Fractional optimal control of compartmental SIR model of COVID-19: Showing the impact of effective vaccination

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Abstract: In this work a compartmental SIR model has been proposed for describing the dynamics of COVID-19 with Caputo's fractional derivative(FD). SIR compartmental model has been used here with fractional differential equations(FDEs). The mathematical model of the pandemic consists of three compartments namely susceptible, infected and recovered individuals. The dynamics of the pandemic COVID-19 with FDEs for showing the effect of memory as most of the cell biological systems can be described accurately by FDEs Time dependent control(Effective vaccination) has been applied model to formulated fractional optimal control problem(FOCP) to reduce the viral load. Pontryagin's Maximum Principle(PMP) has been used to formulate FOCP. An effective vaccination is very helpful for controlling the pandemic, which is observed through the numerical simulation via Grunwald-Letnikov(G-L) approximation. All numerical simulation work has been carried in MATLAB platform.

Keywords: COVID-19; Mathematical Modelling; Compartmental model, FDE, FOCP; Caputo's FD, G-L approximation

1. INTRODUCTION

Mathematical model of epidemics can provide various aspects for understanding the transmission dynamics of COVID-19 and it needs suggestions for effective strategies of control. The insight of transmission dynamics of an endemic can be easily formulated by mathematical modelling, back to the year 1766 when a work has been published by Daniel Bernoulli where the authors described the effects of variolation of smallpox on the expectancy of life [Dietz, K. and Heesterbeek, J.A.P., 2000]. Other mathematical models in epidemiology have been introduced in the year 1927 when Kermack and McKendrick published papers which described the transmission dynamics of epidemics with differential equations [Kermack, W.O. and McKendrick, A.G., 1991]. Since it has been observed that the mathematical models have been used for the purpose of public health, lot of researchers with huge number of studies as well as publications have been made on the modelling and analysis of epidemic models. However, the large number of studies were restricted to ordinary differential equations(ODEs). For instance, the global stability of compartmental SEIR and SEIAHR models with classical derivatives and various rates of saturated contact has been studied in [Zhang, J. and Ma, Z., 2003, Li, X.Z. and Zhou, L.L., 2009]. A compartmental SIRS model with delay and classical derivative, which involves saturation incidence and temporary immunity have been investigated in [Xu, R., Ma, Z. and Wang, Z., 2010]. An stochastic epidemic model with integer order derivative has been used for predicting the spread of coronavirus disease and it is shown in [Sene, N., 2020]. Borah et al. in their work [Borah, M.J. et al.,2020] used a mathematical model for

studying the correlation between the weather conditions and COVID-19 in India. In recent years FDEs are extensively used to model different phenomena in different fields of science and technology including epidemiology [Dietz, K. and Heesterbeek, J.A.P., 2000.]. Mathematical modelling of epidemics found in the literature depicts that the nonlinear dynamical equations are capable of give very crucial insight into the transmission dynamics of the disease.

The recent outbreak of COVID-19 around the globe has attracted huge interest in the field of mathematical modelling of this contagious disease, it has been observed that by constructing nonlinear compartmental data driven models for understanding the dynamics of transmission of the epidemics [Asamoah, J.K.K. et al.,2020, Asamoah, J.K.K.et al.,2021, Postnikov, E.B., 2020]. A new compartmental deterministic model consists of eight compartments for COVID-19 which is able to capture the awareness programs and various control strategies for severe as well as mild cases of infections in Nigeria has been proposed by Musa et al. in [Musa, S.S. et al.,2021]. They showed the fitting of proposed model to the cumulative cases of the infection in Nigeria from the period 29th March to 12th June 2020. From the results it is clear that, if the campaigns of awareness are not implemented properly then there is a chance for the infection to go upward. Memon et al. in their work [Memon, Z. et al.,2021] showed the formulation and analysis of a compartmental SEQIJR model which can assess the role of quarantine as well as isolation as measures of control measures for the prevention of the spread of COVID-19 in case of Pakistan.

FDEs are proved to be very handy and strong tool to model the epidemics for studying the biological systems. This is due to

the effect of memory, most of the biological systems [Baleanu, D. et al.,2020, Podlubny, I., 1998]. Caputo's FD has been used for studying and analysing various type of infectious diseases including HIV/AIDS [Ding, Y. and Ye, H., 2009], Malaria [Atangana, A. and Qureshi, S., 2020], Zika [Rakkiyappan, R.,2019].Javidi et al. in their work [Javidi, M. and Ahmad, B., 2014] extended the cholera model, which was formulated by the researcher in [Codeço, C.T., 2001] using Caputo's FD. The author in the work [Vargas-De-León, C., 2015] showed the use of Lyapunov functions which are known as Volterra–type and they investigated asymptotic stability of various compartmental models (SIS,SIR, SIRS) and Ross vector-borne diseases in the sense of Caputo. A compartmental model structure (normal weight-overweight-obese) has been constructed for exploring the dynamics of obesity in a variable population by Caputo's FD[Demirci, E., 2017]. The basic compartmental SEIR using Caputo derivative where the population dynamics is variable and has been studied in [Özalp, N. and Demirci, E., 2011]. The authors demonstrated an analysis of qualitative stability for novel as well as realistic deterministic model. In a recent model of the infection caused by Zika virus the researchers in [Rezapour, S. et al.,2020] used a nonlinear FDE with Caputo's FD of which the total population of humans as well as mosquitoes have been grouped into two classes of compartment (susceptible people, infected people; susceptible mosquitoes, infected mosquitoes). In [Qureshi, S., 2020], the researchers developed a compartmental SEIR model by both classical and Caputo's FD for describing the dynamics of transmission of Rubella in Pakistan. Because of the strong nature of FDEs and their derivative operators has been used for the construction of realistic as well as mathematical representations of the problems in the real-world in the field of science, finance, and engineering [Carvalho, A.R. et al.,2018], some current studies have been considered the mathematical modelling of COVID-19 by various derivative operators. A deterministic model of COVID-19 with Caputo's FD has been developed and studied in [Baba, I.A. and Nasidi, B.A., 2021]. The authors used the Banach contraction mapping for establishing the existence and uniqueness of the solution of the model. In [Bahloul, M.A. et al.,2020], the authors proposed a compartmental SEIQRDP fractional deterministic model with Caputo's FD for examining the epidemic caused by novel coronavirus. A compartmental SEIPAHRF model with Caputo's FD has been proposed for analysing the transmission dynamics of COVID-19 in Wuhan [Ahmad, S. et al.,2020]. They solved the proposed model numerically and made a comparison of it with the initial reported data of 67 days on confirmed cases of infection and death cases in Wuhan. In other current study, in [Ahmed, I. et al.,2020] authors proposed a nonlinear COVID-19 model with Caputo's FD for exploring the importance of the dynamics of lockdown for controlling the spread of the pandemic. In [Owusu-Mensah, I. et al.,2020], authors studied a novel COVID-19 model using Caputo's FD. The powerful and generalized form of Adams Bashforth–Moulton numerical scheme has been applied for solving the problem numerically. In [Tuan, N.H.,2020] authors used an extended version of the traditional compartmental SEIR model for constructing a data-driven modelling of COVID-19 with Caputo's FD. The applications of optimal control have been found in different

fields of science and engineering[Kostylenko, O. et al.,2018, Lemos-Paião et al.,2020]. Epidemic models are extensively used for studying the control strategy like vaccination [Brandeau, M.L.,2003]. Here in this work, FOCP has been formulated for showing the effect of effective vaccination to fight against the pandemic.

Our motivation is based on the literature on the application of FDE in modelling nonlinear systems extensively used in real-world problems and epidemiology. Recent researches and studies found in literature can demonstrate that mathematical modelling of nonlinear systems with FDE provides more realistic outcomes as compared to the ODE based models, see, e.g., [Qureshi, S., 2020, Sardar, T. et al.,2015]. The application of Caputo's FD has been shown and the extension of an existing COVID-19 model [Mwalili, S. et al.,2020] which has been characterized by the classical derivative. The aim of the current paper is two parts, first, the mathematical and epidemiological well-possedness of the model based pn classical derivative proposed in [Mwalili, S.et al.,2020] has been established and an approximate analytical technique has been employed for obtaining the long-term dynamics of the disease. Second, the existing epidemic model has been extended and modified by Caputo's FD which is demonstrated in the literature to be one of the useful derivative operators for describing the effects of memory effectively. Our study has been motivated by the recent work found in [Erturk, V.S. and Kumar, P., 2020].

This work has been arranged as follows: In 'Preliminaries', some definitions of FD have been discussed. In section 3, proposed mathematical model of novel coronavirus disease in terms FD has been discussed. In section 4, FOCP has been formulated. In section 5, we discussed about the simulation and results of the problem. In section 6, the conclusion of the work has been included.

2. PRELIMINARIES

Some preliminary definitions have been discussed here. Definitions of FD are not unique, there are various definitions of FD but Riemann-Liouville(R-L) and Caputo's FD have been extensively used in mathematical modelling.

Gamma function of α, where $\alpha > 0$ is defined as

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt \tag{1}$$

Left-sided R-L FD is written as

$$^{RL}_a D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds \tag{2}$$

Left-sided Caputo's FD is written as

$$^C_a D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds \tag{3}$$

where the order of the FD has been denoted by α and the condition $n-1 < \alpha < n$ follows, where n is an integer, Γ is called Euler's gamma function and $a > 0$ is constant. Throughout this work Caputo's FD has been used where $^C_a D_t^\alpha$

signifies left-sided Caputo's FD operator.

3. MATHEMATICAL MODEL OF COVID-19 USING FD

For showing the dynamics of COVID-19 a compartmental model has been proposed. It is seen that the susceptible people are infected by carriers of COVID-19. The total population at time t is denoted as $N(t)$ where, $N(t) = S(t) + I(t) + R(t)$ the total population is not constant. The susceptible people are recruited at a rate of λ, the susceptible individuals are transferred to exposed at a rate of β, d is known as rate of activation of recovered individuals, the mortality rate of the susceptible individuals is denoted by μ_1, the killing rate of the infected individuals by the vaccine has been denoted by ρ, the mortality rate of the infected individuals have been denoted by μ_2, the proliferation of the recovered individuals has been denoted by $dSIR$, the mortality rate of the recovered individuals has been denoted by μ_3. After following the assumptions, COVID-19 transmission dynamics has been presented by the following FDEs as

$$\begin{aligned} {}_0^C D_t^\alpha S &= \lambda - \beta SI - \mu_1 S \\ {}_0^C D_t^\alpha I &= \beta SI - \rho IR - \mu_2 I \\ {}_0^C D_t^\alpha R &= dSIR - \mu_3 R \end{aligned} \tag{4}$$

The initial conditions are given as $S(0) = S_0$, $I(0) = I_0$ and $R(0) = R_0$.

Values of the parameters are estimated and they are given in the table below

Table 1. Estimated Parameters

Parameters	Values
λ	9
β	0.0029
μ_1	0.15
ρ	2
μ_2	0.24
d	0.2
μ_3	0.14
w	0.011

4. FORMULATION OF FOCP

Here FOCP has been formulated, control strategy $u(t)$, and it has been included in the mathematical model of COVID-19. The control strategy $u(t)$ has been designed for showing the effective vaccination to avoid the infection. After including the time dependent control, the system of equations are become

$$\begin{aligned} {}_0^C D_t^\alpha S &= \lambda - (1-u)\beta SI - \mu_1 S \\ {}_0^C D_t^\alpha I &= \beta SI(1-u) - \rho IR - \mu_2 I \\ {}_0^C D_t^\alpha R &= dSIR - \mu_3 R \end{aligned} \tag{5}$$

The initial conditions are given as $S(0) = S_0$, $I(0) = I_0$ and $R(0) = R_0$.

The objective of including the control in COVID-19 model is to find out the strategy of optimal control, which is required for reducing the spread of COVID-19. The optimal control has been obtained by minimizing the objective functional, which is subject to the system (5). The objective functional is

$$J(u) = \int_0^{t_f} \left[S + R - \frac{1}{2} w u^2 \right] dt \tag{6}$$

It needs to be minimized, in this objective functional t_f signifies the final time, w is the weight cost of control strategy.

4.1 Hamiltonian and equations of optimality

Here the necessary condition of optimal control has been obtained using PMP[Pontryagin, L.S. et al.,2018].

The Hamiltonian function for the model in (5) has been obtained as

$$\begin{aligned} H(S,I,R,u) &= S + R - \frac{1}{2} w u^2 + \lambda_1 \left[\lambda - (1-u)\beta SI - \mu_1 S \right] \\ &+ \lambda_2 \left[\beta SI(1-u) - \rho IR - \mu_2 S \right] + \lambda_3 \left[dSIR - \mu_3 R \right] \end{aligned}$$

$$\tag{7}$$

where $\lambda_1, \lambda_2, \lambda_3$ are co-state variables with S, I, R to be determined using PMP [Pontryagin, L.S. et al.,2018].

$$\begin{aligned} {}_t^C D_{t_f}^\alpha \lambda_1 &= 1 - \mu_1 \lambda_1 I(1-u)(\beta\lambda_1 - \beta\lambda_2) + dIR\lambda_3 \\ {}_t^C D_{t_f}^\alpha \lambda_2 &= -S(1-u)(\beta\lambda_1 - \beta\lambda_2) - b\lambda_2 + \rho R\lambda_2 + dSR\lambda_3 \\ {}_t^C D_{t_f}^\alpha \lambda_3 &= dSIR - \mu_3 R \end{aligned} \tag{8}$$

With the transversality condition $\lambda_i\left(t_f\right)=0, i=1,...,3.$

The optimal control set has been characterized

$$u^* = \frac{SI\left(\beta\lambda_1 - \beta\lambda_2\right)}{w}$$

Proof The co-state equations and the transversality conditions are standard results, which are obtained from PMP[Pontryagin, L.S. et al.,2018]. The co-state equations have been obtained as

$$_t^C D_{t_f}^\alpha \lambda_1 = -\frac{\partial H}{\partial S} = 1 - \mu_1\lambda_1 I\left(1-u\right)\left(\beta\lambda_1 - \beta\lambda_2\right) + dIR\lambda_3$$

$$_t^C D_{t_f}^\alpha \lambda_2 = -\frac{\partial H}{\partial I} = -S\left(1-u\right)\left(\beta\lambda_1 - \beta\lambda_2\right) - b\lambda_2 + \rho R\lambda_2 + dSR\lambda_3$$

$$_t^C D_{t_f}^\alpha \lambda_3 = -\frac{\partial H}{\partial R} = dSIR - \mu_3 R \qquad (9)$$

With the transversality condition $\lambda_i\left(t_f\right)=0, i=1,...,3.$ By

PMP the equation of optimality $\dfrac{\partial H}{\partial u} = 0$ From this we get

$$wu^* - SI\left(\beta\lambda_1 - \beta\lambda_2\right) = 0$$

$$u^* = \frac{SI\left(\beta\lambda_1 - \beta\lambda_2\right)}{w} \qquad (10)$$

After considering the boundedness of optimal control we get

$$u^* = \max\left\{\min\left(\frac{SI\left(\beta\lambda_1 - \beta\lambda_2\right)}{w},1\right),0\right\} \qquad (11)$$

5. SIMULATION AND RESULTS

Here the numerical approximation of the FOCP is demonstrated. There are various numerical as well as analytical methods for approximating the FDEs for finding the solution of it. FOCP is solved such a way that the state equations have been solved in forward in time and the co-state equations have been solved backward in time simultaneously.

Here in this work for approximating and solving the nonlinear FDEs. G-L approximation is used [Noupoue, Y.Y.Y. et al.,2019].

The formula for numerically approximating the αth derivative at the point kh in terms of G-L approximation has been expressed when the FDE is given by

$\left(_{GL}D_{a^+}^\alpha g\right)(t) = f(g(t),t)$ along with the initial condition

$$_{(k-\frac{I_m}{h})GL}D_{t_k}^\alpha g(t) \approx \frac{1}{h^\alpha}\sum_{j=0}^{k}(-1)^j\binom{\alpha}{j}g(t_{k-j}) \qquad (12)$$

Here in this equation $^I{}_m$ signifies the length of the memory and $t_k = kh, (k=1,2,...)$, h signifies the step size and the binomial coefficients are signified by $(-1)^j\binom{\alpha}{j}$.

Generally, the binomial coefficients have been calculated as

$$p_0^\alpha = 1,$$
$$p_j^\alpha = \left(1 - \frac{1+\alpha}{j}\right)p_{j-1}^\alpha \qquad (13)$$

So the nonlinear FDE with initial condition has been approximated using G-L approximation has been expressed as

$$g(t_k) = f(g(t_k),t_k)h^\alpha - \sum_{j=1}^{k}p_j^\alpha f(t_{k-j}) \qquad (14)$$

The simulation is carried out for the values of α lies between 0.7 to 1.

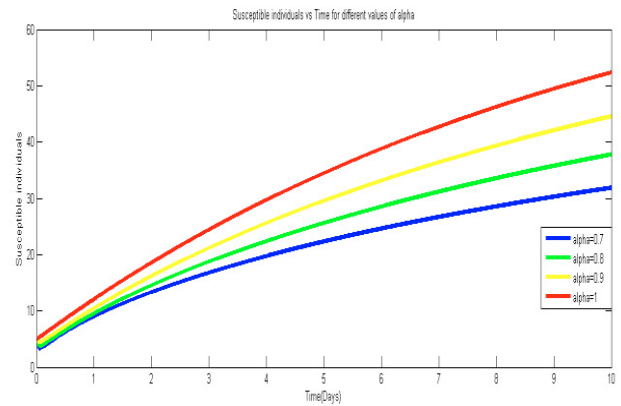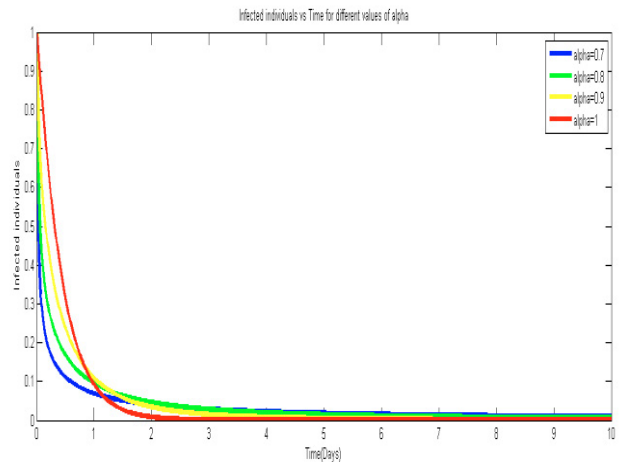Figure 1.Susceptible individuals vs Time for various values of alpha

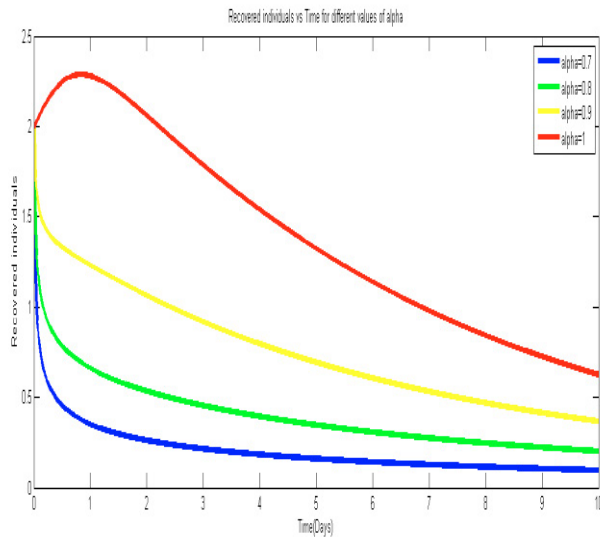Figure 2.Infected individuals vs Time for various values of alpha

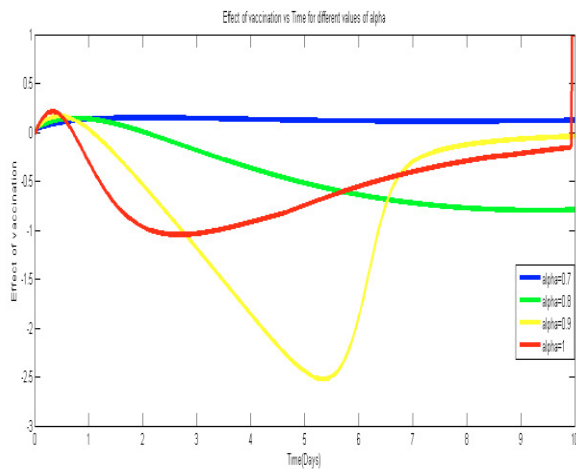Figure 3. Recovered individuals vs Time for various values of alpha

Figure 4. Effect of vaccination vs Time for various values of alpha

It has been observed from the simulation results that FOCP is formulated for keeping the leve of both susceptible individuals and recovered individuals high while minimising the vaccination cost. The optimal control has been described in terms of susceptible and infected individuals and the corresponding co-state variables. From the result it is clear that mínimum vaccination is required when the infection is controlled to low level as shown in Figure 2. In this situation the side effect of the vaccination is also reduced. It has been observed from the Figure 3 that the recovered individuals are always kept at positive level. It has not been eradicated. When the infection is found to be low from the Figure 2, then the recovered individuals are not required to be kept at high level so it kept very low. The initial reduction in the control(effect of vaccination) has been occured in Figure 4, at the same time when the population density of recovered individuals is high. This shows that during the periods of effective vaccination less vaccine is required for combating the pandemic. The control strategy helps to enhance an infected person's natural immune response. The value of α is varied between 0.7 to 1, in Figure 1 the density of susceptoible individuals have been

shown. From the Figure 4 it is said that máximum leve lof vaccination is applied for various values of α. From Figure 1, it can be said that susceptible population returns to its target value and from Figure 2, it is said that the viral load reduces after applying effective vaccination which leads to improve the immune response, i.e. the increase in the number of recovered individuals, which is shown in Figure 3.

6. CONCLUSIONS

In this work FOCP for COVID-19 has been done formulated. The mathematical model of COVID-19 has been proposed. Biological systems are memory based system, which can be described accurately by using FDEs. It helps to describe the system more accurately. Here the transmission dynamics of the COVID-19 pandemic is described by FDEs and the effect of effective vaccination is shown by formulating FOCP. Fractional model of COVID-19 has been constructed where Caputo's FD has been taken. Control in terms of effective vaccination has been included in the fractional modelling of the pandemic. The condition of optimality has been obtained by PMP. From the numerical simulation it is evident that effective vaccination is good for controlling the spread of the pandemic. It can also be said that FD has been considered to be the best to deal with the complex models in the real world. From the simulation corresponding to different orders indicates that as the order of the FD reduces from 1 then the spread of the disease will be slower. From the simulation it is clear that the designed control strategy is operative while reducing the number of cases in various compartments of the mathematical model. Moreover, from the simulation the optimal profile shows when order of FD is reducing from 1 then the need for the application of control strategy has been prominent at the maximum possible level and need to maintain it for the maximum period of the pandemic. It can be said that if COVID-19 can be controlled to very low levels, then the dosage of the vaccine can be reduced and the side effects can also be reduced, which can be the success of FOCP to improve the quality of the treatment of COVID-19. The future scope of the study includes the use of multiple control i.e. maintain social distance and maintain good hygiene by using face mask and sanitizer to combat against this dangerous pandemic.

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