Research article

Solutions of the Schrödinger equation and thermodynamic properties of a combined potential

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ABSTRACT

The solution of the radial Schrödinger equation was obtained using the methodology of supersymmetric approach with a combination of modified generalized Pöschl-Teller potential and inversely quadratic Yukawa potential model. The non-relativistic ro-vibrational energy spectra and the corresponding wave functions were obtained and numerical results were generated for some states. The variation of energy of the combined potential and the subsets potentials with the screening parameter for various quantum number were graphically studied. The effect of the potential parameters on the energy for different states was also studied numerically. For more usefulness and applications of the work, the vibrational partition function and the various thermal properties like mean energy, Helmholtz energy, heat capacity and entropy were calculated. The behaviour of the thermodynamic properties with respect to temperature change for various quantum number and maximum quantum states were examined in detail. The temperature has positive effect on all the thermal properties except the free energy.

1. Introduction

Over the years, there has been an intense studies on the relativistic and non-relativistic wave equations with different quantum potential terms of interest. In most of the studies, the authors focused on the solutions of the Dirac equation under spin and pseudospin symmetries and the Klein-Gordon equation for spinless particle in the relativistic quantum mechanics while in the non-relativistic quantum mechanics, the attention is on the solutions of Schrödinger equation. The details of their results showed the calculations of the energy eigenvalues and their corresponding unnormalized wave functions. It is noted that the solutions of these wave equations in the presence of some physical potential models cannot be obtained for $\ell = 0$ due to the attachment of the centrifugal term. To obtain the solution for $\ell \neq 0$, it becomes difficult using some traditional methodologies especially, those involves the transformation of variable such the Nikiforov-Uvarov method [1], asymptotic iteration method [2], Formula method for bound state problems [3], and others. To address this problem, certain approximation schemes like the Pekeris approximation-type [4], Qiang et al. [5] approximation scheme and Greene-Aldrich approximation scheme [6] were developed for the solutions involving $\ell \neq 0$.

The approximation scheme adopted in any study, depends on the nature or type of the adopted interacting potential.

Recently, the studies of the solutions of the wave equations have been extended to different areas of studies such as the thermodynamic properties [7, 8, 9, 10, 11, 12], optical properties [13, 14, 15], scattering state and phase shift [16, 17, 18, 19, 20], entropic systems [21, 22, 23, 24, 25, 26], Fisher information [27, 28, 29, 30, 31, 32]. As part of the extensional studies, Onate et al. [33], examined the stability of the Black Hole temperature by employing the generalized Morse potential model.

In the present work, we want to investigate the solutions of the Schrödinger equation and the thermodynamic properties with a combination of modified generalized Pöschl-Teller type potential and inversely quadratic Yukawa-like potential model. The inversely quadratic Yukawa potential has been reported under the bound state solutions of both relativistic and non-relativistic regime. This potential appears to be like double Yukawa potential. The generalized Pöschl-Teller potential has also been reported in the domain of bound state. In this work, the generalized Pöschl-Teller potential is modified and added to the inversely quadratic Yukawa potential. The propose combined potential is physically given as

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$$V(r) = \frac{V_0 + \sigma_0 V_1^2 \sinh(\alpha r)}{\sinh^2(\alpha r)} + \frac{\sigma_1}{r^2}, \tag{1}$$

where V_0, V_1 and α are the parameters of the potential and screening parameter respectively that characterized the depth and range of the potential. The parameters $\sigma_0 = \frac{e^{-\alpha r}}{V_1}$ and $\sigma_1 = -\frac{V_2}{e^{2\alpha r}}$ maintained the physical fitness of the proposed potential. Apart from the two combined potentials, the solutions of other potentials such as the main Yukawa potential, can be obtain from potential (1) which has applications in different areas of physics. The scheme of this work is organize as follows: In the next section, we present the bound state solution of the radial Schrödinger equation with the combined potential via supersymmetric approach, in section 3, the thermodynamic properties were presented. The discussion and conclusion were given in section 4 and 5 respectively.

2. Bound state solutions

To obtain the energy eigenvalues of the Schrödinger equation with potential (1), we consider the original Schrödinger equation given by

$$\left[-\frac{\hbar^2}{2\mu} \left(\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \right) + V(r) - E \right] \psi(r) = 0. \tag{2}$$

Setting the wave function $\psi(r) = \frac{U_{n,\ell}(r) Y_{m,\ell}(\theta,\varphi)}{r}$, and consider the radial part of the Schrödinger equation, we have

$$\left[-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V(r) - E_{n,\ell} + \frac{\hbar^2}{2\mu} \frac{\ell(\ell+1)}{r^2} \right] U_{n,\ell}(r) = 0, \tag{3}$$

where $V(r)$ is the interacting potential given in Eq. (1), $E_{n,\ell}$ is the non-relativistic energy of the system, $\hbar$ is the reduced Planck's constant, μ is the reduced mass, n is the quantum number, ℓ is the angular momentum quantum number $U_{n,\ell}(r)$ is the wave function and $\frac{1}{r^2}$ is the centrifugal term that is introduced in Eq. (3). This term is introduced to enable us adopt a suitable approximation scheme that could deal with the same term in the proposed potential given in Eq. (1). For a short potential range, we employ the following approximation scheme to the centrifugal term

$$\frac{\ell(\ell+1)}{r^2} \approx \frac{4\alpha^2 \ell(\ell+1) e^{-2\alpha r}}{(1 - e^{-2\alpha r})^2}. \tag{4}$$

The shape of the approximation scheme and its validity is shown below.

Substituting Eqs. (1) and (4) into Eq. (3), we have

$$\frac{d^2 U_{n,\ell}(r)}{dr^2}$$
$$= \left[-\frac{2\mu E_{n,\ell}}{\hbar^2} + \frac{\frac{4\mu(V_1 + 2V_2\alpha^2)e^{-2\alpha r}}{\hbar^2}}{1 - e^{-2\alpha r}} + \frac{\left(\frac{8\mu(V_0 - V_2\alpha^2)}{\hbar^2} + 4\alpha^2 \ell(\ell+1) \right) e^{-2\alpha r}}{(1 - e^{-2\alpha r})^2} \right] U_{n,\ell}(r) \tag{5}$$

At this point, we employ the methodology of supersymmetry. This

supersymmetry method appeared many years ago [34, 35, 36, 37] and has been used to solve the wave equations with several potentials. To use the basic concepts of the supersymmetric quantum mechanics formalism and shape invariance technique [38, 39, 40, 41, 42, 43] of supersymmetric approach, we first write the ground state wave function as

$$U_{0,\ell}(r) = exp\left(-\int W(r)dr \right), \tag{6}$$

where $W(r)$ is called superpotential function in supersymmetric quantum mechanics. The superpotential function is a solution to the differential equation given in Eq. (5). Considering the interacting potential given in Eq. (1) and the differential equation given in Eq. (5), we propose our superpotential function in the form

$$W(r) = \Phi_0 + \frac{\Phi_1 e^{-2\alpha r}}{1 - e^{-2\alpha r}}, \tag{7}$$

where Φ_0 and Φ_1 are superpotential constants that will be determined subsequently. Substituting Eq. (6) into Eq. (5), we have a non-linear Riccati equation of the form

$$W^2(r) - \frac{dW(r)}{dr} = -\frac{2\mu E_{n,\ell}}{\hbar^2} + \frac{\frac{4\mu(V_1 + 2V_2\alpha^2)e^{-2\alpha r}}{\hbar^2}}{1 - e^{-2\alpha r}} + \frac{\left(\frac{8\mu(V_0 - V_2\alpha^2)}{\hbar^2} + 4\alpha^2 \ell(\ell+1) \right) e^{-2\alpha r}}{(1 - e^{-2\alpha r})^2} \tag{8}$$

To determine the two constants in Eq. (7), we substitute Eq. (7) into Eq. (8), and by comparison, we have

$$\Phi_0^2 = -\frac{2\mu E_{n,\ell}}{\hbar^2} \tag{9}$$

$$\Phi_1 = \alpha \left(-1 \pm \sqrt{(2\ell+1)^2 + \frac{8\mu(V_0 - V_2\alpha^2)}{\alpha^2 \hbar^2}} \right) \tag{10}$$

$$\Phi_0 = \frac{\frac{2\mu(V_1 + 2V_2\alpha^2)}{\hbar^2} + \Phi_1^2}{2\Phi_1} \tag{11}$$

In this work, the consideration of the bound state is ideal when the radial part of the wave function $\psi(r)$ clearly satisfies the boundary conditions that $U_{n,\ell}(r)/r$ becomes zero as r tends to infinity and $U_{n,\ell}(r)/r$ is finite as r is zero. In order to make $U_{n,\ell}(r)/r$ satisfy the regularity conditions, we obtain $\Phi_0 < 0$. To achieve every desirable result, it becomes necessary to construct a pair of partner potentials $V_{\pm}(r) = W^2(r) \pm \frac{dW(r)}{dr}$, using the superpotential function given in Eq. (7). Thus,

$$V_+(r) = \Phi_0^2 + \frac{\Phi_1(2\Phi_0 - \Phi_1)e^{-2\alpha r}}{1 - e^{-2\alpha r}} + \frac{\Phi_1(\Phi_1 - 2\alpha)e^{-2\alpha r}}{(1 - e^{-2\alpha r})} \tag{12}$$

$$V_-(r) = \Phi_0^2 + \frac{\Phi_1(2\Phi_0 - \Phi_1)e^{-2\alpha r}}{1 - e^{-2\alpha r}} + \frac{\Phi_1(\Phi_1 + 2\alpha)e^{-2\alpha r}}{(1 - e^{-2\alpha r})} \tag{13}$$

Eqs. (12) and (13) satisfied a condition for partner potentials and thus, establish a relationship of the form

$$V_+(r, a_0) = V_+(r, a_1) + R(a_1) \tag{14}$$

via mapping of the form $\Phi_0 \to \Phi_0 + 2\alpha$, where $\Phi_0 = a_0$. In Eq. (14) above, a_0 is an old set of parameter while a_1 is a new set of parameter uniquely determined from the old set of parameter a_0. The term $R(a_1)$ is called a remainder or residual term which is independent of the variable r. From the mapping, we can write the following recurrence relation that involves the two sets of parameters $a_1 = a_0 + 2\alpha$, $a_2 = a_0 + 2(2\alpha)$. $a_3 = a_0 + 3(2\alpha)$, $a_4 = a_0 + 4(2\alpha)$, hence, $a_n = a_0 + 2\alpha n$. In view of the recurrence relation, Eq. (14) can be transform as

$$R(a_1) = \left(\frac{2\mu\left(\frac{V_1}{2}+V_2\alpha^2\right)}{\hbar^2}+\frac{a_0}{2}\right)^2 - \left(\frac{2\mu\left(\frac{V_1}{2}+V_2\alpha^2\right)}{\hbar^2}+\frac{a_1}{2}\right)^2 \tag{15}$$

$$R(a_2) = \left(\frac{2\mu\left(\frac{V_1}{2}+V_2\alpha^2\right)}{\hbar^2}+\frac{a_1}{2}\right)^2 - \left(\frac{2\mu\left(\frac{V_1}{2}+V_2\alpha^2\right)}{\hbar^2}+\frac{a_2}{2}\right)^2 \tag{16}$$

$$R(a_3) = \left(\frac{2\mu\left(\frac{V_1}{2}+V_2\alpha^2\right)}{\hbar^2}+\frac{a_2}{2}\right)^2 - \left(\frac{2\mu\left(\frac{V_1}{2}+V_2\alpha^2\right)}{\hbar^2}+\frac{a_3}{2}\right)^2 \tag{17}$$

$$R(a_n) = \left(\frac{2\mu\left(\frac{V_1}{2}+V_2\alpha^2\right)}{\hbar^2}+\frac{a_{n-1}}{2}\right)^2 - \left(\frac{2\mu\left(\frac{V_1}{2}+V_2\alpha^2\right)}{\hbar^2}+\frac{a_n}{2}\right)^2 \tag{18}$$

Therefore, the energy eigenvalues can be determined by using the shape invariance approach via

$$E_0^{(-)} = 0 \tag{19}$$

$$E_n^{(-)} = \sum_{\kappa=1}^{n} R(a_\kappa) = R(a_1) + R(a_2) + R(a_3) + - - - + R(a_n) \tag{20}$$

$$\overline{E} = E_0^{(-)} + E_n^{(-)} = \left(\frac{2\mu\left(\frac{V_1}{2}+V_2\alpha^2\right)}{\hbar^2}+\frac{a_0}{2}\right)^2 \tag{21}$$

Base on the parameters of the system, we deduce the full energy eigenvalue equation as

$$E_{n,\ell} = -\frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\frac{2\mu\left(\frac{V_1}{2}+V_2\alpha^2\right)}{\alpha^2\hbar^2}+\left(n+\frac{1}{2}+\frac{1}{2}\sqrt{(2\ell+1)^2+\frac{8\mu(V_0-V_2\alpha^2)}{\alpha^2\hbar^2}}\right)^2}{n+\frac{1}{2}+\frac{1}{2}\sqrt{(2\ell+1)^2+\frac{8\mu(V_0-V_2\alpha^2)}{\alpha^2\hbar^2}}}\right]^2 \tag{22}$$

To obtain the radial wave function, we define $y = e^{-2\alpha r}$, and substitute it into Eq. (5) to have

$$\left[\frac{d^2}{dy^2}+\frac{1}{y}\frac{d}{dy}+\frac{\frac{\mu E_{n,\ell}}{2\alpha^2\hbar^2}}{y^2}-\frac{\frac{\mu\left(V_1+2V_2\alpha^2\right)y}{\alpha^2\hbar^2}}{y^2(1-y)}-\frac{\left(\frac{8\mu(V_0-V_2\alpha^2)}{\alpha^2\hbar^2}+4\alpha^2\ell\left(\ell+1\right)\right)y}{y^2(1-y)^2}\right]U_{n,\ell}\left(r\right)=0 \tag{23}$$

Analyzing the asymptotic behaviour of Eq. (23) at origin and at infinity, it can be tested when $r \to 0 (y \to 1)$ and when $r \to \infty (y \to 0)$, Eq. (23) has a solution $U_{n,\ell}(y) = (1-y)^{\Upsilon_1}$ and $U_{n,\ell}(y) = y^{\Upsilon_0}$, where

$$\Upsilon_0 = \sqrt{-\frac{\mu E_{n,\ell}}{2\alpha^2\hbar^2}} \tag{24}$$

$$\Upsilon_1 = \sqrt{(2\ell+1)^2+\frac{8\mu(V_0-V_2\alpha^2)}{\alpha^2\hbar^2}} \tag{25}$$

Hence taking a trial wave function of the form

$$U_{n,\ell}(y) = y^{\Upsilon_0}(1-y)^{\Upsilon_1}f(y) \tag{26}$$

and substitute it into Eq. (23) to have

$$f''(y)+f'(y)\left(\frac{(2\Upsilon_0+1)-(2\Upsilon_0+2\Upsilon_1+1)y}{y(1-y)}\right)-f(y)\left(\frac{(\Upsilon_0+\Upsilon_1)^2+\frac{\mu E_{n,\ell}+2\mu\left(V_1+2V_2\alpha^2\right)}{2\alpha^2\hbar^2}}{y(1-y)}\right)=0 \tag{27}$$

Eq. (27) is a well-known differential equation satisfied by the hypergeometric function. Hence, its solution is given by

$$f(y) = {}_2F_1(-n, n+2(\Upsilon_0+\Upsilon_1); 2\Upsilon_0+1, y) \tag{28}$$

Using the hypergeometric function to replace the function $f(y)$, the complete wave function is given as

$$U_{n,\ell}(y) = N y^{\Upsilon_0}(1-y)^{\Upsilon_{12}}\, {}_2F_1(-n, n+2(\Upsilon_0+\Upsilon_1); 2\Upsilon_0+1, y) \tag{29}$$

3. Combined potential and thermodynamic properties

In this section, we calculate the thermodynamic properties such as mean energy (U), Helmholtz energy (F), heat capacity (C) and entropy (S) via partition function (Z). The thermodynamic properties in study are computed using MAPLE PROGRAMME. To compute the partition function, we write the energy given in Eq. (22) in a compact form that is suitable (vibrational energy) for the computation of the partition function. Thus, Eq. (22) is written as

$$E_n = \frac{-\alpha^2\hbar^2}{8\mu}\left\{2B + \frac{2B^2}{(n+\tau)} + \frac{(n+\tau)^2}{2}\right\}^2, n = 0, 1, 2, \ldots\ldots < n_{max} = [\lambda] \tag{30}$$

where

$$B = \frac{2\mu\left(\frac{V_1}{2} + V_2\alpha^2\right)}{\alpha^2\hbar^2} \tag{31}$$

$$\tau = \frac{1}{2}\left(1 + \sqrt{\frac{8\mu(V_0 - V_2\alpha^2)}{\alpha^2\hbar^2}}\right) \tag{32}$$

$$A = \frac{\alpha^2\hbar^2}{2\mu} \tag{33}$$

and we have introduced $\lambda = -\tau + \sqrt{B}$ for mathematical simplicity so that $[\lambda]$ is the largest integer inferior to λ. At this point, we define the partition function is obtained as

$$Z(\beta, \lambda) = \sum_{n=0}^{\lambda} e^{-\beta E_n}, \quad \beta = \frac{1}{KT} \tag{34}$$

where K is the Boltzmann constant. On substituting Eq. (30) into Eq. (34) yields

$$Z(\beta, \lambda) = \sum_{n=0}^{\lambda} e^{-\frac{\beta\alpha^2\hbar^2 B^2}{2\mu(n+\tau)^2} - \frac{\beta\alpha^2\hbar^2 B}{\mu} - \frac{\beta\alpha^2\hbar^2}{2\mu}(n+\tau)^2} \tag{35}$$

In the classical limit, the sum is replaced by an integral and as such Eq. (35) becomes

$$Z(\beta, \lambda) = \int_0^{\lambda} e^{-\frac{\beta\alpha^2\hbar^2 B^2}{2\mu\rho^2} - \frac{\beta\alpha^2\hbar^2 B}{\mu} - \frac{\beta\alpha^2\hbar^2}{2\mu}\rho^2} d\rho, \quad \rho = (n+\tau) \tag{36}$$

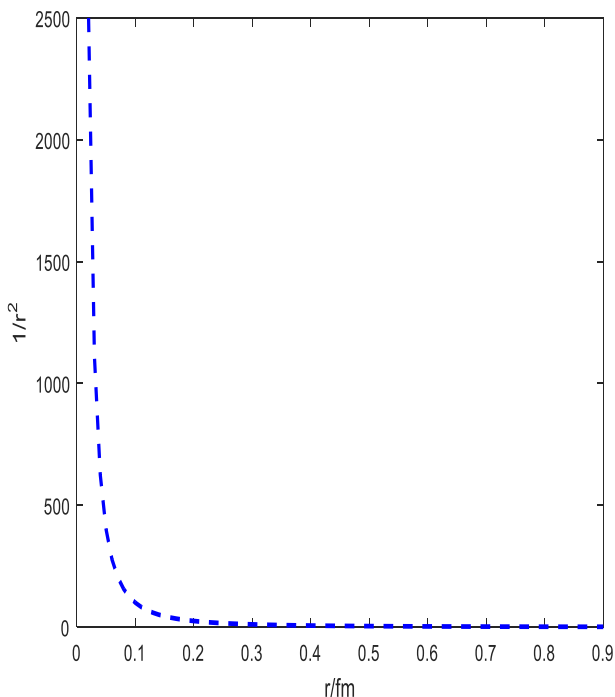Using Maple programme, we have the partition function as

Figure 1. The approximation scheme used for this work.

$$Z(\beta, \lambda) = \frac{1}{2}\Upsilon_2\left(\Upsilon_3 + \Upsilon_4 - 2\sqrt{\pi}\right) \tag{37}$$

$$\Upsilon_2 = e^{-\beta A\rho^2 - 2\beta AB}\sqrt{\beta AB^2} \tag{38}$$

$$\Upsilon_3 = \frac{2\lambda e^{-\frac{\beta AB^2}{\lambda^2}}}{\sqrt{\beta AB^2}} \tag{39}$$

$$\Upsilon_4 = \frac{2\sqrt{AB}\sqrt{\beta}\sqrt{\pi}\,erf\left(\frac{B\sqrt{\beta A}}{\lambda}\right)}{\sqrt{\beta AB^2}} \tag{40}$$

3.1. Mean energy

The mean energy is given as

$$U(\beta, \lambda) = -\frac{\partial\ln Z(\beta, \lambda)}{\partial\beta} \tag{41}$$

Substituting Eq. (37) into Eq. (41), we have the mean energy as

$$U\left(\beta, \lambda\right) = \frac{1}{\Upsilon_2\phi_2}\left[2\left(AB\Upsilon_2\phi_2 + \frac{\Upsilon_2}{2}\left(-\frac{1}{2(AB\beta)}AB\Upsilon_3 - \frac{\Upsilon_4}{2\beta} - 2\phi_3\right)\right)\right] \tag{42}$$

$$\phi_2 = \Upsilon_3 - \Upsilon_4 - 2\sqrt{\pi} \tag{43}$$

$$\phi_3 = \frac{ABe^{\frac{\beta AB^2}{\lambda^2}}}{\lambda\sqrt{\beta AB^2}} \tag{44}$$

3.2. Helmholtz energy

The Helmholtz energy is given as

$$F(\beta, \lambda) = -\frac{1}{\beta}\ln Z(\beta, \lambda) \tag{45}$$

Substituting Eq. (37) into Eq. (45), we have the Helmholtz energy as

$$F(\beta, \lambda) = -KT\ln\frac{\Upsilon_2\Phi_2}{2} \tag{46}$$

3.3. Entropy

The entropy as one of the thermodynamic properties is given as

$$S(\beta, \lambda) = K\ln Z(\beta, \lambda) - K\beta\frac{\partial\ln Z(\beta, \lambda)}{\partial\beta} \tag{47}$$

Substituting Eq. (37) into Eq. (47), we have the entropy as

$$S(\beta, \lambda) = K\ln\frac{\Phi_3}{2} - \frac{1}{\Phi_3}2K\beta\left[-AB\Phi_3 + \frac{\Upsilon_2}{2}(\Phi_{1-0} - 2\Phi_3)\right] \tag{48}$$

$$\Phi_{1-0} = -\frac{AB\Upsilon_3}{(AB\beta)} - \frac{\Upsilon_4}{\lambda} \tag{49}$$

3.4. Heat capacity

The heat capacity is given as

$$C(\beta, \lambda) = K_\beta\beta^2\frac{\partial^2\ln Z(\beta, \lambda)}{\partial\beta^2} \tag{50}$$

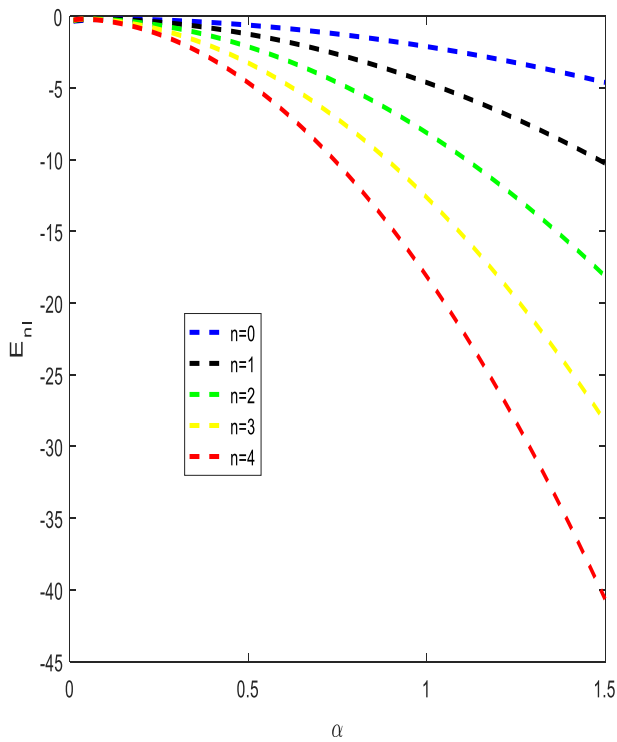On substituting Eq. (37) into Eq. (50), we have the heat capacity as

Figure 2. Energy $(E_{n,\ell})$ of the combined potential against the screening parameter (α) with $\mu = \hbar = \ell = 1$, $V_0 = 0.01$, $V_1 = 0.1$ and $V_2 = 0.001$.

$$C(\beta,\lambda) = \frac{1}{\Phi_2}[2\Phi_A] + \frac{1}{\Phi_2}[4\Phi_B AB] - 2\Phi_C - \frac{\Phi_C}{\Phi_2} \tag{51}$$
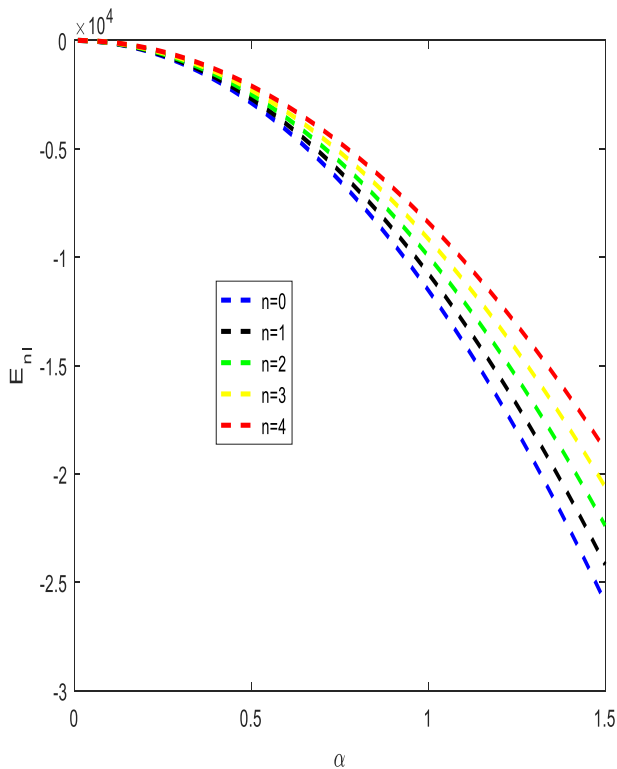

Figure 3. Energy $(E_{n,\ell})$ of the modified generalized Pöschl-Teller potential against the screening parameter (α) with $\mu = \hbar = \ell = V_0 = V_1 = 1$.
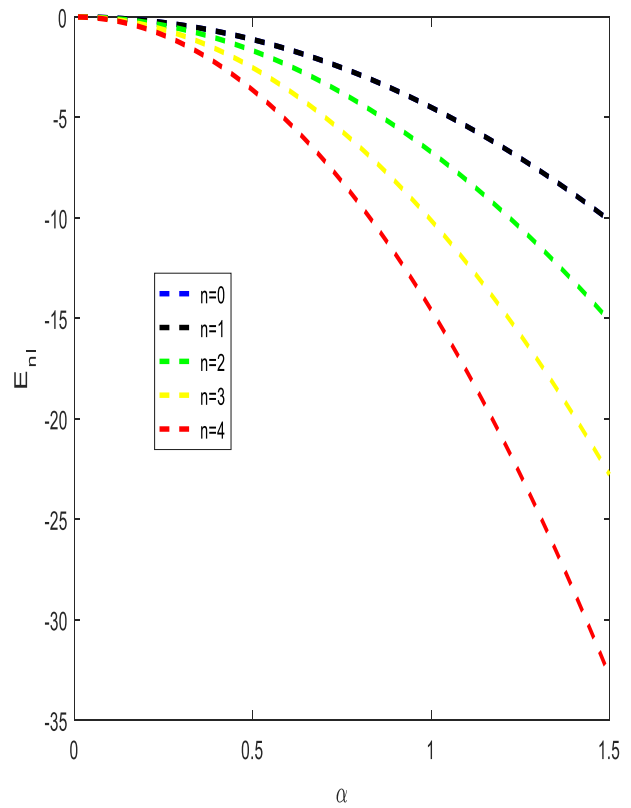

Figure 4. Energy $(E_{n,\ell})$ of the inversely quadratic Yukawa potential against the screening parameter (α) with $\mu = \hbar = \ell = V_2 = 1$.

$$\Phi_4 = -AB\Upsilon_3\left(\frac{1}{2AB\beta} + \frac{1}{\lambda^2}\right) - \frac{\Upsilon_4}{2\beta} - 2\Phi_3 \tag{52}$$

$$\Phi_5 = \frac{3AB^2\Upsilon_3(AB\beta)^2}{4} + \frac{AB^2\Upsilon_3}{\lambda^2}\left(\frac{1}{(AB\beta)} + \frac{1}{\lambda^2}\right) \tag{53}$$

$$\Phi_6 = \frac{\Upsilon_4}{4\beta^2} - \frac{AB\Upsilon_3}{\lambda^2}\left(\frac{1}{2\beta} + \frac{B}{\lambda^2}\right) \tag{54}$$

$$\lambda_0 = -\frac{\Upsilon_4}{2\beta^2} - \frac{AB^2\Upsilon_3}{\lambda^2}\left(\frac{1}{2(AB\beta)} + \frac{1}{\lambda^2}\right) \tag{55}$$

$$\lambda_1 = -\frac{\Upsilon_4}{2\beta^2} - AB\Upsilon_3\left(\frac{1}{2(AB\beta)} + \frac{1}{\lambda^2}\right) \tag{56}$$

$$\Phi_A = \left(2AB^2\Phi_2 - 2AB\Upsilon_2\Phi_4 + \frac{\Upsilon_2}{2}(\Phi_5 + \Phi_6)\right) \tag{57}$$

$$\Phi_B = \left(-AB\Phi_2 + \frac{\Upsilon_2}{2}(\lambda_0 - 2\Phi_3)\right) \tag{58}$$

$$\Phi_C = \left(-AB\Phi_2 + \frac{\Upsilon_2}{2}(\lambda_1 - 2\Phi_3)\right) \tag{59}$$

4. Discussion

The shape of the approximation used in this work is shown in Figure 1. The variation of the energies for the combined potential, modified generalized Pöschl-Teller-like potential and inversely quadratic Yukawa potential respectively against the screening parameter are examined in Figures 2, 3, and 4. In each case, the screening parameter varies inversely with the energy eigenvalues. The behaviour of the thermodynamic properties of the combined potential has been examined
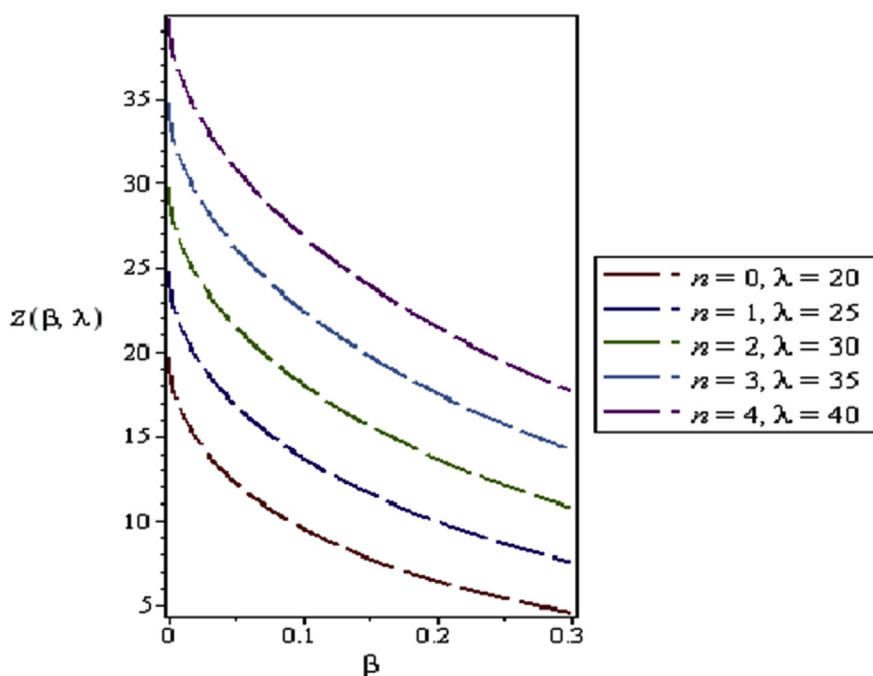
Figure 5. The vibrational partition function with temperature parameter for the combined potential.

graphically. In Figure 5, we showed the variation of the vibrational partition function with temperature parameter. As it is observed, the partition function decreases exponentially as temperature parameter increase for the ground state and the first four excited states studied in this work. In Figure 6, we plotted mean energy against the temperature parameter. The mean energy decreases monotonically as the temperature parameter increases steadily. The decrease in the mean energy is sharp at temperatures 0K to 0.05K. After this range, the decrease in mean energy

reduces drastically and almost becoming constant. In Figure 7, we presented the variation of vibrational heat capacity against the temperature parameter. It is observed that as the temperature parameter rises gradually, the heat capacity decreases monotonically for various quantum numbers. At a temperature of about 0K, the vibrational heat capacity at all quantum states are almost equal. As the temperature increases beyond 0K, the heat capacity at various quantum states separate gradually. The variation of the vibrational Helmholtz energy with the temperature
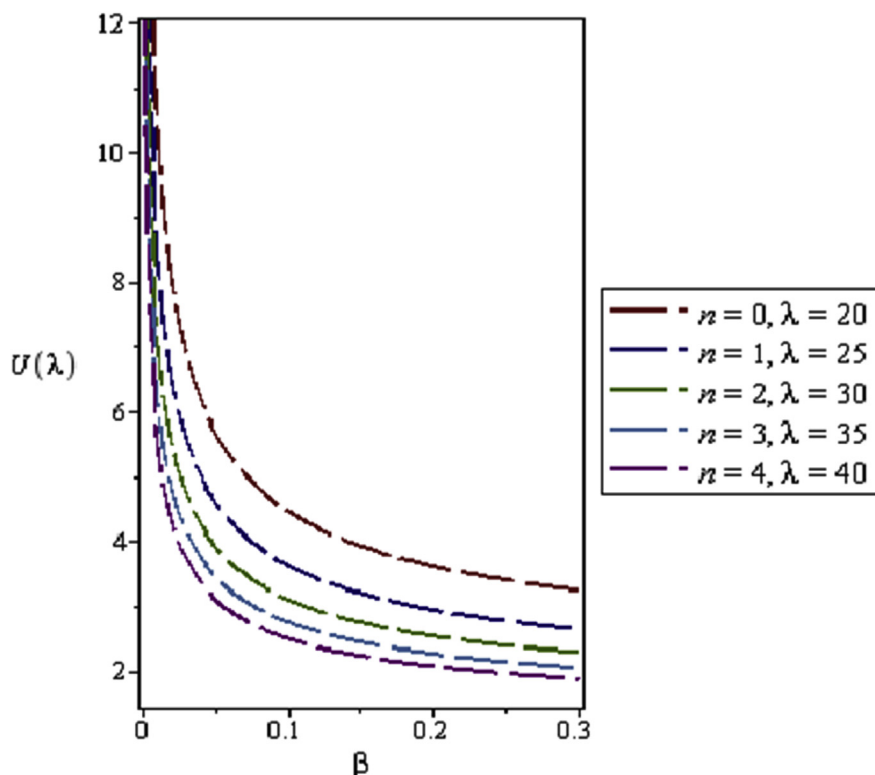

Figure 6. The vibrational mean energy against the temperature parameter for the combined potential.
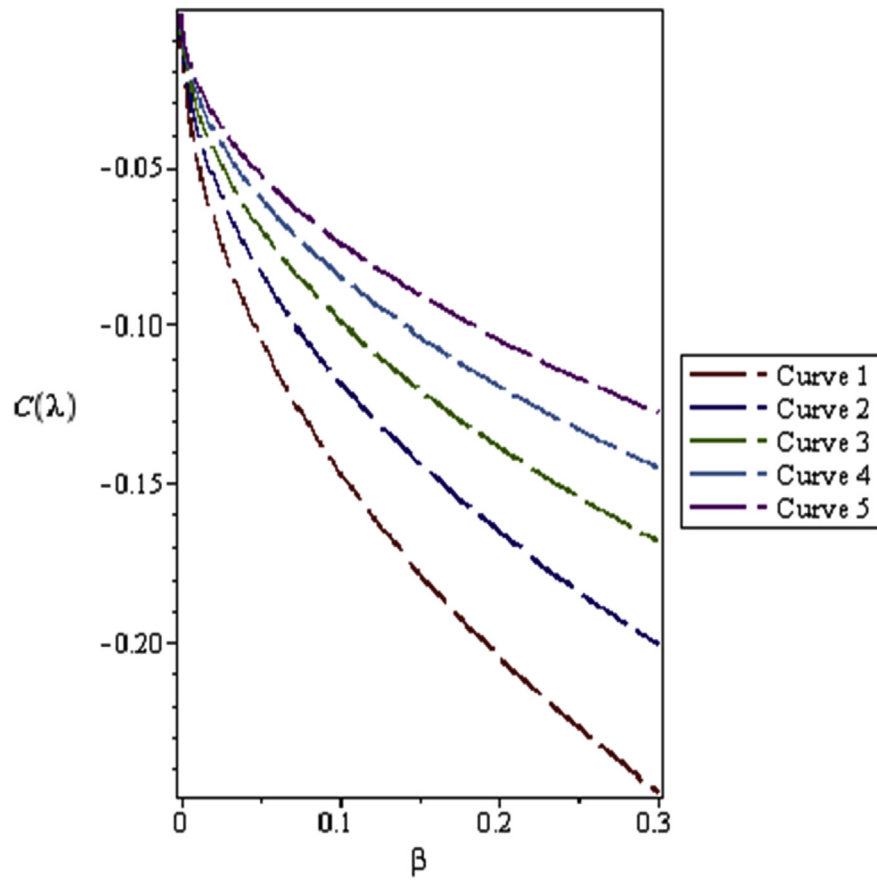
Figure 7. The vibrational heat capacity with the temperature parameter for the combined potential.
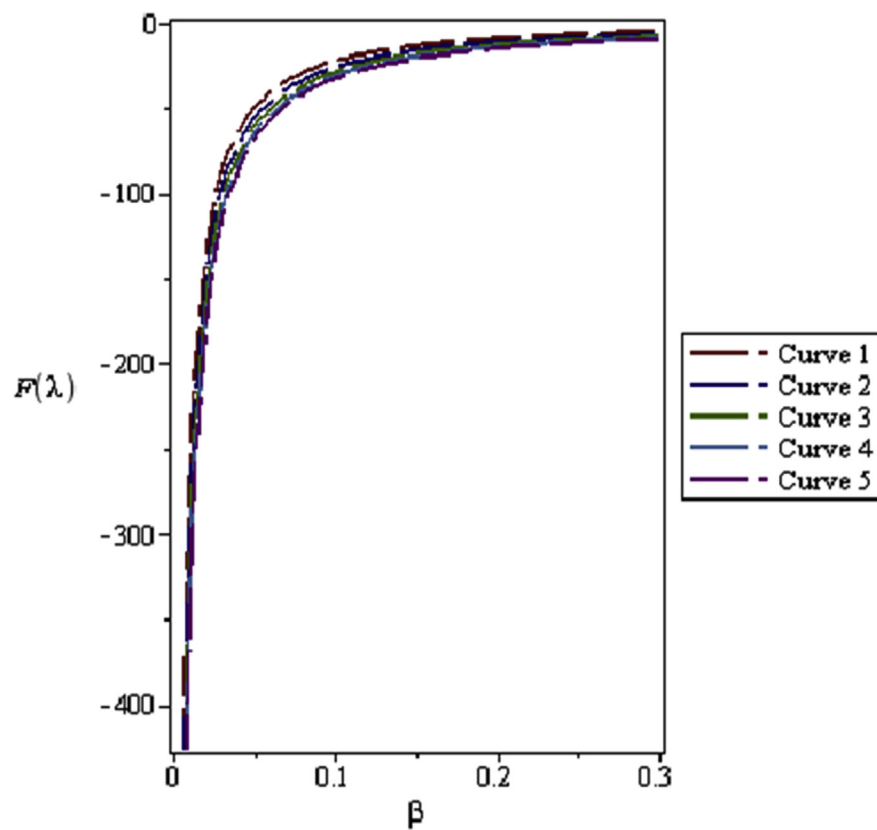

Figure 8. The vibrational Helmholtz energy with the temperature parameter for the combined potential.
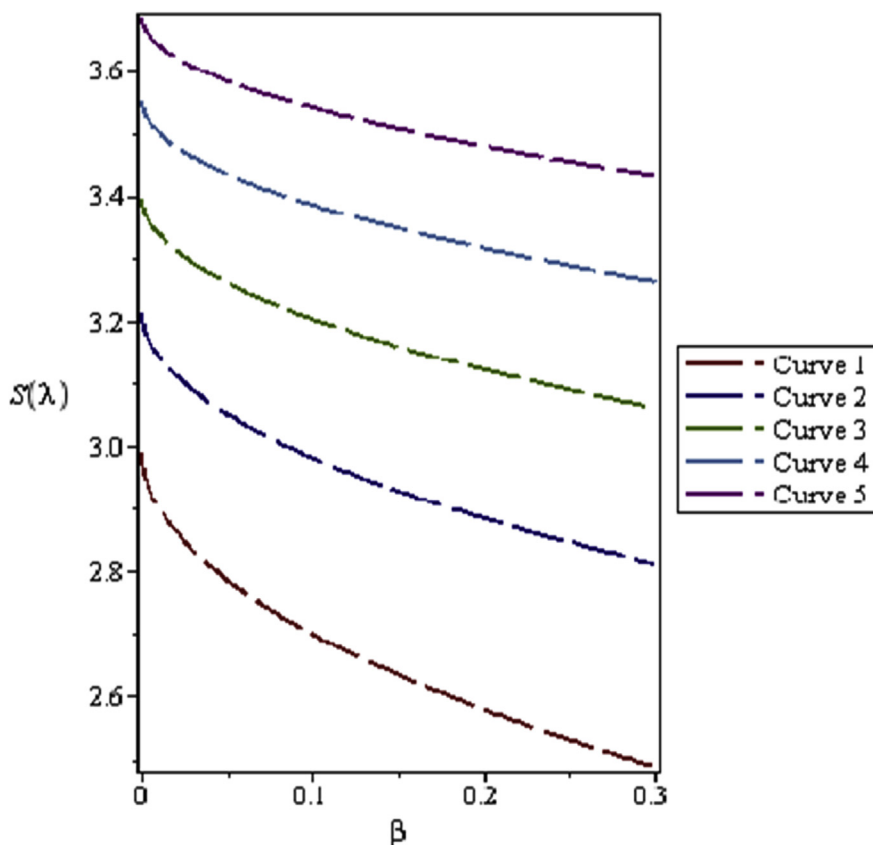
Figure 9. The vibrational entropy against the temperature parameter for the combined potential.

parameter is shown in Figure 8. It is noted that as the temperature parameter increases gradually, the vibrational Helmholtz energy increases. At temperature of 0K to 0.05K, there is a sharp increase in the Helmholtz energy. However, at temperatures greater than 0.05K, the increase in the Helmholtz energy becomes gradual and tending to be constant. Figure 9 shows the variation of the vibrational entropy against the temperature parameter. The behaviour observed for the partition function is was observed for the entropy.

The numerical results of eigenvalue equation for the combined potential for three values of the screening parameter with various states has been presented in Table 1. It is observed that the energy eigenvalues

increase as both the quantum number and the angular momentum quantum number increases respectively. However, the energy decreases as the screening parameter increases. To test the validity of the approximation scheme, we presented the numerical results for $\ell-$ wave solution (results with approximation scheme) and $s-$ wave solution (results without approximation scheme). There is a great discrepancy between the numerical values of the solution for $\ell-$ wave and $s-$ wave for every quantum state. The discrepancy increases as both the screening parameter and the angular momentum quantum number increases. Thus, the approximation scheme not very suitable even though it has been popularly used (see Table 2).

Table 1. Energy eigenvalues of a combined potential of various states with three values of the screening parameter for different values of the potential parameters.

state	α	$E_{n,\ell}^{V_0=V_1=V_2=0.1}$	$E_{n,\ell}^{V_0=V_1=0.1}$	$E_{n,\ell}^{V_0=V_2=0.1}$	$E_{n,\ell}^{V_1=V_2=0.1}$
2s	0.15	-0.343319003	-0.367645521	-2.343611220	-1.450693583
	0.20	-0.399216617	-0.435294138	-2.193341574	-1.580459179
	0.25	-0.462667116	-0.508899728	-2.105219119	-1.717093552
2p	0.15	-0.284323541	-0.322277359	-2.714370405	-1.240066373
	0.20	-0.321749309	-0.387806894	-2.482901337	-1.300741192
	0.25	-0.368973029	-0.471544874	-2.308595767	-1.369248902
3s	0.15	-0.451603175	-0.463385085	-2.087554171	-1.707559066
	0.20	-0.565191670	-0.572153045	-2.022297761	-1.943050682
	0.25	-0.696527564	-0.683292150	-2.034037194	-2.195468866
3p	0.15	-0.374999972	-0.397991302	-2.234712544	-1.476713782
	0.20	-0.461755578	-0.495653411	-2.088926122	-1.628793959
	0.25	-0.568727271	-0.613186828	-2.032250083	-1.795810313
3d	0.15	-0.331593258	-0.363865167	-2.395654839	-1.287303812
	0.20	-0.410452888	-0.463120719	-2.168964086	-1.386018840
	0.25	-0.512491438	-0.589753175	-2.058353311	-1.504416526

Table 2. Energy eigenvalues for $\ell-$ wave solution and $s-$ wave solution for three values of the screening parameter with $V_0 = 0.4, V_1 = 0.3, V_2 = 0.3$ and $\mu = \hbar = 1$.

$\ell-$ wave solution			$s-$ wave solution		
$n = \ell$	α	$E_{n,\ell}$	n	α	E_n
1	0.20	-41.17741095	1	0.20	-34.26553534
	0.25	-29.97089075		0.25	-22.77117649
	0.30	-24.09621015		0.30	-16.54476369
2	0.20	-59.06922490	2	0.20	-35.93350091
	0.25	-49.82678345		0.25	-24.47734337
	0.30	-46.35263557		0.30	-18.29759371
3	0.20	-90.63307640	3	0.20	-37.64146883
	0.25	-87.44530310		0.25	-26.24601876
	0.30	-91.37133226		0.30	-20.14044746
4	0.20	-142.1089761	4	0.20	-39.38943891
	0.25	-152.5764234		0.25	-28.07720153
	0.30	-173.1921855		0.30	-22.07332071

4.1. Special cases

I: When $\sigma_1 = 0$, the combined potential reduces to modified generalized Pöschl-Teller potential of the form

$$V(r) = \frac{4V_0 e^{-2\alpha r}}{(1 - e^{-2\alpha r})^2} + \frac{2V_1 e^{-2\alpha r}}{(1 - e^{-2\alpha r})} \quad (60)$$

with energy equation as

$$E_{n,\ell} = -\frac{\alpha^2 \hbar^2}{2\mu} \left[\frac{\frac{\mu V_1}{\alpha^2 \hbar^2} + \left(n + \frac{1}{2} + \frac{1}{2}\sqrt{\left(2\ell+1\right) + \frac{8\mu V_0}{\alpha^2 \hbar^2}} \right)^2}{n + \frac{1}{2} + \frac{1}{2}\sqrt{\left(2\ell+1\right) + \frac{8\mu V_0}{\alpha^2 \hbar^2}}} \right]^2 \quad (61)$$

II: When $V_0 = \sigma_0 = 0$, the combined potential becomes an inversely quadratic Yukawa potential of the form

$$V(r) = -\frac{4\alpha^2 V_2 e^{-2\alpha r}}{(1 - e^{-2\alpha r})^2} \quad (62)$$

and the energy equation reduces to

$$E_{n,\ell} = -\frac{\alpha^2 \hbar^2}{2\mu} \left[\frac{\frac{2\mu V_2}{\hbar^2} + \left(n + \frac{1}{2} + \frac{1}{2}\sqrt{\left(2\ell+1\right) + \frac{8\mu V_2}{\hbar^2}} \right)^2}{n + \frac{1}{2} + \frac{1}{2}\sqrt{\left(2\ell+1\right) + \frac{8\mu V_2}{\hbar^2}}} \right]^2 \quad (63)$$

III: When $V_0 = \sigma_1 = 0$, the combined potential turns

$$V(r) = \frac{2V_1 e^{-2\alpha r}}{1 - e^{-2\alpha r}} \quad (64)$$

The energy equation of this potential is equivalent to energy equation of a Yukawa potential model

$$E_{n,\ell} = -\frac{\alpha^2 \hbar^2}{2\mu} \left[\frac{\frac{\mu V_1}{\alpha^2 \hbar^2} + (1 + n + \ell)^2}{2(1 + n + \ell)} \right]^2 \quad (65)$$

5. Conclusion

In this work, we studied the radial Schrödinger equation with a combination of potentials. We have also studied the thermodynamic properties of the combined potential model with a temperature dependent partition function being determined first. Using the partition function, the various thermodynamic properties such as mean energy, Helmholtz energy, heat capacity and entropy were obtained. We observed that the behaviour of the energy for the combined potential with respect to the screening parameter is similar to the behaviour of energy for the two subset potentials of the combined potential with the same screening parameter. We also noticed that the vibrational partition function and the vibration mean energy exhibit similar features. However, the approximation that has been popularly used is not very valid.

Declarations

Author contribution statement

C.A. Onate: Conceived and designed the analysis; Analyzed and interpreted the data; Wrote the paper.

M.C. Onyeaju: Analyzed and interpreted the data; Wrote the paper.

Data availability statement

No data was used for the research described in the article.

Declaration of interests statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

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