Modelling the Effects of Selection Temperature and Mutation on the Prisoner's Dilemma Game on a Complete Oriented Star

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Abstract

This paper models the prisoner's dilemma game based on pairwise comparison in finite populations on a complete oriented star (COS). First, we derive a linear system on a COS for calculating the corresponding fixation probabilities that imply dependence of the selection temperature and mutation. Then we observe and analyze the effects of two parameters on fixation probability under different population sizes. In particular, it is found through the experimental results that (1) high mutation is more sensitive to the fixation probability than the low one when population size is increasing, while the opposite is the case when the number of cooperators is increasing, and (2) selection temperature demotes the fixation probability.

Introduction

As a standard approach to describing the evolutionary dynamics of population, evolutionary game theory has drawn considerable attentions ([1], [2], [3], [4], [5], [6], [7], [8], [9]). In this context, an ideal assumption is that individuals interact randomly with each other in large, well-mixed population. In real world, however, the organization of population is usually highly structured rather than uniform, where the dynamics between individuals depend on both the strategy and the population configuration ([10], [11], [12], [13], [14], [15], [16], [17], [18]). The population configuration can be modeled by a weighted digraph. A *complete oriented star* (COS) is a complete bipartite digraph, where one partition consists of a single vertex known as the central vertex, and the other is a collection of vertices known as the peripheral vertices. COSs are a popular class of network topologies ([19], [20]) and have a wide-ranging applied background, such as computer network topology and a social organization, where the leader and employees can be seen as the central vertex and the peripheral vertices, respectively. The communications between them can be represented by the weight. To our knowledge, the evolutionary game with mutation based on the pairwise comparison process with COS structure has yet to be investigated.

Taking the prisoner's dilemma ([21], [22], [23], [24]) as an example, this paper addresses the evolutionary game of population with COS structure, where the mutation is based on pairwise comparison ([25], [26], [27]). First, we derive a linear system for calculating the fixation probabilities. Then we observe and analyze such phenomena appeared in the game as the effects of selection temperature and mutation rates and population size on fixation probability.
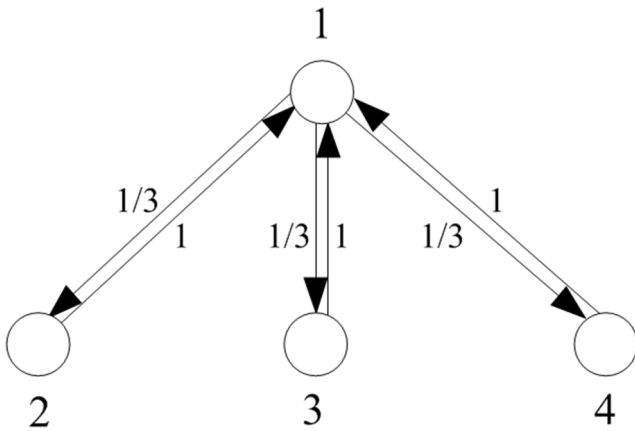
The organization of this paper is as follows. In the next section, we present preliminary knowledge to support our study. In section 3, we derive analytic solutions of fixation probabilities by converting our task into a linear system on a COS. In Section 4, we give some simulations. Finally, we end the paper in Section 5.

Preliminary Knowledge

A *complete oriented star* (COS) of size N, denoted S_N, is a digraph with vertex set $V = \{1, 2, ..., N\}$ and edge set $E = \{\langle 1, i \rangle, \langle i, 1 \rangle | 2 \le i \le N\}$. We call vertex 1 as the *central vertex*, and vertices 2 through N as *peripheral vertices*. The *intrinsic weights* for S_N are defined as $w(1, i) = 1/(N-1)$ and $w(i, 1) = 1$ for $2 \le i \le N$. Figure 1 (S_4) and Figure 2 (S_5) depict two COSs with intrinsic weights. In the sequel, the term "COS" means "COS with intrinsic weights".

In a Prisoner's Dilemma game, individuals can choose one of two strategies: *cooperation* (C) and *defection* (D), and the corresponding individuals are called *cooperator* and *defector*, respectively. The payoff matrix is given below.

$$
\begin{array}{cc}
 & \begin{array}{cc} C & D \end{array} \\
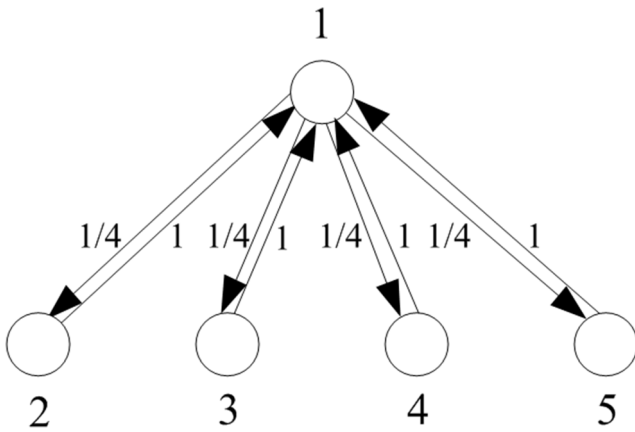\begin{array}{c} C \\ D \end{array} & \begin{pmatrix} R & S \\ T & P \end{pmatrix}
\end{array}
$$

Figure 1. A complete oriented star S_4 with intrinsic weights.
doi:10.1371/journal.pone.0107417.g001

where T, R, P and S stand for *temptation to defect*, *reward for mutual cooperation*, *punishment* and *sucker payoff*, respectively. It is always assumed that $T > R > P > S$. In this paper, we adopt the following payoff matrix suggested by Nowak and May [28]. This model preserves the essentials of the Prisoner's Dilemma game and $1 \leq b \leq 2$ is the only tunable parameter.

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix} \end{array} \qquad (2)$$

We should emphasize that our observations are not restricted to the present weak dilemma strength as the model, but remain fully valid also for the strong strength (strictly satisfying T>R>P>S).

Consider a homogeneous population with S_N structure, whose individuals play a Prisoner's Dilemma game with the payoff matrix (2). Suppose that, initially, there are m randomly chosen cooperators and $N - m$ defectors. A *central cooperator* (respectively, *defector*) is a cooperator (respectively, defector) occupying the central vertex. A *peripheral cooperator* (respectively, *defector*) is a cooperator (respectively, defector) occupying a peripheral vertex.

Figure 2. A complete oriented star S_5 with intrinsic weights.
doi:10.1371/journal.pone.0107417.g002
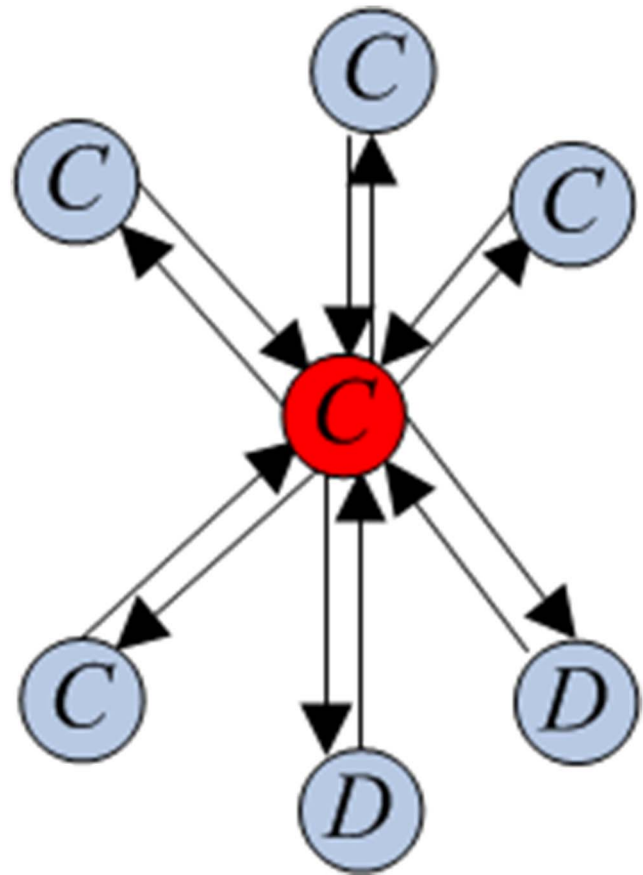
Figure 3. A complete oriented star with central cooperator.
doi:10.1371/journal.pone.0107417.g003

The sketches with central cooperator and central defector are given in Figure 3 and Figure 4, respectively.

Let π_{CC} and π_{CD} denote the mean payoffs of the central cooperator and central defector, respectively, and let π_{PC} and π_{PD} denote the payoffs of a peripheral cooperator and a peripheral defector provided the central vertex is a cooperator and defector, respectively. A straightforward calculation will give us

$$\pi_{CC} = \frac{(m-1) \times 1}{N-1}, \ \pi_{PD} = \frac{(N-m) \times b}{N-1},$$
$$\pi_{CD} = \frac{m \times b}{N-1}, \ \pi_{PC} = 0. \qquad (3)$$

Accordingly, the payoff differences are given by

$$\Delta \pi_{CC,PD} = \pi_{CC} - \pi_{PD} = \frac{(m-1) \times 1 + (N-m) \times b}{N-1},$$
$$\Delta \pi_{CD,PC} = \pi_{CD} - \pi_{PC} = \frac{m \times b}{N-1}. \qquad (4)$$

In this paper, we adopt a process based on pairwise comparison between individuals. In each step of the process, an random (focal) individual i is selected for reproducing an offspring, which means an ancestor reproduces an offspring rather than one individual breaks down into two individuals, then the offspring compares its payoff π_i to the payoff π_j of a randomly chosen neighbor j, and
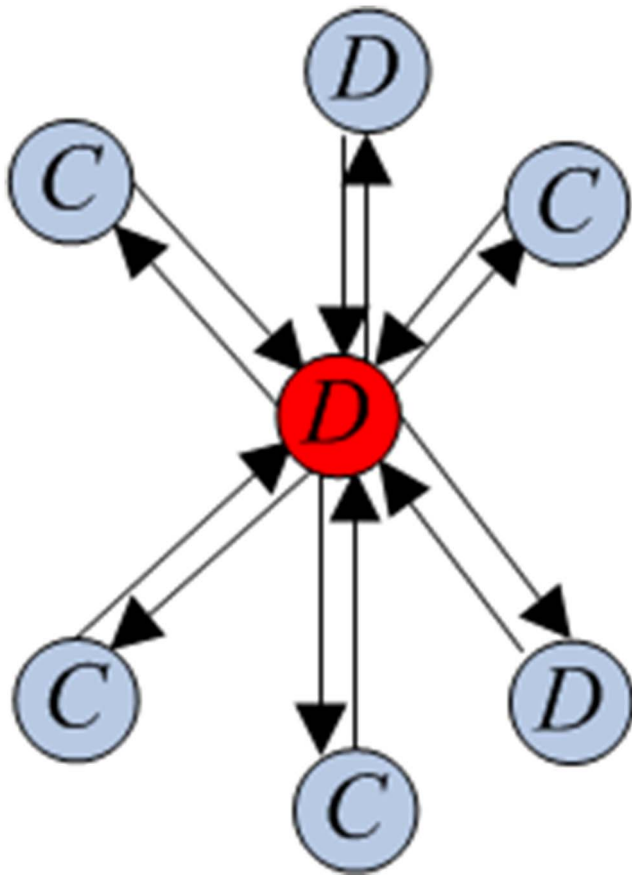
Figure 4. A complete oriented star with central defector.
doi:10.1371/journal.pone.0107417.g004

adopts the strategy of that neighbor with probability $\dfrac{1}{1+e^{\beta(\pi_i-\pi_j)}}$ ([25], [26], [27]), where $i,j \in \{CC, CD, PC, PD\}$ and the parameter $\beta \geq 0$, which corresponds to an inverse temperature in statistical physics, controls the intensity of selection. Small β (high temperature) means that selection is almost neutral, whereas for large β (low temperature) selection can become arbitrarily strong. This process occurs with probability $1-\mu$; with probability μ a mutation occurs, which means that the focal individual produces an offspring with random strategy, C or D.

The mth-order *fixation probability* on S_N, denoted ρ_m, is defined as the possibility of the event that m cooperators could eventually take over the entire population.

Formulas for Fixation Probabilities

In this section, we derive the formulas for the fixation probabilities on S_N.

At time t, the configuration of a population on S_N is described by a vector $\mathbf{M}(t) = (m_1(t), m_2(t))$, where $m_1(t) = 1$ or 0 according as a cooperator occupies vertex 1 or not, and $m_2(t)$ denotes the number of cooperators staying at vertices 2 through N. Let $m(t)$ denote the total number of cooperators at time t, then $m(t) = m_1(t) + m_2(t)$. Let ρ_{m_1,m_2} denote the probability that, starting with $\mathbf{M}(0) = (m_1, m_2)$, the cooperation finally fixates. For brevity, let $P_{i,j,k,l} = P(\mathbf{M}(1)=(k,l)|\mathbf{M}(0)=(i,j))$ denote the conditional probability, where $i,k \in \{0,1\}$ and $j,l \in \{0,1,...,N-1\}$. The probability of an offspring of central cooperator adopting strategy D ($P_{1,m-1,0,m-1}$) can be derived from the two sources: (1) the

offspring compares its payoff to the payoff of a peripheral defector and adopts its strategy with probability $\dfrac{N-m}{N} \times \dfrac{1-\mu}{1+e^{+\beta(\Delta\pi_{CC,PD})}}$; (2) the offspring mutates into defector with probability $\dfrac{\mu}{2}$. Analogously, The probability of an offspring of a selected peripheral defector adopting strategy C ($P_{1,m-1,1,m}$) can be derived from the two sources: (1) the offspring compares its payoff to the payoff of a central cooperator and adopts its strategy with probability $\dfrac{N-m}{N} \times \dfrac{1-\mu}{1+e^{-\beta(\Delta\pi_{CC,PD})}}$; (2) the offspring mutates into cooperator with probability $\dfrac{N-m}{N} \times \dfrac{\mu}{2}$. It is easy to calculate the following transition probabilities:

$$P_{1,m-1,0,m-1} = \frac{N-m}{N} \times \frac{1-\mu}{1+e^{+\beta(\Delta\pi_{CC,PD})}} + \frac{\mu}{2} \qquad 1 \leq m \leq N \quad (5a)$$

$$P_{1,m-1,1,m} = \frac{N-m}{N} \times \left(\frac{1-\mu}{1+e^{-\beta(\Delta\pi_{CC,PD})}} + \frac{\mu}{2}\right) \qquad (5b)$$
$$1 \leq m \leq N-1$$

$$P_{1,m-1,1,m-1} =$$
$$1 - \frac{N-m}{N}\left(\frac{1-\mu}{1+e^{\beta(\Delta\pi_{CC,PD})}} + \frac{1-\mu}{1+e^{-\beta(\Delta\pi_{CC,PD})}} + \frac{\mu}{2}\right) - \frac{\mu}{2} \qquad (5c)$$
$$1 \leq m \leq N,$$

By a similar arguments, the transition probabilities $P_{0,m,1,m}$ and $P_{0,m,0,m-1}$ are given by the following

$$P_{0,m,1,m} = \frac{m}{N-1} \times \frac{1-\mu}{1+e^{\beta(\Delta\pi_{CD,PC})}} + \frac{\mu}{2} \qquad 0 \leq m \leq N-1, \quad (5d)$$

$$P_{0,m,0,m-1} = \frac{m}{N-1}\left(\frac{1-\mu}{1+e^{-\beta(\Delta\pi_{CD,PC})}} + \frac{\mu}{2}\right) \qquad 1 \leq m \leq N-1, \quad (5e)$$

$$P_{0,m,0,m} =$$
$$1 - \frac{m}{N-1}\left(\frac{1-\mu}{1+e^{+\beta(\Delta\pi_{CD,PC})}} + \frac{1-\mu}{1+e^{-\beta(\Delta\pi_{CD,PC})}} + \frac{\mu}{2}\right) - \frac{\mu}{2} \qquad (5f)$$
$$0 \leq m \leq N-1,$$

By the total probability formula we have

$$\rho_{1,m-1} = P_{1,m-1,0,m-1}\rho_{0,m-1} + P_{1,m-1,1,m}\rho_{1,m}$$
$$+ P_{1,m-1,1,m-1}\rho_{1,m-1}, \qquad (6a)$$
$$1 \leq m \leq N-1,$$

$$\rho_{0,m} = P_{0,m,1,m}\rho_{1,m} + P_{0,m,0,m-1}\rho_{0,m-1}$$
$$+ P_{0,m,0,m}\rho_{0,m}. \qquad (6b)$$
$$1 \leq m \leq N-1,$$

Substituting equations (5a)-(5f) into (6a) and (6b) and simplifying, we get

$$\rho_{1,m-1} = a_m\rho_{0,m-1} + b_m\rho_{1,m}, \qquad 1 \le m \le N-1, \quad (6c)$$

$$\rho_{0,m} = c_m\rho_{1,m} + d_m\rho_{0,m-1}, \qquad 1 \le m \le N-1, \quad (6d)$$

where

$$a_m = \frac{2(N-m)(1-\mu) + N\mu(1+e^{\beta(\Delta\pi_{CC,PD})})}{(N-m)[2(1-\mu)e^{\beta(\Delta\pi_{CC,PD})} + \mu(1+e^{\beta(\Delta\pi_{CC,PD})})]},$$

$$b_m = \frac{[2(N-m)+m\mu](1+e^{\beta(\Delta\pi_{CC,PD})})}{(N-m)[2(1-\mu)e^{\beta(\Delta\pi_{CC,PD})} + \mu(1+e^{\beta(\Delta\pi_{CC,PD})})]},$$

$$c_m = \frac{2m(1-\mu) + \mu(N-1)[1+e^{\beta(\Delta\pi_{CD,PC})}]}{[2m(1-\mu) + \mu(N-1+m)][1+e^{\beta(\Delta\pi_{CD,PC})}]},$$

$$d_m = \frac{2m(1-\mu)e^{\beta(\Delta\pi_{CD,PC})} + m\mu[1+e^{\beta(\Delta\pi_{CD,PC})}]}{[2m(1-\mu) + \mu(N-1+m)][1+e^{\beta(\Delta\pi_{CD,PC})}]}.$$

We transform equations (6c) and (6d) into linear equations:

$$\begin{bmatrix} \begin{array}{cccc|cccc} d_1 & 1 & 0.........0 & & 0 & c_1 & ....0 \\ 1 & d_2 & 1..........0 & & 0 & 0 & c_2 & 0...0 \\ \multicolumn{4}{c|}{......................} & \multicolumn{4}{c}{......................} \\ 0 & .... & d_{N-1} & 1 & 0 & 0 & ....c_{N-1} \\ \hline a_1 & 0 & ....0 & 0 & 1 & b_1 & ....0 \\ 0 & a_2 & 0...0 & 0 & 0 & 1 & b_2 & ....0 \\ \multicolumn{4}{c|}{......................} & \multicolumn{4}{c}{......................} \\ 0 & 0 & ...a_{N-1} & 0 & 0 & ....1 & b_{N-1} \end{array} \end{bmatrix} \begin{bmatrix} \rho_{0,1} \\ \rho_{0,2} \\ \vdots \\ \rho_{0,N-1} \\ \rho_{1,0} \\ \rho_{1,1} \\ \vdots \\ \rho_{1,N-2} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (7)$$

According to Cramer rule, we can obtain

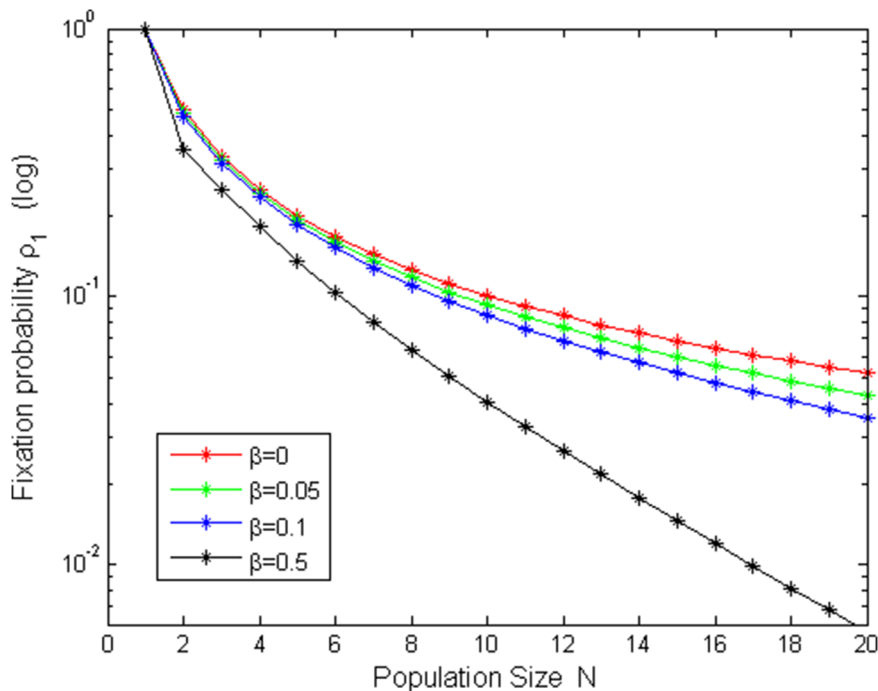$$\rho_{0,m} = \frac{D_m}{D}, \qquad m = 1,2,...N-1,$$

$$\rho_{1,m-1} = \frac{D_{m+N}}{D}, \qquad m = 1,2,..N-1,$$
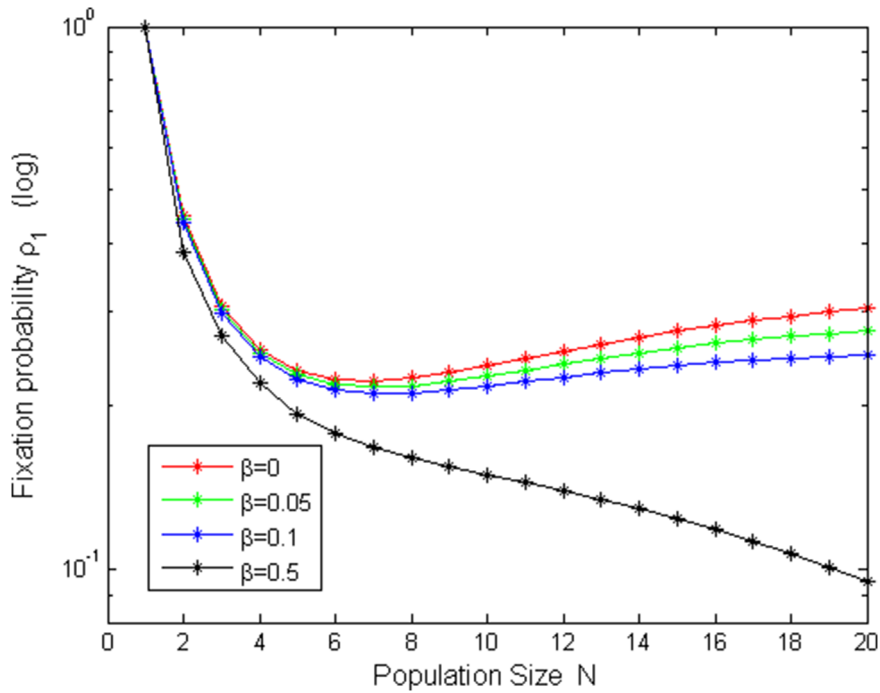
$$\rho_{0,0} = 0, \qquad \rho_{1,N-1} = 1.$$

where D is the determinant for the first term left hand side of linear equations (7), D_m and D_{m+N} are the determinants by replacing the mth-column, $(m+N)$th-column in D by the term on the right hand side of liner equations (7).

$$\rho_m = \frac{m}{N}\rho_{1,m-1} + (1-\frac{m}{N})\rho_{0,m} = \frac{mD_{m+N} + (N-m)D_m}{DN} \quad (8)$$

Figure 5. ρ_1 vs N with $\mu = 1/4000$, b = 1.2 and different β values.
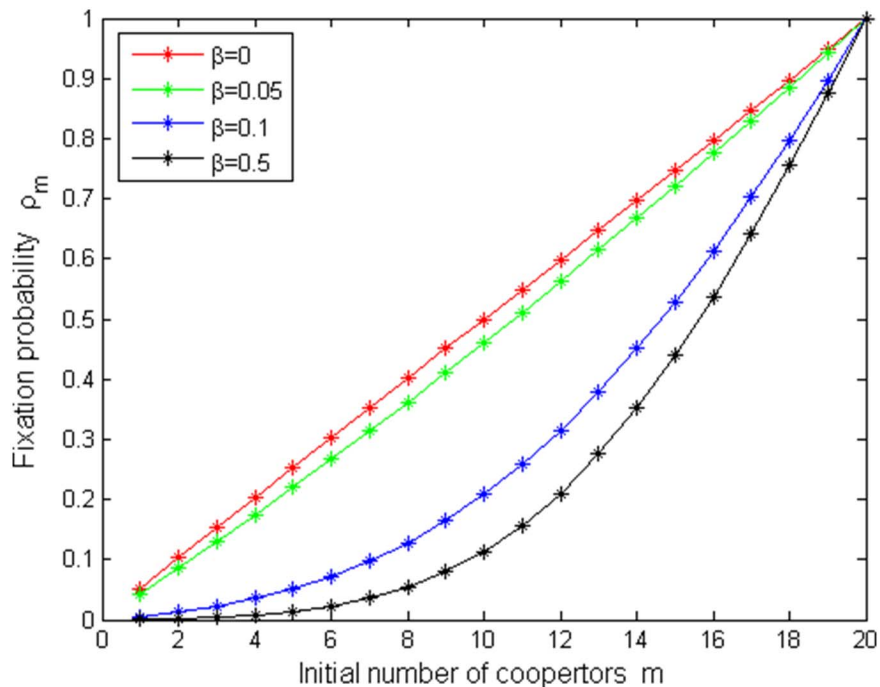doi:10.1371/journal.pone.0107417.g005

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Figure 6. ρ_1 vs N with $\mu = 0.95$, b = 1.2 and different β values.
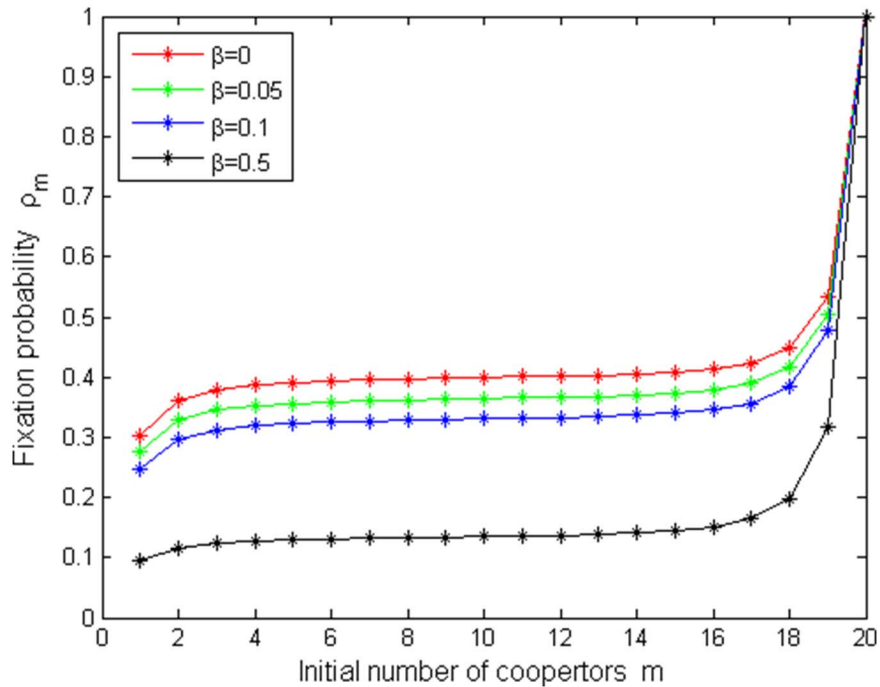doi:10.1371/journal.pone.0107417.g006

Numerical Examples

When $N \geq 3$, it is more difficult to investigate the properties of $\rho_1(\mu, \beta, b)$ (see appendix S1). In this section, through numerical examples we investigate how the parameters affect the evolution of the Prisoner's Dilemma game on a complete oriented star.

First, we will focus our attention on how fixation probability is affected by population size N *and* selection temperature β *and* initial number of cooperators m in given population size (let $N = 20$) *under the low mutation rates* ($\mu << N^{-2}$) *and high mutation rates* ($\mu \rightarrow 1$) ([29], [30], [31], [32], [33]). The simulations are as follows:


Figure 7. ρ_m vs N with $\mu = 1/4000$, b = 1.2 and different β values.
doi:10.1371/journal.pone.0107417.g007

Figure 8. ρ_m vs N with $\mu = 0.95$, b = 1.2 and different β values.
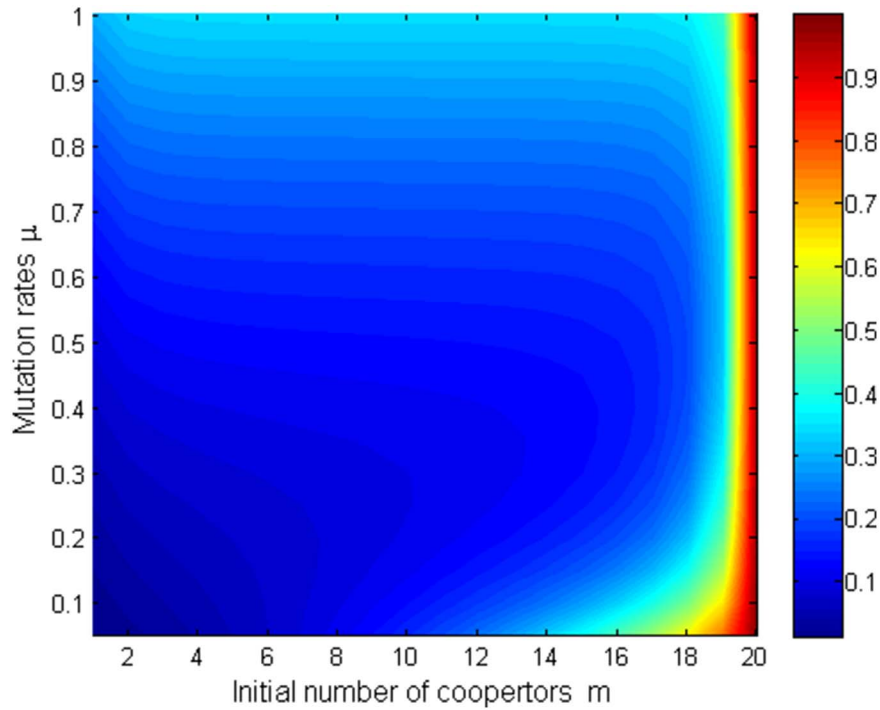doi:10.1371/journal.pone.0107417.g008

Figure 5–8 shows fixation probabilities in the pairwise comparison process under *the low mutation rates* and *high mutation rates* and different selection temperatures β. Figure 5–6 describes the first-order fixation probability ρ_1 as a function of population size N under the different selection temperatures β. Obviously, in Figure 6, there is a slight turning of ρ_1 with the increment of N under small β but circumstances turn out to be different in Figure 5, in which ρ_1 is decreasing with N, which accords with a fact: the larger the population size, the more difficultly to taken over it for a single cooperator with given strategy. Figure 7–8 depicts mth-order fixation probability $\rho_m(1 < m \leq 20)$ as a function of initial number of cooperators m under the different selection
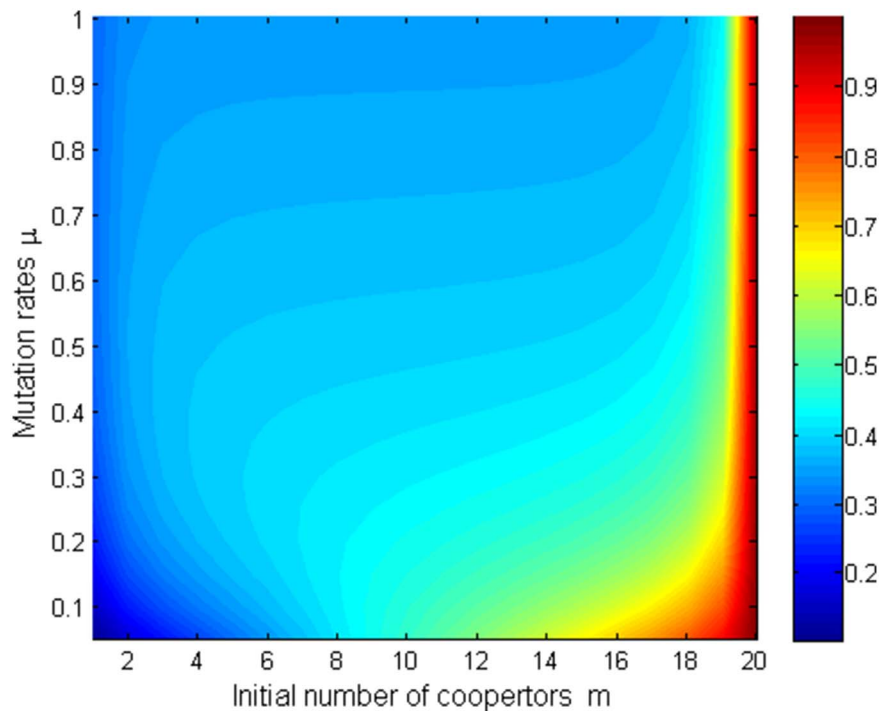
Figure 9. μ, m vs ρ_m with $\beta = 0.01$, b = 1.2.
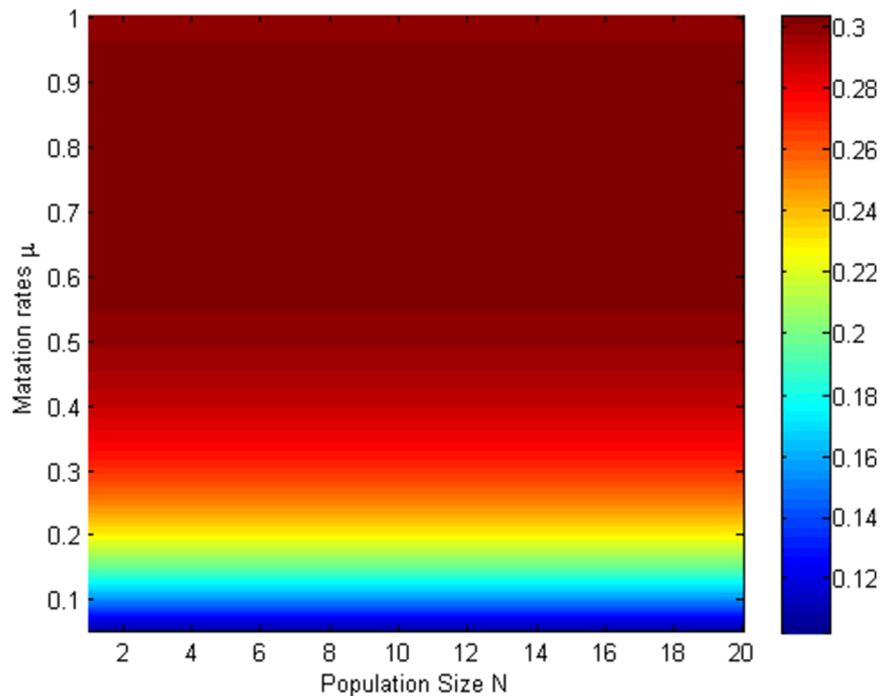doi:10.1371/journal.pone.0107417.g009

Figure 10. μ, m vs ρ$_m$ with β = 0.5, b = 1.2.
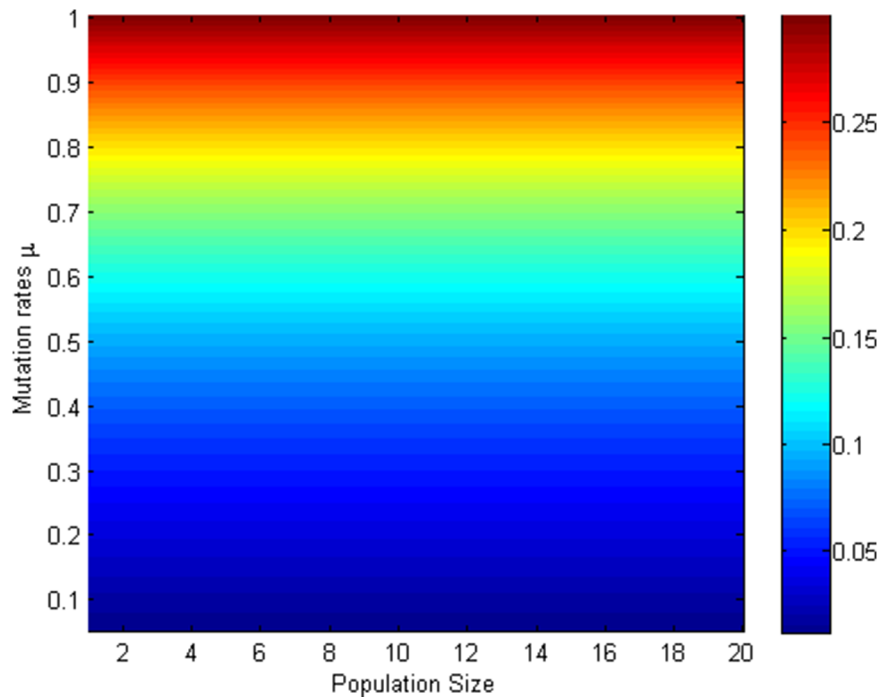doi:10.1371/journal.pone.0107417.g010

temperatures β. $\beta = 0$ (red circle) means neutral selection and ρ_m is given by the fraction of cooperators in Figure 7, which is similar as done in the evolutionary game in well-mixed populations ([26]). However, that is not the case in Figure 8 because of high mutation rates. One can see that with the increment of m, ρ_m increases in Figure 7, but in Figure 8, only for a high initial number of

cooperators, they have reasonable changes. $\rho_m (1 \leq m \leq 20)$ decreases with the increment of β if fix N in Figure 7–8.

Since Figure 5–8 shows that mutation rates μ plays a key role on the fixation probabilities, to further demonstrate the effects of mutation rates μ, next we will investigate the relationship of μ, and (i) mth-order probability $\rho_m (1 < m \leq 20)$ and initial number of

Figure 11. μ, m vs ρ$_1$ with β = 0.01, b = 1.2.
doi:10.1371/journal.pone.0107417.g011

Figure 12. μ, m vs ρ₁ with β = 0.5, b = 1.2.
doi:10.1371/journal.pone.0107417.g012

cooperators in Figure 9 ($\beta = 0.01$, $b = 1.2$) and Figure10 ($\beta = 0.5$, $b = 1.2$) (ii) the first-order fixation probability ρ_1 and population size in Figure 11 ($\beta = 0.01$, $b = 1.2$) and Figure 12 ($\beta = 0.5$, $b = 1.2$) under the different selection temperatures β. In Figure 9–10, one can see roughly that ρ_m can be mainly divided into parts obviously and the corresponding parameters space are given following the order of from little to great. When m is relatively large, ρ_m has a remarkable increment. In particular, In Figure 10, but only for a high initial number of cooperators and no matter under what values of μ, ρ_m can change obviously and it is not without that, ρ_m can maintain a more distinct increment. A clear description in which ρ_1 increases with the increment of μ is displayed in Figure 11–12, but the degree of increment is smaller in Figure 12 than that in Figure 11.

Concluding Remarks

In this paper, we have chosen a complete oriented star (COS) to study how the mutation rates and selection temperature and population size affects the prisoner's dilemma game based on pairwise comparison in finite populations. A method has been derived to calculate the fixation probabilities. Then we observe and analyze effects of selection temperature and mutation rates and population size on fixation probability. We will also attempting to modify the model under study to adapt the evolution of the trustworthiness of large-scale distributed systems. It is also worth extending this work to, say, the cooperation on a pair of graphs, or on hyper graph ([34]).

Supporting Information

Appendix S1 The expression of first-order fixation probability for S_3.
(DOCX)

Author Contributions

Conceived and designed the experiments: JGR. Performed the experiments: JGR. Analyzed the data: JGR. Contributed reagents/materials/analysis tools: YHX. Contributed to the writing of the manuscript: JGR YHX.

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