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Research article

Addressing a decision problem through a bipolar Pythagorean fuzzy approach: A novel methodology applied to digital marketing

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ABSTRACT

Decision-making in real-world scenarios faces uncertainty. Fuzzy theory has been a means to represent such uncertainty. In this study, we propose an approach that incorporates bipolarity into multi-criteria decision-making processes applied to digital marketing. The proposal considers both the positive and negative dimensions of data, leading to better-informed decisions. Our contribution integrates bipolarity into Pythagorean fuzzy matrices, a framework that broadens the utility of bipolar fuzzy theory. Through computational experimentation, we identify the most effective strategy for digital marketing platforms. When compared to existing techniques, our approach shows advantages, underlining its potential to improve decision-making in uncertain scenarios and offering insights for businesses that refine their digital marketing strategies.

1. Introduction

In a world characterized by uncertainty, the usage of fuzzy theory provides a framework to handle this uncertainty, as initially presented in [1]. The subsequent exploration of the role of fuzzy sets in topology, crucial for grasping their functionality, was tackled later [2]. A groundbreaking advance was the evolution of bipolar fuzzy sets, a mechanism to differentiate between positive and negative data, as highlighted in [3]. Moreover, the fuzzy sets have been explored within broader systems, particularly emphasizing their impact in bio-economic and industrial sectors through control methodologies [4].

Building upon bipolar fuzzy sets, research directions have been diversified. For instance, certain studies have stated numerical solutions of differential systems by considering a hybrid fuzzy theory, so highlighting the broad applications of the fuzzy theory in complex mathematical computations [5]. Furthermore, an extension of bipolar fuzzy sets emphasizes its generalized range, allowing membership degrees to span the interval [-1, 1] from the traditional interval [0,1].

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The dynamic field of digital marketing is intrinsically laden with uncertainties and bipolar tendencies. The task of pinpointing the ideal social media platform (SMP) often encounters hurdles due to the uncertainties surrounding aspects such as advertising expenses, audience demographics, engagement metrics, return on investment (ROI), and user behavior. These uncertainties necessitate methodologies capable of integrating fuzzy and crisp solutions, particularly in evolving scenarios such as those modeled by fractional pandemic models [6,7].

Traditional methodologies, based on precise data, occasionally fall short in modeling such uncertainties, as outlined in [8]. One can champion a methodology that seamlessly integrates the mentioned uncertainties, leveraging the bipolar valued fuzzy sets discussed in [9]. The methodology can consider intuitionistic fuzzy soft matrices and introduce operations in the soft set paradigm [10–14]. An intuitionistic fuzzy matrix is a pair of fuzzy matrices, with membership and non-membership functions namely, that capture the universal elements in relation to certain attributes (characteristics).

The pioneering work presented in [15] applied bipolar intuitionistic fuzzy soft sets to decision-making problems. The emergence of the complex proportional assessment (COPRAS) method, proposed in [16], provided a new strategy for handling decision-making frameworks. The COPRAS method was later adapted to intuitionistic fuzzy soft sets in multi-criteria decision-making problems, as discussed in [17,18]. Similarly, the introduction of the intercriteria correlation (CRITIC) method, stated in [19], offered a strategy for determining attribute weights in bipolar Pythagorean fuzzy (BPF) environments, explored further in [20–22]. A Pythagorean fuzzy set is a generalization of an intuitionistic fuzzy set.

Considering alternative approaches, the compromise solution (MARCOS) method, developed in [23], provides a valuable tool for measurement and ranking. Additionally, the multi-attribute ideal-real comparative analysis (MAIRCA) method, introduced in [24], serves as an effective technique for establishing the gap between expected and observed scenarios. Also, the growing exploration of topological structures in information sciences reveals new insights, as demonstrated in [25,26].

Therefore, our study establishes the following four objectives: (i) to introduce the BPF soft regular generalized matrices; (ii) to propose a hybrid fuzzy methodology, expanding the horizons of the existing techniques; (iii) to delineate operations designed for these matrices, priming them for real-world applications; and (iv) to create algorithms based on the proposed approach with a spotlight in digital marketing. A distinctive feature of our research is the exploration of BPF soft regular generalized matrices and their operations, which has been unexplored in prior studies. Our approach becomes even more pronounced within the context of digital marketing, a highly uncertain framework. The proposed approach and the resultant algorithms redefine the decision-making in digital marketing with complex data.

In summary, our approach offers a decision-making paradigm in digital marketing, integrating fuzzy theory, bipolarity, and soft settings. This approach allows for the handling of uncertain by means of imprecise data and the accommodation of multiple criteria simultaneously through matrix operations, providing a contribution to decision-making strategies. Thus, our research addresses a crucial gap in the literature and advances the current understanding of decision-making under uncertainty.

The article is planned as follows. In Section 2, we introduce the preliminaries and notations to ensure clarity. In Section 3, four algorithms are described tailored for digital marketing, as applications of the bipolar Pythagorean fuzzy soft matrices (BPFSM). Section 4 presents a multi-criteria decision-making utilizing the bipolar Pythagorean fuzzy soft regular generalized matrices based on methods infused with BPF data. Lastly, Section 5 offers a discussion on our exploration of bipolarity in Pythagorean fuzzy soft regular matrices, drawing conclusions, limitations, and potential future research.

2. Preliminaries

The present article draws upon topological concepts, particularly from fuzzy set theory and its bipolar extensions. To ensure clarity, this section begins by introducing a selection of notations, as detailed in Table 1, which are fundamental to our work. Subsequently, we provide concepts that are consistently employed throughout the document.

2.1. Notations

Next, we layout the notations which are helpful to our discussions. These notations provide the scaffolding for the terminologies used in the proposed algorithms. Table 1 offers a concise guide, introducing concepts like the BPFSM, soft regular closed and open matrices, and the normalized decision matrix (NDM). Each notation is relevant for the effective conceptualization of the algorithms and their applications.

2.2. Topological concepts

Topology, which is intrinsically concerned with continuity and boundaries, is fundamental for understanding various mathematical and information science structures. Advancements in the recent literature highlight the significance of topological concepts in information systems. In the present work, understanding topology is essential to fully grasp bipolar fuzzy matrices. To further assist the reader, in the next subsection, we present all requisite contents, particularly emphasizing the topological concepts. We recommend that readers familiarize with these contents to enhance their understanding of the subsequent sections.

Table 1
Acronyms used in the present article.

Acronym	Definition
AAI	Anti-ideal (solution)
AI	Ideal (solution)
BPF	Bipolar Pythagorean fuzzy
BPFSM	Bipolar Pythagorean fuzzy soft matrix
$\text{BPFSM}_{m\times n}$	Set of bipolar Pythagorean fuzzy soft matrices of $m \times n$ order
$\text{BFSM}_{m\times n}$	Set of bipolar fuzzy soft matrices of $m \times n$ order
BPFSCM	Bipolar Pythagorean fuzzy soft closed matrix
BPFSMTS	Bipolar Pythagorean fuzzy soft matrix topological space
BPFSRCM	Bipolar Pythagorean fuzzy soft regular closed matrix
BPFSRGCM	Bipolar Pythagorean fuzzy soft regular generalized closed matrix
BPFSRGOM	Bipolar Pythagorean fuzzy soft regular generalized open matrix
BPFSOM	Bipolar Pythagorean fuzzy soft open matrix
BPFSROM	Bipolar Pythagorean fuzzy soft regular open matrix
NDM	Normalized decision matrix
ROI	Return on investment
SMP	Social media platform
WNDM	Weighted normalized decision matrix

2.3. Definitions

Next, some definitions are provided to facilitate the reading. Let $U = \{u_1, \ldots, u_m\}$ be the universal set, $E = \{e_1, \ldots, e_n\}$ a set of characteristics (attributes or criteria), A a subset of E ($A \subseteq E$), $\emptyset$ the empty (null) set, τ a topology on $\{U, E\}$, and the trinity $\{X, \tau, E\}$ be a topological space over the space set X where the attributes E are defined. For example, X can be five SMPs as $X = \{$Facebook; Instagram; LinkedIn; WhatsApp; YouTube$\}$ with attributes $E = \{$advertisement cost; demography; marketing goal; monthly active users; product cost$\}$.

Definition 2.1 *(Fuzzy soft set and its matrix form).* The pair $\{\mathcal{F}, A\}$ is called a fuzzy soft set over U whenever $\mathcal{F}$ is a mapping given by $\mathcal{F}\colon A \mapsto I^U$, with I^U denoting the collection of all fuzzy subsets of U. If $\{\mathcal{F}, A\}$ is a fuzzy soft set in the fuzzy soft class $\{U, E\}$, then $\{\mathcal{F}, A\}$ can be represented in a matrix form as $A = [a_{ij}]$, where $a_{ij} = \mu_j(u_i) = \mu_{ij}$, if $e_j \in A$, with u_i being an element of U; otherwise, that is, if $e_j \notin A$, $a_{ij} = 0$, with μ_{ij} being the membership of u_i in the fuzzy set $\mathcal{F}(e_j)$, for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

Definition 2.2 *(Intuitionistic fuzzy soft set and its matrix form).* Let $\{\mathcal{F}, A\}$ be an intuitionistic fuzzy soft set in the intuitionistic fuzzy soft class $\{U, E\}$, that is, $\{\mathcal{F}, A\}$ is a pair of fuzzy matrices with membership and non-membership functions. Then, $\mathcal{F}$ is an intuitionistic fuzzy matrix that maps elements of A to membership and non-membership values μ and ν, capturing elements of U in relation to attributes of A. Thus, $\{\mathcal{F}, A\}$ can be represented in a matrix form as $A = [a_{ij}]$, where $a_{ij} = \{\mu_{ij}, \nu_{ij}\}$, if $e_j \in A$, with $\nu_{ij} = \nu_j(u_i)$; otherwise, $a_{ij} = \{0, 1\}$. Hence, for any $u_i \in U$ and $e_j \in A$, μ_{ij} is the membership of u_i in the intuitionistic fuzzy set $\mathcal{F}(e_j)$, while ν_{ij} is its non-membership, with $0 \le \mu_{ij} + \nu_{ij} \le 1$, for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

Definition 2.3 *(Pythagorean fuzzy soft set and its matrix form).* Let $\{\mathcal{F}, A\}$ be a Pythagorean fuzzy soft set in the Pythagorean class $\{U, E\}$, that is, $\{\mathcal{F}, A\}$ is a generalization of an intuitionistic fuzzy set. Then, $\{\mathcal{F}, A\}$ can be represented in a matrix form as $A = [a_{ij}]$, where $a_{ij} = \{\mu_{ij}, \nu_{ij}\}$, if $e_j \in A$; otherwise, $a_{ij} = \{0, 1\}$, with μ_{ij} being the membership of u_i in the Pythagorean fuzzy set $\mathcal{F}(e_j)$ and ν_{ij} the associated non-membership, holding $0 \le \mu_{ij}^2 + \nu_{ij}^2 \le 1$, for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

Definition 2.4 *(Bipolar fuzzy soft set and its matrix form).* Let $\{\mathcal{F}, A\}$ be a bipolar fuzzy soft set in the bipolar fuzzy soft class $\{U, E\}$, that is, $\{\mathcal{F}, A\}$ possesses membership degrees that span the interval [-1, 1] departing from the traditional interval [0,1]. Thus, $\{\mathcal{F}, A\}$ can be represented in a matrix form as $A = [a_{ij}]$, where $a_{ij} = \{\mu_{ij}^-, \mu_{ij}^+\}$, if $e_j \in A$; otherwise, $a_{ij} = \{0, 0\}$. Here, μ_{ij}^- denotes the negative membership of u_i in the bipolar fuzzy set (e_j), and μ_{ij}^+ the positive membership, considering $-1 \le \mu_{ij}^- \le 0$ and $0 \le \mu_{ij}^+ \le 1$, for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

Definition 2.5 *(Bipolar Pythagorean fuzzy soft set and its matrix form).* Let $\{\mathcal{F}, A\}$ be a bipolar Pythagorean fuzzy soft set in the fuzzy soft class $\{U, E\}$. Then, $\{\mathcal{F}, A\}$ may be formulated in a matrix form as $A = [a_{ij}]$, where $a_{ij} = \{\mu_{ij}^+, \nu_{ij}^+, \mu_{ij}^-, \nu_{ij}^-\}$, if $e_j \in A$; otherwise, $a_{ij} = \{0, 1, 0, -1\}$, with μ_{ij}^+ being the positive membership of u_i in the BPF set (e_i) and μ_{ij}^- its negative membership, whereas ν_{ij}^+ is the positive non-membership of $u_i \in \mathcal{F}(e_i)$ and ν_{ij}^- its negative non-membership, holding $0 \le (\mu_{ij}^-)^2 + (\nu_{ij}^-)^2 \le 1$ and $0 \le (\mu_{ij}^+)^2 + (\nu_{ij}^+)^2 \le 1$, for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

Definition 2.6 *(Bipolar fuzzy soft null and universal matrices).* Given matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ in $\text{BPFSM}_{m\times n}$, a matrix A is termed a BPF soft submatrix of B, denoted by $A \subseteq B$, if $\mu_{ij}(A) \le \mu_{ij}(B)$ and $\nu_{ij}(A) \ge \nu_{ij}(B)$, for all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$. Within $\text{BPFSM}_{m\times n}$, we have the following: (i) an empty (null) BPFSM, denoted as $\emptyset_{m\times n}$, is a matrix whose elements are $\{0, 1, 0, -1\}$, indicating no positive membership as well as full negative non-membership; and (ii) a universal BPFSM, denoted by $U_{m\times n}$, is a matrix whose elements are $\{1, 0, -1, 0\}$, implying full positive membership and negative non-membership.

Definition 2.7 *(Operations on the bipolar Pythagorean fuzzy soft matrix set).* Let $A = [a_{ij}]$ and $B = [b_{ij}] \in \text{BPFSM}_{m\times n}$, with $\mu_{ij}(A)$ and $\mu_{ij}(B)$ being their corresponding membership functions, for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$. Then, we define the following operations on the BPFSM set:

[Addition] $A + B = C = [c_{ij}]$, with $c_{ij} = \big\{ \max\{\mu^+_{ij}(A), \mu^-_{ij}(A), \mu^+_{ij}(B), \mu^-_{ij}(B)\},\ \min\{\nu^+_{ij}(A), \nu^-_{ij}(A), \nu^+_{ij}(B), \nu^-_{ij}(B)\} \big\}$.

[Subtraction] $A - B = C = [c_{ij}]$, with $c_{ij} = \big\{ \min\{\mu^+_{ij}(A), \mu^-_{ij}(A), \mu^+_{ij}(B), \mu^-_{ij}(B)\},\ \max\{\nu^+_{ij}(A), \nu^-_{ij}(A), \nu^+_{ij}(B), \nu^-_{ij}(B)\} \big\}$.

[Product] $A \times B = C = [c_{ij}]$, with

$$c_{ij} = \Big\{ \max_{i,j} \min\{\mu^+_{ij}(A), \mu^+_{ij}(B)\}, \min_{i,j} \max\{\nu^+_{ij}(A), \nu^+_{ij}(B)\}, \min_{i,j} \max\{\mu^-_{ij}(A), \mu^-_{ij}(B)\}, \max_{i,j} \min\{\nu^-_{ij}(A), \nu^-_{ij}(B)\} \Big\},$$

where, for each element c_{ij}, the indices i and j run through the entire set of row and column indices of the matrices A and B, meaning that each c_{ij} considers the combination of all elements from both matrices.

[Complement] For $A = [a_{ij}]$, with $a_{ij} = \{\mu^+_{ij}, \nu^+_{ij}, \mu^-_{ij}, \nu^-_{ij}\}$, its complement is $A^c = [a^c_{ij}]$, where $a^c_{ij} = \{\nu^+_{ij}, \mu^+_{ij}, \nu^-_{ij}, \mu^-_{ij}\}$.

[Union] $A \cup B = C = [c_{ij}]$, with $c_{ij} = \{\mu^+_{ij}(C), \nu^+_{ij}(C), \mu^-_{ij}(C), \nu^-_{ij}(C)\}$, where

$$\mu^+_{ij}(C) = \max\{\mu^+_{ij}(A), \mu^+_{ij}(B)\}, \quad \nu^+_{ij}(C) = \min\{\nu^+_{ij}(A), \nu^+_{ij}(B)\}, \quad \mu^-_{ij}(C) = \min\{\mu^-_{ij}(A), \mu^-_{ij}(B)\}, \quad \nu^-_{ij}(C) = \max\{\nu^-_{ij}(A), \nu^-_{ij}(C)\}.$$

[Intersection] $A \cap B = C = [c_{ij}]$, with $c_{ij} = \{\mu^+_{ij}(C), \nu^+_{ij}(C), \mu^-_{ij}(C), \nu^-_{ij}(C)\}$, where

$$\mu^+_{ij}(C) = \min\{\mu^+_{ij}(A), \mu^+_{ij}(B)\}, \quad \nu^+_{ij}(C) = \max\{\nu^+_{ij}(A), \nu^+_{ij}(B)\}, \quad \mu^-_{ij}(C) = \max\{\mu^-_{ij}(A), \mu^-_{ij}(B)\}, \quad \nu^-_{ij}(C) = \min\{\nu^-_{ij}(A), \nu^-_{ij}(B)\}.$$

Definition 2.8 *(Bipolar Pythagorean fuzzy soft matrix topological space).* A topology τ on $\{U, E\}$ is the family of $\text{BPFSM}_{m\times n}$ over $\{U, E\}$, $\tau_{m\times n}$ say, satisfying the following properties: (i) $\emptyset$, $U \in \tau_{m\times n}$; (ii) if $A, B \in \tau_{m\times n}$, then $A \cup B \in \tau_{m\times n}$; and (iii) if $A, B \in \tau_{m\times n}$, then $A \cap B \in \tau_{m\times n}$. The trinity $\{X, \tau_{m\times n}, E\}$ is said to be a BPFSMTS over the space set X.

Next, we introduce the bipolar Pythagorean fuzzy soft open matrices (BPFSOMs), fundamental in the advanced topological domain, whose roles of certain sets in such a domain are the following:

[Set V] Analogous to subsets in traditional topology, this is a foundational set subject to interior and closure operations.

[Set G] It represents the open conditions within the topology and defines the interior of set V by forming unions that adhere to specific criteria, such as open sets in traditional topology.

[Set K] It has similar functions to closed sets in traditional topology, establishing the boundaries or closure for sets like V, whose closure is discerned by intersections of sets within K under defined closed conditions.

[Set W] This is a distinctive set with no direct analogue in traditional topology and serves as a reference, ensuring that the sets continue to satisfy certain topological properties.

The mentioned roles, extended from usual concepts, often consider other topological relationships.

Definition 2.9 *(Bipolar Pythagorean fuzzy soft open matrix).* Let A be an $m \times n$ matrix where each element signifies a bipolar fuzzy state within a topological space. This matrix, crucial for our subsequent operations and relations, is identified as a BPFSOM.

Definition 2.10 *(Pythagorean fuzzy soft interior and closure).* In a BPFSMTS, $\{X, \tau_{m\times n}, E\}$ namely, for a set $\{V, E\}$, we have the following: (i) the "interior" is defined as $\text{int}\{V, E\} = \big\{\{G, E\} \mid \{G, E\} \subseteq \{V, E\}\big\}$, where each $\{G, E\}$ is a bipolar Pythagorean fuzzy soft regular open matrix (BPFSRGOM); and (ii) the "closure" is stated as $\text{cl}\{V, E\} = \big\{\{K, E\} \mid V \subseteq K\big\}$, where each $\{K, E\}$ is a bipolar Pythagorean fuzzy soft regular generalized closed matrix (BPFSRGCM). In this context, "int" and "cl" refer to the interior and closure operations, respectively.

Definition 2.11 *(Bipolar Pythagorean fuzzy soft regular open matrix).* In a BPFSMTS, $\{X, \tau_{m\times n}, E\}$ say, a set $\text{BPFS}_{m\times n}$ defined by $\{K, E\}$ is called a BPFSRGOM if it satisfies the condition $\{K, E\} = \text{int}\big\{\text{cl}\{K, E\}\big\}$.

Definition 2.12 *(Bipolar Pythagorean fuzzy soft regular closed matrix).* In a BPFSMTS, $\{X, \tau_{m\times n}, E\}$ namely, a set $\text{BPFS}_{m\times n}$ defined by $\{K, E\}$ is called a BPFSRGCM if it holds the condition $\{K, E\} = \text{cl}\big\{\text{int}\{K, E\}\big\}$.

Definition 2.13 *(Bipolar Pythagorean fuzzy soft closed matrices —BPFSCM).* In a BPFSMTS, $\{X, \tau_{m\times n}, E\}$ say, a set $\text{BPFS}_{m\times n}$ defined by $\{K, E\}$ is considered a BPFSCM if $\text{cl}\{K, E\} \subseteq W$ under the condition that $\{K, E\} \subseteq W$, where W is a BPFSROM within the space $\{X, \tau_{m\times n}, E\}$.

Table 2
Proposed algorithms and their associated method.

Algorithm	Method
1	BPF CRITIC
2	BPF CRITIC COPRAS
3	BPF CRITIC MARCOS
4	BPF CRITIC MAIRCA

2.4. Properties and operations on bipolar Pythagorean fuzzy soft matrices

Theorem 2.14. *Let* $\{V,E\}$ *and* $\{W,E\}$ *be* $BPFSRGCM_{m\times n}$*. Then, the disjunction of* $\{V,E\}$ *and* $\{W,E\}$ *is also a BPFSRGCM in the space* $\{X,\tau_{m\times n},E\}$.

Remark 2.15. The conjunction of two $\text{BPFSRGCM}_{m\times n}$ is not guaranteed to be a BPFSRGCM in the space $\{X,\tau_{m\times n},E\}$.

Theorem 2.16. *Let* $\{V,E\}$ *be a BPFSRGCM and* $\{V,E\}\subseteq\{W,E\}\subseteq \text{cl}\{V,E\}$*. Then,* $\{W,E\}$ *is a BPFSRGCM in the space* $\{X,\tau_{m\times n},E\}$.

After understanding the dynamics of the closed matrices, we define the case of open matrices. In essence, their definitions provide a contrasting perspective that enhances our topological elements.

Definition 2.17 *(Bipolar Pythagorean fuzzy soft regular generalized open matrix)*. Consider a BPFSMTS, $\{X,\tau_{m\times n},E\}$ namely. A set $\{K,E\}$ within this space is termed a BPFSRGOM if its complement $\{K,E\}^{\text{c}}$ is a BPFSRGCM, and for any $m\times n$ matrix G that is a BPFSRCM satisfying $\{K,E\}\supseteq G$, we have $\text{int}\{K,E\}\supseteq G$, indicating that the interior of $\{K,E\}$ contains G.

The properties of the BPFSRGOM present interesting dynamics, especially when combined with other similar matrices.

Theorem 2.18. *Let* $\{V,E\}$ *and* $\{W,E\}$ *be two BPFSRGOM in the space* $\{X,\tau_{m\times n},E\}$*. Then, the conjunction of* $\{V,E\}$ *and* $\{W,E\}$ *is also a BPFSRGOM in the space* $\{X,\tau_{m\times n},E\}$.

Remark 2.19. The disjunction of any two $\text{BPFSRGOM}_{m\times n}$ is not guaranteed to be a BPFSRGOM in the space $\{X,\tau_{m\times n},E\}$.

Beyond the properties and operations inherent to the matrices presented, it is also essential to understand how we can evaluate them quantitatively, providing a tangible measure of their characteristics. To do this, we define the score function below.

Definition 2.20 *(Score function)*. For any BPFSM, the $m\times n$ score matrix of $A=[a_{ij}]=[\{\mu_{ij}^+,\nu_{ij}^+,\mu_{ij}^-,\nu_{ij}^-\}]$ is defined as $S=S(A)=S([a_{ij}])=[s_{ij}]$, with elements given by

$$s_{ij}=1-\left|\frac{(\mu_{ij}^+)^2-(\nu_{ij}^+)^2+(\mu_{ij}^-)^2-(\nu_{ij}^-)^2}{2}\right|,$$

where $s_{ij}\in[-1,1]$, for $i\in\{1,\dots,m\}$ and $j\in\{1,\dots,n\}$.

This section introduced definitions, theorems, and remarks on topological concepts related to the BPFSMs. The introduced concepts form the core theoretical basis for the algorithms discussed in the next section, which demonstrate the practical application of these topological concepts.

3. Algorithms for BPF soft matrices topology

To prepare the discussions and analyses that follow, we incorporated a comprehensive summary and background in Section 2. The summary and background defined and elaborated recurring terms and concepts throughout the article, ensuring its fluid and accessible reading. This section articulates four algorithms, which are proposed as applications of a BPFSMTS [27,28] within the domain of digital marketing. Specific methods embedded within each algorithm are illustrated in Table 2.

3.1. Context

Next, we introduce algorithms rooted in advanced topological concepts, which have been emphasized for their significance in information systems [25,26]. Within the scope of the present work, these mathematical concepts are applied to digital marketing.

The algorithms offer innovative solutions for tackling real-world challenges such as customer segmentation, advertising optimization, and ROI calculations.

Consider the complex environments of a digital marketing ecosystem, populated by numerous touchpoints – ranging from social media engagement and web analytics to service experience. Traditional data analysis methods [29] often fail under the weight of the high volume, velocity, and variety of data (often named as big data [30]) generated in such environments. In contrast, the algorithms discussed in this section provide a robust, data-driven framework to manage this complexity effectively. When applied considering a range of criteria –both benefit and non-benefit– to assess the effectiveness of each alternative, these algorithms allow for more informed decision-making processes in the field of digital marketing.

In the subsequent sections, we detail the mentioned algorithms, each of which utilizes topological structures to optimize various facets of digital marketing. As summarized in Table 2, we outline four specific algorithms based on the BPF CRITIC, BPF CRITIC COPRAS, BPF CRITIC MARCOS, and BPF CRITIC MAIRCA methods. To cater to a diverse audience, each algorithm is provided into three detailed formats: either as a textual algorithm, as a flowchart, and as an accompanying breakdown to assist in understanding their underlying mechanics. These formats enable readers to identify the complexities of each algorithm in the manner that they find most accessible.

3.2. BPF CRITIC algorithm

Presented in Algorithm 1, the BPF CRITIC method offers a framework for analyzing data in digital marketing. Specifically designed for user-friendliness and efficiency, it excels in managing complex datasets. One of its core functionalities is the normalization of the decision matrix and the estimation of attribute weights, a frequently encountered aspect in the domain of marketing analytics, thereby providing a structured approach to handle BPFSMs. This enables a more sophisticated interpretation of customer behavior and engagement metrics. Consequently, it empowers marketers to make strategically informed decisions based on data insights.

Algorithm 1 BPF CRITIC method [19].

1: BEGIN
2: **Input** BPFSOM$_{m\times n}$, $A=[a_{ij}]$ say, where each a_{ij} is a four-tuple $\{\mu^+_{ij}, \nu^+_{ij}, \mu^-_{ij}, \nu^-_{ij}\}$ indicating positive and negative feedbacks, for $i\in\{1,\dots,m\}$ and $j\in\{1,\dots,n\}$.
3: Construct a BPFSMTS, $\{X, \tau_{m\times n}, E\}$ namely, such that $\{E, A\}$ is a BPFSRGOM in the space $\{X, \tau_{m\times n}, E\}$.
4: Compute the score values, s_{ij} say, for each element of the matrix A using Definition 2.20.
5: Generate the NDM, $S^*=[s^*_{ij}]$ namely, for $i\in\{1,\dots,m\}$ and $j\in\{1,\dots,n\}$, with elements given by

$$s^*_{ij}=\begin{cases}\frac{s_{ij}-\min_i\{s_{ij}\}}{\max_i\{s_{ij}\}-\min_i\{s_{ij}\}}, & \text{for benefit criteria;}\\ \frac{\min_i\{s_{ij}\}-s_{ij}}{\min_i\{s_{ij}\}-\max_i\{s_{ij}\}}, & \text{for non-benefit criteria.}\end{cases}$$

6: Calculate the correlation matrix $R=[r_{jk}]$ for the attributes e_j and e_k with elements stated as

$$r_{jk}=\frac{\sum_{i=1}^m(s^*_{ij}-\overline{s}^*_j)(s^*_{ik}-\overline{s}^*_k)}{\sqrt{\left(\sum_{i=1}^m(s^*_{ij}-\overline{s}^*_j)^2\right)\left(\sum_{i=1}^m(s^*_{ik}-\overline{s}^*_k)^2\right)}},$$

where $\overline{s}^*_j=(1/m)\sum_{i=1}^m s^*_{ij}$ and $\overline{s}^*_k=(1/m)\sum_{i=1}^m s^*_{ik}$, for $j,k\in\{1,\dots,n\}$.
7: Determine the vector $\text{sd}=[\text{sd}_j]$ of standard deviations calculated as $\text{sd}_j=\left((1/m)\sum_{i=1}^m(s^*_{ij}-\overline{s}^*_j)^2\right)^{1/2}$, for $j\in\{1,\dots,n\}$.
8: State the vector $\Phi=[\phi_j]$ of deviation degrees of criterion e_j from any other criterion e_k by using $\phi_j=\text{sd}_j\sum_{k=1}^n(1-r_{jk})$, for $j\in\{1,\dots,n\}$.
9: Estimate the vector $\omega=[\omega_j]$ of weights of the attributes by considering $\omega_j=\phi_j/\sum_{j=1}^n\phi_j$, for $j\in\{1,\dots,n\}$.
10: **Output** Weights ω_j for $j\in\{1,\dots,n\}$.
11: END

To assist in understanding the mechanics of the BPF CRITIC method, a breakdown of Algorithm 1 is provided as follows:

[Input definition] The algorithm starts by receiving the BPFSOM, which incorporates both positive and negative feedbacks and serves as the foundational data from which the rest of the algorithm operates.

[Topology construction] Here, a BFS matrix topology is fashioned, ensuring compatibility with the BPFSRGOM.

[Score valuation] This step calculates the score values for each element in the matrix, allowing us to gauge variances and correlations.

[Data normalization] This is a fundamental step where the decision matrix is normalized, providing uniformity and ensuring a balanced data treatment.

[Correlation computation] This step measures the inter-dependencies and relationships within the matrix, giving an understanding of how different attributes interact with each other.

[Standard deviation] This step provides insights into the data variability.

[Deviation degree] Here, it is evaluated how each criterion differs from the others, providing a measure of distinctiveness.

[Attribute weighting] This step determines the importance of each attribute assigning weights and indicating a hierarchy.

[Output presentation] Lastly, the computed attribute weights are presented as the algorithm output, giving a concise summary of the attributes relative importance.

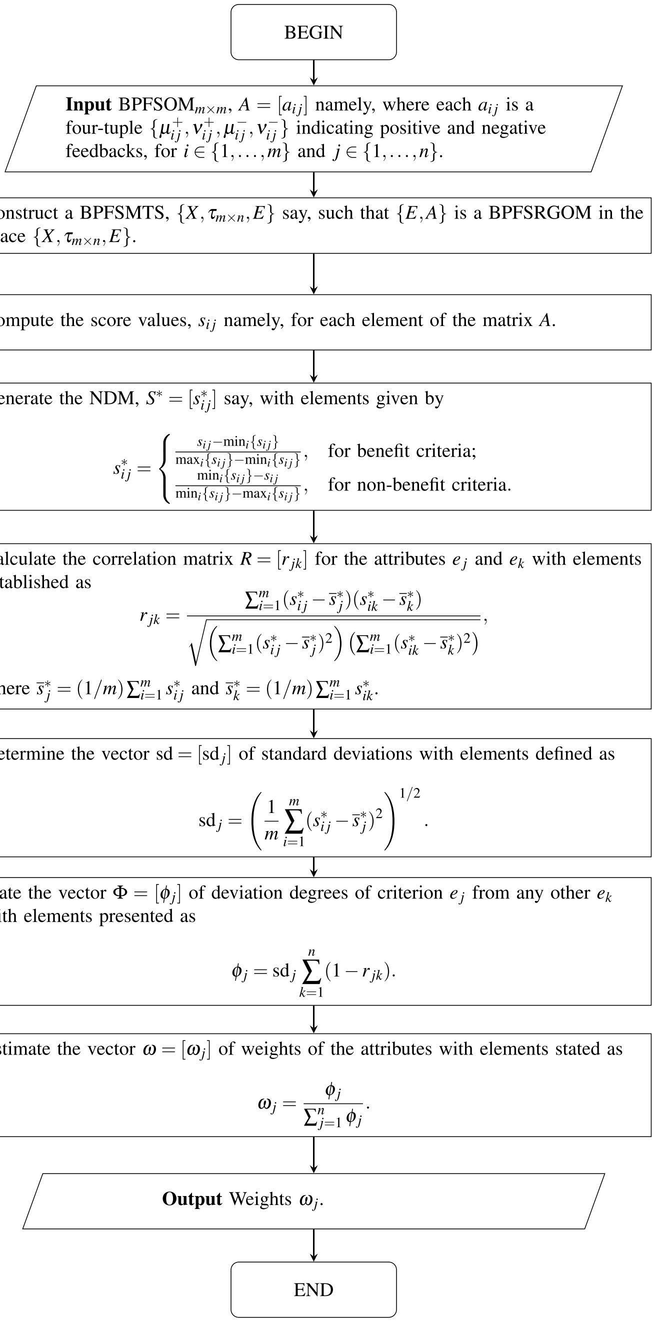

Fig. 1. BPF CRITIC method.

Algorithm 1 ensures comprehensive data processing, highlighting its significance in making informed digital marketing decisions. For further clarity and visualization, refer to the schematic representation of the BPF CRITIC method workflow in Fig. 1.

3.3. BPF CRITIC COPRAS algorithm

Expanding on the BPF CRITIC algorithm, the BPF CRITIC COPRAS method presented in Algorithm 2 integrates optimization indexes for positive and negative criteria, essential for decision-making in digital marketing. The BPF CRITIC COPRAS method is particularly helpful in scenarios where decisions need to be made considering importance of various attributes.

Algorithm 2 BPF CRITIC COPRAS method [16,31].

BEGIN
Input $\text{BPFSOM}_{m\times n}$, $A=[a_{ij}]$ say, where a_{ij} comprises a four-tuple $\{\mu_A^+, \nu_A^+, \mu_A^-, \nu_A^-\}$ related to positive and negative feedbacks, for $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$.
Invoke Algorithm 1 to determine the attribute weight ω_j for each criterion j, for $i \in \{1,\dots,m\}$
Generate the weighted normalized decision matrix (WNDM) as $\Omega=[s_{ij}^*][\omega_j]$, where the multiplication is performed element-wise with s_{ij}^* being the elements of the NDM, S^* namely, obtained in Step 5 of Algorithm 1, for $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$.
Calculate the vector $\Gamma=[\gamma_i]$ of optimization indexes using the weights stated as $\gamma_i^+=\sum_{j=1}^{r} s_{ij}^*\omega_j$ for the positive feedbacks and $\gamma_i^-=\sum_{j=r+1}^{n} s_{ij}^*\omega_j$ for the negative feedbacks, with $i \in \{1,\dots,m\}$.
Estimate the priority vector $P=[p_i]$ considering both positive and negative feedbacks by means of its elements defined as

$$p_i=\gamma_i^+ + \left(\frac{\min\{\gamma_i^-\}}{\gamma_i^-}\right)\left(\frac{\sum_{i=1}^{m}\gamma_i^-}{\sum_{i=1}^{m}\frac{\min\{\gamma_i^-\}}{\gamma_i^-}}\right),$$

for $i \in \{1,\dots,m\}$.
Compute the utility degree vector $\Psi=[\psi_i]$ with elements given by

$$\psi_i=\left(\frac{p_i}{p_{\max}}\right)\times 100\%,$$

indicating the performance of alternative i in comparison to the best performer $p_{\max}$, for $i \in \{1,\dots,m\}$.
Order values ψ_i forming priority (ranking) of alternative i, with $i \in \{1,\dots,m\}$, and make decisions based on this ranking.
Output Values ψ_i for each alternative i, with $i \in \{1,\dots,m\}$, providing insights for decision-making.
END

To elucidate the workflow of the BPF CRITIC COPRAS method, a concise breakdown of Algorithm 2 is presented as follows:

[Input definition] The method starts by receiving the BPFSOM containing the foundational data for subsequent processing.
[Topology construction] At this step, the BPFSMTS is constructed, ensuring that a BPFSRGOM is formed in the prescribed space.
[Score valuation] In this step, score values are computed for the BPFSOM.
[Attribute weighting] Here, the CRITIC COPRAS method invokes Algorithm 1 to determine the weights of attributes.
[Matrix normalization] In this step, the score matrix is normalized using the determined weights to get the WNDM.
[Optimization index computation] Here, the optimization indexes for both positive and negative feedbacks of criteria are calculated using the derived weights.
[Priority value estimation] Now, this step estimates a priority value (ranking) for each alternative, considering both the positive and negative feedbacks of the criteria.
[Utility degree calculation] Here, the method computes the utility degree for each alternative, which measures its performance compared to the best performer.
[Ranking and decision-making] In this step, the alternatives are ranked employing the utility values and then decisions are made based on this ranking.
[Output presentation] The algorithm concludes by outputting the utility values for each alternative, providing a comprehensive perspective for informed decision-making.

The BPF CRITIC COPRAS method presented in Algorithm 2 offers an approach to decision-making by not only taking into account the weights of the criteria but also by providing a comprehensive ranking system based on both positive and negative feedbacks of the criteria. For further clarity and visualization, refer to the schematic representation of the BPF CRITIC COPRAS method workflow in Fig. 2.

3.4. BPF CRITIC MARCOS algorithm

Building on the BPF CRITIC approach, Algorithm 3 provides the BPF CRITIC MARCOS method, which is helpful for the intricacies of digital marketing challenges. By integrating both the optimal and non-optimal solutions in a matrix representation, it offers a comprehensive view of decision outcomes. The BPF CRITIC MARCOS is crucial in digital marketing where professionals must consider complex performance indicators and market dynamics. This method permits us to evaluate both the best and worst scenarios facilitating efficient strategic decision-making.

Algorithm 3 BPF CRITIC MARCOS method [23,32].

1: BEGIN
2: **Input** $\text{BPFSOM}_{m\times n}$, $A=[a_{ij}]$ say, with a_{ij} being the four-tuple $\{\mu_A^+, v_A^+, \mu_A^-, v_A^-\}$ related to positive and negative feedbacks, for $i\in\{1,\dots,m\}$ and $j\in\{1,\dots,n\}$.
3: Apply Algorithm 1 to ascertain the attribute weight ω_j for each criterion j, thereby differentiating its influence on the final outcome, for $j\in\{1,\dots,n\}$.
4: Define a matrix $\widehat{S}=[\widehat{s}_{ij}]$ with elements $\widehat{s}_{ij}$ to include ideal (AI) and anti-ideal (AAI) solutions as the first and last rows of S, respectively, extracted from the extreme negative and positive scores, providing references to the best and worst possible scenarios under each criterion j, for $i\in\{1,\dots,m\}$ and $j\in\{1,\dots,n\}$.
5: Normalize the elements $\widehat{s}_{ij}$ of $\widehat{S}$ obtaining the matrix $\widetilde{S}=[\widetilde{s}_{ij}]$ with $\widetilde{s}_{ij}=\widehat{s}/\max_i\{\widehat{s}_{ij}\}$, if j pertains to non-benefit criteria; otherwise, $\widetilde{s}_{ij}=\max_i\{\widehat{s}_{ij}\}/\widehat{s}_{ij}$, that is, if j pertains to benefit criteria; ensuring scores to be dimensionless and comparable, considering the nature of each criterion, for $i\in\{1,\dots,m\}$ and $j\in\{1,\dots,n\}$.
6: Derive the matrix $Y=[y_{ij}]$ with elements $y_{ij}=\widetilde{s}_{ij}\omega_j$, incorporating the weight ω_j of criterion j to obtain a weighted matrix that shows the compounded influence of criteria preferences and performance scores, for $i\in\{1,\dots,m\}$ and $j\in\{1,\dots,n\}$.
7: Evaluate $t_i^-=Y_i/Y_{\text{AAI}}$ and $t_i^+=Y_i/Y_{\text{AI}}$ to determine the proximity of alternative i to the AAI and AI solutions, respectively, so indicating their overall desirability or undesirability, where $Y_i=[\widetilde{s}_{i1}\omega_1,\dots,\widetilde{s}_{in}\omega_n]$, $Y_{\text{AI}}=[\widetilde{s}_{\text{AI}_1}\omega_1,\dots,\widetilde{s}_{\text{AI}_n}\omega_n]$, and $Y_{\text{AAI}}=[\widetilde{s}_{\text{AAI}_1}\omega_1,\dots,\widetilde{s}_{\text{AAI}_n}\omega_n]$, for $i\in\{1,\dots,m\}$ and $j\in\{1,\dots,n\}$, with

$$\widetilde{s}_{\text{AI}_j}=\begin{cases}\max_i\{\widetilde{s}_{ij}\}, & \text{for benefit criteria;}\\ \min_i\{\widetilde{s}_{ij}\}, & \text{for non-benefit criteria;}\end{cases}\qquad \widetilde{s}_{\text{AAI}_j}=\begin{cases}\min_i\{\widetilde{s}_{ij}\}, & \text{for benefit criteria;}\\ \max_i\{\widetilde{s}_{ij}\}, & \text{for non-benefit criteria.}\end{cases}$$

8: Establish utility functions $f(t_i^+)$ and $f(t_i^-)$ reflecting the desirability –for $f(t_i^+)$– and undesirability –for $f(t_i^-)$– of alternative i through the equations stated as $f(t_i^+)=t_i^+/(t_i^++t_i^-)$ and $f(t_i^-)=t_i^-/(t_i^++t_i^-)$, highlighting the trade-off between optimal and suboptimal performance, for $i\in\{1,\dots,m\}$.
9: Compute the aggregated utility function $f(t_i)$ considering both desirability and undesirability, facilitating a balanced evaluation as

$$f(t_i)=\frac{t_i^++t_i^-}{1+(1-f(t_i^+))/f(t_i^+)+(1-f(t_i^-))/f(t_i^-)},$$

and organize the alternatives based on $f(e_i)$, with a higher value indicating a more favorable alternative, for $i\in\{1,\dots,m\}$.
10: **Output** Utility values $f(t_i)$ for each alternative i, with $i\in\{1,\dots,m\}$, providing a ranked list that signals their suitability grounded on the cumulative assessment from multi-criteria.
11: END

Breaking down the workings of the BPF CRITIC MARCOS method, Algorithm 3 consists of:

[Input definition] The process begins by acquiring the BPFSOM, with each entry in this matrix representing feedback for a respective criterion, offering a comprehensive evaluation of the platform.

[Topology construction] At this step, a topological structure is constructed, ensuring it aligns with the input matrix dimensions.

[Significance score computation] Here, each element in the BPFSOM is assigned to a score that captures the relative importance of each feedback within the decision matrix, which improves the accuracy of the subsequent evaluations.

[Attribute weighting] In this step, Algorithm 1 is employed to determine the weight of each criterion, helping in differentiating the influence of each criterion on the eventual outcome.

[Inclusion of AI solutions] Here, the BPFSOM is expanded to integrate both the AI and AAI solutions, whose solutions, derived from the feedback extremes act as reference points that exemplify the best and worst outcomes for each criterion.

[Matrix normalization] In this step, normalizing on the BPFSOM is conducted, which ensures that the scores are dimensionless and can be compared, whose normalization also takes into account the nature of each criterion.

[Weighted matrix derivation] Here, the score matrix is weighted, emphasizing the compounded influence of criteria preferences and performance scores.

[Desirability computation] In this step, the proximity of each alternative to the AI and AAI solutions is ascertained, whose solutions show the overall attractiveness or unattractiveness of each alternative.

[Utility function establishment] Here, the utility functions state the desirability and undesirability of each alternative, whose functions show the trade-off between optimal and suboptimal performance.

[Aggregated utility calculation] In this step, the aggregated utility is computed, considering both desirability and undesirability, whose utility helps in organizing the alternatives, with higher values indicating better alternatives.

[Output presentation] Concluding the algorithm, the utility value for each alternative are presented, whose output provides a ranked list based on the combined evaluation from multi-criteria, indicating the suitability of each alternative.

The BPF CRITIC MARCOS method offers an approach for decision-making by holistically assessing both the AI and AAI scenarios. This in-depth evaluation aids in differentiating the alternatives, presenting a ranked list of them based on a multi-criteria.

3.5. BPF CRITIC MAIRCA algorithm

Building upon the foundation of the BPF CRITIC approach, Algorithm 4 introduces the BPF CRITIC MAIRCA method, specifically designed for the multifaceted environment of digital marketing. Recognizing that not all decision criteria hold equal importance, this algorithm emphasizes the priority of certain indicators, adapting to the dynamic needs of digital marketing where priorities shift based on metrics such as customer engagement, ROI, or brand visibility.

A breakdown of the BPF CRITIC MAIRCA method, as described in Algorithm 4, is provided as:

[Input definition] The algorithm initiates by accepting the BPFSOM, where each element of this matrix depicts a specific feedback characterized by the four-tuple associated.

Algorithm 4 BPF CRITIC MAIRCA method [24].

1: BEGIN
2: **Input** $\text{BPFSOM}_{m\times n}$, $A=[a_{ij}]_{m\times n}$ say, with a_{ij} representing a specific feedback, defined by the four-tuple $\{\mu_A^+, \nu_A^+, \mu_A^-, \nu_A^-\}$, for $i\in\{1,\dots,m\}$ and $j\in\{1,\dots,n\}$.
3: Call Algorithm 1 to derive ω_j, the weighting factor expressing the importance of each criterion j, for $j\in\{1,\dots,n\}$.
4: Evaluate J_{e_j} as the priority of each indicator j, established using a uniform distribution for equity given by $J_{e_j}=1/n$, for $j\in\{1,\dots,n\}$.
5: Determine the matrix $Z=[z_{ij}]$ with elements z_{ij} by integrating the priority and weight of each indicator as $z_{ij}=J_{e_j}\omega_j$, for $i\in\{1,\dots,m\}$ and $j\in\{1,\dots,n\}$.
6: Generate the adjusted matrix $Z^*=[z^*_{ij}]$ recalculating each z_{ij} based on the type of criterion j, for $i\in\{1,\dots,m\}$ and $j\in\{1,\dots,n\}$, where the criteria are considered either benefit or non-benefit to the outcome stated as follows, considering the elements s_{ij} of the matrix S obtained in Step 4 of Algorithm 1:
- For a benefit criterion, where higher values are preferable, adjust z_{ij} using the formula defined as

$$z^*_{ij}=z_{ij}\left(\frac{\max_i\{s_{ij}\}-s_{ij}}{\max_i\{s_{ij}\}-\min_i\{s_{ij}\}}\right).$$

- Conversely, for a non-benefit criterion, where lower values are preferable, use the alternative formulation stated as

$$z^*_{ij}=z_{ij}\left(\frac{s_{ij}-\max_i\{s_{ij}\}}{\min_i\{s_{ij}\}-\max_i\{s_{ij}\}}\right).$$

7: Calculate the disparity matrix $\Upsilon=[\rho_{ij}]$ between the expected and observed performance, with its elements expressed as $\rho_{ij}=z_{ij}-z^*_{ij}$, for $i\in\{1,\dots,m\}$ and $j\in\{1,\dots,n\}$.
8: Obtain θ_i as the consolidated metric reflecting overall performance or suitability, derived from $\theta_i=\sum_{j=1}^n\rho_{ij}$, for $i\in\{1,\dots,m\}$.
9: Rank the alternatives via θ_i, identifying alternative i as the most suitable if θ_i is the minimum value, succeeded by other alternatives as per ascending values θ_i, for $i\in\{1,\dots,m\}$.
10: **Output** A ranking of alternatives using metric θ_i, with $i\in\{1,\dots,m\}$, providing a basis for selection or prioritization in subsequent decision-making processes.
11: END

[Topology formation] At this step, a BPFSMTS is formulated, certifying that the BPFSOM operates under the BPFSRGOM regime.
[Relevance determination] Here, the scores are evaluated, indicating the relevance of every feedback within the BPFSOM.
[Criterion significance extraction] In this step, the method employs Algorithm 1 to distill weights that convey the importance of each criterion.
[Indicator priority assessment] This step evaluates the priority of the indicator, underpinned by a uniform distribution for equity.
[Merging weight with priority] Here, the algorithm integrates the stated priority and weight for each indicator.
[Criterion-based adjustments] In this step, every merged element undergoes modifications as per the type of criterion, where these criteria can be classified as either benefit or non-benefit.
[Discrepancy measurement] This stage calculates the disparity values, showing the discrepancy between the expected and observed outcomes.
[Comprehensive metric formation] Here, the algorithm formulates a metric encapsulating the overall performance.
[Hierarchy establishment] At this step, utilizing the mentioned metric, the alternatives are organized in an ascending sequence.
[Output generation] The method culminates by delivering a ranking of alternatives via the mentioned metric, guiding subsequent decision-making processes.

Fig. 2 states a visual representation of the BPF CRITIC COPRAS, MARCOS, and MAIRCA methods. The BPF CRITIC MAIRCA method offers a systematic approach to multi-criteria decision analysis. By integrating different evaluation dimensions like feedback parameters, criteria weights, and indicator priorities, the method outputs a ranked list of alternatives that can guide stakeholders in their decision-making processes. The discussed algorithms share a common goal: to provide efficient solutions for decision-making in complex scenarios, with a special focus on digital marketing. They are grounded in the BPFSMTS framework, which enables systematic interpretation and data analysis, catering to the varied demands of this rapidly changing field. While these methods vary in their specific methodologies, they are designed to offer comprehensive views across digital marketing indicators. Each introduces unique considerations and approaches within its structure, collectively shaping a versatile toolbox for decision-makers in digital marketing. This ensemble of algorithms aids in key processes such as normalization, weighting, and ranking, ultimately enhancing the decision-making efficiency in a dynamic landscape.

4. Application of the BPF approach to digital marketing

In this section, we present a multi-criteria decision-making application of the BPF soft regular generalized matrix using the BPF CRITIC COPRAS, BPF CRITIC MAIRCA and BPF CRITIC MARCOS methods that are equipped with BPF data. We illustrate our proposed methods with a numerical example.

4.1. Social media platform

An SMP is an internet-based media of communication that encompasses applications or websites where users can create and share content and connect with others. SMPs include Facebook, Instagram, Linkedin, Pinterest, Twitter, WhatsApp, and YouTube. In our application, we consider generic platforms. Today, the majority of individuals engage with some form of social media. An SMP not only allows us to communicate with friends and family but also enables corporations to interact with their users, gather customer feedback, and enhance their brands. There are various types of social media sites available that enrich the digital marketing.

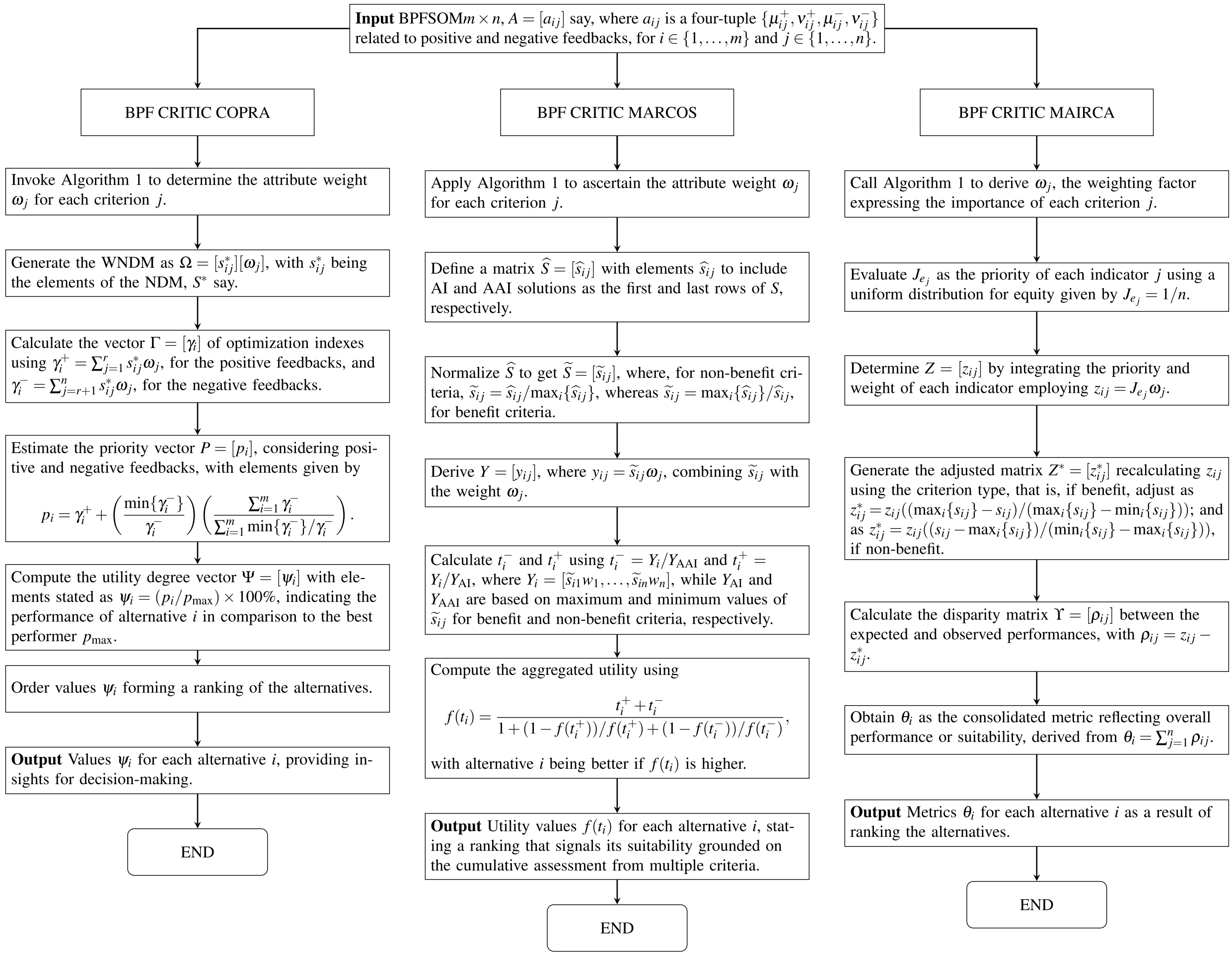

Fig. 2. Visual representation of the BPF CRITIC COPRAS, MARCOS, and MAIRCA methods.

4.2. Empirical example

Next, we apply the BPFSRGOM to address an empirical decision-making scenario, focusing on a company objective to identify the optimal SMP to maximize its ROI. We consider five generic SMPs as alternatives: [x_1] Platform 1; [x_2] Platform 2; [x_3] Platform 3; [x_4] Platform 4; and [x_5] Platform 5. The goal is to pinpoint the most effective platform for business development that also offers substantial reach within the target audience. The process begins by evaluating each platform based on certain criteria. One can think of this as rating each social media on various aspects that matter to a business, like how many users they can reach or how much it costs to advertise.

In our case, the decision-making process involves five evaluation criteria (attributes): [e_1] marketing goals; [e_2] demographics; [e_3] cost of advertisement; [e_4] monthly active users; and [e_5] cost of product. Among them, e_1, e_2, and e_4 are considered benefit criteria, while e_3 and e_5 are viewed as cost (non-benefit) criteria. A point worth noting here is that not all criteria are equal: some are benefit (higher is better), while others, like costs, are the opposite (lower is better).

Given the mentioned evaluation criteria, the BPFSMTS approach offers a distinct advantage: it maintains the integrity of the original data during the decision-making process. Unlike traditional methods where the union or intersection of various opinions may distort the original dataset, the BPFSMTS approach ensures that the resulting set remains within the initial data space. This is crucial for making a sound and accurate decision. In simpler terms, the BPFSMTS method respects the original data given and do not let them to get "watered down" or distort as we process the data.

In the following, we outline the steps for constructing a BPFSMTS and achieving the most effective platform for business development:

[Step 1] Arrange the given BPFSOM organizing all the ratings and feedbacks to have about each SMP into a neat table (or matrix) which is the foundation of our analysis, given in our case by:

$$A=\begin{bmatrix}\{0.3,0.7,-0.4,-0.8\} & \{0.9,0.1,-0.9,-0.2\} & \{0.5,0.4,-0.7,-0.7\} & \{0.6,0.6,-0.5,-0.7\} & \{0.4,0.8,-0.4,-0.7\}\\ \{0.4,0.8,-0.3,-0.7\} & \{0.8,0.1,-0.8,-0.1\} & \{0.4,0.4,-0.6,-0.6\} & \{0.5,0.5,-0.6,-0.6\} & \{0.5,0.9,-0.4,-0.5\}\\ \{0.5,0.3,-0.7,-0.5\} & \{0.3,0.7,-0.5,-0.7\} & \{0.6,0.3,-0.3,-0.3\} & \{0.3,0.4,-0.3,-0.8\} & \{0.4,0.3,-0.2,-0.3\}\\ \{0.3,0.3,-0.2,-0.7\} & \{0.1,0.4,-0.6,-0.4\} & \{0.2,0.7,-0.6,-0.6\} & \{0.5,0.4,-0.9,-0.3\} & \{0.3,0.4,-0.5,-0.5\}\\ \{0.3,0.5,-0.5,-0.3\} & \{0.4,0.3,-0.7,-0.8\} & \{0.7,0.6,-0.3,-0.6\} & \{0.4,0.3,-0.8,-0.6\} & \{0.4,0.4,-0.3,-0.4\}\end{bmatrix}.$$

[Step 2] Express the score values S of A, giving to each platform a score based on our criteria, with higher scores being better, stated as:

$$S=\begin{bmatrix}0.560 & 0.215 & 0.955 & 0.880 & 0.595\\ 0.800 & 0.680 & 0.865 & 0.690 & 0.990\\ 0.560 & 0.370 & 1.000 & 1.000 & 0.675\\ 0.775 & 0.975 & 0.775 & 0.595 & 0.965\\ 1.000 & 0.960 & 0.930 & 0.825 & 0.965\end{bmatrix}.$$

[Step 3] Normalize the decision matrix converting scores from different origins to a common scale so they can be compared easily, reaching:

$$S^*=\begin{bmatrix}0.0000 & 0.0000 & 0.8000 & 0.7037 & 0.0000\\ 0.0000 & 0.2039 & 1.0000 & 1.0000 & 0.7975\\ 0.5455 & 0.6118 & 0.4000 & 0.2346 & 1.0000\\ 0.4886 & 1.0000 & 0.0000 & 0.0000 & 0.9367\\ 1.0000 & 0.9803 & 0.3111 & 0.5679 & 0.9367\end{bmatrix}.$$

[Step 4] Obtain the correlation coefficient of the attributes e_1 to e_5 to understand how each criterion relates to the others, where if two criteria often move together (for example, cost and quality), they have a high correlation, that in our case is presented as:

$$R=\begin{bmatrix}1.0000 & 0.8761 & -0.4094 & -0.4897 & 0.6358\\ 0.8761 & 1.0000 & -0.7005 & -0.7023 & 0.7665\\ -0.4094 & -0.7005 & 1.0000 & 0.9820 & -0.4286\\ -0.4897 & -0.7023 & 0.9820 & 1.0000 & -0.4214\\ 0.6358 & 0.7665 & -0.4286 & -0.4214 & 1.0000\end{bmatrix}.$$

[Step 5] Calculate the standard deviations for each attribute, that in our case is obtained as:

$$\mathrm{sd}=\begin{bmatrix}0.4209 & 0.4509 & 0.3890 & 0.3926 & 0.4170\end{bmatrix}.$$

[Step 6] Determine the deviation degrees from one criterion in relation to the other criteria, whose step is about understanding how different each criterion is from the others, it being another way to measure their uniqueness, and in our case established as:

$$\phi=\begin{bmatrix}1.4258 & 1.6955 & 1.7727 & 1.8183 & 1.4378\end{bmatrix}.$$

[Step 7] State the weights of attributes, after understanding the importance and uniqueness of each criterion, assigning them, where the weights tell us which criteria are more important in our decision-making process, that in our case are presented as:

$$\omega=\begin{bmatrix}0.1749 & 0.2080 & 0.2175 & 0.2231 & 0.1764\end{bmatrix}.$$

Thus, the weightage of each criterion has been evaluated using the BPF CRITIC method. Next, for the BPF CRITIC COPRAS method, follow Step 1, Step 2 and Step 3 as in BPF CRITIC method and then:
[Step 4] Compute the WNDM, adjusting the scores based on the weights of each criterion and giving extra importance to criteria more relevant, with WNDM being computed in our case as:

$$\Omega=\begin{bmatrix}0.0000 & 0.0000 & 0.1740 & 0.1570 & 0.0000\\ 0.0000 & 0.0424 & 0.2175 & 0.2231 & 0.1407\\ 0.0954 & 0.1273 & 0.0870 & 0.0523 & 0.1764\\ 0.0855 & 0.2080 & 0.0000 & 0.0000 & 0.1653\\ 0.1749 & 0.2039 & 0.1498 & 0.1267 & 0.1653\end{bmatrix}.$$

[Step 5] Establish the optimization indexes γ_i^+ and γ_i^- given as in Table 3, where these indexes help us to understand the performance of each platform compared to the best and worst possible scenarios.
[Steps 6 and 7] Compute the priority values p_i, utility degrees ψ_i, and rankings, where now we rank our SMP based on all the previous steps, whose platform with the highest ranking is our best alternative, which are obtained using the BPF CRITIC COPRAS method and whose results are presented in Table 4.

Proceeding with our analysis, we apply the BPF CRITIC MARCOS method. Notably, the initial stages (Steps 1, 2, and 3) of the BPF CRITIC MARCOS method are consistent with those of the BPF CRITIC method, which was designed to ensure the resulting set

Table 3
Indexes γ_i^+ and γ_i^- obtained with the BPF CRITIC COPRAS method for the indicated SMP.

SMP	γ_i^+	γ_i^-
x_1	0.1570	0.1740
x_2	0.2655	0.3582
x_3	0.2750	0.2634
x_4	0.2935	0.1653
x_5	0.5056	0.3151

Table 4
Priority values p_i, utility degrees ψ_i, and rankings obtained with the BPF CRITIC COPRAS method for the indicated SMP.

SMP	p_i	ψ_i	$\psi_i \times 100\%$	Ranking
x_1	0.4848	0.6055	60.55	5
x_2	0.5884	0.7349	73.49	3
x_3	0.5833	0.7285	72.85	4
x_4	0.5995	0.7488	74.88	2
x_5	0.8006	1.0000	100.00	1

remained within the initial data space. This method assists decision-makers in identifying the optimal SMP for digital marketing. The following steps associated with the BPF CRITIC MARCOS method are introduced:

[Step 4] Represent an expanded matrix by adding an AI solution and its AAI solution expressed in our case as:

$$\widehat{S} = \begin{matrix} \text{AAI} \\ \\ \\ \\ \\ \\ \text{AI} \end{matrix} \begin{bmatrix} 0.560 & 0.215 & 1.000 & 0.595 & 0.990 \\ 0.560 & 0.215 & 0.955 & 0.880 & 0.595 \\ 0.560 & 0.370 & 1.000 & 1.000 & 0.675 \\ 0.800 & 0.680 & 0.865 & 0.690 & 0.990 \\ 0.775 & 0.975 & 0.775 & 0.595 & 0.965 \\ 1.000 & 0.960 & 0.930 & 0.825 & 0.965 \\ 1.000 & 0.975 & 0.775 & 1.000 & 0.595 \end{bmatrix}.$$

This matrix extends the existing data to consider an additional set of criteria, enabling a more comprehensive decision analysis.
[Step 5] Normalize the elements of $\widehat{S}$ ensuring scores to be dimensionless and comparable, considering the nature of each criterion, in our case reaching:

$$\widetilde{S} = \begin{matrix} \text{AAI} \\ \\ \\ \\ \\ \\ \text{AI} \end{matrix} \begin{bmatrix} 0.5600 & 0.2205 & 1.0000 & 0.5950 & 1.0000 \\ 0.5600 & 0.2205 & 0.8115 & 0.8800 & 1.0000 \\ 0.5600 & 0.3795 & 0.7750 & 1.000 & 0.8815 \\ 0.8000 & 0.6974 & 0.8960 & 0.690 & 0.6010 \\ 0.7750 & 1.0000 & 1.0000 & 0.595 & 0.6166 \\ 1.0000 & 0.9846 & 0.8333 & 0.8250 & 0.6166 \\ 1.0000 & 1.0000 & 0.7750 & 1.0000 & 0.6010 \end{bmatrix}.$$

[Step 6] Obtain the weighted normalized expanded matrix, assigning weights to the criteria to emphasize the importance of certain factors over others, whose expanded matrix is established in our case as:

$$Y = \begin{matrix} \text{AAI} \\ \\ \\ \\ \\ \\ \text{AI} \end{matrix} \begin{bmatrix} 0.0980 & 0.0459 & 0.2175 & 0.1327 & 0.1060 \\ 0.0980 & 0.0459 & 0.1765 & 0.1963 & 0.1764 \\ 0.0980 & 0.0789 & 0.1686 & 0.2231 & 0.1555 \\ 0.1400 & 0.1451 & 0.1949 & 0.1539 & 0.1060 \\ 0.1356 & 0.2080 & 0.2175 & 0.1327 & 0.1088 \\ 0.1749 & 0.2048 & 0.1813 & 0.1841 & 0.1088 \\ 0.1749 & 0.2080 & 0.1686 & 0.2231 & 0.1060 \end{bmatrix}.$$

[Step 7] Rank the SMPs so that the decision-maker can recognize which platform best suits the company needs, with the ranking of the SMP being computed using the BPF CRITIC MARCOS method, and the results being summarized in Table 5.

Table 5
Results obtained with the BPF CRITIC MARCOS method for the indicated SMP.

SMP	Sum	t_i^+	t_i^-	$f(t_i^+)$	$f(t_i^-)$	$f(t_i)$	Ranking
x_1	0.6931	0.7870	1.1549	0.5947	0.4053	0.6167	4
x_2	0.7241	0.8222	1.2066	0.5947	0.4053	0.6443	5
x_3	0.7399	0.8401	1.2329	0.5947	0.4053	0.6583	3
x_4	0.8026	0.9114	1.3375	0.5947	0.4053	0.7142	2
x_5	0.8539	0.9696	1.4228	0.5947	0.4053	0.7600	1

Table 6
Values of θ_i and rankings obtained with the BPF CRITIC MAIRCA method for the indicated SMP.

SMP	θ_i	Ranking
x_1	0.1338	5
x_2	0.0753	2
x_3	0.0923	3
x_4	0.1082	4
x_5	0.0359	1

Table 7
Comparative analysis of the methods.

Method	Ranking	Best SMP
BPF CRITIC COPRAS	x_5, x_4, x_3, x_1, x_2	x_5
BPF CRITIC MARCOS	x_5, x_4, x_3, x_2, x_1	x_5
BPF CRITIC MAIRCA	x_5, x_2, x_3, x_4, x_1	x_5

Continuing our analysis, we now consider the BPF CRITIC MAIRCA method. Similar to the previous procedures, the initial stages (Steps 1, 2, and 3) of this method align with those of the BPF CRITIC method. The additional stages of this method are as follows:

[Step 4] Evaluate the priority for an attribute using $J_{e_j} = 1/5 = 0.2$ in our case, with $j \in \{1, \ldots, 5\}$.
[Step 5] Determine the ranking matrix stated in our case as:

$$Z = \begin{bmatrix} 0.0350 & 0.0416 & 0.0435 & 0.0446 & 0.0353 \\ 0.0350 & 0.0416 & 0.0435 & 0.0446 & 0.0353 \\ 0.0350 & 0.0416 & 0.0435 & 0.0446 & 0.0353 \\ 0.0350 & 0.0416 & 0.0435 & 0.0446 & 0.0353 \\ 0.0350 & 0.0416 & 0.0435 & 0.0446 & 0.0353 \end{bmatrix}.$$

[Step 6] Adjust the ranking matrix Z obtaining in our case that:

$$Z^* = \begin{bmatrix} 0.0000 & 0.0000 & 0.0348 & 0.0314 & 0.0000 \\ 0.0000 & 0.0085 & 0.0435 & 0.0446 & 0.0281 \\ 0.0191 & 0.0255 & 0.0174 & 0.0105 & 0.0353 \\ 0.0171 & 0.0416 & 0.0000 & 0.0000 & 0.0331 \\ 0.0350 & 0.0408 & 0.0300 & 0.0253 & 0.0331 \end{bmatrix}.$$

[Step 7] Determine the values of ρ_{ij} using $\rho_{ij} = Z_{ij} - Z^*_{ij}$ generating the disparity matrix established in our case as:

$$\Upsilon = \begin{bmatrix} 0.0350 & 0.0416 & 0.0087 & 0.0132 & 0.0353 \\ 0.0350 & 0.0331 & 0.0000 & 0.0000 & 0.0071 \\ 0.0159 & 0.0161 & 0.0261 & 0.0342 & 0.0000 \\ 0.0179 & 0.0000 & 0.0435 & 0.0446 & 0.0022 \\ 0.0000 & 0.0008 & 0.035 & 0.0193 & 0.0022 \end{bmatrix}.$$

[Steps 8 and 9] Compute the metric $\theta_i = \sum_{i=1}^{m} \rho_{ij}$ and allocate the rankings of the SMPs, which can be seen in Table 6.

The results derived from the application of the three distinct methods are compiled in Table 7. As illustrated in this table and Fig. 3, x_5 (representing Platform 5) consistently stands out as the premier choice across all methods, highlighting the robustness of the proposed methodology for devising digital marketing strategies.

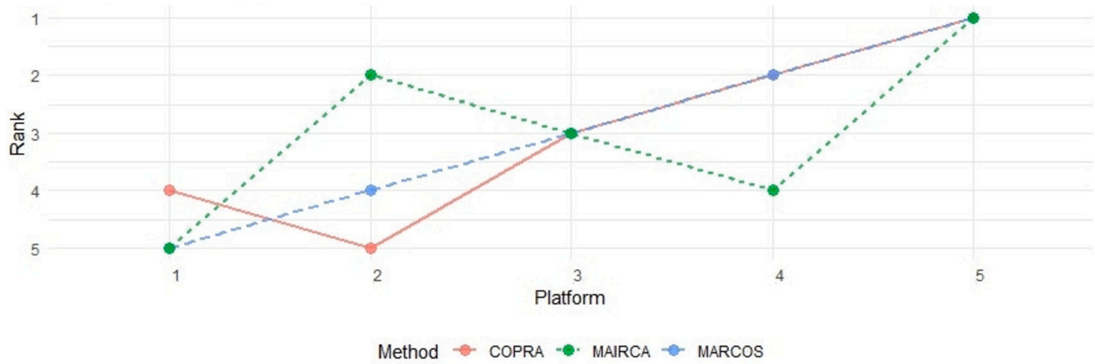

Fig. 3. Behavior of five SMP with the indicated method.

To further elucidate the significance of our proposed method, assume a global digital marketing agency considering various international markets, employing multiple SMPs to cater to diverse audience segments. The selection criteria would encompass audience demographics, user behavior, advertising expenses, engagement rates, and ROI. Considering the cultural variability across regions, data become fraught with uncertainties. Existing methodologies often falter amidst such uncertainties, demanding precise data, which may be elusive or costly, especially on a global scale.

Our approach manages ambiguous or incomplete data. It operates within a spectrum of possibilities, proving invaluable in scenarios characterized by complex data. By harnessing matrix operations, it evaluates multi-criteria simultaneously, offering a better decision-making framework compared to traditional methods. Our approach leads to informed decisions, contributing to possibly successful marketing campaigns and enhanced ROI. In essence, our methodology is tailored for the complexities of global SMP selection, overcoming the shortcomings of traditional methodologies.

5. Conclusions

In this study, we explored the potential of bipolarity in Pythagorean fuzzy soft regular generalized closed matrices within the framework of their corresponding topological spaces. This exploration presented an approach to problem-solving in the field of digital marketing.

To select optimal social media strategies, we leveraged three decision-making methods: bipolar Pythagorean fuzzy CRITIC COPRAS, BPF CRITIC MAIRCA, and BPF CRITIC MARCOS, namely. Through our analysis, one of the generic platforms consistently emerged as the most suitable one for digital marketing, showing the reliability of our approach. The contributions of the present research are twofold and detailed as follows.

Firstly, our work provides a comprehensive understanding and application of bipolar Pythagorean fuzzy soft matrix topological spaces and bipolar Pythagorean fuzzy soft regular generalized open matrices, demonstrating their potential in decision-making scenarios. Secondly, we introduced groundbreaking algorithms tailored for digital marketing based on the bipolar Pythagorean fuzzy CRITIC, CRITIC COPRAS, CRITIC MARCOS, and CRITIC MAIRCA methods. Together, these methods showed the essence of our study in reshaping digital marketing strategies. However, we must recognize certain limitations. The deployment of our approach requires an in-depth grasp of bipolar Pythagorean fuzzy sets and soft matrix topological spaces, which might pose challenges for those unfamiliar with these concepts. Furthermore, while our methodology is tailored for digital marketing, its application to other domains might necessitate modifications. The efficacy of our methodology is intrinsically tied to the quality of input data, emphasizing the significance of meticulous data collection and analysis.

Looking ahead, we see potential in integrating fuzzy set theory with recent advancements in the field. Models such as (2,1)-fuzzy sets, SR-fuzzy sets, (3,2)-fuzzy sets, and new generalizations of fuzzy soft sets could enhance the applicability and utility of decision-making methodologies across various fields, including image processing, pattern recognition, and artificial intelligence [33–35]. Also, the use of other statistical distributions can be considered instead of triangular distributions when applying fuzzy theory [36]. Additionally, we plan to develop an `R` package or a `Python` library to encapsulate the proposed algorithms, offering a practical resource for researchers and practitioners. In conclusion, this work combined theoretical advancements with practice, providing tools for making optimal decisions in uncertain environments. Despite its limitations, this study shows a path for further research and applications in fuzzy set theory and digital marketing.

CRediT authorship contribution statement

Vishalakshi Kuppusamy: Writing – original draft, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Maragathavalli Shanmugasundaram:** Writing – original draft, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Prasantha Bharathi Dhandapani:** Writing – original draft, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Carlos Martin-Barreiro:** Writing – original draft, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Xavier Cabezas:** Writing – original draft, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Víctor Leiva:** Writing – review & editing, Visualization, Supervision, Methodology, Investigation, Formal analysis, Conceptualization. **Cecilia Castro:** Writing – review & editing, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this article.

Data availability

Data will be made available on request from the authors.

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