

Alcohol calculations and their uncertainty

John Searle¹

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Abstract

A dilution model is widely used to link blood alcohol concentration and the quantity of alcohol consumed. Whilst some authors use the total body water formulation of that model, others use the Widmark Factor formulation. A paper by Forrest gave a table of example values of the Widmark Factor and Barbour, based on Forrest's work and using Forrest's computer program, subsequently presented Forrest's results by way of a chart. Whilst the results of Forrest and Barbour are often used interchangeably, there is a significant difference between them on the factors for women. This paper examines the source of the unexpected discrepancy. It is essential to quote an error range, in blood alcohol concentration calculations, for the results. The extent of that error range was investigated by Gullberg who also employed the Widmark Factor formulation. Gullberg concluded that when reporting a calculated blood alcohol concentration, a coefficient of variation of $\pm 21\%$ should be applied. Similarly, Gullberg concluded that when calculating the volume of drink, a coefficient of variation of $12\frac{1}{2}\%$ should be applied. The present paper derives and publishes the formulae for calculating this coefficient of variation. It is then shown that Gullberg's conclusions are mistaken: the coefficient of variation is not some fixed percentage but must be calculated in each case.

Keywords

alcohol, calculations, error, uncertainty, Widmark

The basis of alcohol calculations

Blood alcohol calculations originated in the 1920s with the pioneering work of Widmark,¹ who noticed, whilst developing the micro-analysis of alcohol, that the results were always higher than might be expected from a simple dilution calculation. In other words, a dose of m grams of alcohol, in a subject of mass M kilograms would always produce a blood alcohol concentration (BAC) higher than m/M .

Widmark realised that this was due to the proportion of water in the body as a whole being less than the proportion of water in blood. Bones and fat contained little water and so absorbed only a low amount of the alcohol. That was raising the concentration in the blood. To allow for this difference, Widmark proposed to incorporate an empirical factor r , so that the calculated BAC C was given by

$$C = \frac{m}{rM} \times 100 \text{ milligrams per } 100 \text{ millilitres}$$

The factor r , which has the units of litres per kilogram, became known as the Widmark Factor and has been shown to depend on the gender of the subject, as well as a number of anthropometric factors, of which body mass index (BMI) is perhaps the most important. Many other researchers have similarly given

subjects a dose of alcohol and measured the BAC, creating a database of figures from which the Widmark Factor can be estimated for any subject. Zuba and Piekoszewsk² commented that the Widmark procedure was the most popular method of making alcohol calculations.

In 1981, Watson et al.³ suggested an alternative formulation, changing the terminology to bring greater clarity and make the dilution equation easier to understand. Instead of thinking of the body water as a proportion of the body constituents, the volume of body water can be estimated directly. According to Watson:

Men

$$V = 2.447 + 0.3362 \times \text{Weight} + 10.74 \times \text{Height} - 0.09516 \times \text{Age}$$

Women

$$V = 2.097 + 0.2466 \times \text{Weight} + 10.69 \times \text{Height}$$

¹Road Accident Analysis, United Kingdom

Corresponding author:

John Searle, Road Accident Analysis, Mead House Hill Street, Hinckley, LE10 1DS United Kingdom.

Email: johnsearle@meadh.freeserve.co.uk

If the proportion of water in the blood is P litres per litre, then the dilution equation becomes:

$$C = \frac{mP}{V} \times 100 \text{ milligrams per 100 millilitres}$$

The change is a conceptual one, bringing out the underlying dilution model, but mathematically it is a change of terminology. The Widmark Factor is the total body water divided by the product of the body mass and the proportion of water in blood, both of which are known quantities.

Although total body water is more easily understood, ultimately either formulation can be used. One can write either rM or V/P , because those are equivalent and have the same definition: the mass of alcohol in grams necessary, in the absence of elimination, to create in the subject a BAC of 100 milligrams per 100 millilitres. They have the same value for any subject, a value which can be measured experimentally by giving the subject a dose of alcohol and taking a sample of blood.

The present paper discusses the investigations by Forrest⁴ who conducted a large number of such tests. The paper also discusses the uncertainty of blood alcohol calculations, including the examples of uncertainty calculation presented by Gullberg⁵ and by Zuba and Piekoszewsk.² In all those papers, the authors use the Widmark Factor formulation rather than the total body water formulation. Of necessity therefore, this paper also uses the Widmark Factor formulation.

Scope of paper

Blood alcohol calculations are widely presented in court. Such calculations often rely upon the investigations by Forrest, whose paper⁴ tabulates examples of the Widmark factor whilst Barbour,⁶ to whom Forrest made available his computer program, presented those results as charts. When such a calculation has been made, the conclusions of Gullberg⁵ are then often used to estimate the uncertainty of the calculated result.

The present paper is concerned with two problems which have arisen in this process. First, the factors for women which are given as examples by Forrest ought to coincide with the values given by Barbour's chart for those same examples. There appears to be a significant difference. Second, Gullberg did not publish the derivation of any formula and his method of estimating the uncertainty is based on fixed percentages. That appears to be in contradiction to the estimation of uncertainty suggested by Widmark¹ and Alha.⁷

These two topics form the basis of the present paper.

The dilution model

Using metric units, when the alcohol consumed is diluted in the body then the BAC may be written as:

$$C = \frac{100m}{rM} - \beta t \text{ milligrams per 100 millilitres}$$

(abbreviated here as mg%)

where

C is the calculated BAC at the relevant time
 m is the mass of alcohol consumed during the drinking session, in grams

M is the mass of the subject, in kilograms

r is the subject's Widmark Factor in litres per kilogram.

β is the subject's elimination rate, in mg% per hour

t is the duration in hours from the start of the session to the relevant time.

The 'relevant time' is the time at which an estimation of the BAC is required, for example the time when an accident took place. As an abbreviation, C_o will be used to denote the calculated level of BAC had there been no elimination, that is

$$C_o = \frac{100m}{rM}$$

The above formula for BAC is based upon elimination occurring at its full rate from the start of drinking to the relevant time. There are three ways in which that may not be the case:

- (a) At the start of the session the rate of drinking may have been so slow that elimination was at less than full rate.
- (b) The drinking may have been in two sessions, say lunchtime and evening, and in between the blood alcohol reached zero so that elimination stopped for a while. The calculation must be restricted to the current session.
- (c) After drinking ceased, the blood alcohol may have reached zero before the relevant time.

Following Gullberg, the mass of alcohol taken into the body may be expressed in further parameters:

$$m = vzad$$

where

v is the volume of drink consumed in millilitres
 z is the strength of the drink as percentage ABV $\div 100$

a is the proportion of the alcohol absorbed

d is the density of alcohol (= 0.789 grams per millilitre, constant)

One may therefore write:

$$C = \frac{100 \text{ v} \text{zad}}{rM} - \beta t$$

This formula calculates the BAC from a past history of alcohol consumption, a form which may be called the Forward Widmark calculation. The formula may of course be re-arranged to make *v* the subject and so calculate, from a measured level of blood alcohol *B*, the volume of drink consumed. That may be called the Reverse Widmark calculation. Widmark gave examples of both directions of calculation.

The Widmark Factor

The Widmark Factor, denoted by *r*, is not a simple constant but depends on anthropometric parameters. The influence of such parameters was explored by Forrest, who found that gender and BMI were the most important.

Other parameters such as age and stature³ have been suggested, and BMI has its limitations in characterising body build.⁸ However, Forrest's results are widely used. Forrest published examples of what the average factor would be, for men and for women, at different levels of BMI. Those examples, and interpolations between them, are often used in calculations presented in Court.

Barbour subsequently obtained from Forrest the computer program which had been used to calculate the Widmark Factor from the BMI. Barbour then ran the program to obtain extensive results, which he published in the form of two charts, one for men and one for women. When those charts are applied to the examples Forrest gave, it is found that the two authors agree entirely on the results for men, but for women the results differ:

Figure 1 shows the two sets of results. Forrest⁹ gave a mathematical relationship which may be simplified to the following form:

$$\begin{aligned} \text{Widmark Factor for men} & \quad r = 1.0181 - 0.01213 \times \text{BMI} \\ \text{Widmark Factor for women} & \quad r = 0.9367 - 0.01240 \times \text{BMI} \end{aligned}$$

For men, the tabulated examples published by Forrest and the chart by Barbour follow closely this simple relationship. For women, the charts of Barbour follow the relationship but the table by Forrest does not. Forrest's examples, for women, appear to be erroneous.

Zuba et al.⁵ comment that the procedure developed by Forrest is practical and appears to encompass the current state of knowledge relating to upgrading Widmark's equation. That appears to be the case, but one must work from the simple mathematical expression of Forrest's results and not from the table of examples he gave.

Uncertainty of the calculated result

With the BAC formula, as with any mathematical formula, errors in the input parameters will produce an error in the calculated result. That error can be estimated, by the method of error propagation, from the contribution of each input parameter.

Suppose in general terms that a result *y* is to be calculated from a formula

$$y = f(x_1, x_2, \dots, x_n)$$

where the first input variable *x*₁ is subject to an error of standard deviation *S*₁, the second input variable *x*₂ is subject to an error of standard deviation *S*₂ and so on. If those input errors are normally distributed then they will, according to the method of error propagation, create in *y* an error which has a standard

Widmark Factor for men			Widmark Factor for women		
BMI	Forrest	Barbour	BMI	Forrest	Barbour
17.9	0.80	0.80	15.6	0.74	0.74
21.9	0.75	0.75	20.1	0.69	0.69
24.7	0.72	0.72	22.8	0.61	0.65
27.2	0.69	0.69	25.3	0.58	0.62
29.6	0.66	0.66	27.3	0.53	0.60

This discrepancy for women should not exist. The matter is of practical importance since, for a woman of high BMI, it can lead to a difference of 25% or more in the estimation of BAC.

deviation of:

$$S_y = \sqrt{[\partial y / \partial x_1]^2 S_1^2 + [\partial y / \partial x_2]^2 S_2^2 + \dots + [\partial y / \partial x_n]^2 S_n^2}$$

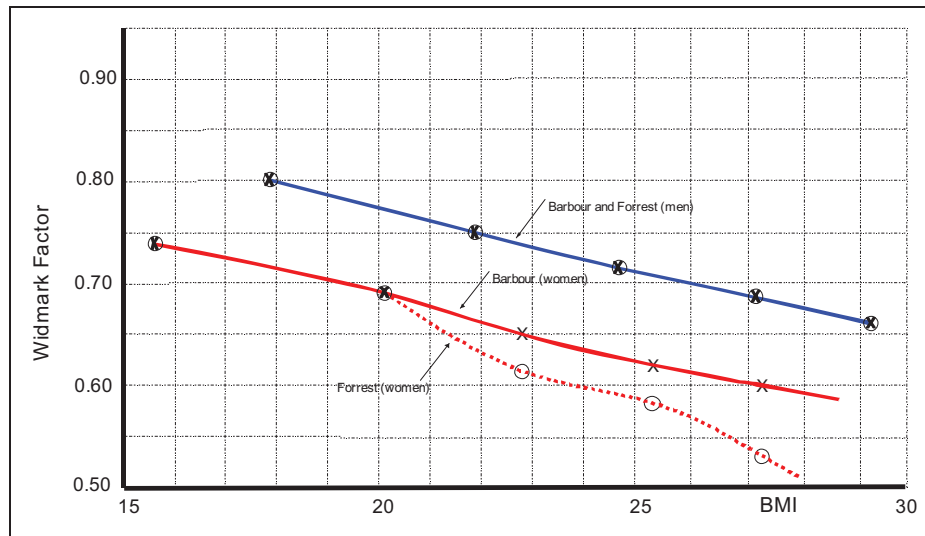


Figure 1. Barbour's and Forrest's results for the Widmark Factor. Note: BMI = Weight in kilograms/Square of height in metres.

The calculation of error propagation is explained in the extensive literature on the subject.¹⁰ The basic principle is that S_1 represents the size of the small errors in x_1 , whilst $\partial y/\partial x_1$ represents the change in y which a unit change in x_1 will produce.

The results of applying this formula may often be simplified by expressing the input errors as coefficients of variation rather than standard deviations. The coefficient of variation of each variable is its standard deviation divided by its mean.

Here, the concentration C is a function of eight variables:

$$C = f(v, z, a, d, r, M, \beta, t)$$

and each of those input variables will contribute uncertainty to the calculated value of the BAC C . However two of the variables, that is the body mass M and the density of alcohol d ($=0.789$), are known with some precision. Their contributions to the uncertainty are ignored in the present paper, although the formula for uncertainty can easily be extended to encompass them.

The remaining variables each have their own uncertainty, which can be expressed as a standard deviation but is more conveniently expressed as a coefficient of variation, that is the standard deviation divided by the mean:

- e_v coefficient of variation of the volume of drink consumed
- e_z coefficient of variation of the alcoholic strength of the drink (ABV)
- e_a coefficient of variation of the proportion of the alcohol absorbed
- e_r coefficient of variation of the Widmark Factor for the individual

- e_β coefficient of variation of the rate of alcohol elimination for the individual
- e_t coefficient of variation of the duration of the drinking session

Error propagation was first applied to alcohol calculations by Widmark himself, later followed by Alha.⁷ A shortcoming of their analysis is that they appear to be considering a laboratory environment. They assume that the time when the drink was consumed is known with exactitude whereas in real life there is uncertainty, and often there is uncertainty in the strength of the drink and the proportion absorbed as well. Also, Widmark and Alha did not take into account the negative correlation,⁵ that is some -0.135 , between the Widmark Factor r and the rate of elimination β . Gullberg took that into account but stated only the general principle of error propagation, without publishing any formula.

Appendix 1 applies the method of error propagation to the uncertainty of the Forward BAC calculation. The coefficient of variation of the calculated level of BAC is found to be:

$$e_c = \frac{C_o}{C} \sqrt{[e_v^2 + e_a^2 + e_z^2 + e_r^2] + (\beta t/C_o)^2 [e_\beta^2 + e_t^2] - 0.27 (\beta t/C_o) e_r e_\beta} \quad (1)$$

Some of the coefficients of variation relate to the circumstances of the event under investigation and those coefficients must be estimated from the circumstances. If for example the event is in a laboratory, then the time when the alcohol was consumed will be known exactly, as will be the time at which an estimate of BAC is required, such as the time when a blood sample was drawn. The uncertainty in the duration, that is e_t , will therefore be zero. Similarly the value of

e_v , the uncertainty in the volume drunk, is also likely to be zero in these circumstances. In a real-life event, the duration of drinking and the volume drunk may both have significant uncertainty and estimates of those uncertainties must be made from the circumstances.

Other coefficients of variation concern the parameters relating to alcohol dilution and elimination, that is e_r the accuracy to which Widmark's Factor can be determined, e_z the accuracy of manufacturer's values of ABV and e_β the accuracy of the assumed rate of elimination. Those coefficients of variation have been determined by researchers. Gullberg reviews the published literature and suggests suitable values, that is $e_r=0.092$, $e_z=0.03$ and $e_\beta=0.22$. These will be adopted here, because this paper re-works Gullberg's example and the comparison is made easier by adopting the same input. All the variables are assumed uncorrelated except r and β , where the covariance is -0.135 .

Once the coefficients of variation for the input parameters have been established, the uncertainty e_c of the calculated BAC can be obtained from Formula 1.

Appendix 2 presents a similar exercise for the calculation in the other direction, that is the Reverse Widmark calculation of the volume of drink consumed from a measurement of the BAC at a later time. The formula for the volume drunk is:

$$v = \frac{(B + \beta t)}{100 \text{ rM}} \text{ zad} \quad \text{where } B \text{ is the measured level of BAC.}$$

All the variables are assumed uncorrelated except r and β , where the covariance is -0.135 . The coefficient of variation of the calculated value of v , the volume of drink ingested, will be:

$$e_v = \sqrt{\frac{[B/B_0]^2 e_B^2 + [\beta t/B_0]^2 (e_\beta^2 + e_t^2) + (e_r^2 + e_a^2 + e_z^2) - 0.27[\beta t/B_0] e_r e_\beta}{(2)}}$$

where $B_0 = B + \beta t$

Once again the coefficients of variation for the input parameters must be estimated for each of them. The value of e_B , that is the accuracy of blood alcohol analysis, is about 0.0375 in the UK.

Gullberg stated only the general principle of error propagation, without deriving any formula by which the uncertainty could be calculated. He did however give an example and, without showing any working, stated the result he had calculated for it. Gullberg's example is presented here in metric units, but this time giving the formula and calculating through to the result.

In Gullberg's example, a man of mass 81.6 kg having an estimated Widmark Factor of 0.73 ($e_r=0.092$) drinks 3.55 ± 0.178 litres of beer

($e_v=0.05$) with an ABV of $4.0 \pm 0.12\%$ ($e_z=0.03$). It is soon calculated that the mass of alcohol in the drink is $3550 \times 0.040 \times 0.789 = 112$ grams.

It is also soon found that the value of C_0 is

$$\frac{100 \times 112}{0.73 \times 81.6} = 188 \text{ mg\%}.$$

The rate of elimination is 14.8 mg%/hour so that after 5 hours, when all the alcohol has been absorbed and some has been eliminated, the calculated BAC will be:

$$C_0 - \beta t = 188 - 14.8 \times 5 = 188 - 74 = 114 \text{ mg\%}$$

Hence from Formula 1 the coefficient of variation of the calculated BAC will be:

$$e_c = \sqrt{\frac{188}{114} [0.05^2 + 0.03^2 + 0^2 + 0.092^2] + \frac{74^2}{188^2} \times [0.22^2 + 0^2] - 0.27 \times \frac{74}{188} \times 0.092 \times 0.22} = 0.21.$$

The coefficient of variation of the calculated BAC is therefore 0.21, which is the same result as Gullberg obtained (21%). On the basis of that example, Gullberg concluded that:

"When reporting an estimated BAC, a 2CV [i.e. twice the coefficient of variation] uncertainty interval should be approximately $\pm 42\%$ ".

That is simply not so. The coefficient of variation is not a constant 21% for all circumstances, but must be calculated on a case by case basis. That may be seen from Gullberg's own example, by noting that after a further 7 hours the BAC will have fallen to a calculated 10 mg%. It hardly needs saying that the coefficient of variation of that figure is far greater than the 21% of 10 mg%, which would be only 2.1 mg%. A constant percentage as suggested by Gullberg will not do.

Furthermore, Gullberg has chosen an example where uncertainty in absorption (e_a) can be ignored, as can uncertainty in the duration of the drinking session (e_t).

In that same paper, Gullberg gives an example of the uncertainty of a Reverse Widmark calculation, again without giving any formula. It is based on the same data, except now it is the measured blood alcohol B which is given, as 120 mg%, and the volume of drink is to be calculated. That calculation is straightforward and the result is 3662 millilitres. A back calculation of B_0 , the BAC at time zero, gives: $120 + 14.8 \times 5 = 194 \text{ mg\%}$.

By Formula 2 above we have:

$$e_v = \sqrt{\frac{[120/194]^2 \times 0.036^2 + [74/194]^2 [0.22^2 + 0^2] + [0.092^2 + 0^2 + 0.03^2] - 0.27 \times [74/194] \times 0.092 \times 0.22}{\times 0.092 \times 0.22}}$$

$$= 0.122.$$

In this example, the coefficient of variation of the volume of drink is therefore 0.122, that is about 12½%, the same result that Gullberg obtained. However, again Gullberg generalises that example to all circumstances and says:

“A 2CV [i.e. twice the coefficient of variation] uncertainty interval of 25% should be applied when reporting estimates of the number of drinks”.

That is not so at all, as may be seen by considering an example where the measured BAC was 10 mg% but the drinking started 12 hours before. The coefficient of variation of the volume of drink would be about double the value it was in Gullberg’s example.

Zuba et al. also give an example of calculating the uncertainty of alcohol calculation, again without giving any formula. Zuba simplified the Widmark Factor to omit elimination and then used a commercially available program on error propagation. Expressed in the nomenclature of the present paper, Zuba’s example was:

$v = 250$ millilitres	$e_v = 0.04$	Drink volume
$z = 0.40$	$e_z = 0.0125$	Drink strength ABV
$M = 75$ kg	$e_M = 0.0267$	Body mass
$r = 0.70$	$e_r = 0.0714$	Widmark Factor

With those input values it can soon be calculated that $C = 150.3$ mg%, the value obtained by Zuba. The coefficient of variation is found, from Formula 1, to be some 0.08708. That corresponds exactly with the result for twice CV quoted by Zuba from his computer program, that being 17.4% which is twice 0.08708 expressed in percentage terms.

Zuba, like Gullberg, generalises the result of the example and says that the uncertainty of blood alcohol calculations is less than 20%. That generalisation is far from correct, especially when elimination has played a large part.

Conclusions

Although the values of the Widmark Factor tabulated by Forrest are derived from the same data as the charts subsequently published earlier by Barbour, for women they differ. Some values in Forrest’s table appear erroneous. Those errors may be avoided by using the charts published by Barbour or by using the simple formulae given in this paper.

It is important when reporting calculations of BAC from volume of alcohol, or in the reverse direction the volume of drink from a later BAC, to provide an estimate of the uncertainty of the result. Gullberg appears to be mistaken in suggesting that the coefficients of variation are fixed percentages, that is 21% and 12½%, respectively. Formulae are presented, in the body of the paper, whereby the coefficient of variation can be calculated.

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Appendix I

Estimation of the uncertainty of a Forward Widmark calculation

Using the nomenclature of the main paper the Forward Widmark calculation is:

$$C = \frac{100vazd}{rM} - \beta t$$

where C is the BAC at the relevant time

v is the volume of drink consumed in millilitres
 a is the proportion of the alcohol absorbed
 z is the strength of the drink as percentage
 ABV ÷ 100
 d is the density of alcohol (= 0.789 grams per millilitre, constant)
 r is the subject's proportion of body water in litres/kilogram, divided by the proportion of water in blood in litres/litre (Widmark Factor)
 M is the mass of the subject, in kilograms
 β is the subject's elimination rate, in mg% per hour
 t is the duration from the start of the session to the relevant time, in hours

Uncertainty in any of the input parameters will add to the uncertainty in C, that is the calculated BAC. Writing S with a suffix to denote the standard deviation of each parameter, the uncertainty (standard deviation) of the BAC is:

$$S_c = \sqrt{\begin{aligned} & [\partial C/\partial v]^2 S_v^2 + [\partial C/\partial a]^2 S_a^2 + [\partial C/\partial z]^2 S_z^2 \\ & + [\partial C/\partial r]^2 S_r^2 + [\partial C/\partial \beta]^2 S_\beta^2 + [\partial C/\partial t]^2 S_t^2 \\ & + 2[\partial C/\partial r] \cdot [\partial C/\partial \beta] \cdot \text{Cov}(r, \beta) \end{aligned}}$$

All the variables are assumed uncorrelated except r and β, where the covariance is $-0.135 S_r S_\beta$.

Performing the partial differentiations, we obtain:

$$\begin{aligned} \frac{\partial C}{\partial v} &= \frac{C_o}{v} & \frac{\partial C}{\partial a} &= \frac{C_o}{a} & \frac{\partial C}{\partial z} &= \frac{C_o}{z} & \frac{\partial C}{\partial r} &= \frac{C_o}{r} \\ \frac{\partial C}{\partial \beta} &= t & \frac{\partial C}{\partial t} &= \beta \end{aligned}$$

Putting those partial derivatives into the formula for the uncertainty of the BAC, and writing $S_v = ve_v$ and so on for the other variables:

$$e_c = \frac{C_o}{C} \sqrt{\begin{aligned} & [e_v^2 + e_a^2 + e_z^2 + e_r^2] + (\beta t/C_o)^2 [e_\beta^2 + e_t^2] \\ & - 0.27(\beta t/C_o)e_r e_\beta \end{aligned}} \quad (1)$$

Appendix 2

Estimation of the uncertainty of a reverse BAC calculation

With the reverse calculation, the volume of drink that has been consumed is to be calculated from a measured level of blood alcohol, denoted by B, obtained from a sample taken at the relevant time. The uncertainty of the blood alcohol measurement is e_B .

Using the nomenclature of the main paper, also presented in Appendix 1, the reverse BAC calculation is:

$$v = (B + \beta t) \frac{zad}{100 rM}$$

The standard deviation of the calculated value of v, the volume of drink ingested, will be:

$$S_v = \sqrt{\begin{aligned} & [\partial v/\partial C]^2 S_c^2 + [\partial v/\partial \beta]^2 S_\beta^2 + [\partial v/\partial t]^2 S_t^2 \\ & + [\partial v/\partial z]^2 S_z^2 + [\partial v/\partial a]^2 S_a^2 \\ & + [\partial v/\partial r]^2 S_r^2 + 2[\partial v/\partial r] \cdot [\partial v/\partial \beta] \cdot \text{Cov}(r, \beta) \end{aligned}}$$

All variables are assumed uncorrelated except r and β, where the covariance is $-0.135 S_r S_\beta$.

Performing the partial differentiations we obtain

$$\begin{aligned} \frac{\partial v}{\partial C} &= \frac{v}{(B + \beta t)} & \frac{\partial v}{\partial \beta} &= \frac{vt}{(C + \beta t)} & \frac{\partial v}{\partial t} &= \frac{v\beta}{(B + \beta t)} \\ \frac{\partial v}{\partial z} &= \frac{v}{z} & \frac{\partial v}{\partial a} &= \frac{v}{a} & \frac{\partial v}{\partial r} &= \frac{v}{r} \end{aligned}$$

Putting those partial derivatives into the formula for the uncertainty of v and writing $e_v = S_v/v$ and so on for the other variables:

$$e_v = \sqrt{\begin{aligned} & \frac{B^2 e_c^2}{(B + \beta t)^2} + \frac{B^2 t^2 e_\beta^2}{(B + \beta t)^2} + \frac{B^2 t^2 e_t^2}{(B + \beta t)^2} + e_z^2 + e_a^2 + e_r^2 \\ & - 2 \times \frac{0.135 \beta t e_r e_\beta}{(B + \beta t)} \end{aligned}}$$

Denoting $B + \beta t$ by B_o we have:

$$e_v = \sqrt{\begin{aligned} & [B/B_o]^2 e_B^2 + [\beta t/B_o]^2 (e_\beta^2 + e_t^2) + (e_z^2 + e_a^2 + e_r^2) \\ & - 0.27[\beta t/B_o] e_r e_\beta \end{aligned}} \quad (2)$$