

APPENDIX

Detailed derivation of the estimates for the posterior conditionals (parametric)

In the following, we describe the estimate for posterior conditional $\gamma_{sf}^* = \pi(\gamma_{sf}|Y_{sf}, \delta_{sf}^2)$ from prior and posterior parametric distributions of γ_{sf} followed by the derivation of the posterior conditional from $\delta_{sf}^{2*} = \pi(\delta_{sf}^2|Y_{sf}, \gamma_{sf})$ from prior and posterior parametric distributions of δ_{sf}^2 .

We assume the prior distribution of γ_{sf} to follow $\gamma_{sf} \sim N(\hat{\gamma}_s, \tau_s^2)$ and we apply Bayes' Theorem to find the posterior conditional distribution of γ_{sf} denoted $\pi(\gamma_{sf}|Y_{sf}, \delta_{sf}^2)$ which is proportional to the fusion between the likelihood and the prior:

$$\begin{aligned} \pi(\gamma_{sf}|Y_{sf}, \delta_{sf}^2) &\propto L(Y_{sf}|\gamma_{sf}, \delta_{sf}^2)\pi(\gamma_{sf}) \\ &\propto \exp\left\{-\frac{1}{2\delta_{sf}^2} \sum_i (Y_{isf} - \gamma_{sf})^2\right\} \exp\left\{-\frac{1}{2\tau_s^2} (\gamma_{sf} - \gamma_s)^2\right\} \\ &= \exp\left\{-\frac{1}{2\delta_{sf}^2} \left[\sum_i Y_{isf}^2 - 2 \sum_i Y_{isf} \gamma_{sf} + n_s \gamma_{sf}^2\right] - \frac{1}{2\tau_s^2} [\gamma_{sf}^2 - 2\gamma_{sf} \gamma_s + \gamma_s^2]\right\} \\ &\propto \exp\left\{-\frac{1}{2} \left(\frac{n_s \tau_s^2 + \delta_{sf}^2}{\delta_{sf}^2 \tau_s^2}\right) [\gamma_{sf}^2 - 2\left(\frac{\tau_s^2 \sum_i Y_{isf} + \delta_{sf}^2 \gamma_s}{n_s \tau_s^2 + \delta_{sf}^2}\right) \gamma_{sf}]\right\} \end{aligned}$$

By completing the square, we get:

$$\exp\left\{-\frac{1}{2} \left(\frac{n_s \tau_s^2 + \delta_{sf}^2}{\delta_{sf}^2 \tau_s^2}\right) \left[\gamma_{sf} - \left(\frac{\tau_s^2 \sum_i Y_{isf} + \delta_{sf}^2 \gamma_s}{n_s \tau_s^2 + \delta_{sf}^2}\right)\right]^2\right\}, \text{ which describes the kernel of a normal}$$

distribution with expected value:

$$E[\gamma_{sf}|Y_{sf}, \sigma_{sf}^2] = \frac{\tau_s^2 \sum_i Y_{isf} + \delta_{sf}^2 \gamma_s}{n_s \tau_s^2 + \delta_{sf}^2}$$

which, given $\hat{\gamma}_{sf}, \hat{\delta}_{sf}^2, \bar{\gamma}_s$ and $\bar{\tau}_s^2$ as set above, can be re-written to give the following estimate γ_{sf}^* :

$$\gamma_{sf}^* = \hat{E}[\gamma_{sf}|Y_{sf}, \sigma_{sf}^{2*}] = \frac{n_s \bar{\tau}_s^2 \hat{\gamma}_{sf} + \delta_{sf}^2 \bar{\gamma}_s}{n_s \bar{\tau}_s^2 + \delta_{sf}^{2*}}.$$

Next, we describe the derivation of the posterior conditional from $\delta_{sf}^{2*} = \pi(\delta_{sf}^2|Y_{sf}, \gamma_{sf})$ from prior and posterior parametric distributions of δ_{sf}^2 .

For the posterior conditional distribution of δ_{sf}^2 , given the *Inverse* $\Gamma(\lambda_s, \theta_s)$ prior, we set:

$$\begin{aligned}\pi(\delta_{sf}^2 | Y_{sf} \gamma_{sf}) &\propto L(Y_{sf} | \gamma_{sf} \delta_{sf}^2) \pi(\delta_{sf}^2) \\ &\propto (\delta_{sf})^{\frac{-n_i}{2}} \exp\left\{-\frac{1}{2\delta_{sf}^2} \sum_i (Y_{isf} - \gamma_{sf})^2\right\} (\delta_{sf}^2)^{-\lambda_i+1} \exp\left\{-\frac{\theta_i}{\delta_{sf}^2}\right\} \\ &= (\delta_{sf})^{-(\frac{n_i}{2} + \lambda_s) - 1} \exp\left\{-\frac{\theta_s + \frac{1}{2} \sum_i (Y_{isf} - \gamma_{sf})^2}{\delta_{sf}^2}\right\}.\end{aligned}$$

This describes an *Inverse* Γ distribution with the expected value:

$$E[\delta_{sf}^2 | Y_{sf}, \gamma_{sf}] = \frac{\theta_s + \frac{1}{2} \sum_i (Y_{isf} - \gamma_{sf})^2}{\frac{n_i}{2} + \lambda_s - 1}$$

Subsetting this equation with the Method of Moments estimates for $\bar{\theta}_s$ and $\bar{\lambda}_s$, we get the estimate for

δ_{sf}^{2*} :

$$\delta_{sf}^{2*} = \hat{E}[\delta_{sf}^2 | Y_{sf}, \gamma_{sf}^*] = \frac{\bar{\theta}_s + \frac{1}{2} \sum_i (Y_{isf} - \gamma_{sf}^*)^2}{\frac{n_i}{2} + \bar{\lambda}_s - 1}$$