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Research article

Stability and singularity analysis of the cosmologies with different scenarios for deceleration parameter in the presence of torsion

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ABSTRACT

In the current paper, path deviation equations in absolutely parametric parallel geometries are derived. It is considered as a geodesic deviation equation. Additionally, it is modified by a torsion term. It proposes the deviation path equation that describes the trajectory deviation of a particle under the influence of the gravitational field. To examine the singularity of the Cosmological models, the modified version of the Raychaudhuri equation is utilized. The generalized law of the variation of Hubble's parameter is utilized to achieve some Cosmological models.

1. Introduction

Many researchers have studied the stability of models of the universe using Riemannian geometry. Sawicki and Hu investigated the stability of cosmic models in $f(R)$ gravity [1]. De Felice et al. examined the nonlinear stability of the Cosmic models in the massive gravity [2]. Furthermore in Riemannian geometry, many researchers examined the problem of the initial singularity of the universe evolution; for illustration, see Ref. [3]. Several researchers investigated the effect of torsion on Cosmology. Shie et al.considered the scalar torsion mode of the Poincar'e Gauge theory of gravity. They proposed it as a viable model to explain the current status of the universe [4]. Camera et al.studied the influence of the torsion gravity via Galaxy clustering and cosmic shear measurements [5]. Furthermore, one of the wide class of the extended theories of gravity, the so-called $f(T)$ gravity theory, arises. It is a generalization of the teleparallel gravity, where the curvature is replaced by ~~the~~ torsion. As a consequence, the torsion scalar (T) is replaced by the curvature scalar (R) in the Lagrangian; see Refs. [6,7]. To investigate the significant effect of a torsion tensor on the cosmic models, Capozziello et al. studied the $f(R)$ Cosmology with torsion [8]. This study was restricted by homogeneous and isotropic Cosmological models. It led to an accelerated expansion of the universe in the $f(R)$ gravity with torsion (see Tables 6 and 7).

Wanas constructed a parameterized absolute parallelism geometry (PAP) [9]. This geometry may be considered as a generalization of the absolute parallelism geometry (AP) and the Riemannian geometry. From the PAP geometry, Wanas derived the equation of path, also~~,~~ and the modified Raychaudhuri equation was achieved by Wanas and Bakry [10]. These two equations contain an extra term expressing torsion. Through the cancelation of the torsion parameterized, the path equation gives the geodesic one. On the other hand, the modified Raychaudhuri equation yields the ordinary Raychaudhuri one.

This paper, by serving the generalized path of the deviation equation, the Raychaudhuri equation in PAP space, addressed the role of torsion on the two important issues in theoretical cosmology i.e., stability and singularity. More precisely, it is done by including

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torsion in the path deviation equations as well as the original Raychaudhuri equations, which are then applied to four cosmological models with different assumptions about the evolution of the deceleration parameters. Torsion is indeed a non-Riemannian geometric extension of standard general relativity, which allows spins of matter and warping of spacetime to be included. Cosmological models with torsion have been considered in the literature to solve problems of either the very early or the present-day Universe. These questions and questions include the origin of the universe, the existence of singularities in black holes, the nature of inflation and dark energy, the origin of the matter-antimatter asymmetry in the universe, and the nature of dark matter. For this reason, this paper focuses on the stability and singularity of certain cosmological models in PAP and the influence of torsion. Furthermore, the study aims to examine the influence of torsion on space-time in different stages. To clarify the presentation, the remainder of this paper will be organized as follows: a review of PAP geometry, generalized path deviation equations, the Cosmological Models, the stability of the suggested Cosmological models, the singularity problem, and the concluding remarks are in Sections (1–6), respectively.

1.1. A review of the PAP geometry

In the what follows, a review of the PAP geometry is outlined. The building of the conventional AP space is completely defined in the 4-dimensions by a tattered vector λ_i^γ $(i = 1, 2, 3, 4)$ to indicate the vector number. Moreover, $(\gamma = 1, 2, 3, 4)$ indicates the coordinate component, the covariant vector of λ_i^γ is [11]

$$\lambda_i^\gamma \lambda_{i\ \alpha} = \delta_\alpha^\gamma, \lambda_i^\alpha \lambda_{j\ \alpha} = \delta_{i\ j} \tag{1}$$

Consider the metric tensor:

$$g_{\alpha\beta} = \lambda_{i\alpha}\lambda_{i\beta}, g^{\alpha\beta} = \lambda_i^\alpha \lambda_i^\beta, \text{and } g_{\beta\gamma}g^{\alpha\beta} = \delta_\gamma^\alpha \tag{2}$$

The square line element is given by

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta \tag{3}$$

The covariant derivative is defined as follows:

$$A_{\beta;\gamma} = A_{\beta,\gamma} - \begin{Bmatrix} \gamma & \\ \alpha & \beta \end{Bmatrix} A_\alpha, \tag{4}$$

$$\text{where } \begin{Bmatrix} \gamma & \\ \alpha & \beta \end{Bmatrix} = g^{\gamma\varepsilon}\left(g_{\alpha\varepsilon,\beta} + g_{\beta\varepsilon,\alpha} - g_{\alpha\beta,\varepsilon}\right) \Big/ 2 \tag{5}$$

One can define a non-symmetric connection $\Gamma_{\alpha\beta}^\gamma$ as:

$$\Gamma_{\alpha\beta}^\gamma = \lambda_i^\gamma \lambda_{i\ \alpha,\beta} = -\lambda_{i\ \alpha}\lambda_{i,\beta}^\gamma \tag{6}$$

The non-symmetric connection (6) is a consequence of the following condition:

$$\lambda_{i\ \alpha\ |\ \beta} = \lambda_{i\ \alpha,\beta} - \Gamma_{\alpha\beta}^\gamma\ \lambda_{i\ \gamma} \tag{7}$$

Using the affine connection as given by Eq. (6), one can define:

$$\Lambda_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^\gamma - \Gamma_{\beta\alpha}^\gamma = -\Lambda_{\beta\alpha}^\gamma, \tag{8}$$

$$\text{and } \psi_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^\gamma - \begin{Bmatrix} \gamma & \\ \alpha & \beta \end{Bmatrix} = \lambda_i^\gamma \lambda_{i\ \alpha\ ;\ \beta} = -\lambda_{i\ \alpha}\lambda_{i\ ;\ \beta}^\gamma, \tag{9}$$

where $\Lambda_{\alpha\beta}^\gamma$ is the torsion tensor, and $\psi_{\alpha\beta}^\gamma$ is the contortion tensor.

The general absolute derivative is given by [9].

$$A_{\alpha\ \|\ \beta} = A_{\alpha,\beta} - \nabla_{\alpha\beta}^\gamma A_\gamma, \tag{10}$$

$$\text{and } A_{\|\beta}^\alpha = A_{,\beta}^\alpha + \nabla_{\gamma\ \beta}^\alpha A^\gamma \tag{11}$$

In addition, the connection $\nabla_{\alpha\beta}^\gamma$ is given by

$$\nabla_{\alpha\beta}^\gamma = \begin{Bmatrix} \gamma & \\ \alpha & \beta \end{Bmatrix} + b\psi_{\alpha\beta}^\gamma, \tag{12}$$

where b is a non-dimensional parameter.

The path equation in the PAP geometry is given by Wanas as [9].

$$\frac{dZ^\beta}{d\tau} + \begin{Bmatrix} \gamma & \\ \alpha & \beta \end{Bmatrix} Z^\alpha Z^\gamma = -\,b\ \Lambda_{(\alpha\ \gamma)}^\beta Z^\alpha Z^\gamma, \tag{13}$$

where $\Lambda^{\beta}_{(\alpha\ \gamma)}$ is the symmetric part of the torsion tensor, and $Z^{\alpha} = dx^{\alpha}/d\tau$.

Now, consider the following remarks:

1. In the case when $b = 0$, the PAP-space will be reduced to the Riemannian geometry. In this case, Eq. (13) will be concentrated to the geodesic equation. This is clear by substituting the value of the parameter into Eq. (12) to give

$$\nabla^{\gamma}_{\alpha\beta} = \left\{ \begin{matrix} \gamma & \\ \alpha & \beta \end{matrix} \right\} \tag{14}$$

2. In the case when $b = 1$, it can be easily shown, that the geometry will be abridged to the conventional AP geometry with: $\nabla^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta}$.
3. 3. Eq. (13) is a new path in the PAP space. As above, this equation can be converted to the geodesic equation when $b = 0$. This case represents the trajectory of the test particle in the background gravitational field. Furthermore, the torsion term at the right end of Eq. (13) can be represented as a kind of connection between the warping of the background gravitational field and some intrinsic property of the moving particle.

1.2. Generalized path of the deviation equation

In this section, a generalized path of the deviation equation that is valid for any two paths, with arbitrary tangent vectors and not compulsorily parallel, is derived. The assumption of the path deviation utilizes the following conditions:

1 When the two curves represent the following two different paths

$$\frac{DZ_1^{\beta}}{D\tau} = 0, \text{ and } \frac{DZ_2^{\beta}}{D\sigma} = 0, \tag{15}$$

where τ and σ are the affine parameters.

2. The deviation vector between two points on the paths is $\xi^{\beta}(\tau)$. If $d\tau$ and $d\sigma$ are infinitesimal arcs on the paths 1 and 2, respectively, it follows that the deviation vectors between $\xi^{\beta}(\tau)$ and $\xi^{\beta}(\tau + d\tau)$ gives

$$\frac{d\sigma}{d\tau} = 1 + \lambda, \text{ and } \frac{d\lambda}{d\tau} = 0 \tag{16}$$

3. The paths are infinitesimally close in a neighborhood U as

$$x_2^{\beta}(\sigma) = x_1^{\beta} + \varepsilon\, \xi^{\beta}(\tau) \tag{17}$$

4. As a linear analysis, the higher orders in (ε), ξ^{β}, and δZ^{β}, will be disregarded.
5. For simplicity, one may define the deviation vector ξ^{β} as connecting between two points of equal arc lengths. Let $d\tau = d\sigma = ds$ and the tangent vector satisfies the following relations:

$$Z_1^{\beta}(s) Z_{1\ \beta}(s) = Z^2 \text{ where } Z_1^{\beta} = dx_1^{\beta}(s)/ds \tag{18}$$

and

$$Z_2^{\beta}(s) Z_{2\beta}(s) = Z^2 \text{ where } Z_2^{\beta} = dx_2^{\beta}(s)/ds \tag{19}$$

Physically, the elapsed proper time, between two observers moving with two test particles on the paths, is given as
For the equation of path 1, one gets

$$\frac{DZ_1^{\beta}}{Ds} = \frac{dZ_1{}^{\beta}}{ds} + \left\{ \begin{matrix} \gamma & \\ \alpha & \beta \end{matrix} \right\}(x) Z_1{}^{\alpha} Z_1{}^{\gamma} + b\, \Lambda^{\beta}_{(\alpha\ \gamma)}(x)\, Z_1{}^{\alpha} Z_1{}^{\gamma} = 0 \tag{20}$$

For the equation of path 2, one finds

$$\frac{DZ_2^{\beta}}{Ds} = \frac{dZ_2{}^{\beta}}{ds} + \left\{ \begin{matrix} \gamma & \\ \alpha & \beta \end{matrix} \right\}(x_1 + \xi) Z_2{}^{\alpha} Z_2{}^{\gamma} + b\, \Lambda^{\beta}_{(\alpha\ \gamma)}(x_1 + \xi)\, Z_2{}^{\alpha} Z_2{}^{\gamma} = 0 \tag{21}$$

Eq. (20) may be written as:

$$\frac{d^2\left(x_1^{\beta} + \xi^{\beta}\right)}{ds} + \left(\left\{ \begin{matrix} \gamma & \\ \alpha & \beta \end{matrix} \right\}(x_1 + \xi) + b\, \Lambda^{\beta}_{(\alpha\ \gamma)}(x_1 + \xi) \right) \left(\frac{d\left(x_1^{\alpha} + \xi^{\alpha}\right)}{ds} \frac{d\left(x_1^{\gamma} + \xi^{\gamma}\right)}{ds} \right) = 0 \tag{22}$$

Using the Taylor expansion up to first order in, one catch

$$\left\{ \begin{matrix} \gamma \\ \alpha\,\beta \end{matrix} \right\}(x_1+\xi) = \left\{ \begin{matrix} \gamma \\ \alpha\,\beta \end{matrix} \right\}(x_1) + \left\{ \begin{matrix} \gamma \\ \alpha\quad\beta \end{matrix} \right\}(x_1)\,_{,\lambda}\,\xi^{\lambda}, \tag{23}$$

$$\text{and } \Lambda^{\beta}_{(\alpha\,\gamma)}(x_1+\xi) = \Lambda^{\beta}_{(\alpha\,\gamma)}(x_1) + \Lambda^{\beta}_{(\alpha\,\gamma)}(x_1)\,_{,\lambda}\,\xi^{\lambda} \tag{24}$$

Combining equations ((15)–(17) and (20) and (22)–(24), ond finds

$$\frac{d^2\xi^{\beta}}{ds^2} + 2\left[\left\{ \begin{matrix} \gamma \\ \alpha\quad\beta \end{matrix} \right\} + b\Lambda^{\beta}_{(\alpha\,\gamma)}\right]\frac{dx^{\alpha}}{ds}\frac{d\xi^{\gamma}}{ds} + \left[\left\{ \begin{matrix} \gamma \\ \alpha\quad\beta \end{matrix} \right\} + b\Lambda^{\beta}_{(\alpha\quad\gamma)}\right]_{,\lambda}\frac{dx^{\alpha}}{ds}\frac{dx^{\gamma}}{ds}\xi^{\lambda} = 0 \tag{25}$$

2. Cosmological models

Berman and Gomide studied the Cosmic models by using the law of variation of Hubble's parameter, which gives a constant deceleration parameter (CDP) [12,13]. Following this work, some researchers derived some models of the universe by assuming a linearly varying deceleration parameter (LVDP); see Refs. [14,15] for an illustration. The quapic deceleration parameter **(**PUVDP); see Ref. [16]. Moreover, the periodic varying deceleration parameter in $f(R,T)$ gravity (PVDP); see Ref. [17]. Throughout this section, some types of the generalized variation law of Hubble parameter are examined to achieve the deceleration parameter q and scale factor $S(t)$. One may summarize the previous works in the following table:

The deceleration parameter (q) is given by [18]

$$q = \frac{dH^{-1}}{dt} - 1 \tag{26}$$

From (26) and Table 1, one gets.

The physical behavior of these Cosmological models was studied in previous works. For more convenience, the deceleration parameters versus the Cosmic time t will be plotted by using $m = 1.5$, for CDP, $m = 1.6$ and $a = 0.09$ for LVDP, $n = 0.5$ for PUVDP, and $m = 1.55$, $k = 0.1$ for PVDP.

From Fig. 1, we see that in PUVDP model, when $q = -0.73$. Accordingly, this model gives $t_{day} = 13.7\ Gyr$, represents the now time, for the other models, see Refs. [14–17].

In the next section, the stability analysis of cosmological models and the results for singularities will be examined. Particular attention is salaried to those caused by the torsion of the geometric effect in space-time.

3. Stability analysis of the cosmological models

The stability of cosmological models has been studied by many researchers using different methods in different theories; see, for example, Sawicki and Hu elucidating the stability of cosmological solutions in gravitational models [1]. In addition, Gorini et al. reported the stability of some models of perfect fluid cosmology [19]. Dappiaggi et al. studied stable cosmological models driven by free quantum scalar fields [20]. This section is dedicated to studying the behavior of homogeneous and isotropic universal models by using state-of-the-art techniques from stability theory. This technique was first used in the GR by Wanas and Bakry [21]. The study of the stability of cosmological solutions is closely related to the formation of the large-scale structure of the universe. This structure exists in our universe. They cannot form in a stable world model. It would be interesting to see if a world model is capable of generating large-scale structures such as galaxies and galaxy clusters.

Typically, some authors have used opposing components of the deflection vector to achieve stability criteria. They used the covariate/contravariant of the deflection vector in driving stabilizing conditions. On the other hand, one might suggest another procedure which mainly depends on using the magnitude of the deflection vector. This method is independent of the coordinate system. Therefore, the following assumptions can be considered: $e_t = \left|\xi_{\gamma}\,\xi^{\gamma}\right|^{1/2}$ To examine the stability of the model; see Ref. [21] for an explanation. The tetrad gives the structure of AP space and has homogeneity and isotropy, expressed by Robertson in spherical polar coordinates (t, r, θ, φ) as follows [22]:

Table 1
The Hubble parameter and the derivatives of the scale factors.

$\ddot{S}$	$\dot{S}$	$H = \frac{\dot{S}}{S}$	Type Model
$D^2\,(1-m)\,S^{1-2m}$	$D\,S^{1-m}$	$D\,S^{-m}$	CDP $t: 0 \to \infty$
$\frac{4\,S\,(1-m+a\,t)}{t^2\,(2m-at)^2}$	$\frac{2\,S}{t\,(2m-at)}$	$\frac{2}{t\,(2m-at)}$	LVDP $t: 0 \to \frac{2m}{a}$
$\frac{S\,(1-3\,t^2+12\,n\,t-8\,n^2)}{t^2\,(2n-t)^2\,(4n-t)^2}$	$\frac{S}{t\,(2n-t)\,(4n-t)}$	$\frac{1}{t\,(2n-t)\,(4n-t)}$	PUVDP $t: 0 \to 4n$
$\frac{S\,k^2\,(1-\cos kt)}{m\sin^2 kt}$	$\frac{S\,k}{m\sin kt}$	$\frac{k}{m\sin kt}$	PVDP $t: 0 \to 100$

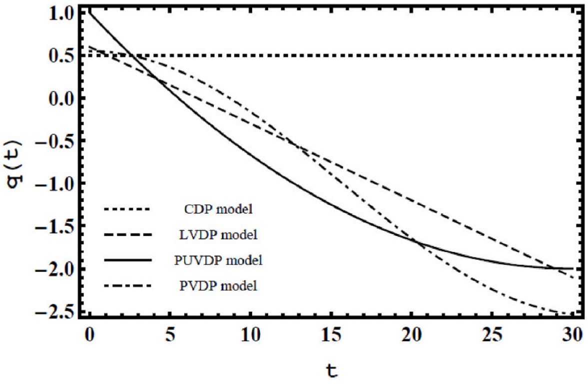

Fig. 1. q versus t.

$$\left.\begin{aligned}
\lambda_0^{\mu} &= \{1,0,0,0\}\\
\lambda_1^{\mu} &= \left\{0, \frac{L^{+}\sin\theta\cos\varphi}{4S}, \frac{L^{-}\cos\theta\cos\varphi - 4\sqrt{K}r\sin\varphi}{4r\,S}, \frac{-\left(L^{-}\sin\varphi + 4\sqrt{K}\,r\cos\theta\cos\varphi\right)}{4rS\sin\theta}\right\}\\
\lambda_2^{\mu} &= \left\{0, \frac{L^{+}\sin\theta\sin\varphi}{4S}, \frac{L^{-}\cos\theta\sin\varphi + 4\sqrt{K}r\cos\varphi}{4r\,S}, \frac{L^{-}\cos\varphi - 4\sqrt{K}\,r\cos\theta\sin\varphi}{4rS\sin\theta}\right\}\\
\lambda_3^{\mu} &= \left\{0, \frac{L^{+}\cos\theta}{4S}, \frac{-L^{-}\sin\theta}{4r\,S}, \frac{\sqrt{K}}{S}\right\}
\end{aligned}\right\} \tag{27}$$

where $\mu = 0,1,2,3$ represents the coordinate components, and $L^{\pm} = 4 \pm Kr^2$, K is the curvature constant $= (-1,0,1)$, and $S(t)$ is the scale factor.

The definition (2) may be replaced by:

$$g_{\alpha\beta} = \eta_{ij}\lambda_{i\,\alpha}\lambda_{i\,\beta}, \tag{28}$$

where, $\eta_{ij} = diag(1, -1, -1, -1)$.

Eq. (28) defines the pseudo-Riemannian structure that is associated with the AP structure. The metric tensor corresponding to the tetrad as given by (27) and (28) is given by:

$$g_{00} = 1, g_{11} = -(4S/L^{+})^2, g_{22} = g_{11}\,r^2, \text{and } g_{33} = g_{11}\,r^2\sin^2\theta \tag{29}$$

Using (3), (27), (28) and (29), one gets

$$ds^2 = dt^2 - \frac{16S^2(t)}{L^{+2}}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right)\right] \tag{30}$$

The Christoffel symbols are given by,

$$\begin{aligned}
&\left\{\begin{matrix}0\\1\ \ 1\end{matrix}\right\} = 16S\dot{S}/L^{+2}, \left\{\begin{matrix}0\\2\ \ 2\end{matrix}\right\} = 16\,r^2S\dot{S}/L^{+2}, \left\{\begin{matrix}0\\3\ \ 3\end{matrix}\right\} = 16\,r^2\sin^2\theta\,S\dot{S}/L^{+2}\\
&\left\{\begin{matrix}1\\1\ \ 1\end{matrix}\right\} = -2K\,r/L^{+}, \left\{\begin{matrix}1\\0\ \ 1\end{matrix}\right\} = \left\{\begin{matrix}2\\0\ \ 2\end{matrix}\right\} = \left\{\begin{matrix}3\\0\,3\end{matrix}\right\} = \frac{\dot{S}}{S}\\
&\left\{\begin{matrix}1\\2\ \ 2\end{matrix}\right\} = \frac{K^2r^5 - 16\,r}{L^{+2}}, \left\{\begin{matrix}1\\3\ \ 3\end{matrix}\right\} = \frac{-L^{-}\,r\sin^2\theta}{L^{+2}}\\
&\left\{\begin{matrix}2\\1\ \ 2\end{matrix}\right\} = \left\{\begin{matrix}3\\1\ \ 3\end{matrix}\right\} = L^{-}\Big/ r\,L^{+}, \left\{\begin{matrix}2\\3\ \ 3\end{matrix}\right\} = -\sin\theta\cos\theta, \left\{\begin{matrix}3\\2\ \ 3\end{matrix}\right\} = \cot\theta,
\end{aligned} \tag{31}$$

where $\dot{S} = dS/dt$.

The non-vanishing contortion components are given by:

$$\psi_{01}^{1} = \psi_{02}^{2} = \psi_{03}^{3} = -\dot{S}/S, \psi_{11}^{0} = -16S\dot{S}/L^{+2}, \psi_{22}^{2} = \psi_{11}^{0}\,r^2, \psi_{33}^{0} = \psi_{11}^{0}\,r^2\sin^2\theta$$

$$\psi^2_{13} = -\psi^2_{31} = 4\sqrt{K}\sin\theta \Big/ L^+, \psi^3_{21} = -\psi^3_{12} = 4\sqrt{K} \Big/ L^+ \sin\theta \tag{32}$$

The non-vanishing symmetric torsion components are listed as:

$$\Lambda^0_{(1\ \ 1)} = 16S\,\dot{S} \Big/ L^{+2}, \Lambda^0_{(2\ 2)} = 16\, r^2 S\,\dot{S} \Big/ L^{+2}, \Lambda^0_{(3\ 3)} = 16\, r^2\sin^2\theta\, S\,\dot{S} \Big/ L^{+2}$$

$$\Lambda^1_{(1\ \ 0)} = \Lambda^2_{(2\ 0)} = \Lambda^3_{(3\ 0)} = -\dot{S} \Big/ 2S \tag{33}$$

Substituting from equations (31) and (33) into Eq. (25), with the moving coordinate's condition, one gets.
$\frac{d^2\xi^0}{dt^2} = 0$ (34)

$$\frac{d^2\xi^\beta}{dt^2} + \frac{(2-b)\,\dot{S}}{S}\frac{d\xi^\beta}{dt} = 0, \tag{35}$$

where $\beta = 1, 2, 3$.

Integrating equations (34) and (35), one finds

$$\xi^0 = C^0\, t., \xi^\beta = C^\beta\, t., (\text{For } b = 2), \tag{36}$$

$$\xi^\beta = C^\beta \int \frac{dt}{S^{2-b}}, (\text{For } b \neq 2), \tag{37}$$

where C^0 and C^β are constants.

It is worthy to note that, in the case of $b = 0$, Eq. (37) is reduced to the deviation vector in Riemannian geometry; see Wanas and Bakry [21]. The current work only considers the cases of $b = 0$ and $b = 2$.

Using the scale factor as given in Table 2and Eq. (37), we get the following Tables (3, 4 and 5).

This integration is evaluated by the Mathematica software (12.0.0.0). It should be noted that the integration constants given in equations (36) and (37) are taken as unity.

In the following, the physical behavior of e_t is plotted against cosmic time t. For this purpose, three figures will be represented by choosing the value as given in Fig. 1, and then they will be compared with the conclusion of Berman [12].

From the above figures, one may reach th following remarks: In Figs. 2 and 3, the CDP and the LVDP models in the case of ($b = 0$ or $b = 2$), in the light of the concept of stability/instability, are an unstable position. This is because the deviation vector increases with time. Throughout Fig. 4, the PUVDP model at ($b = 0$ *or* 2) starts from the unstable at $t \in [0,1[$, because the deviation vector increases with increasing time. Then it comes to a stable position at $t \in [1,2]$, because the deviation vector decrease with the increase of time. Finally, in Fig. 5, the PVDP model at ($b = 0$ or $b = 2$) gives an oscillatory stability. It does not explain the instant of a Big Rip. Also, the behaviors of the stability in the case of $0 < b < 1$ give an unstable models.

4. Singularity of the cosmological models

Many researchers have investigated singularity in different ways. For example, it has been investigated along with the Big Bang model without singularities; for instance, see Ref. [23]. Furthermore, they tested a new type of isotropic Cosmological models and find it is not subjected to singularity. Vacuum theory provided a possible solution to the singularity of Cosmology, see Ref. [24]. For singularity-free Cosmological models, see Refs. [25–27], and many others.

Now, we aim to realize the effect of torsion on the existence of the initial singularity of our proposed cosmological model. For this purpose, a modified version of Raychaudhuri's equation can be defined in the PAP geometry, which includes a torsion term [10], as

$$\frac{d\Theta}{dt} - 2\left(\Omega^2 - \Sigma^2\right) + \frac{\Theta^2}{3} + Z^2 Z^\alpha Z^\beta \left(R_{\alpha\beta} + b\, L_{\alpha\beta}\right) - bZ^\alpha\left(\Omega^\beta_\sigma + \Sigma^\beta_\sigma\right)\Lambda^\sigma_{\alpha\beta} - \frac{b}{3}\Theta Z^\alpha C_\alpha = 0, \tag{38}$$

where Θ is the expansion scalar, Ω is the rotation scalar, and Σ is the shear scalar defined by $\Theta = Z^\alpha_{|\,|\,\alpha}, \Omega_{\alpha\beta} = Z_{[\alpha\,\|\,\beta]}, \Omega^2 = \frac{1}{2}\Omega^{\alpha\beta}\Omega_{\alpha\beta}$,

Table 2
The deceleration parameters and the scale factor.

$S(t)$	q	Type Model
$(Dt)^{1/m}$	$m - 1$	CDP $t: 0 \to \infty$
$\left(\frac{t}{2m - at}\right)^{1/m}$	$m - 1 - at$	LVDP $t: 0 \to 2m/a$
$\left[\frac{t\,(4n - t)}{(2n - t)^2}\right]^{1/8n^2}$	$(8n^2 - 1) - 12nt + 3t^2$	PUVDP $t: 0 \to 4n$
$(\tan kt/2)^{1/m}$	$m\cos kt - 1$	PVDP $t: 0 \to 100$

Table 3
The deviation vector for the suggested cosmological models.

Type model	The deviation vector ξ^{β}	
	$b = 0$	$b = 2$
CDP $t: 0 \to \infty$	$t^{1-2/m}$	t
LVDP $t: 0 \to 33$	$\frac{-4\,t\,(3.2t^{-1} - 0.09)^{1.25} Hypergeometric2F1[-1.25, -0.25, 0.75, 0.03\,t]}{(1 - 0.03\,t)^{1.25}}$	t
PUVDP $t: 0 \to 2$	$\frac{1}{2}\log\left(\frac{t}{2-t}\right) - t$	t
PVDP $t: 0 \to 100$	$\frac{-8.73\cos(0.05\,t)\quad(\sin 0.05\,t)^{1.3}\ Hypergeometric2F1[1.15, 1.15, 2.15, (\cos 0.05\,t)^2]}{(\tan 0.05\,t)^{1.29}}$	t

Table 4
The magnitude of the deviation vector at ($b = 0$).

Type model	The magnitude of the deviation vector $e_t = \lvert\xi_\alpha\,\xi^\alpha\rvert^{1/2}$
CDP $t: 0 \to \infty$	$\left\lvert t^{(m-1)/m}\right\rvert$
LVDP $t: 0 \to 2m/a$	$\left\lvert\frac{4\,t\,(3.2t^{-1} - 0.09)^{0.625} Hypergeometric2F1\,[-1.25, -0.25, 0.75, 0.03\,t]}{(1 - 0.03t)^{1.25}}\right\rvert$
PUVDP $t: 0 \to 2$	$\left\lvert\left(\frac{[t\,(2-t)]^{1/2}}{1-t}\right)\left(\frac{1}{2}\,\log\left(\frac{t}{2-t}\right) - t\right)\right\rvert$
PVDP $t: 0 \to 100$	$\left\lvert\frac{-8.73\cos(0.05\,t)\quad(\sin 0.05\,t)^{1.3}\ Hypergeometric2F1[1.15, 1.15, 2.15, (\cos 0.05\,t)^2]}{(\tan 0.05\,t)^{0.65}}\right\rvert$

Table 5
The magnitude of the deviation vector at ($b = 2$).

Type model	$e_t = \lvert\xi_\alpha\,\xi^\alpha\rvert^{1/2}$
CDP $t: 0 \to \infty$	$\left\lvert t^{(m+1)/m}\right\rvert$
LVDP $t: 0 \to 2m/a$	$\left\lvert\frac{t^{1.625}}{(3.2 - 0.09\,t)^{0.625}}\right\rvert$
PUVDP $t: 0 \to 2$	$\left\lvert\frac{t^{1.5}\,(2-t)^{0.5}}{(1-t)}\right\rvert$
PVDP $t: 0 \to 100$	$\left\lvert t\,(\tan\ 0.05\,t)^{0.65}\right\rvert$

Table 6
The modified Raychaudhuri equation for the cosmological models.

Type model	The modified Raychaudhuri equation
CDP $t: 0 \to \infty$	$d\Theta/dt = 3(b-1)/t^2$
LVDP $t: 0 \to 2m/a$	$d\Theta/dt = \frac{12(b-1)(m-at)}{t^2(2m-at)^2}$
PUVDP $t: 0 \to 4n$	$d\Theta/dt = \frac{3(b-1)(8n^2 - 12nt + 3t^2)}{t^2(4n-t)^2}$
PVDP $t: 0 \to 100$	$d\Theta/dt = \frac{3k^2(b-1)\cos kt}{m\sin^2 kt}$

Table 7
The relation between the cosmic time t, and the redshift z.

Type model	t (z)
CDP	Is not exist
LVDP z : $-1 \to 4$	$\frac{2m}{a + (1+z)^m}$
PUVDP z : $-1 \to 4$	$1 + \frac{1+z}{\sqrt{z^2 + 2\,z + 2}}$
PVDP z : $-1 \to 4$	$\frac{2}{k}\ \tan^{-1}\ \frac{1}{(1+z)^m}$

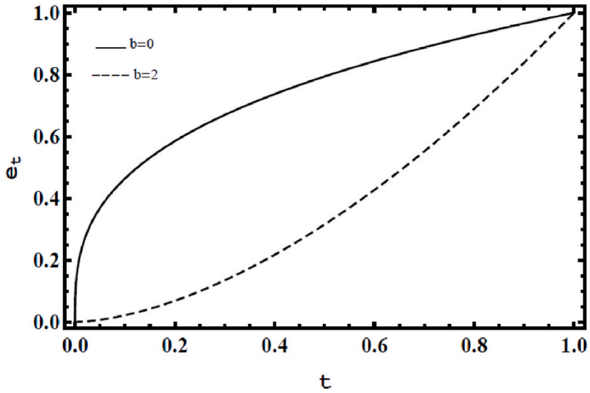

Fig. 2. e_t versus t, for the CDP model.

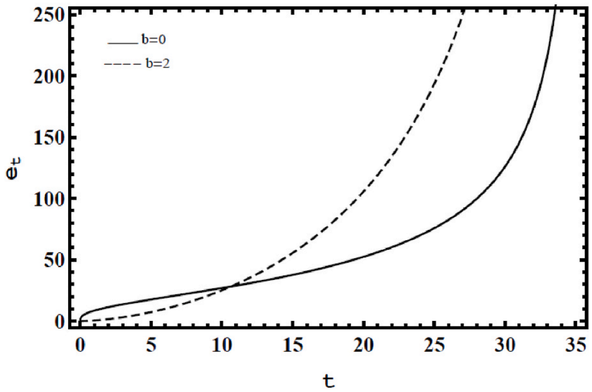

Fig. 3. e_t versus t, for the LVDP model for $m = 1.6$ and $a = 0.09$.

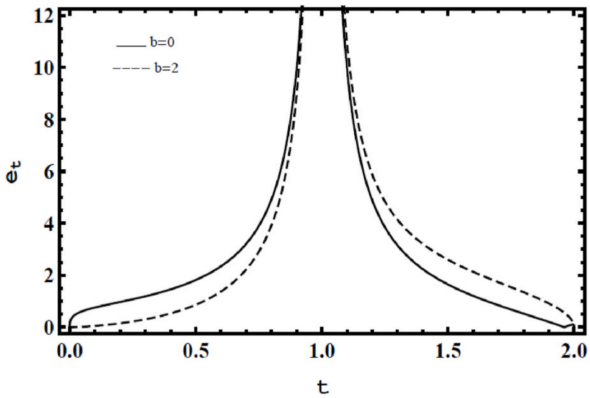

Fig. 4. e_t versus t, for PUVDP model, $n = 0.5$.

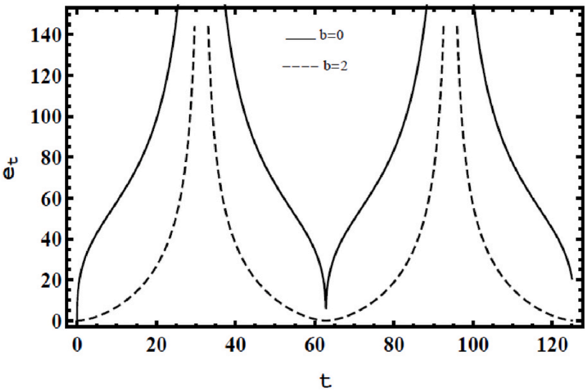

Fig. 5. e_t versus t, for PVDP model, $m = 1.55$ and $k = 0.1$.

$$\Sigma_{\alpha\beta} = Z_{(\alpha \parallel \beta)} - \frac{\Theta}{3Z^2} P_{\alpha\beta}, \Sigma^2 = \frac{1}{2}\Sigma^{\alpha\beta}\Sigma_{\alpha\beta}, P_{\alpha\beta} = Z^2 g_{\alpha\beta} - Z_\alpha Z_\beta$$

$$\text{and } Z^2 = Z^\alpha Z_\alpha, C_\alpha = \psi^{\beta}_{\alpha\ \beta}, Z_\sigma Z^\alpha Z^\beta \Lambda^{\sigma}_{\alpha\ \beta} = 0 \tag{39}$$

The $R_{\alpha\beta}$ is the Ricci tensor and $L_{\mu\nu}$ is defined by

$$L_{\mu\sigma} = \left(\psi^{\alpha}_{\mu\sigma,\alpha} - \psi^{\alpha}_{\mu\alpha,\sigma}\right) + b\ \left(\psi^{\alpha}_{\varepsilon\ \alpha}\psi^{\varepsilon}_{\mu\ \sigma} - \psi^{\alpha}_{\varepsilon\ \sigma}\psi^{\varepsilon}_{\mu\ \alpha}\right) + \begin{Bmatrix} \varepsilon \\ \mu \quad \sigma \end{Bmatrix}\psi^{\alpha}_{\varepsilon\ \alpha} - \begin{Bmatrix} \varepsilon \\ \mu \quad \sigma \end{Bmatrix}\psi^{\alpha}_{\varepsilon\ \sigma} + \begin{Bmatrix} \varepsilon \\ \mu \quad \sigma \end{Bmatrix}\psi^{\varepsilon}_{\mu\ \sigma} - \begin{Bmatrix} \varepsilon \\ \mu \quad \sigma \end{Bmatrix}\psi^{\varepsilon}_{\mu\ \alpha}. \tag{40}$$

After a long but straightforward calculation, the result of the modified Raychaudhuri equation as given in (38), in the PAP geometry, is obtained as

$$\frac{d\Theta}{dt} = 3\ (b-1)\left\{\frac{\dot{S}^2}{S^2} - \frac{\ddot{S}}{S}\right\} \tag{41}$$

This result has been previously shown by Wanas et al. [28]. The term containing the parameter b in Eq. (41) comes from the torsion effect. Some researchers have generalized the Raychaudhuri equation by adding a new term expressing the torsion in the space-time; for illustration, see Wanas and Bakry [10,29]. Comparing the modified version of the Raychaudhuri equation as given in Eq. (38) with the original Raychaudhuri equation that developed in Riemannian geometry, we found some additional terms as given by Raychaudhuri [30]. The disappearance of the parameter b indicates that the interaction between quantum spin and twist is switched off. In short, the disappearance of parameter b indicates that the moving particles are non-rotating. The combination of Eq. (41) and Table 1 results in the following table:

The existence of the initial singularity depends mainly on the solution of Eq. (41). Particularly, it depends on the resulting sign on the left-hand side of the term ($d\Theta/dt$). The same conclusion can be obtained by using the standard conventions of the singularity theorems of GR. In other words, the singular models are characterized by $d\Theta/dt < 0$, which is a necessary condition, but not sufficient. In what follows, the function $d\Theta/dt$ will be plotted versus the Cosmic time t, for the considered Cosmological models.

In Fig. 6, the CDP model, for ($b = 0$), starts with a singular case. Then, it comes to a non-singular case. Simultaneously, at ($b = 2$), the model has always non-singular case.

In Fig. 7, the LVDP model for ($b = 0$) starts with a singular case. Then, it enters into a non-singular case. Consequently, it ends with a non-singular, at a Big Rip time at $t = 33$ (Gyr). As ($b = 2$), it starts with a non-singular case, and then ends with a singularity, at a Big Rip time $t = 33$ (Gyr).

In Fig. 8, the PUVDP model ($b = 0$) has an oscillatory singularity. It starts with a singular case, and then ends with a non-singular case, at a Big Rip time $t = 1$ Gyr, and finally ends with a singular case at a time $t = 2$ Gyr. By contrast, at ($b = 2$), it starts with a non-singular case, and then comes to a singularity at a Big Rip time at $t = 1$ Gyr. Subsequently, it ends with a non-singular case, at the time $t = 2$ Gyr.

In Fig. 9, the PVDP model, for ($b = 0$), has a periodic singularity. It starts with a singular case, and ends with a non-singular case. As ($b = 2$), it has a periodic singularity; it starts with a non-singular case, and then ends with a singular case.

It can be seen from Figs. (10–13) that the cosmological model approaches non-singularity as the value of b increases.

According to the analysis of these cosmological observations, it is said that the transition redshift of accelerated expansion occurs at $z_t = 0.43 \pm 0.07$; for instance, see Refs. [31,32]. The behavior of the function $d\Theta/dt$ is graphically studied versus to the transition redshift z. Therefore, For the considered cosmological model, the relationship between cosmic time t and redshift z is given in the following table:

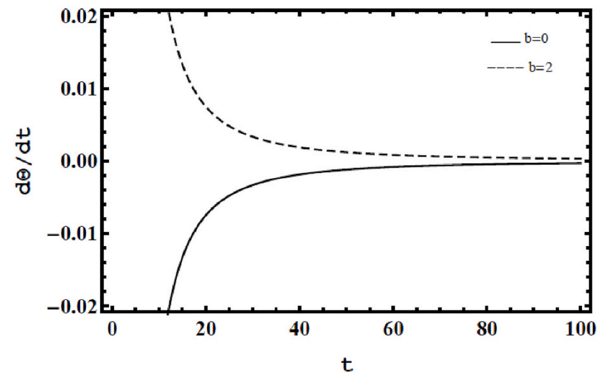

Fig. 6. $d\Theta/dt$ versus t, for the CDP model.

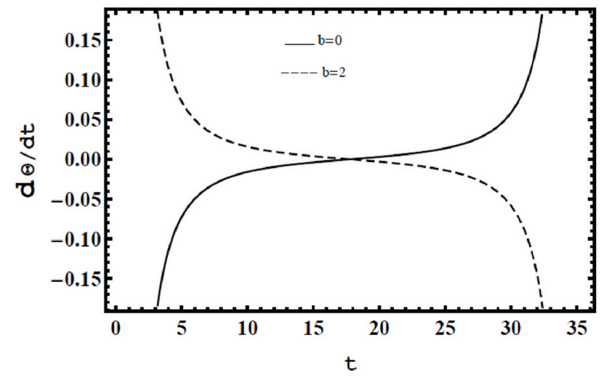

Fig. 7. $d\Theta/dt$ versus t, for the LVDP model, where $m = 1.6$ and $a = 0.09$.

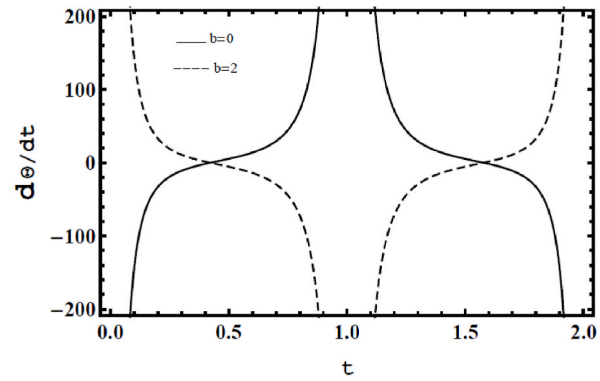

Fig. 8. $d\Theta/dt$ versus t, for the PUVDP model, where $n = 0.5$.

In what follows, the function $d\Theta/dt$ will be plotted versus the red shift z, for the considered Cosmological models as seen in Fig. 13.

In Fig. 14, the LVDP model, for ($b = 0$), starts with a non-singular case at $z = -1$ and enters to a singular case at $z \geq 0.4$ (which is an accelerating expansion consistent with the observational data). As ($b = 2$), it starts with a singular case, and then ends with a non-singular one.

In Figs. 15 and 16 the PUVDP and PVDP models, have the same physical behavior as the LVDP model.

It is clear from Figs. 6–12 that the influence of the term that represents the torsion in the Raychaudhuri equation, reverses the behavior of the function $d\Theta/dt$.

5. Concluding remarks

The current paper investigates a path deviation in the absolute parameterized parallelism geometry to study the stability criteria for

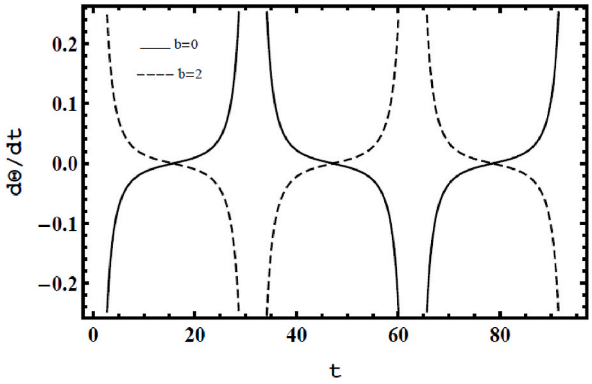

Fig. 9. $d\Theta/dt$ versus t, for PVDP model, where $m = 1.55$ and $k = 0.1$.

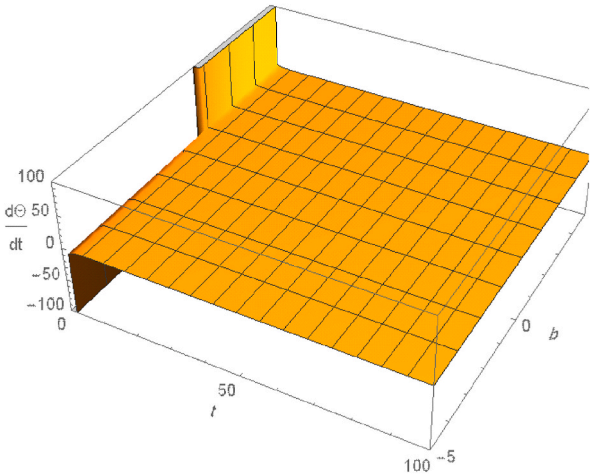

Fig. 10. $d\Theta/dt$ versus t and b for the CDP model.

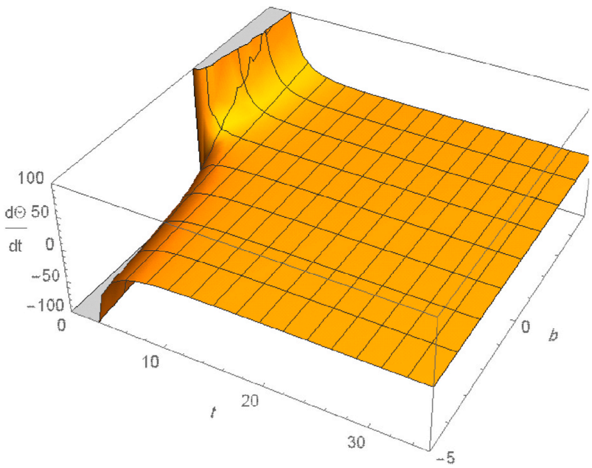

Fig. 11. $d\Theta/dt$ versus t and b for the LVDP model, where $m = 1.6$ and $a = 0.09$.

some Cosmological models. Additionally, the singularity cases are included in the modified Raychaudhuri equation. The analysis focuses on the torsion influence in both cases. At this end, from the above mentioned results the following conclusions may be drawn as follows:

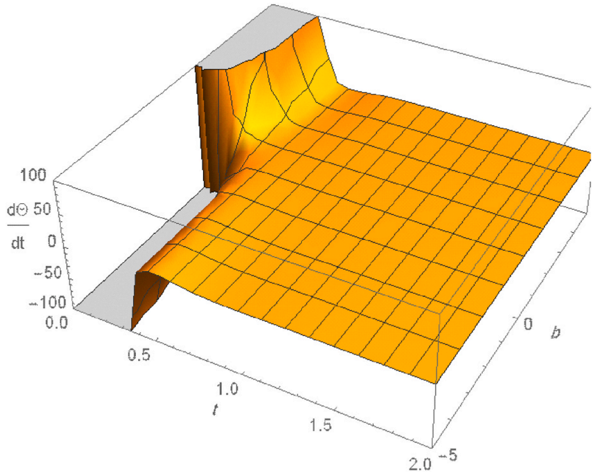

Fig. 12. $d\Theta/dt$ versus t and b for the PUVDP model, where $n = 0.5$.

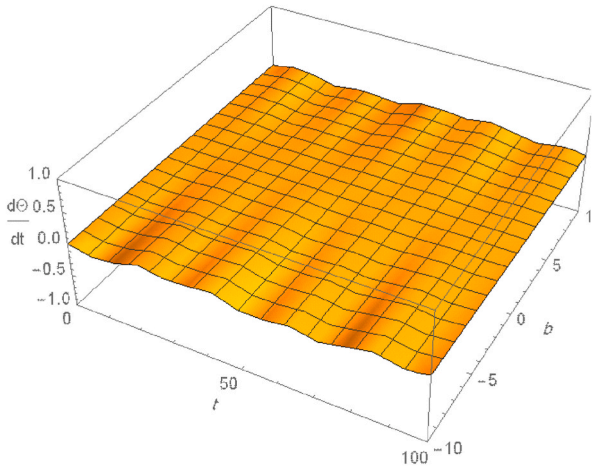

Fig. 13. $d\Theta/dt$ versus t and b for PVDP model, where $m = 1.55$ and $k = 0.1$.

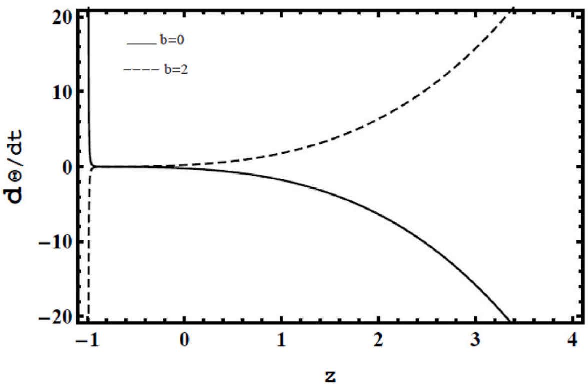

Fig. 14. Displays the $d\Theta/dt$ versus the redshift z, for the LVDP model, where $m = 1.6$ and $a = 0.09$.

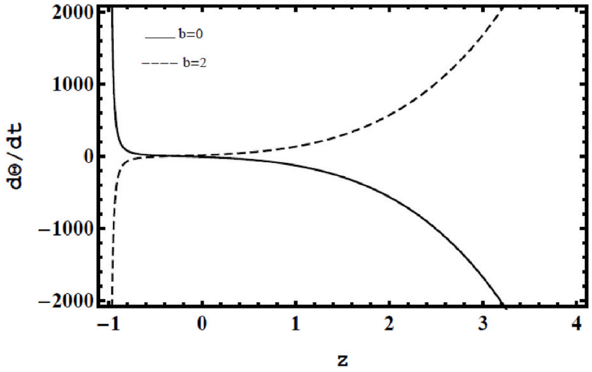

Fig. 15. Displays the $d\Theta/dt$ versus the redshift z, for the PUVDP model, where $n = 0.5$.

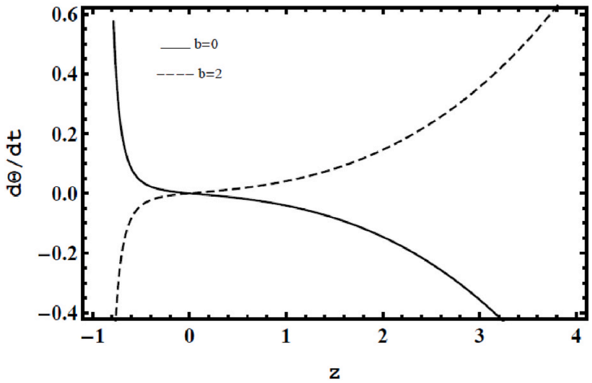

Fig. 16. Displays the $d\Theta/dt$ versus the against redshift z, for PVDP model, where $m = 1.55$ and $k = 0.1$.

• The Parameter b exists on the right side of the equation of motion (13) by which we study the stability of motion, and it is also found in the modified Raychaudhuri equation (41), which explains to us the existence of the initial singularities in the cosmic models. The physical meaning of this parameter has previously been interpreted in many works of literature as having a quantum nature, with effects occurring at the beginning (the Big Bang) and the end (the Big Rip) of the universe. For more information, see [9,33,34,35].

• The torsion term in the path deviation equation and the modified Raychaudhuri equation can be expressed as a connection between the torsion of the background gravitational field and some inherent properties of the universe.

- The torsion term does not affect the stability/instability of the Cosmological models. The Universe started with the stable case at the Big Bang. Simultaneously, it gradually moved to the instability case.
- A modified version of Raychaudhuri's equations is used to check for singularities in models of the universe.
- In the Riemannian geometry, there is an initial singularity in Cosmological models. As $b = 0$, the Universe began with a singularity at the Big Bang, and ended up with a non-singularity at a Big Rip. This problem, with GT, needs more clarification.
- In PAP geometry with torsion terms, the universe begins with the non-singularity of the Big Bang and ends with the singularity of the Big Rip. This result is physically compatible with the evolution of the Universe; for illustration, see Ref. [36]. In general, if $b > 1$ then all FRW models will be free from the initial singularities. The exact experimental determination of the parameter (b) will judge the problem of initial singularities. The space-time torsion generates gravitational repulsion in the early universe full of quarks and leptons, preventing cosmic singularities [37]. Concerning the singular free cosmological models, there is a lot of literature confirming the results of the current study, for illustration, see Refs. [38–41].
- The Cosmological model comes from a non-singularity state at the redshift $z = -1$ and ends with a singularity state for ($b = 0$), while reflecting its behavior for ($b = 2$)
- In the end, the results (stability and singularity) are model dependent, since it depends on the scaling factors describing the model under study.

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Author contribution statement

M. A. Bakry: Conceived and designed the analysis; Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper.

A. Eid: Conceived and designed the analysis; Analyzed and interpreted the data; Wrote the paper.

Data availability statement

No data was used for the research described in the article.

Declaration of competing interest

Authors declare that have no conflict of interest.

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