Fig. 2. Graphical Representation of Alternative ranking.

Table-13
GRC between *PIS* and *NIS*.

U	u_1	u_2	u_3	u_4	u_5	u_6
G_i^+	0.6064	0.7238	0.5678	0.8051	0.5037	0.7537
G_i^-	0.7365	0.6206	0.8808	0.5841	0.5256	0.5913

$$u_j^+ = \begin{pmatrix} \langle S_{6\langle Q_{2.94}\rangle}, S_{0\langle Q_{1.26}\rangle}\rangle, \langle S_{3.4\langle Q_{2.57}\rangle}, S_{1.62\langle Q_{2}\rangle}\rangle, \langle S_{2.57\langle Q_{2.19}\rangle}, S_{2.13\langle Q_{2.1}\rangle}\rangle, \\ \langle S_{2.77\langle Q_{2.04}\rangle}, S_{2.13\langle Q_{1.6}\rangle}\rangle, \langle S_{6\langle Q_{1.2}\rangle}, S_{0\langle Q_{3.33}\rangle}\rangle, \langle S_{3.42\langle Q_{2.47}\rangle}, S_{1.6\langle Q_{1.69}\rangle}\rangle \end{pmatrix}$$

$$u_j^- = \begin{pmatrix} \langle S_{1.77\langle Q_{1.98}\rangle}, S_{1.57\langle Q_{2.63}\rangle}\rangle, \langle S_{1.52\langle Q_{1.88}\rangle}, S_{2\langle Q_{1.62}\rangle}\rangle, \langle S_{1.78\langle Q_{1.79}\rangle}, S_{2.1\langle Q_{0}\rangle}\rangle, \\ \langle S_{2.69\langle Q_{1.82}\rangle}, S_{0\langle Q_{0}\rangle}\rangle, \langle S_{0.88\langle Q_{3.12}\rangle}, S_{2.3\langle Q_{0}\rangle}\rangle, \langle S_{1.47\langle Q_{1.52}\rangle}, S_{1.8\langle Q_{2.36}\rangle}\rangle \end{pmatrix}$$

Calculate the grey relational coefficient (GRC) by equations (50) and (52) on the *jth* criterion among u_i and PIS^+, NIS^- and the degree of grey relational coefficient of each alternative from PIS^+, NIS^- are determined by using equations (51) and (53) in Table 13.

$$g_{ij}^+ = \frac{\min\limits_{1 \le i \le n}\left(\min\limits_{1 \le j \le n}\right) d_{ij}^+ + \zeta \max\limits_{1 \le i \le n}\left(\max\limits_{1 \le j \le n}\right) d_{ij}^+}{d_{ij}^+ + \zeta \max\limits_{1 \le i \le n}\left(\max\limits_{1 \le j \le n}\right) d_{ij}^+} \tag{50}$$

Table-14
RRD and its Conditional probability.

U	u_1	u_2	u_3	u_4	u_5	u_6
$\mathscr{T}_i$	0.4515	0.5384	0.3919	0.5795	0.4894	0.5604
$Pr(X\|u_i)$	0.4515	0.5384	0.3919	0.5795	0.4894	0.5604

Table-15
Score functions of expected losses.

U	u_1	u_2	u_3	u_4	u_5	u_6
$Sc(R(a_P\|u_i))$	0.2722	0.2634	0.2732	0.2406	0.2217	0.2693
$Sc(R(a_B\|u_i))$	0.2319	0.3013	0.2092	0.1687	0.2449	0.2684
$Sc(R(a_N\|u_i))$	0.2679	0.2860	0.2533	0.2741	0.2760	0.2902

Table-16
Conditional probability of MADM methods.

Methods	$Pr(X\|u_1)$	$Pr(X\|u_2)$	$Pr(X\|u_3)$	$Pr(X\|u_4)$	$Pr(X\|u_5)$	$Pr(X\|u_6)$
GRA Method [48]	0.4515	0.5384	0.3919	0.5795	0.4894	0.5604
WAM [39]	0.4558	0.5597	0.3899	0.6092	0.5149	0.5901
BP method [58,59]	0.4495	0.5383	0.3163	0.6040	0.4871	0.5361
Our method	0.4125	0.6145	0.1591	0.6654	0.4815	0.5675

Table-17
Conditional probability wise ranking of the objects.

Methods	Ranking
GRA Method [48]	$u_4 \succ u_6 \succ u_2 \succ u_5 \succ u_1 \succ u_3$
WAM [39]	$u_4 \succ u_6 \succ u_2 \succ u_5 \succ u_1 \succ u_3$
BP method [58,59]	$u_4 \succ u_2 \succ u_6 \succ u_5 \succ u_1 \succ u_3$
Our method	$u_4 \succ u_2 \succ u_6 \succ u_5 \succ u_1 \succ u_3$

$$G_i^+ = \sum_{j=1}^{m} \omega g_{ij}^+ \tag{51}$$

$$g_{ij}^- = \frac{\min\limits_{1\le i\le n}\left(\min\limits_{1\le j\le n}\right)d_{ij}^- + \zeta \max\limits_{1\le i\le n}\left(\max\limits_{1\le j\le n}\right)d_{ij}^-}{d_{ij}^- + \zeta \max\limits_{1\le i\le n}\left(\max\limits_{1\le j\le n}\right)d_{ij}^-} \tag{52}$$

$$G_i^- = \sum_{j=1}^{m} \omega g_{ij}^- \tag{53}$$

$d_{ij}^+ = (u_{ij}, u_j^+)$, $d_{ij}^- = (u_{ij}, u_j^-)$ and $(i = 1, 2, ..., m)(j = 1, 2, ..., n)$.

The relative relational degree (RRD) denoted by $\mathscr{F}_i$ and conditional probability (Pr) of an object belongs to the state X based on GRA method are determined by equations (54) and (55) as

$$\mathscr{F}_i = \frac{G_i^+}{G_i^+ + G_i^-} \tag{54}$$

$$Pr(X/u_i) = \mathscr{F}_i \tag{55}$$

The relative relational degree (RRD) and Conditional probability based on GRA method is evaluated in Table 14.

Next, we assume the same loss function as obtained in Table-11, based on IDHLTS operational laws also we determine the expected loss based on LF as shown in Table-15.

Hence the final result of each object's decision can be determined according to the minimum loss principle, (1), (2) and (3) that as $POS(X) = \{u_2, u_5\}$, $BND(X) = \{u_1, u_3, u_4, u_6\}$ and $NEG(X) = \varphi$. From Table-15, shows that the GRA method makes nearly identical decisions as our suggested method. In this way, the effectiveness of our proposed method can be demonstrated. The difference between the results of the GRA and our method is the object u_4, u_6 is divided into $POS(X)$ to $BND(X)$ and the object u_5 is divided by $BND(X)$ to $POS(X)$. The reasons can be conculuded due to the distance formula of the GRA method cannot accurately reflect the position relationship of each scheme, and there may be a situation that the scheme is close to both positive and negative ideal solutions, so it cannot fully reflect the advantages and disadvantages of the comprehensive level of each assessed object. There are other conditional probability findings that are calculated using various methodologies and operators. Hence it is analyzed that our proposed method is efficient and practical to solve the ambiguity and uncertainty to solve the DM problems. There are more results, ranking, and conditional probabilities we calculated based on different methods and aggregation operators, as a result, u_4 is our best result as shown in Table-16, 17.

The TWD as a key component, the conditional probabilities can also be used in ranking schemes. Conditional probabilities are computed using the weighted aggregation method in the TWDs model proposed by Ye et al. [39] The bidirectional projective (BP) [58, 59] technique considers the connection between the project and the ideal solution, making conditional probability computation more objective. Considering the same weights of attributes the conditional probability is obtained in Table-16, based on MADM methods. And Table-17, shows the ordering of alternatives based on conditional probability. Hence from Table-16, we analyzed that the best result is u_4 obtained from other MADM methods which are similar to that of the proposed method, which shows the practicability of the proposed methods.

8.1. Discussions on the limitations and advantages

The DRTS model, as an aspect of TWDs, may present the loss caused by various actions at various stages and select the action that produces the least loss. The conditional probability is the other components of the TWDs technique. In all existing TWDs models [60, 61], the conditional probability are directly determined by DMs, making the decision results less rigorous. Estimate the weight of each attribute using the entropy measure, and the TOPSIS method is used to convert the RRC of the computed object into a conditional probability. Linguistic terms are more compatible with human expression tendencies when expressing the problem of quality. IDHLTSs provides more versatile methods to describe quality information than a collection of single Linguistic term set. IDHLTSs give TWDs a new way to express diagnostic data. While DMs analyses project attribution data, they may employ IDHLTSs to make the assessment's value more comprehensible, reducing decision time tremendously. The model we present is created in the IDHLT environment. The TWDs model is a brand-new research tool based on IDHLT data systems. As a result, it has a high research value.

A particular parameter is included in Hamacher t-norm and t-conorm, which allows for more adaptable data processing and better modelling of real decision-making tasks. They consider the various decision-making attitudes of DMs, making the decision more practical. This article has also demonstrated some of the IDHLHWA operator's needed qualities. The crucial advantages of the proposed method are shown are as below:

(1) The diagnosis of more flexible DMs in the process of TWDs may be expressed using IDHLE, which is made up of FHLT and SHLT. As a result, the TWD technique based on IDHLEs can help in decision-making.
(2) The conditional probability is derived using the TOPSIS technique, which improves the GRA model by changing the distance as a measure of distance using a weighted grey relational degree. Hamacher operators, LFs accumulation takes into consideration DMs' diverse decisive attitudes. They make the DM process better acceptable. The proposed technique, however, has several restrictions.
(3) This work does not cover the status of group choices or the weight of various experts to simplify computations. In future work, we'll expand this concept to group choices and make it more valuable.

9. Conclusions

In this article a three way decision approach is proposed for the selection of Logistics supplier based on the loss function with the hybrid study IFS and DHLTSs set model. Therefore this study first examine the hybrid study of IFS and DHLTSs set and develop a new theory called intuitionistic fuzzy double hierarchy set (IDHLTSs) to handle the uncertainty TWD process. Also proposes the basic operational laws and aggregation operators for IDHLTSs to aggregate the DM process. IDHLTSs are a powerful tool as a combination of first-level and second-level linguistic term sets to more flexibly describe uncertainty and ambiguity. The weight of experts and criteria are determined by distance a measure and entropy measure. To make the decision making process more rational conditional probability is determined using TOPSIS method. The attribute values and LFs in TWDs are described using IDHLTSs. The step-wise detail of the proposed method is constructed. Finally, an illustrative example in 3 PL supplier selection domains and comparison is given to verify the developed approach and to demonstrate its practicality and effectiveness.

10. Limitations and future direction

The proposed operators are introduced in IDHLTSs, which are an efficient generalization, and they conveys appropriate data more conveniently in complex expressions than single LTSs. But, this theory has its own limitations, as they do not work in the case of $((\alpha/\tau)+(\beta/\tau))>1$ and $((k/\delta)+(l/\delta))>1$. To overcome the shortcomings of the proposed work, this work can be further extended to future studies by using fuzzy extensions, such as the Pythagorean double hierarchy linquistic term set, Fermatean double hierarchy linguistic term set, and various aggregation operators like Yager, Einstein aggregations operators for solving different MCGDM problems.

Author contribution statement

All authors listed have significantly contributed to the development and the writing of this article.

Data availability statement

No data was used for the research described in the article.

Additional information

No additional information is available for this paper.

Declaration of competing interest

The author declare that they have no known competing financial interest or personal relationship that could have appeared to influence the work repeated in this paper.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through large group Research Project under grant number. RGP 2/113/44.

References

[1] J. Chai, J.N. Liu, E.W. Ngai, Application of decision-making techniques in supplier selection: a systematic review of literature, Expert Syst. Appl. 40 (10) (2013) 3872–3885.
[2] M.S. Spencer, D.S. Rogers, P.J. Daugherty, JIT systems and external logistics suppliers, Int. J. Oper. Prod. Manag. 14 (6) (1994) 60–74.
[3] A.B. Maltz, The relative importance of cost and quality in the outsourcing of warehousing, J. Bus. Logist. 15 (2) (1994) 45.
[4] K.H. Lai, E.W. Ngai, T.C.E. Cheng, Measures for evaluating supply chain performance in transport logistics, Transport. Res. E Logist. Transport. Rev. 38 (6) (2002) 439–456.
[5] A. Aguezzoul, Third-party logistics selection problem: a literature review on criteria and methods, Omega 49 (2014) 69–78.
[6] B. Kizielewicz, et al., Study towards the time-based mcda ranking analysis–a supplier selection case study, Facta Univ. – Ser. Mech. Eng. 19 (3) (2021) 381–399.
[7] S. Perçin, H. Min, A hybrid quality function deployment and fuzzy decision-making methodology for the optimal selection of third-party logistics service providers, Int. J. Logist. Res. Appl. 16 (5) (2013) 380–397.
[8] R.E. Gergin, I. Peker, A.C.G. Kısa, Supplier selection by integrated IFDEMATEL-IFTOPSIS Method: a case study of automotive supply industry, Decision Making: App. Manag. Eng. 5 (1) (2022) 169–193.
[9] M. Riaz, H.M.A. Farid, Picture fuzzy aggregation approach with application to third-party logistic provider selection process, Rep. Mech. Eng. 3 (1) (2022) 227–236.
[10] L.A. Zadeh, Fuzzy Sets, Information And Control, 8(3), 1965, pp. 338–353.
[11] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Set Syst. 20 (1986) 87–96.
[12] L.A. Zadeh, What is computing with words (CWW), in: Computing with Words, Springer, Berlin, Heidelberg, 2013, pp. 3–37.
[13] Z. Xu, H. Wang, On the syntax and semantics of virtual linguistic terms for information fusion in decision making, Inf. Fusion 34 (2017) 43–48.
[14] J. Tang, F. Meng, Linguistic intuitionistic fuzzy Hamacher aggregation operators and their application to group decision making, Gran. Comp. 4 (1) (2019) 109–124.
[15] F. Herrera, L. Martínez, A 2-tuple fuzzy linguistic representation model for computing with words, IEEE Trans. Fuzzy Syst. 8 (6) (2000) 746–752.
[16] Z. Xu, A method based on linguistic aggregation operators for group decision making with linguistic preference relations, Inf. Sci. 166 (1–4) (2004) 19–30.
[17] X.J. Gou, H. Liao, Z. Xu, F. Herrera, Double hierarchy hesitant fuzzy linguistic term set and MULTIMOORA method: a case of study to evaluate the implementation status of haze controlling measures, Inf. Fusion 38 (2017) 22–34.
[18] X.J. Gou, H. Liao, Z. Xu, F. Herrera, Multiple criteria decision making based on distance and similarity measures under double hierarchy hesitant fuzzy linguistic environment, Comp. Indus. Eng. 126 (2018) 516–530.
[19] X. Li, Z. Xu, H. Wang, Three-way decisions based on some Hamacher aggregation operators under double hierarchy linguistic environment, Int. J. Intell. Syst. 36 (12) (2021) 7731–7753.
[20] K. Menger, Statistical metrics, Proc. Natl. Acad. Sci. USA 28 (12) (1942) 535–537.
[21] B. Schweizer, A. Sklar, Probabilistic Metric Spaces, North Holland, New York, NY, USA, 1983.
[22] H.J. Zimmermann, P. Zysno, Latent connectives in human decision making, Fuzzy Set Syst. 4 (1) (1980) 37–51.
[23] J. Dombi, Basic concepts for a theory of evaluation: the aggregative operator, Eur. J. Oper. Res. 10 (3) (1982) 282–293.
[24] H. Hamachar, Uber logische Verknupfungen unscharfer Aussagen und deren zugehorige Bewertungs-Funktionen, Prog. Cyber. Sys. Res. 3 (1978) 276–288.
[25] A. Ashraf, K. Ullah, A. Hussain, M. Bari, Interval-valued picture fuzzy Maclaurin symmetric mean operator with application in multiple attribute decision-making, Rep. Mech. Eng. 3 (1) (2022) 210–226.
[26] D. Pamucar, M. Zivzovic, D. Duricic, Modification of the CRITIC method using fuzzy rough numbers, Decision Making: App. Manag. Eng. 5 (2) (2022) 362–371.
[27] Y.Y. Yao, S.K.M. Wong, P. Lingras, A decision-theoretic rough set model, in: Proceeding of ISMIS, 1990, pp. 17–25.
[28] Y.Y. Yao, Three-way decision: an interpretation of rules in rough set theory, Rough Set Knowl. Technol. 5589 (2009) 642–649.
[29] Y.Y. Yao, X. Deng, Sequential three-way decisions with probabilistic rough sets, in: IEEE 10th International Conference on Cognitive Informatics and Cognitive Computing (ICCI-CC'11), IEEE, 2011, pp. 120–125.
[30] Z. Pawlak, Rough sets, Int. J. Com. Infor. Sci. 11 (5) (1982) 341–356.
[31] Y. Chien, Pattern classification and scene analysis, IEEE Trans. Automat. Control 19 (4) (1974) 462–463.
[32] J. Hu, Y. Yang, X. Chen, A novel TODIM method-based three-way decision model for medical treatment selection, Int. J. Fuzzy Syst. 20 (4) (2018) 1240–1255.
[33] P. Wang, P. Zhang, P.Z. Li, A three-way decision method based on Gaussian kernel in a hybrid information system with images: an application in medical diagnosis, Appl. Soft Comput. 77 (2019) 734–749.
[34] X. Li, X. Huang, A novel three-way investment decisions based on decision-theoretic rough sets with hesitant fuzzy information, Int. J. Fuzzy Syst. 22 (8) (2020) 2708–2719.
[35] X. Li, H. Wang, Z. Xu, Work resumption after epidemic using three-way decisions, Int. J. Fuzzy Syst. 23 (3) (2021) 630–641.
[36] D. Liang, D. Liu, Deriving three-way decisions from intuitionistic fuzzy decision-theoretic rough sets, Inf. Sci. 300 (2015) 28–48.
[37] D.C. Liang, Z.S. Xu, D. Liu, Three-way decisions based on decision-theoretic rough sets with dual hesitant fuzzy information, Inf. Sci. 396 (2017) 127–143.
[38] D.C. Liang, et al., Three-way decisions based on decision-theoretic rough sets under linguistic assessment with the aid of group decision making, Appl. Soft Comput. 29 (2015) 256–269.
[39] J. Ye, J. Zhan, Z. Xu, A novel decision-making approach based on three-way decisions in fuzzy information systems, Inf. Sci. 541 (2020) 362–390.
[40] D.C. Liang, et al., Triangular fuzzy decision-theoretic rough sets, Int. J. Approx. Reason. 54 (8) (2013) 1087–1106.
[41] J.P. Herbert, J.T. Yao, Game-theoretic rough sets, Fundam. Inf. 108 (3–4) (2011) 267–286.
[42] X. Jia, Z. Tang, W. Liao, L. Shang, On an optimization representation of decision-theoretic rough set model, Int. J. Approx. Reason. 55 (1) (2014) 156–166.
[43] F. Jia, P. Liu, A novel three-way decision model under multiple-criteria environment, Inf. Sci. 471 (2019) 29–51.
[44] D. Liang, Z. Xu, D. Liu, Y. Wu, Method for three-way decisions using ideal TOPSIS solutions at Pythagorean fuzzy information, Inf. Sci. 435 (2018) 282–295.
[45] Y.M. Wang, Using the method of maximizing deviations to make decision for multiindicies, Syst. Eng. Electron. 20 (7) (1998) 24–26.

[46] T. Wang, H. Li, X. Zhou, D. Liu, B. Huang, Three-way decision based on third-generation prospect theory with Z-numbers, Inf. Sci. 569 (2021) 13–38.
[47] G.W. Wei, Gray relational analysis method for intuitionistic fuzzy multiple attribute decision making, Expert Syst. Appl. 38 (9) (2011) 11671–11677.
[48] P.D. Liu, H.Y. Yang, Three-way decisions with single-valued neutrosophic decision theory rough sets based on grey relational analysis, Math. Probl Eng. (2019) 1–12.
[49] W.Z. Wang, X.W. Liu, Intuitionistic fuzzy geometric aggregation operators based on Einstein operations, Int. J. Intell. Syst. 26 (2011) 1049–1075.
[50] W.Z. Wang, X.W. Liu, Intuitionistic fuzzy information aggregation using Einstein operations, IEEE Trans. Fuzzy Syst. 20 (2012) 923–938.
[51] C.Q. Tan, W.T. Yi, X.H. Chen, Hesitant fuzzy Hamacher aggregation operators for multicriteria decision making, Appl. Soft Comput. 26 (2015) 325–349.
[52] Z. Yue, An avoiding information loss approach to group decision making, Appl. Math. Model. 37 (1–2) (2013) 112–126.
[53] R. Verma, B.D. Sharma, Exponential entropy on intuitionistic fuzzy sets, Kybernetika 49 (1) (2013) 114–127.
[54] D.C. Liang, D. Liu, A novel risk decision making based on decision-theoretic rough sets under hesitant fuzzy information, IEEE Trans. Fuzzy Syst. 23 (2) (2015) 237–247.
[55] S.L. Vargo, P.P. Maglio, M.A. Akaka, On value and value co-creation: a service systems and service logic perspective, Eur. Manag. J. 26 (3) (2008) 145–152.
[56] A. Yazdanparast, I. Manuj, S.M. Swartz, Co-creating logistics value: a service-dominant logic perspective, Int. J. Logist. Manag. 21 (3) (2010) 375–403.
[57] Y. Ju, H. Hou, J. Yang, Integration quality, value co-creation and resilience in logistics service supply chains: moderating role of digital technology, Ind. Manag. Data Syst. 121 (2) (2021) 364–380.
[58] L.N. Wang, H. Wang, Z.S. Xu, Z.L. Ren, A bi-projection model based on linguistic terms with weakened hedges and its application in risk allocation, Appl. Soft Comput. J 87 (2020), 105996.
[59] X.D. Liu, J.J. Zhu, S.F. Liu, Bidirectional projection method with hesitant fuzzy information, Sys. Eng. Theory Pract. 34 (10) (2014) 2637–2644.
[60] D. Liu, Y.Y. Yao, T.R. Li, Three-way investment decisions with decision-theoretic rough sets, Int. J. Comput. Intell. Syst. 4 (1) (2011) 66–74.
[61] D. Liu, T. Li, D. Liang, Three-way government decision analysis with decision-theoretic rough sets, Int. J. Uncertain. Fuzziness Knowledge-Based Syst. (2012) 119–132.