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A new alpha logarithmic-generated class to model precipitation data with theory and inference

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A B S T R A C T

Precipitation, or rainfall, is a central feature of the weather cycle and plays a vital role in sustaining life on Earth. However, existing models such as the Poisson, exponential, normal, log-normal, generalized Pareto, gamma, generalized extreme value, lognormal, beta, Gumbel, Weibull, and Pearson type III distributions used for predicting precipitation are often inadequate for precisely representing the complex pattern of rainfall. This study proposes a novel and innovative approach to address these limitations through the new alpha logarithmic-generated (NAL-G) class of distributions. The study authors thoroughly examine the NAL-G class and a unique model, the NAL-Exponential (NAL-Exp) distribution, with a focus on analyzing mathematical properties such as moments, quantile function, entropy, order statistics, and more. Six recognized classical estimation methods are employed, and extensive simulations are conducted to determine the best one. The NAL-Exp distribution demonstrates its high adaptability and value through its superior performance in modeling four distinct rainfall data sets. The results show that the NAL-Exp distribution outperforms other commonly used distribution models, highlighting its potential as a valuable tool in hydrological modeling and analysis. The increased versatility and flexibility of this new approach hold great potential for enhancing the accuracy and reliability of future rainfall predictions.

1. Introduction

The analysis of precipitation (also known as rainfall) data has a long history and has been the focus of a significant amount of research and development throughout the course of its existence. It is a significant and active field of research and development that is being pursued with the intention of producing an appropriate model. The analysis of rainfall data may be carried out using a wide variety of approaches and technologies, such as statistical methods, mathematical models, and techniques including machine learning. Studying numerous elements of rainfall is possible with the use of these methodologies, such as its temporal and geographical distribution, the frequency and severity of severe occurrences, and the link between rainfall and other factors such as temperature, atmospheric pressure, and land use. Modeling the frequency and pattern of rainfall events over a certain amount of time may be used to establish a relationship between precipitation data and a probability distribution, whether the distribution is discrete or continuous. Rainfall analysis frequently makes use of the following probability distributions: the Poisson distribution (PD), the exponential distribution (ED), the normal distribution (ND), the log-normal distribution (LND), the generalized Pareto distribution (GPD), the gamma distribution (GD), the Weibull distribution (WD), the Gumbel distribution (GD), the Pearson type III distribution (PIIID), the

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generalized extreme value distribution (GEVD), the log-Pearson type III distribution (LPIIID), and the findings of analyzing rainfall data have essential uses in a wide variety of sectors, such as meteorology, hydrology, agriculture, water resource management, and the evaluation of natural hazards.

In addition, severe rainfall events are uncommon and complex to forecast, which makes it tough to precisely estimate the possible implications they may have. The type and properties of the rainfall data that are being studied should guide the selection of the most suitable probability distribution to use in the analysis. The advantages of employing multiple probability distributions to describe daily rainfall may be better understood by looking at changes in precipitation patterns that occur in different locations of the globe, which can give a helpful insight into those benefits. Extensive research on this subject has been conducted by a community of scientists from all over the world in locations such as Bangladesh (Alam et al. [1]), Ethiopia (Liyew and Melese [2]), Costa Rica (Altman et al. [3]), Nagpur (India) (Mohanty et al. [4]), China (Chen et al. [5]), Pakistan (Yonus et al. [6]), Europe (Wouters et al. [7]), Colombia (Correa et al. [8]), Saudi Arabia (Hasanean and Almazroui [9]), Brazil (Beskow et al. [10]), Pakistan (Amin et al. [11]), Egypt (Gado et al. [12]), and many others.

Furthermore, the authors of the study draw the readers' attention to several other prominent works in the area of generated classes (G-classes), including those by Marshall and Olkin [13], Shaw and Buckley [14], Eugene et al. [15], Pourreza et al. [16], Cordeiro and de Castro [17], Alzaatreh et al. [18], Bourguignon et al. [19], Cordeiro et al. [20], Al-Shomrani et al. [21], Al Mutairi et al. [22], Al-Babtain et al. [23], Alghamdi and Abd El-Raouf [24], and numerous others.

The aim of this study is to introduce the NAL-G class as a promising and novel approach, and to evaluate its basic and advanced features to demonstrate the superior flexibility and performance of the proposed NAL-Exp distribution in modeling lifetime data compared to several established models. It is important to note that, the study authors have chosen the exponential distribution, usually its cumulative distribution function (CDF) is denoted $(1 - e^{-\theta x})$, as a reference model to investigate the NAL-G class, which is a suitable approach for modeling precipitation data. This choice is based on its well-established suitability for a broad range of applications, its simplicity, and its significant statistical properties, which allow for adequate modeling and prediction of rainfall events.

Moreover, the proposed NAL-Exp distribution offers an alternative to several established distributions in the literature, such as the logistic exponential, alpha power exponential, exponentiated exponential, half logistic exponential, Nadarajah-Haghighi exponential, exponential, log-normal, generalized extreme value, and Gumbel distributions. One of the primary motivations for the development of the NAL-G class is the ease of acquiring the reliability function and hazard rate function from its simple closed-form CDF, which can be used to analyze the behavior of precipitation data and make predictions about future rainfall events.

The paper is organized as follows: In Section 2, we introduce the NAL-G class and explore its significant structural properties. Section 3 delves into a sub-model of this class known as the NAL-Exp distribution. To assess the effectiveness of six classical estimators, we conduct Monte Carlo simulations in Sections 4 and 5. Additionally, in Section 6, we present four real-world applications of the sub-model. The novelty and contributions of our study are discussed in Section 7, while Section 8 summarizes the key findings and draws conclusions. Lastly, Section 9 outlines potential avenues for future research.

2. The NAL-G class of distributions

Definition 1. If $X \sim$NAL-G $(x^{'}; \alpha, \Delta)$ where $\alpha > 0, \alpha \neq 1$ and Δ are scale parameters and vector space, respectively, then the cdf of X, denoted by $(F|_{X;\alpha,\Delta})$ for the NAL-G class can be represented as follows:

$$F|_{X;\alpha,\Delta} = \frac{log\left[1 + \alpha - \alpha^{1-G(x;\Delta)}\right]}{log\alpha}, x \in \Re. \tag{1}$$

It's worth mentioning that the for any baseline model, denoted as $G(x; \Delta)$, the NAL-G class CDF satisfies the following conditions:

- $g(x; \Delta) = dG(x; \Delta)/dx$,
- The survival function $1 - G(x; \Delta)$,
- $\lim_{x\to 0} G(x; \Delta) = 0$ and $\lim_{x\to 1} G(x; \Delta) = 1$, and
- $F|_{X;\alpha,\Delta}$ is not defined for $\alpha = 1$.

These conditions provide sufficient evidence that the NAL-G class possesses a valid CDF.

Definition 2. If $X \sim$NAL-G $(x; \alpha, \Delta)$ where $\alpha > 0, \alpha \neq 1$ and Δ are scale parameters and vector space, respectively, then the pdf of X, denoted by $(f|_{X;\alpha,\Delta})$ for the NAL-G class can be represented as follows:

$$f|_{X;\alpha,\Delta} = \frac{g(x; \Delta)\alpha^{1-G(x;\Delta)}}{1 + \alpha - \alpha^{1-G(x;\Delta)}}, x \in \Re\ . \tag{2}$$

The study authors defined the NAL-G class as $X \sim$NAL-G $(x; \alpha, \Delta)$ where $\alpha > 0, \alpha \neq 1$ and Δ are scale parameters and vector space, respectively, hereafter.

2.1. Special representation

The exponentiated generated (exp-G) representation of a probability density function (PDF) as an infinite linear combinations using binomial expansion

$$(1-m)^l = \sum_{n=0}^{\infty} (-1)^n \binom{l}{n} m^n,$$

and McLaurin series expansion of α^u.

$$\alpha^u = \sum_{l=0}^{\infty} \left[\frac{(log\alpha)^l u^l}{l!} \right],$$

is given in (2.3) as

$$f|_{X;\alpha,\Delta} = \sum_{k,l=0}^{\infty} \frac{(-1)^l (1+k)^l (log\alpha)^l}{l!} \left(\frac{\alpha}{1+\alpha} \right)^{k+1} g(x;\Delta)[G(x;\Delta)]^l, \tag{3}$$

$$f|_{X;\alpha,\Delta} = \sum_{k,l=0}^{\infty} \psi_{k,l} [P(x;\Delta)]^l,$$

where $\psi_{k,l} = \frac{(-1)^l (1+k)^l (log\alpha)^l}{l!} \left(\frac{\alpha}{1+\alpha} \right)^{k+1}$ and $[P(x;\Delta)]^l = g(x;\Delta)[G(x;\Delta)]^l$ is the exp-G density with power parameter l.

Definition 3. If $X \sim$ NAL-G $(x;\alpha,\Delta)$ where $\alpha > 0, \alpha \neq 1$ and Δ are scale parameters and vector space, respectively, then the quantile function of X, denoted by $(QF|_{U;\alpha,\Delta} = F^{-1}|_{U;\alpha,\Delta})$ for the NAL-G class can be represented as follows:

$$QF|_{U;\alpha,\Delta} = G^{-1} \left[\frac{log\left[\alpha(1+\alpha-\alpha^u)^{-1}\right]}{log\alpha} \right], u \in (0,1). \tag{4}$$

Definition 4. If $X \sim$ NAL-G $(x;\alpha,\Delta)$ where $\alpha > 0, \alpha \neq 1$ and Δ are scale parameters and vector space, respectively, then the survival function of X, denoted by $(SF|_{X;\alpha,\Delta} = 1 - F|_{X;\alpha,\Delta})$ for the NAL-G class can be represented as follows:

$$SF|_{X;\alpha,\Delta} = \frac{log\alpha - log\left[1+\alpha-\alpha^{1-G(x;\Delta)}\right]}{log\alpha}, x \in \Re. \tag{5}$$

Definition 5. If $X \sim$ NAL-G $(x;\alpha,\Delta)$ where $\alpha > 0, \alpha \neq 1$ and Δ are scale parameters and vector space, respectively, then the hazard rate function of X, denoted by $(hrf|_{X;\alpha,\Delta} = f|_{X;\alpha,\Delta} / SF|_{X;\alpha,\Delta})$ for the NAL-G class can be represented as follows:

$$hrf|_{X;\alpha,\Delta} = \frac{g(x;\Delta)\alpha^{1-G(x;\Delta)} log\alpha}{\left[1+\alpha-\alpha^{1-G(x;\Delta)}\right]\left[log\alpha - log\left(1+\alpha-\alpha^{1-G(x;\Delta)}\right)\right]}, x \in \Re. \tag{6}$$

Definition 6. If $X \sim$ NAL-G $(x;\alpha,\Delta)$ where $\alpha > 0, \alpha \neq 1$ and Δ are scale parameters and vector space, respectively, then the r-th ordinary moment of X, defined as $\mu_r'|_{X;\alpha,\Delta} = \int_{-\infty}^{\infty} x^r f|_{X;\alpha,\Delta} dx$ for $r = 1, 2, 3, 4\ldots$, can be derived by using (3), which is expressed as follows:

$$E|_{X^r} = \mu_r'|_{X;\alpha,\Delta} = \sum_{k,l=0}^{\infty} \psi_{k,l} \int_{-\infty}^{\infty} x^r [P(x;\Delta)]^l dx, x \in \Re .$$

Corollary 1. The r-th incomplete moment of X can be calculated using (3) as follows:

$$E|_{X^r|X \le v} = \mu_r'|_{X,v;\alpha,\Delta} = \sum_{k,l=0}^{\infty} \psi_{k,l} \int_{-\infty}^{v} x^r [P(x;\Delta)]^l dx, x \in \Re.$$

Furthermore, by placing $r = 1$ in the previous expression yields the first incomplete moment of X, denoted as $\mu_1'|_{X,v;\alpha,\Delta}$. This moment is a valuable tool for calculating Bonferroni and Lorenz curves. It is worth noting that numerical computation of the integrals can be

Table 1
New sub-model's CDF* correspond to classical models CDF.

Model	CDF	CDF*	Δ	Support
Exponential	$1-e^{-\theta x}$	$\tau\log[1+\alpha-\alpha^{e^{-\theta x}}]$	$\alpha,\theta>0,\alpha\neq1$	$0<x<\infty$
Lomax	$1-(1+x/\theta)^{-\rho}$	$\tau\log[1+\alpha-\alpha^{(1+x/\theta)^{-\rho}}]$	$\alpha,\theta,\rho>0,\alpha\neq1$	$0<x<\infty$
Weibull	$1-e^{-\theta x^{\rho}}$	$\tau\log[1+\alpha-\alpha^{e^{-\theta x^{\rho}}}]$	$\alpha,\theta,\rho>0,\alpha\neq1$	$0<x<\infty$
Pareto	$1-(x_0/x)^{\theta}$	$\tau\log[1+\alpha-\alpha^{(x_0/x)^{\theta}}]$	$\alpha,\theta>0,\alpha\neq1$	$x_0\leq x<\infty$
Burr	$1-(1+x^{\theta})^{-\rho}$	$\tau\log[1+\alpha-\alpha^{(1+x^{\theta})^{-\rho}}]$	$\alpha,\theta,\rho>0,\alpha\neq1$	$0<x<\infty$
Rayleigh	$1-e^{x^2/\theta^2}$	$\tau\log[1+\alpha-\alpha^{e^{x^2/\theta^2}}]$	$\alpha,\theta>0,\alpha\neq1$	$0<x<\infty$
Gompertz	$1-e^{-\theta(e^{\rho x}-1)}$	$\tau\log[1+\alpha-\alpha^{e^{-\theta(e^{\rho x}-1)}}]$	$\alpha,\theta,\rho>0,\alpha\neq1$	$0<x<\infty$
Kumaraswamy	$1-(1-x^{\theta})^{\rho}$	$\tau\log[1+\alpha-\alpha^{(1-x^{\theta})^{\rho}}]$	$\alpha,\theta,\rho>0,\alpha\neq1$	$0<x\leq1$
Uniform	x/x_M	$\tau\log[1+\alpha-\alpha^{(1-x/x_M)}]$	$\alpha>0,\alpha\neq1$	$0<x\leq x_M$
Power Function	$(x/x_M)^{\theta}$	$\tau\log[1+\alpha-\alpha^{(1-(x/x_M)^{\theta})}]$	$\alpha,\theta>0,\alpha\neq1$	$0<x\leq x_M$

employed for the baseline models.

In Table 1, the authors employ classical models with CDFs. Furthermore, they introduce new sub-model's distribution functions CDF* that include specifications for parameters defined as (Δ), and support to enhance the analysis.

where $\tau=1/\log\alpha$.

2.2. Asymptotics

Proposition 1. $\lim_{x\to-\infty}F|_{X;\alpha,\Delta}=0$ and $\lim_{x\to+\infty}F|_{X;\alpha,\Delta}=1$.

Proposition 2. $\lim_{x\to-\infty}f|_{X;\alpha,\Delta}=g(x;\Delta)\alpha$ and $\lim_{x\to+\infty}f|_{X;\alpha,\Delta}=g(x;\Delta)/\alpha$.

Proposition 3. $\lim_{x\to-\infty}SF|_{X;\alpha,\Delta}=1$ and $\lim_{x\to+\infty}SF|_{X;\alpha,\Delta}=0$.

Proposition 4. $\lim_{x\to-\infty}hrf|_{X;\alpha,\Delta}=g(x;\Delta)\alpha$ and $\lim_{x\to+\infty}hrf|_{X;\alpha,\Delta}=$ undefined.

2.3. Entropy

This section will cover the concept of entropy measures, which provide a means of quantifying the amount of disorder or uncertainty within a particular system. The standard notation used in literature to denote entropy is $H|_X$.

Theorem 1. If $X\sim$ NAL-G $(x;\alpha,\Delta)$ where $\alpha>0,\alpha\neq1$ and Δ are scale parameters and vector space, respectively, then the Rényi entropy of X, for the NAL-G class can be represented as follows:

$$H|_{X;\alpha,\Delta}=\frac{1}{1-\nabla}log\sum_{v,m=0}^{\infty}\psi_{v,m,\nabla}\int_{-\infty}^{+\infty}[g(x;\Delta)]^{\nabla}[G(x;\Delta)]^{m}dx,\nabla>0,\nabla\neq1.$$

Proof: The Rényi [25] entropy of X is defined as

$$H|_{X;\alpha,\Delta}=\frac{1}{1-\nabla}log\int_{-\infty}^{+\infty}\left[f|_{X;\alpha,\Delta}\right]^{\nabla}dx,\nabla>0,\nabla\neq1. \tag{7}$$

To simplify the expression (2) in terms of ∇, we can rewrite it as follows:

$$\left[f|_{X;\alpha,\Delta}\right]^{\nabla}=\left(\frac{\alpha}{1+\alpha}\right)^{\nabla}\alpha^{-\nabla G(x;\Delta)}[g(x;\Delta)]^{\nabla}\left[1-\frac{\alpha}{1+\alpha}\alpha^{-G(x;\Delta)}\right]^{-\nabla}. \tag{8}$$

After simplifying $\left[1-\frac{\alpha}{1+\alpha}\alpha^{-G(x;\Delta)}\right]^{-\nabla}$ using binomial expansion, we get

$$\left[1-\frac{\alpha}{1+\alpha}\alpha^{-G(x;\Delta)}\right]^{-\nabla}=\sum_{v=0}^{\infty}(-1)^{v}\binom{-\nabla}{v}\left(\frac{\alpha}{1+\alpha}\right)^{v}\alpha^{-vG(x;\Delta)},$$

and substitute it into (8) to obtain

$$\left[f|_{X;\alpha,\Delta}\right]^{\nabla}=\sum_{v=0}^{\infty}\binom{-\nabla}{v}\left(\frac{\alpha}{1+\alpha}\right)^{\nabla+v}(-1)^{v}[g(x;\Delta)]^{\nabla}\alpha^{-(\nabla+v)G(x;\Delta)}. \tag{9}$$

We can simplify $\alpha^{-(\nabla+v)G(x;\Delta)}$ further by using exponential expansion as

$$\alpha^{-(\nabla+v)G(x;\Delta)}=e^{-(\nabla+v)G(x;\Delta)log\alpha}=\sum_{m=0}^{\infty}\frac{(-1)^{m}(\nabla+v)^{m}(log\alpha)^{m}}{m!}[G(x;\Delta)]^{m},$$

and substitute it into (9) to obtain (10) as

$$\left[f|_{X;\alpha,\Delta}\right]^{\nabla} = \sum_{v,m=0}^{\infty}\binom{-\nabla}{v}\frac{(-1)^{m+v}(\nabla+v)^{m}(log\alpha)^{m}}{m!}\left(\frac{\alpha}{1+\alpha}\right)^{\nabla+v}[g(x;\Delta)]^{\nabla}[G(x;\Delta)]^{m}. \tag{10}$$

Hence, substituting (10) into (7) yields the final expression (11) for the Rényi entropy of X as

$$H|_{X;\alpha,\Delta} = \frac{1}{1-\nabla}log\sum_{v,m=0}^{\infty}\psi_{m,v,\nabla}\int_{-\infty}^{+\infty}[g(x;\Delta)]^{\nabla}[G(x;\Delta)]^{m}dx, \nabla>0, \nabla\neq 1, \tag{11}$$

where $\psi_{m,v,\nabla} = \binom{-\nabla}{v}\frac{(-1)^{m+v}(\nabla+v)^{m}(log\ \alpha)^{m}}{m!}\left(\frac{\alpha}{1+\alpha}\right)^{\nabla+v}$ ∎.

It is important to note that expression in (10) can be utilized to derive various well-known entropy measures. Among these, the expressions for the final forms of Tsallis [26] defined as:

$$H1|_{X;\alpha,\Delta} = [1/(1-\nabla)]log\int_{-\infty}^{+\infty}\left[f|_{X;\alpha,\Delta}\right]^{\nabla}dx - 1, \nabla>0, \nabla\neq 1, \tag{12}$$

and Havrda-Charvat [27] defined as:

$$H2|_{X;\alpha,\Delta} = \left[1/\left(2^{1-\nabla}-1\right)\right]log\int_{-\infty}^{+\infty}\left[f|_{X;\alpha,\Delta}\right]^{\nabla}dx - 1, \nabla>0, \nabla\neq 1, \tag{13}$$

entropies are presented in corollary 2 and corollary 3, respectively:

Corollary 2. The Tsallis entropy

$$H1|_{X;\alpha,\Delta} = \frac{1}{1-\nabla}\sum_{v,m=0}^{\infty}\psi_{v,m,\nabla}\int_{-\infty}^{+\infty}[g(x;\Delta)]^{\nabla}[G(x;\Delta)]^{m}dx - 1, \nabla>0, \nabla\neq 1. \tag{14}$$

Corollary 3. The Havrda-Charvat entropy

$$H2|_{X;\alpha,\Delta} = \frac{1}{2^{1-\nabla}-1}\sum_{v,m=0}^{\infty}\psi_{v,m,\nabla}\int_{-\infty}^{+\infty}[g(x;\Delta)]^{\nabla}[G(x;\Delta)]^{m}dx - 1, \nabla>0, \nabla\neq 1. \tag{15}$$

2.4. Order statistics (OS)

Let us consider random variables denoted by $X_1|_{\alpha,\Delta}, X_2|_{\alpha,\Delta}, X_3|_{\alpha,\Delta}, \ldots, X_n|_{\alpha,\Delta}$, which follow the NAL-G class of distributions. The PDF of the p-th OS say $f|_{X_{p:n};\alpha,\Delta}$, can be expressed as:

$$f|_{X_{p:n};\alpha,\Delta} = \tau(a,b)\sum_{l=0}^{n-p}(-1)^{l}\binom{n-p}{l}\left(F|_{X;\alpha,\Delta}\right)^{l+p-1}f|_{X;\alpha,\Delta}.$$

The analytical expression of p-th OS PDF is

$$f|_{X_{p:n};\alpha,\Delta} = \tau(a,b)\frac{g(x;\Delta)\alpha^{1-G(x;\Delta)}}{1+\alpha-\alpha^{1-G(x;\Delta)}}\sum_{l=0}^{n-p}(-1)^{l}\binom{n-p}{l}\left[\frac{log\left[1+\alpha-\alpha^{1-G(x;\Delta)}\right]}{log\alpha}\right]^{l+p-1},$$

where $\tau(a,b) = 1/B(p,n-p+1)$.

2.5. Inference

If $X_l \sim$NAL-G $(x_l;\alpha,\Delta)$ where $\alpha>0, \alpha\neq 1$ and $\omega=(\alpha,\Delta)^{T}$ are scale parameters and $(m\times 1)$ vector space, respectively, the log-likelihood function denoted as $L(\omega)$ for a random sample of size n for X can be expressed as follows:

$$L(\omega) = \sum_{l=1}^{n}log[g(x_l;\Delta)] + log\alpha\sum_{l=1}^{n}[1-G(x_l;\Delta)] - \sum_{l=1}^{n}log\left[1+\alpha-\alpha^{1-G(x_l;\Delta)}\right]. \tag{16}$$

In the R programming language, the $L(\omega)$ can be computed by solving the nonlinear equations derived by differentiating (16). The score function components denoted by $U|_{\omega} = [\partial L(\omega)/\partial\alpha, \partial L(\omega)/\partial\Delta]^{T}$ are expressed as follows:

$$\frac{\partial L(\omega)}{\partial \alpha} = \frac{1}{\alpha}\sum_{l=1}^{n}[1-G(x_l;\Delta)] - \sum_{l=1}^{n}\frac{1-\left[(1-G(x_l;\Delta))\alpha^{-G(x_l;\Delta)}g(x_l;\Delta)\right]}{\left[1+\alpha-\alpha^{1-G(x_l;\Delta)}\right]},$$

$$\frac{\partial L(\omega)}{\partial \Delta_m} = \sum_{l=1}^{n}\frac{g^{'}(x_l;\Delta)}{g(x_l;\Delta)} - G^{'}(x_l;\Delta)\sum_{l=1}^{n}\log\alpha - \sum_{l=1}^{n}\frac{\alpha^{1-G(x_l;\Delta)}\log\alpha G^{'}(x_l;\Delta)}{\left[1+\alpha-\alpha^{1-G(x_l;\Delta)}\right]},$$

where $g^{'}(x_l;\Delta) = \partial g(x_l;\Delta)/\partial\Delta_m$, $G^{'}(x_l;\Delta) = \partial G(x_l;\Delta)/\partial\Delta_m$, and the notation Δ_m represents the m-th component of the parameter vector Δ. The maximum likelihood estimate (MLE) of the parameter ω, denoted as $\widehat{\omega}$, is obtained through the solution of the nonlinear system $U|_{\omega} = 0$. Numerical methods using iterative algorithms can be employed with statistical software to solve these equations.

3. Mathematical properties of the NAL-Exp distribution

This section focuses on a distinct member of the NAL-G class, which is known as the NAL-Exp distribution. We conduct a comprehensive examination of this distribution by defining the analytical expressions of its CDF, denoted by $(F|_{X;\alpha,\theta})$, PDF, denoted by $(f|_{X;\alpha,\theta})$, survival function, denoted by $(SF|_{X;\alpha,\theta} = 1 - F|_{X;\alpha,\theta})$, and hazard rate function, denoted by $(hrf|_{X;\alpha,\theta} = f|_{X;\alpha,\theta}/SF|_{X;\alpha,\theta})$. These analytical expressions are given as follows:

$$F|_{X;\alpha,\theta} = \frac{\log\left(1+\alpha-\alpha^{e^{-\theta x}}\right)}{\log\alpha}, \tag{17}$$

$$f|_{X;\alpha,\theta} = \frac{\theta e^{-\theta x}\alpha^{e^{-\theta x}}}{1+\alpha-\alpha^{e^{-\theta x}}}, \tag{18}$$

$$SF|_{X;\alpha,\theta} = \frac{\log\alpha - \log\left(1+\alpha-\alpha^{e^{-\theta x}}\right)}{\log\alpha},$$

$$hrf|_{X;\alpha,\theta} = \frac{\theta e^{-\theta x}\alpha^{e^{-\theta x}}\log\alpha}{\left[1+\alpha-\alpha^{e^{-\theta x}}\right]\left[\log\alpha - \log\left(1+\alpha-\alpha^{e^{-\theta x}}\right)\right]},$$

where $\alpha > 0, \alpha \neq 1$, θ are scale and shape parameters, respectively. It is worth noting that the PDF of NAL-Exp distribution can be represented as infinite linear (IL) combinations using binomial and exponential expansions, followed by (3). Hence, the IL form is presented as follows:

$$f|_{X;\alpha,\theta} = \sum_{k,l,m=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^{k+1}\frac{(-1)^l(1+k)^l(\log\alpha)^l}{l!}\binom{l}{m}\theta e^{-\theta x(1+m)}. \tag{19}$$

3.1. Quantiles

If $X \sim \textit{NAL-Exp}\ (x;\alpha,\theta)$ where $\alpha > 0, \alpha \neq 1$ and θ are scale and shape parameters, respectively, then the QF denoted by $(QF|_{p;\alpha,\theta})$ which is the inverse of CDF can be obtained and is presented in (20) as

$$QF|_{p;\alpha,\theta} = \log\left[\frac{\log\alpha}{\log(1+\alpha-\alpha^{p})}\right]^{\frac{1}{\theta}}. \tag{20}$$

Additionally, to calculate the first quartile (denoted as $Q_1|_{0.25;\alpha,\theta}$), median (denoted as $Q_2|_{0.50;\alpha,\theta}$), and third quartile (denoted as $Q_3|_{0.75;\alpha,\theta}$), substitute $p = 0.25$, 0.50, and 0.75, respectively, into (20), as $p \in (0,1)$. The analytical expressions are presented as follows:

$$Q_1|_{0.25;\alpha,\theta} = \log\left[\frac{\log\alpha}{\log(1+\alpha-\alpha^{0.25})}\right]^{\frac{1}{\theta}},$$

$$Q_2|_{0.50;\alpha,\theta} = \log\left[\frac{\log\alpha}{\log(1+\alpha-\alpha^{0.50})}\right]^{\frac{1}{\theta}},$$

$$Q_3|_{0.75;\alpha,\theta} = \log\left[\frac{\log\alpha}{\log(1+\alpha-\alpha^{0.75})}\right]^{\frac{1}{\theta}}.$$

In addition to analyses, readers may utilize the Bowley [28] and Moors [29] approaches to study the skewness and kurtosis, defined as $Sk|_{Bowley}$ and $Kr|_{Moors}$, respectively. These approaches are defined as follows:

$$Sk|_{Bowley} = \frac{Q_3|_{0.75;\alpha,\theta} + Q_1|_{0.25;\alpha,\theta} - 2Q_2|_{0.50;\alpha,\theta}}{Q_3|_{0.75;\alpha,\theta} - Q_1|_{0.25;\alpha,\theta}},$$

$$Kr|_{Moors} = \frac{Q|_{0.875;\alpha,\theta} - Q|_{0.125;\alpha,\theta} - Q|_{0.625;\alpha,\theta} + Q|_{0.375;\alpha,\theta}}{Q_3|_{0.750;\alpha,\theta} - Q_1|_{0.250;\alpha,\theta}}.$$

3.2. Mode

To determine the mode, we can take the logarithm of the expression given in (18), resulting in the following equation:

$$ln\left(f|_{X;\alpha,\theta}\right) = ln(\theta) - \theta x + e^{-\theta x}\ ln\ \alpha - ln\left(1+\alpha-\alpha^{e^{-\theta x}}\right).$$

By differentiating the last function with respect to x, simplifying the result, and subsequently setting it equal to zero, we can derive the following equation:

$$\theta + \theta e^{-\theta x}\ ln\ \alpha + \frac{\theta e^{-\theta x}\alpha^{e^{-\theta x}}\ ln\ \alpha}{1+\alpha-\alpha^{e^{-\theta x}}} = 0.$$

The nonlinear nature of the final function precludes obtaining an analytical solution. Therefore, numerical methods such as the Newton-Raphson method can be employed to arrive at a solution.

3.3. Statistical moments

Moments are commonly used in probability theory, statistics, and data analysis to characterize and compare different distributions, and can also be used to derive important properties such as the ordinary moments (OMs), moment-generating function (MGF), characteristic function (CF), cumulants generating function (CGF), factorial generating function (FGF), and others.

Theorem 2. If $X \sim$NAL-Exp $(x;\alpha,\theta)$ where $\alpha > 0, \alpha \neq 1$ and θ are scale and shape parameters, respectively, then the r-th OMs of X is

$$E|_{X^r} = \mu_r^{'}|_{X;\alpha,\theta} = \sum_{k,l,m=0}^{\infty} \psi_{k,l,m}\frac{\Gamma(r+1)}{[\theta(1+m)]^{r+1}}.$$

Proof: The r-th OMs of X can be expressed as

$$E|_{X^r} = \mu_r^{'}|_{X;\alpha,\theta} = \int_0^{\infty} x^r f|_{X;\alpha,\theta}dx.$$

$\mu_r^{'}|_{X;\alpha,\theta}$ can be represented as immediately following (19)

$$\mu_r^{'}|_{X;\alpha,\theta} = \sum_{k,l,m=0}^{\infty} \psi_{k,l,m}\int_0^{\infty} x^r\theta e^{-\theta x(1+m)}dx. \tag{21}$$

Suppose $t = x\theta(1+m)$. Solving for x, we get $x = t/\theta(1+m)$. Taking derivatives on both sides with respect to t, we get $dx = dt/\theta(1+m)$. By substituting these expressions into (21) and performing some mathematical manipulations, we can simplify the expression for $\mu_r^{'}|_{X;\alpha,\theta}$ in terms of the gamma function. Precisely, the simplified form is given by

$$E|_{X^r} = \mu_r^{'}|_{X;\alpha,\theta} = \sum_{k,l,m=0}^{\infty} \psi_{k,l,m}\frac{\Gamma(r+1)}{[\theta(1+m)]^{r+1}}, \tag{22}$$

where $\Gamma(\eta) = \int_0^{\infty} z^{\eta-1}e^{-z}dz$ and $\psi_{k,l,m} = \binom{1}{m}\frac{(-1)^{l+m}(1+k)^l(\log\ \alpha)^l}{l!}\left(\frac{\alpha}{1+\alpha}\right)^{k+1}$ ∎.

Additionally, expression (22) significantly contributes to enabling authors to determine various statistical measures.

Corollary 4. The mean of X is defined as $E|_X = \mu_1^{'}|_{X;\alpha,\theta})$ and it can be presented as immediately following (3.6):

$$E|_X = \sum_{k,l,m=0}^{\infty} \psi_{k,l,m}\frac{\Gamma(2)}{[\theta(1+m)]^2}.$$

Corollary 5. The fractional positive moments of X is defined as: $E|_{X^{(p|_i,p|_j)}} = \mu_{(p|_i,p|_j)}^{'}\Big|_{X;\alpha,\theta}$ and it can be presented as immediately following (22):

Table 2
Numerical results of statistics.

Set	$E\|_X$	$E\|_{X^2}$	$E\|_{X^3}$	$E\|_{X^4}$	$Var\|_X$	$Sk\|_X$	$K_r\|_X$
I	1.0687	2.3526	6.9666	24.037	2.0994	2.4289	3.5311
II	0.8445	1.4054	3.3191	9.6593	1.1954	2.4658	4.0985
III	0.1891	0.6567	2.5217	10.242	0.6463	20.336	22.221
IV	0.2706	0.1796	0.1974	0.3021	0.1258	4.0256	8.0116

$$E|_{X^{(p|_i,p|_j)}} = \sum_{k,l,m=0}^{\infty} \psi_{k,l,m} \frac{\Gamma\left[\left(p|_i,p|_j\right)+1\right]}{[\theta(1+m)]^{\left(p|_i,p|_j\right)+1}}.$$

Corollary 6. The variance of X is defined as $\mathrm{Var}|_X = \mu_2^{'}|_{X;\alpha,\theta} - [\mu_1^{'}|_{X;\alpha,\theta}]^2$ and it can be presented as immediately following (22):

$$Var|_X = \left[\begin{array}{c} \sum_{k,l,m=0}^{\infty} \psi_{k,l,m} \frac{\Gamma(3)}{[\theta(1+m)]^3} \\ -\left(\sum_{k,l,m=0}^{\infty} \psi_{k,l,m} \frac{\Gamma(2)}{[\theta(1+m)]^2}\right)^2 \end{array}\right].$$

Corollary 7. The skewness and kurtosis of X are defined as $\mathrm{Sk}|_X = \mu_3^{'2}|_{X;\alpha,\theta} / \mu_2^{'3}|_{X;\alpha,\theta}$, and $\mathrm{Kr}|_X = \mu_4^{'}|_{X;\alpha,\theta} / \mu_2^{'2}|_{X;\alpha,\theta}$, respectively and they can be presented as immediately following (22):

$$Sk|_X = \frac{\left[\sum_{k,l,m=0}^{\infty} \psi_{k,l,m} \frac{\Gamma(4)}{[\theta(1+m)]^4}\right]^2}{\left[\sum_{k,l,m=0}^{\infty} \psi_{k,l,m} \frac{\Gamma(3)}{[\theta(1+m)]^3}\right]^3},$$

and

$$Kr|_X = \frac{\sum_{k,l,m=0}^{\infty} \psi_{k,l,m} \frac{\Gamma(5)}{[\theta(1+m)]^5}}{\left[\sum_{k,l,m=0}^{\infty} \psi_{k,l,m} \frac{\Gamma(3)}{[\theta(1+m)]^3}\right]^2}.$$

Corollary 8. The s-th central moments of X is defined as $\mu_s|_X = \sum_{\lambda=0}^{s} \binom{s}{\lambda} (-1)^{\lambda} \mu_1^{'\lambda}|_{X;\alpha,\theta} \mu_{s-\lambda}^{'}|_{X;\alpha,\theta}$ and it can be presented as immediately following (22):

$$\mu_s|_X = \left[\begin{array}{c} \sum_{\lambda=0}^{s} \binom{s}{\lambda} (-1)^{\lambda} \left[\sum_{k,l,m=0}^{\infty} \psi_{k,l,m} \frac{\Gamma(2)}{[\theta(1+m)]^2}\right]^{\lambda} \\ \sum_{k,l,m=0}^{\infty} \psi_{k,l,m} \frac{\Gamma(s-\lambda+1)}{[\theta(1+m)]^{s-\lambda+1}} \end{array}\right].$$

Corollary 9. The r-th cumulants is defined as $K_r|_X = \mu_r^{'}|_{X;\alpha,\theta} - \sum_{i=0}^{r-1} \binom{r-1}{i-1} C|_i \mu_{r-i}^{'}|_{X;\alpha,\theta}$. The analytical expression for X is immediately following (22):

$$\left[\begin{array}{c} \sum_{k,l,m=0}^{\infty} \psi_{k,l,m} \frac{\Gamma(r+1)}{[\theta(1+m)]^{r+1}} - \sum_{i=0}^{r-1} \binom{r-1}{i-1} C|_i \\ \sum_{k,l,m=0}^{\infty} \psi_{k,l,m} \frac{\Gamma(r-i+1)}{[\theta(1+m)]^{r-i+1}} \end{array}\right].$$

Table 2 provides a comprehensive overview of the behavior of moments and other statistics for different values of I, II, III, and IV.

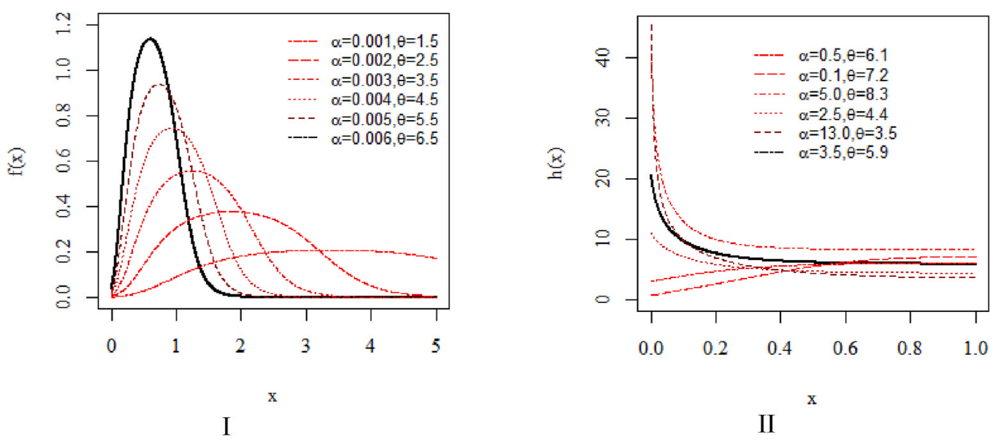

Fig. 1. Possible curves of density (left side-I) and hazard rate (right side-II) for different choices of parameters.

The table displays numerical values for I = ($\alpha = 2.1, \theta = 0.5$), II = ($\alpha = 1.1, \theta = 1.1$), III = ($\alpha = 0.1, \theta = 0.1$), and IV = ($\alpha = 2.1, \theta = 2.5$).

Fig. 1 displays the possible PDF curves, with the left side of the figure showing upside-down increasing curves and the right side showing increasing failure rate HRF curves. The figure displays these curves for different parameter selections.

Corollary 10. The MGF of X is defined as $MGF_p|_{X;\alpha,\theta} = \sum_{r=0}^{\infty}\delta|_{p,r!}\mu_r^{'}|_{X;\alpha,\theta}$ and it can be presented as immediately following (22):

$$MGF_p|_{X;\alpha,\theta} = \sum_{r=0}^{\infty}\delta|_{p,r!}\sum_{k,l,m=0}^{\infty}\psi_{k,l,m}\frac{\Gamma(r+1)}{[\theta(1+m)]^{r+1}}.$$

Corollary 11. The CF of X is defined as $CF_p|_{X;\alpha,\theta} = \sum_{r=0}^{\infty}\delta|_{pi,r!}\mu_r^{'}|_{X;\alpha,\theta}$ and it can be presented as immediately following (22):

$$CF_p|_{X;\alpha,\theta} = \sum_{r=0}^{\infty}\delta|_{pi,r!}\sum_{k,l,m=0}^{\infty}\psi_{k,l,m}\frac{\Gamma(r+1)}{[\theta(1+m)]^{r+1}}.$$

Corollary 12. The CGF of X is defined as $CGF_p|_{X;\alpha,\theta} = \log MGF_p|_{X;\alpha,\theta}$ and it can be presented as immediately following (22):

$$CGF_p|_{X;\alpha,\theta} = log\left[\sum_{r=0}^{\infty}\delta|_{p,r!}\sum_{k,l,m=0}^{\infty}\psi_{k,l,m}\frac{\Gamma(r+1)}{[\theta(1+m)]^{r+1}}\right].$$

Corollary 13. The FGF of X is defined as $FGF_p|_{X;\alpha,\theta} = \sum_{r=0}^{\infty}\delta|_{logp,r!}\mu_r^{'}|_{X;\alpha,\theta}$ and it can be presented as immediately following (22):

$$FGF_p|_{X;\alpha,\theta} = \sum_{r=0}^{\infty}\delta|_{logp,r!}\sum_{k,l,m=0}^{\infty}\psi_{k,l,m}\frac{\Gamma(r+1)}{[\theta(1+m)]^{r+1}}.$$

where $\delta|_{p,r!} = \frac{p}{r!}$, $\delta|_{pi,r!} = \frac{(ip)^r}{r!}$, $\delta|_{logp,r!} = \frac{[\log(1+p)]^r}{r!}$, and $i = \sqrt{-1}$.

Corollary 14. The r-th incomplete moment of X denoted as $\varnothing_r|_{X,v;\alpha,\theta}$ can be calculated using (22) as follows:

$$\varnothing_r|_{X,v;\alpha,\theta} = \sum_{k,l,m=0}^{\infty}\psi_{k,l,m}\frac{\Gamma(r+1,v)}{[\theta(1+m)]^{r+1}}, \quad (23)$$

where $\Gamma(\eta,v) = \int_0^v z^{\eta-1}e^{-z}dz$. Furthermore, by placing $r = 1$ in the previous expression yields the first incomplete moment of X denoted as $\varnothing_1|_{X,v;\alpha,\theta}$ and the simplified form is given by

$$\varnothing_1|_{X,v;\alpha,\theta} = \sum_{k,l,m=0}^{\infty}\psi_{k,l,m}\frac{\Gamma(2,v)}{[\theta(1+m)]^2}. \quad (24)$$

It is worth noting that the use of $\varnothing_1|_{X,v;\alpha,\theta}$ is a valuable tool for calculating inequalities such as the Bonferroni and Lorenz curves. Specifically, these inequalities can be calculated using $B|_{p,\alpha,\theta} = \varnothing_1|_{v;\alpha,\theta} / \mu_1'|_{X;\alpha,\theta}$ and $L|_{p,\alpha,\theta} = \varnothing_1|_{p;\alpha,\theta} / p\mu_1'|_{X;\alpha,\theta}$, respectively.

3.4. Residual life functions (RLFs)

The RLFs provide valuable information about the expected remaining lifetime of a system after a given time. Suppose $X \sim$ *NAL-Exp* $(x; \alpha, \theta)$ where $\alpha > 0, \alpha \neq 1$ and θ are scale and shape parameters, respectively, then we can determine the residual life (RL) and reverse RL functions (RLFs) of X using the following expressions $R|_{X;\alpha,\theta} = S|_{X+p} / S|_p$ and $\overline{R}|_{X;\alpha,\theta} = S|_{X-p} / S|_p$.

$$R|_{X;\alpha,\theta} = \frac{log\alpha - log\left(1 + \alpha - \alpha^{e^{-\theta(x+p)}}\right)}{log\alpha - log(1 + \alpha - \alpha^{e^{-\theta p}})},$$

$$\overline{R}|_{X;\alpha,\theta} = \frac{log\alpha - log\left(1 + \alpha - \alpha^{e^{-\theta(x-p)}}\right)}{log\alpha - log(1 + \alpha - \alpha^{e^{-\theta p}})}.$$

It should be noted that the mean RLF defined as $MRL|_{X;\alpha,\theta} = [1 - \varnothing_1|_{X,v;\alpha,\theta}] / [SF|_{X;\alpha,\theta} - X]$, mean inactivity time denoted as $MIT|_{X;\alpha,\theta} = X - \varnothing_1|_{X,v;\alpha,\theta} / F|_{X;\alpha,\theta}$, and median RLF defined as $MdRL|_{X;\alpha,\theta} = (1/2) MRL|_{X;\alpha,\theta}$, for X can also be determined using $\varnothing_1|_{X,v;\alpha,\theta}$. The analytical expressions for $MRL|_{X;\alpha,\theta}$, $MIT|_{X;\alpha,\theta}$, and $MdRL|_{X;\alpha,\theta}$ are given as follows:

$$MRL|_{X;\alpha,\theta} = \left[1 - \sum_{k,l,m=0}^{\infty} \psi_{k,l,m} \frac{\Gamma(2, v)}{[\theta(1+m)]^2}\right] / \left[SF|_{X;\alpha,\theta} - X\right],$$

$$MIT|_{X;\alpha,\theta} = X - \sum_{k,l,m=0}^{\infty} \psi_{k,l,m} \frac{\Gamma(2, v)}{[\theta(1+m)]^2} / F|_{X;\alpha,\theta},$$

$$MdRL|_{X;\alpha,\theta} = \left(\frac{1}{2}\right) \left[1 - \sum_{k,l,m=0}^{\infty} \psi_{k,l,m} \frac{\Gamma(2, v)}{[\theta(1+m)]^2}\right] / \left[SF|_{X;\alpha,\theta} - X\right].$$

3.5. Entropy

Theorem 3. If $X \sim$ NAL-Exp $(x; \alpha, \theta)$ where $\alpha > 0, \alpha \neq 1$ and θ are scale and shape parameters, respectively, then the Rényi entropy of X, for the NAL-Exp distribution can be represented as follows:

$$H|_{X;\alpha,\theta} = \frac{1}{1 - \nabla} log \sum_{v,m,l=0}^{\infty} \psi_{v,m,l,\nabla}, \nabla > 0, \nabla \neq 1.$$

Proof: The Rényi entropy of X is defined as

$$H|_{X;\alpha,\theta} = \frac{1}{1 - \nabla} log \int_0^{+\infty} \left[f|_{X;\alpha,\theta}\right]^{\nabla} dx, \nabla > 0, \nabla \neq 1. \tag{25}$$

The simplified version of the $[f|_{X;\alpha,\theta}]^{\nabla}$ can be re-write as immediately following (10)

$$\left[f|_{X;\alpha,\theta}\right]^{\nabla} = \sum_{v,m=0}^{\infty} \binom{-\nabla}{v} \frac{(-1)^{v+m} (\nabla + v)^m (log\alpha)^m \theta^{\nabla}}{m!} \left(\frac{\alpha}{1+\alpha}\right)^{\nabla + v} e^{-\theta\nabla x} \left[1 - e^{-\theta x}\right]^m. \tag{26}$$

After simplifying $[1 - e^{-\theta x}]^m$ using binomial expansion, we get

$$\left[1 - e^{-\theta x}\right]^m = \sum_{l=0}^{\infty} \binom{m}{l} (-1)^l e^{-m\theta x},$$

and substitute it into (26) to obtain

$$\left[f|_{X;\alpha,\theta}\right]^{\nabla} = \sum_{v,m,l=0}^{\infty} \binom{-\nabla}{v} \binom{m}{l} \frac{(-1)^{v+m+l} (\nabla + v)^m (log\alpha)^m \theta^{\nabla}}{m!} \left(\frac{\alpha}{1+\alpha}\right)^{\nabla + v} e^{-x(\theta\nabla + \theta m)}. \tag{27}$$

Hence, substituting (27) into (25) yields the final expression for the Rényi entropy of X as

$$H|_{X;\alpha,\theta} = \frac{1}{1 - \nabla} log \sum_{v,m,l=0}^{\infty} \psi_{v,m,l,\nabla}, \nabla > 0, \nabla \neq 1.$$

Table 3
Numerical results of entropy measures.

–	Parameter	Entropy					
		Rényi		Tsallis		Havrda-Charvat	
I	$\nabla = 1.1$	$\theta = 0.1$	$\theta = 0.9$	$\theta = 0.1$	$\theta = 0.9$	$\theta = 0.1$	$\theta = 0.9$
	$\alpha = 0.1$	98.2107	37.3205	7.2012	3.1139	-0.1424	-0.1203
	$\alpha = 0.2$	84.1974	29.8247	6.3597	2.5342	-0.1414	-0.1109
	$\alpha = 0.3$	76.0409	26.2203	5.8446	2.2466	-0.1403	-0.1047
	$\alpha = 0.4$	70.3077	24.2237	5.4709	2.0834	-0.1392	-0.1003
	$\alpha = 0.9$	54.7599	21.9972	4.4068	1.8829	-0.1342	-0.0891
II	$\nabla = 2.5$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 0.5$	$\alpha = 0.9$
	$\theta = 1.1$	3.2050	3.2887	0.9846	0.9011	−3.2289	−2.9059
	$\theta = 2.0$	3.3909	4.1818	0.8134	0.7258	−2.5755	−1.8055
	$\theta = 2.5$	4.0606	5.0711	0.7484	0.6525	−2.1451	−1.0694
	$\theta = 3.0$	4.9424	6.1066	0.6916	0.5871	−1.6695	−0.2556
	$\theta = 3.5$	5.9661	7.2425	0.6400	0.5272	−1.1525	0.6291

Table 4
Simulation outcomes for ($\alpha = 0.5, \theta = 0.5$).

n	Est	Par	$ML\|_I$	$AD\|_{II}$	$CvM\|_{III}$	$MPS\|_{IV}$	$OLS\|_V$	$WLS\|_{VI}$
20	|Bias|	$\widehat{\alpha}$	0.2799	0.8035	0.4720	0.8347	0.9073	0.3689
		$\widehat{\theta}$	0.1484	0.1911	0.2064	0.1614	0.1938	0.1738
	MSE	$\widehat{\alpha}$	0.1676	1.9320	1.0510	3.6865	7.0046	0.3376
		$\widehat{\theta}$	0.0498	0.0737	0.0777	0.0412	0.0593	0.0882
	MRE	$\widehat{\alpha}$	0.5599	1.6071	0.9440	1.6694	1.8147	0.7377
		$\widehat{\theta}$	0.2968	0.3822	0.4128	0.3228	0.3877	0.3476
40	|Bias|	$\widehat{\alpha}$	0.1795	0.2881	0.3198	0.3161	0.2751	0.2605
		$\widehat{\theta}$	0.1059	0.1294	0.1464	0.1241	0.1278	0.1140
	MSE	$\widehat{\alpha}$	0.0517	0.1833	0.2937	0.1992	0.1525	0.1293
		$\widehat{\theta}$	0.0179	0.0279	0.0404	0.0206	0.0270	0.0202
	MRE	$\widehat{\alpha}$	0.3590	0.5762	0.6397	0.6322	0.5502	0.5209
		$\widehat{\theta}$	0.2118	0.2589	0.2929	0.2483	0.2555	0.2280
80	|Bias|	$\widehat{\alpha}$	0.1641	0.1737	0.1989	0.2179	0.1907	0.2041
		$\widehat{\theta}$	0.0904	0.0866	0.0906	0.0820	0.0873	0.0867
	MSE	$\widehat{\alpha}$	0.0436	0.0559	0.0684	0.1271	0.0684	0.0820
		$\widehat{\theta}$	0.0118	0.0121	0.0137	0.0110	0.0110	0.0117
	MRE	$\widehat{\alpha}$	0.3282	0.3473	0.3978	0.4359	0.3813	0.4083
		$\widehat{\theta}$	0.1807	0.1732	0.1812	0.1639	0.1747	0.1734
100	|Bias|	$\widehat{\alpha}$	0.1187	0.1298	0.1451	0.1512	0.1759	0.1519
		$\widehat{\theta}$	0.0651	0.0587	0.0727	0.0623	0.0883	0.0777
	MSE	$\widehat{\alpha}$	0.0211	0.0350	0.0330	0.0411	0.0574	0.0405
		$\widehat{\theta}$	0.0067	0.0059	0.0096	0.0060	0.0120	0.0085
	MRE	$\widehat{\alpha}$	0.2374	0.2596	0.2901	0.3023	0.3518	0.3039
		$\widehat{\theta}$	0.1302	0.1173	0.1454	0.1246	0.1767	0.1554
150	|Bias|	$\widehat{\alpha}$	0.1275	0.1180	0.1277	0.1225	0.1202	0.1280
		$\widehat{\theta}$	0.0560	0.0608	0.0679	0.0490	0.0575	0.0584
	MSE	$\widehat{\alpha}$	0.0282	0.0259	0.0277	0.0240	0.0263	0.0281
		$\widehat{\theta}$	0.0047	0.0062	0.0069	0.0038	0.0050	0.0058
	MRE	$\widehat{\alpha}$	0.2550	0.2360	0.2555	0.2450	0.2404	0.2561
		$\widehat{\theta}$	0.1119	0.1217	0.1358	0.0980	0.1151	0.1167
500	|Bias|	$\widehat{\alpha}$	0.0639	0.0608	0.0648	0.0597	0.0667	0.0711
		$\widehat{\theta}$	0.0306	0.0270	0.0320	0.0280	0.0346	0.0352
	MSE	$\widehat{\alpha}$	0.0067	0.0056	0.0074	0.0061	0.0072	0.0084
		$\widehat{\theta}$	0.0015	0.0012	0.0019	0.0013	0.0019	0.0021
	MRE	$\widehat{\alpha}$	0.1279	0.1216	0.1297	0.1193	0.1334	0.1421
		$\widehat{\theta}$	0.0611	0.0541	0.0640	0.0560	0.0693	0.0705

where $\psi_{v,m,l,\nabla} = \binom{-\nabla}{v}\binom{m}{l}\frac{(-1)^{m+v+l}(\nabla+v)^m(\log\alpha)^m\theta^\nabla}{m!}\left(\frac{\alpha}{1+\alpha}\right)^{\nabla+v}(\theta\nabla+\theta m)^{-1}$ ∎.

It is important to note that expression in (27) can be utilized to derive the final forms of Tsallis (defined as $H1|_{X;\alpha,\theta}$) and Havrda-Charvat (defined as $H2|_{X;\alpha,\theta}$) entropies, which are presented in corollary 2 and corollary 3, respectively:

Corollary 15. The Tsallis entropy

Table 5
Simulation outcomes for $(\alpha = 0.5, \theta = 1.5)$.

n	Est	Par	$ML|_I$	$AD|_{II}$	$CvM|_{III}$	$MPS|_{IV}$	$OLS|_V$	$WLS|_{VI}$
20	|Bias|	$\hat{\alpha}$	0.3289	0.7787	0.4775	0.4087	0.6649	1.1308
		$\hat{\theta}$	0.5220	0.4673	0.6780	0.4546	0.6193	0.6383
	MSE	$\hat{\alpha}$	0.2021	5.6480	0.6448	0.3908	1.8650	59.8796
		$\hat{\theta}$	0.5837	0.4173	1.1723	0.2882	0.5723	1.0794
	MRE	$\hat{\alpha}$	0.6579	1.5574	0.9550	0.8174	1.3299	2.2616
		$\hat{\theta}$	0.3480	0.3115	0.4520	0.3031	0.4129	0.4255
40	|Bias|	$\hat{\alpha}$	0.2324	0.2766	0.2639	0.2764	0.4040	0.2628
		$\hat{\theta}$	0.3113	0.3731	0.4644	0.3112	0.3798	0.3562
	MSE	$\hat{\alpha}$	0.0900	0.1548	0.1214	0.1948	0.7424	0.1338
		$\hat{\theta}$	0.1508	0.2299	0.3459	0.1566	0.2174	0.2201
	MRE	$\hat{\alpha}$	0.4648	0.5532	0.5278	0.5529	0.8080	0.5255
		$\hat{\theta}$	0.2075	0.2487	0.3096	0.2075	0.2532	0.2375
80	|Bias|	$\hat{\alpha}$	0.1570	0.1922	0.2306	0.1640	0.2136	0.1827
		$\hat{\theta}$	0.2275	0.2258	0.2996	0.2075	0.2711	0.2334
	MSE	$\hat{\alpha}$	0.0354	0.0807	0.1058	0.0460	0.0888	0.0653
		$\hat{\theta}$	0.0894	0.0790	0.1591	0.0709	0.1192	0.0853
	MRE	$\hat{\alpha}$	0.3139	0.3843	0.4612	0.3280	0.4272	0.3655
		$\hat{\theta}$	0.1517	0.1506	0.1997	0.1383	0.1807	0.1556
100	|Bias|	$\hat{\alpha}$	0.1562	0.1588	0.1587	0.1646	0.2059	0.1599
		$\hat{\theta}$	0.2246	0.2257	0.2501	0.2206	0.2423	0.2154
	MSE	$\hat{\alpha}$	0.0386	0.0462	0.0406	0.0406	0.1153	0.0453
		$\hat{\theta}$	0.0885	0.0799	0.1194	0.0756	0.0898	0.0789
	MRE	$\hat{\alpha}$	0.3123	0.3177	0.3174	0.3293	0.4118	0.3199
		$\hat{\theta}$	0.1498	0.1504	0.1667	0.1471	0.1615	0.1436
150	|Bias|	$\hat{\alpha}$	0.1175	0.1377	0.1285	0.1413	0.1127	0.1305
		$\hat{\theta}$	0.1537	0.1818	0.2151	0.1941	0.1586	0.1740
	MSE	$\hat{\alpha}$	0.0224	0.0342	0.0277	0.0367	0.0225	0.0287
		$\hat{\theta}$	0.0391	0.0533	0.0667	0.0516	0.0407	0.0492
	MRE	$\hat{\alpha}$	0.2349	0.2754	0.2571	0.2826	0.2254	0.2611
		$\hat{\theta}$	0.1025	0.1212	0.1434	0.1294	0.1058	0.1160
500	|Bias|	$\hat{\alpha}$	0.0510	0.0692	0.0701	0.0606	0.0709	0.0678
		$\hat{\theta}$	0.0860	0.0881	0.0973	0.0933	0.1171	0.1053
	MSE	$\hat{\alpha}$	0.0045	0.0067	0.0080	0.0062	0.0079	0.0072
		$\hat{\theta}$	0.0127	0.0120	0.0135	0.0127	0.0217	0.0179
	MRE	$\hat{\alpha}$	0.1020	0.1385	0.1403	0.1211	0.1419	0.1356
		$\hat{\theta}$	0.0573	0.0587	0.0648	0.0622	0.0780	0.0702

$$H1|_{X;\alpha,\theta} = \frac{1}{1-\nabla} log \sum_{v,m,l=0}^{\infty} \psi_{v,m,l,\nabla} - 1, \nabla > 0, \nabla \neq 1. \tag{28}$$

Corollary 16. The Havrda-Charvat entropy

$$H2|_{X;\alpha,\theta} = \frac{1}{2^{1-\nabla}-1} log \sum_{v,m,l=0}^{\infty} \psi_{v,m,l,\nabla} - 1, \nabla > 0, \nabla \neq 1. \tag{29}$$

Since the Rényi, Tsallis, and Havrda-Charvat entropies do not have closed-form expressions, as shown above. To better understand the behavior of these entropies, Table 3 provides a detailed overview.

Remark I. The Rényi and Tsallis entropies exhibit a decreasing trend with varying α but fixed θ, whereas the Havrda-Charvat entropy displays an increasing trend.

Remark II. The Rényi entropy demonstrates an increasing trend, while the Tsallis entropy exhibits a decreasing trend with a fixed α and varying θ. On the other hand, the Havrda-Charvat entropy displays an increasing trend.

3.6. Order statistics (OS)

Let us consider random variables (RVs) denoted by $X_1|_{\alpha,\theta}, X_2|_{\alpha,\theta}, X_3|_{\alpha,\theta}, \ldots, X_n|_{\alpha,\theta}$, which follow the NAL-G class of distributions. The PDF of the p-th OS say $f|_{X_{p:n};\alpha,\theta}$ can be defined as:

Table 6
Simulation outcomes for ($\alpha = 0.1, \theta = 2.5$).

n	Est	Par	$ML\|_I$	$AD\|_{II}$	$CvM\|_{III}$	$MPS\|_{IV}$	$OLS\|_V$	$WLS\|_{VI}$
20	|Bias|	$\hat{\alpha}$	0.0935	0.0954	0.0973	0.1686	0.1348	0.1001
		$\hat{\theta}$	1.0916	0.8574	1.1241	0.7004	0.8087	0.8450
	MSE	$\hat{\alpha}$	0.0170	0.0211	0.0243	0.0746	0.0430	0.0218
		$\hat{\theta}$	2.7982	1.7928	2.6363	0.7571	1.1800	1.4797
	MRE	$\hat{\alpha}$	0.9353	0.9544	0.9725	1.6862	1.3480	1.0009
		$\hat{\theta}$	0.4366	0.3430	0.4496	0.2801	0.3235	0.3380
40	|Bias|	$\hat{\alpha}$	0.0591	0.0650	0.0772	0.0710	0.0911	0.0823
		$\hat{\theta}$	0.5366	0.5019	0.6709	0.4644	0.6160	0.6274
	MSE	$\hat{\alpha}$	0.0058	0.0085	0.0117	0.0092	0.0180	0.0152
		$\hat{\theta}$	0.6365	0.4961	1.0479	0.3237	0.7331	0.7840
	MRE	$\hat{\alpha}$	0.5914	0.6501	0.7715	0.7096	0.9112	0.8234
		$\hat{\theta}$	0.2146	0.2008	0.2684	0.1858	0.2464	0.2510
80	|Bias|	$\hat{\alpha}$	0.0422	0.0521	0.0610	0.0498	0.0594	0.0530
		$\hat{\theta}$	0.3336	0.3252	0.4409	0.2665	0.4240	0.3837
	MSE	$\hat{\alpha}$	0.0030	0.0052	0.0077	0.0053	0.0059	0.0068
		$\hat{\theta}$	0.2870	0.1650	0.2990	0.1217	0.3120	0.2835
	MRE	$\hat{\alpha}$	0.4222	0.5211	0.6100	0.4983	0.5938	0.5296
		$\hat{\theta}$	0.1335	0.1301	0.1764	0.1066	0.1696	0.1535
100	|Bias|	$\hat{\alpha}$	0.0386	0.0395	0.0483	0.0462	0.0535	0.0439
		$\hat{\theta}$	0.2872	0.2958	0.3658	0.2725	0.3596	0.3119
	MSE	$\hat{\alpha}$	0.0025	0.0027	0.0038	0.0044	0.0049	0.0034
		$\hat{\theta}$	0.1406	0.1385	0.2658	0.1177	0.2335	0.1714
	MRE	$\hat{\alpha}$	0.3859	0.3953	0.4832	0.4622	0.5347	0.4392
		$\hat{\theta}$	0.1149	0.1183	0.1463	0.1090	0.1438	0.1247
150	|Bias|	$\hat{\alpha}$	0.0345	0.0363	0.0380	0.0334	0.0378	0.0382
		$\hat{\theta}$	0.2308	0.2508	0.2877	0.2338	0.2577	0.2513
	MSE	$\hat{\alpha}$	0.0021	0.0022	0.0023	0.0025	0.0024	0.0024
		$\hat{\theta}$	0.0809	0.0970	0.1158	0.0921	0.1044	0.1019
	MRE	$\hat{\alpha}$	0.3449	0.3629	0.3801	0.3337	0.3782	0.3819
		$\hat{\theta}$	0.0923	0.1003	0.1151	0.0935	0.1031	0.1005
500	|Bias|	$\hat{\alpha}$	0.0199	0.0227	0.0214	0.0176	0.0225	0.0188
		$\hat{\theta}$	0.1333	0.1510	0.1827	0.1039	0.1697	0.1214
	MSE	$\hat{\alpha}$	0.0006	0.0008	0.0007	0.0005	0.0009	0.0007
		$\hat{\theta}$	0.0275	0.0366	0.0510	0.0241	0.0459	0.0246
	MRE	$\hat{\alpha}$	0.1989	0.2265	0.2140	0.1763	0.2245	0.1877
		$\hat{\theta}$	0.0533	0.0604	0.0731	0.0415	0.0679	0.0486

$$f|_{X_{p:n};\alpha,\theta} = \tau(a,b)\sum_{l=0}^{n-p}(-1)^l\binom{n-p}{l}\left(F|_{X;\alpha,\theta}\right)^{l+p-1} f|_{X;\alpha,\theta},$$

and the analytical expression of th $f|_{X_{p:n};\alpha,\theta}$ is presented as follows:

$$f|_{X_{p:n};\alpha,\theta} = \tau(a,b)\sum_{l=0}^{n-p}(-1)^l\binom{n-p}{l}\left[\frac{log\left(1+\alpha-\alpha^{e^{-\theta x}}\right)}{log\alpha}\right]^{l+p-1}\left[\frac{\theta e^{-\theta x}\alpha^{e^{-\theta x}}}{1+\alpha-\alpha^{e^{-\theta x}}}\right],$$

where $\tau(a,b) = 1/B(p, n-p+1)$.

The minimum (first) and maximum (largest) OS presented as $f|_{X_{p:1};\alpha,\theta} = n[1-(F|_{X;\alpha,\theta})^{l+p-1}]^{n-1} f|_{X;\alpha,\theta}$ and $f|_{X_{p:n};\alpha,\theta} = n[(F|_{X;\alpha,\theta})^{l+p-1}]^{n-1} f|_{X;\alpha,\theta}$, respectively, are highly valuable for statistical analyses that involve identifying extreme values within a dataset. The analytical expression are, respectively, presented as follows:

$$f|_{X_{p:1};\alpha,\theta} = n\left[1-\left[\frac{log\left(1+\alpha-\alpha^{e^{-\theta x}}\right)}{log\alpha}\right]^{l+p-1}\right]^{n-1}\left[\frac{\theta e^{-\theta x}\alpha^{e^{-\theta x}}}{1+\alpha-\alpha^{e^{-\theta x}}}\right],$$

and

Table 7
Simulation outcomes for $(\alpha = 0.1, \theta = 0.9)$.

n	Est	Par	$ML\|_{I}$	$AD\|_{II}$	$CvM\|_{III}$	$MPS\|_{IV}$	$OLS\|_{V}$	$WLS\|_{VI}$
20	|Bias|	$\hat{\alpha}$	0.0986	0.0993	0.0977	0.1430	0.1394	0.1187
		$\hat{\theta}$	0.3300	0.2623	0.3488	0.2383	0.2798	0.3286
	MSE	$\hat{\alpha}$	0.0201	0.0220	0.0217	0.0461	0.0374	0.0307
		$\hat{\theta}$	0.2749	0.1260	0.2593	0.0850	0.1373	0.2297
	MRE	$\hat{\alpha}$	0.9861	0.9928	0.9767	1.4297	1.3939	1.1873
		$\hat{\theta}$	0.3666	0.2914	0.3876	0.2648	0.3109	0.3651
40	|Bias|	$\hat{\alpha}$	0.0585	0.0681	0.0743	0.0785	0.0856	0.0735
		$\hat{\theta}$	0.1840	0.1837	0.2779	0.1567	0.2078	0.1961
	MSE	$\hat{\alpha}$	0.0064	0.0079	0.0095	0.0115	0.0148	0.0100
		$\hat{\theta}$	0.0675	0.0657	0.1901	0.0359	0.0853	0.0840
	MRE	$\hat{\alpha}$	0.5852	0.6812	0.7435	0.7851	0.8562	0.7354
		$\hat{\theta}$	0.2045	0.2042	0.3088	0.1741	0.2309	0.2179
80	|Bias|	$\hat{\alpha}$	0.0424	0.0412	0.0502	0.0691	0.0681	0.0499
		$\hat{\theta}$	0.1188	0.1218	0.1279	0.1358	0.1583	0.1313
	MSE	$\hat{\alpha}$	0.0030	0.0025	0.0044	0.0082	0.0102	0.0037
		$\hat{\theta}$	0.0214	0.0250	0.0339	0.0271	0.0485	0.0294
	MRE	$\hat{\alpha}$	0.4241	0.4123	0.5015	0.6908	0.6813	0.4993
		$\hat{\theta}$	0.1320	0.1353	0.1421	0.1508	0.1759	0.1459
100	|Bias|	$\hat{\alpha}$	0.0412	0.0504	0.0441	0.0384	0.0520	0.0462
		$\hat{\theta}$	0.1107	0.1276	0.1145	0.0817	0.1396	0.1228
	MSE	$\hat{\alpha}$	0.0028	0.0037	0.0036	0.0025	0.0059	0.0036
		$\hat{\theta}$	0.0213	0.0265	0.0229	0.0110	0.0490	0.0255
	MRE	$\hat{\alpha}$	0.4123	0.5042	0.4408	0.3842	0.5196	0.4618
		$\hat{\theta}$	0.1230	0.1417	0.1272	0.0907	0.1551	0.1365
150	|Bias|	$\hat{\alpha}$	0.0321	0.0403	0.0388	0.0392	0.0363	0.0353
		$\hat{\theta}$	0.0791	0.0916	0.0921	0.0952	0.1000	0.0886
	MSE	$\hat{\alpha}$	0.0017	0.0025	0.0026	0.0026	0.0022	0.0023
		$\hat{\theta}$	0.0097	0.0132	0.0138	0.0133	0.0159	0.0123
	MRE	$\hat{\alpha}$	0.3214	0.4030	0.3882	0.3917	0.3626	0.3526
		$\hat{\theta}$	0.0879	0.1017	0.1024	0.1058	0.1111	0.0985
500	|Bias|	$\hat{\alpha}$	0.0180	0.0181	0.0211	0.0165	0.0195	0.0168
		$\hat{\theta}$	0.0455	0.0474	0.0553	0.0456	0.0567	0.0483
	MSE	$\hat{\alpha}$	0.0005	0.0006	0.0007	0.0005	0.0006	0.0004
		$\hat{\theta}$	0.0031	0.0034	0.0049	0.0034	0.0048	0.0036
	MRE	$\hat{\alpha}$	0.1795	0.1812	0.2105	0.1654	0.1949	0.1679
		$\hat{\theta}$	0.0506	0.0526	0.0615	0.0507	0.0630	0.0537

$$f|_{X_{p:n};\alpha,\theta} = n\left[\left[\frac{log\left(1+\alpha-\alpha^{e^{-\theta x}}\right)}{log\alpha}\right]^{l+p-1}\right]^{n-1}\left[\frac{\theta e^{-\theta x}\alpha^{e^{-\theta x}}}{1+\alpha-\alpha^{e^{-\theta x}}}\right].$$

Additionally, the CDF of the p-th OS say $F|_{X_{p:n};\alpha,\theta}$, can be defined as:

$$F|_{X_{p:n};\alpha,\theta} = \sum_{l=j}^{n}\binom{n}{l}\left(F|_{X;\alpha,\theta}\right)^{l}\left(1-F|_{X;\alpha,\theta}\right)^{n-l},$$

and the analytical expression of the $F|_{X_{p:n};\alpha,\theta}$ is presented as follows:

$$F|_{X_{p:n};\alpha,\theta} = \sum_{l=j}^{n}\binom{n}{l}\left[\frac{log\left(1+\alpha-\alpha^{e^{-\theta x}}\right)}{log\alpha}\right]^{l}\left[\frac{log\alpha-log\left(1+\alpha-\alpha^{e^{-\theta x}}\right)}{log\alpha}\right]^{n-l}.$$

These statistics find applications in a variety of fields, such as reliability analysis, extreme value theory, quality control, and hypothesis testing.

4. Inference

In the discipline of statistical inference, there are three apparent techniques: point (classical) estimation, interval estimation, and hypothesis testing. The scholarly literature presents various techniques for the point estimation of parameters. Nonetheless, the estimation methods that are most commonly employed are expounded upon in Sections 4.1 Through 4.6.

Table 8
Simulation outcomes for ($\alpha = 0.8, \theta = 0.5$).

n	Est	Par	$ML\|_I$	$AD\|_{II}$	$CvM\|_{III}$	$MPS\|_{IV}$	$OLS\|_V$	$WLS\|_{VI}$
20	|Bias|	$\widehat{\alpha}$	0.5627	1.0732	0.9405	0.9325	2.7020	1.3326
		$\widehat{\theta}$	0.2215	0.2050	0.2751	0.1573	0.2742	0.2019
	MSE	$\widehat{\alpha}$	0.8380	9.5583	1.1814	2.6893	71.2003	1.1086
		$\widehat{\theta}$	0.1432	0.0710	0.1940	0.0424	0.1795	0.0840
	MRE	$\widehat{\alpha}$	0.7034	1.3415	1.1756	1.1657	3.3775	1.6657
		$\widehat{\theta}$	0.4430	0.4100	0.5503	0.3146	0.5484	0.4039
40	|Bias|	$\widehat{\alpha}$	0.3128	0.4165	0.4805	0.5287	0.8685	0.6066
		$\widehat{\theta}$	0.1245	0.1481	0.1482	0.1074	0.1728	0.1341
	MSE	$\widehat{\alpha}$	0.1694	0.3204	0.6132	1.0996	3.8260	1.0707
		$\widehat{\theta}$	0.0275	0.0336	0.0360	0.0182	0.0475	0.0313
	MRE	$\widehat{\alpha}$	0.3910	0.5207	0.6007	0.6609	1.0856	0.7583
		$\widehat{\theta}$	0.2490	0.2961	0.2963	0.2148	0.3456	0.2683
80	|Bias|	$\widehat{\alpha}$	0.2524	0.2999	0.3587	0.2132	0.3909	0.2444
		$\widehat{\theta}$	0.0870	0.0852	0.1217	0.0713	0.1074	0.0876
	MSE	$\widehat{\alpha}$	0.1009	0.2165	0.2637	0.0763	0.6037	0.0953
		$\widehat{\theta}$	0.0125	0.0115	0.0219	0.0071	0.0202	0.0122
	MRE	$\widehat{\alpha}$	0.3154	0.3749	0.4484	0.2665	0.4886	0.3055
		$\widehat{\theta}$	0.1741	0.1704	0.2434	0.1427	0.2149	0.1753
100	|Bias|	$\widehat{\alpha}$	0.2061	0.2953	0.3565	0.2881	0.2647	0.2352
		$\widehat{\theta}$	0.0739	0.0796	0.1065	0.0801	0.0916	0.0797
	MSE	$\widehat{\alpha}$	0.0697	0.2040	0.2505	0.1447	0.1337	0.0939
		$\widehat{\theta}$	0.0094	0.0112	0.0174	0.0103	0.0134	0.0103
	MRE	$\widehat{\alpha}$	0.2576	0.3691	0.4456	0.3602	0.3309	0.2941
		$\widehat{\theta}$	0.1477	0.1593	0.2130	0.1602	0.1832	0.1594
150	|Bias|	$\widehat{\alpha}$	0.1533	0.1938	0.2271	0.1851	0.2819	0.2068
		$\widehat{\theta}$	0.0576	0.0626	0.0797	0.0660	0.0850	0.0672
	MSE	$\widehat{\alpha}$	0.0406	0.0742	0.1056	0.0689	0.1700	0.0856
		$\widehat{\theta}$	0.0055	0.0066	0.0094	0.0061	0.0104	0.0071
	MRE	$\widehat{\alpha}$	0.1916	0.2422	0.2839	0.2314	0.3523	0.2585
		$\widehat{\theta}$	0.1152	0.1253	0.1594	0.1321	0.1699	0.1343
500	|Bias|	$\widehat{\alpha}$	0.0909	0.1136	0.1135	0.1084	0.1052	0.1151
		$\widehat{\theta}$	0.0318	0.0391	0.0426	0.0346	0.0406	0.0391
	MSE	$\widehat{\alpha}$	0.0134	0.0216	0.0191	0.0211	0.0178	0.0212
		$\widehat{\theta}$	0.0015	0.0024	0.0027	0.0019	0.0025	0.0023
	MRE	$\widehat{\alpha}$	0.1137	0.1421	0.1419	0.1354	0.1315	0.1439
		$\widehat{\theta}$	0.0637	0.0782	0.0851	0.0692	0.0813	0.0781

4.1. Maximum likelihood estimation ($ML|_I$)

If X_v ~NAL-Exp $(x_v; \alpha, \theta)$ with $\alpha, \theta > 0$, then we are determined to derive the NAL-Exp distribution estimates $(\widehat{\alpha}, \widehat{\theta})$ if we can successfully minimize the following analytical expression:

$$ML|_I = n\log\theta - \theta\sum_{v=1}^{n} x_v - e^{-\left(\theta\sum_{v=1}^{n} x_v\right)}\log\alpha - \sum_{v=1}^{n}\log\left(1 + \alpha - \alpha^{e^{-\theta x_v}}\right).$$

The partial derivative of $ML|_I$ with respect to α and θ is shown in the appendix.

4.2. Anderson-darling estimation ($AD|_{II}$)

If X_v ~NAL-Exp $(x_v; \alpha, \theta)$ with $\alpha, \theta > 0$, then we are determined to derive the NAL-Exp distribution estimates $(\widehat{\alpha}, \widehat{\theta})$ if we can successfully minimize the following analytical expression:

$$AD|_{II} = -n - \frac{1}{n}\sum_{v=1}^{n}(2v-1)\left[\log F_{v:n}|_{X;\alpha,\theta} - \log\left(1 - F_{v:n}|_{X;\alpha,\theta}\right)\right].$$

4.3. Cramer-von Mises estimation ($CvM|_{III}$)

If X_v ~NAL-Exp $(x_v; \alpha, \theta)$ with $\alpha, \theta > 0$, then we are determined to derive the NAL-Exp distribution estimates $(\widehat{\alpha}, \widehat{\theta})$ if we can successfully minimize the following non-linear analytical expression:

Table 9
Simulation outcomes for ($\alpha = 0.15, \theta = 0.25$).

n	Est	Par	$ML\|_I$	$AD\|_{II}$	$CvM\|_{III}$	$MPS\|_{IV}$	$OLS\|_V$	$WLS\|_{VI}$
20	|Bias|	$\hat{\alpha}$	0.1191	0.1170	0.1514	0.2217	0.1283	0.2018
		$\hat{\theta}$	0.0945	0.0738	0.1189	0.0681	0.0826	0.0812
	MSE	$\hat{\alpha}$	0.0361	0.0302	0.0552	0.1475	0.0366	0.1087
		$\hat{\theta}$	0.0267	0.0111	0.0341	0.0069	0.0140	0.0127
	MRE	$\hat{\alpha}$	0.7943	0.7798	1.0094	1.4780	0.8551	1.3450
		$\hat{\theta}$	0.3781	0.2950	0.4755	0.2723	0.3306	0.3247
40	|Bias|	$\hat{\alpha}$	0.0808	0.1082	0.1034	0.0832	0.1255	0.0916
		$\hat{\theta}$	0.0447	0.0542	0.0614	0.0409	0.0661	0.0514
	MSE	$\hat{\alpha}$	0.0119	0.0235	0.0240	0.0139	0.0312	0.0157
		$\hat{\theta}$	0.0039	0.0058	0.0073	0.0027	0.0090	0.0055
	MRE	$\hat{\alpha}$	0.5387	0.7212	0.6893	0.5546	0.8369	0.6106
		$\hat{\theta}$	0.1786	0.2169	0.2456	0.1636	0.2644	0.2056
80	|Bias|	$\hat{\alpha}$	0.0504	0.0707	0.0603	0.0582	0.0687	0.0760
		$\hat{\theta}$	0.0315	0.0358	0.0400	0.0290	0.0357	0.0383
	MSE	$\hat{\alpha}$	0.0040	0.0095	0.0053	0.0072	0.0096	0.0096
		$\hat{\theta}$	0.0017	0.0020	0.0027	0.0012	0.0024	0.0023
	MRE	$\hat{\alpha}$	0.3363	0.4715	0.4019	0.3881	0.4578	0.5067
		$\hat{\theta}$	0.1260	0.1431	0.1599	0.1160	0.1428	0.1532
100	|Bias|	$\hat{\alpha}$	0.0532	0.0578	0.0578	0.0529	0.0655	0.0539
		$\hat{\theta}$	0.0286	0.0297	0.0344	0.0271	0.0326	0.0310
	MSE	$\hat{\alpha}$	0.0044	0.0052	0.0053	0.0052	0.0083	0.0048
		$\hat{\theta}$	0.0014	0.0014	0.0019	0.0011	0.0017	0.0015
	MRE	$\hat{\alpha}$	0.3545	0.3852	0.3852	0.3527	0.4369	0.3594
		$\hat{\theta}$	0.1145	0.1187	0.1376	0.1083	0.1302	0.1240
150	|Bias|	$\hat{\alpha}$	0.0429	0.0501	0.0512	0.0546	0.0506	0.0489
		$\hat{\theta}$	0.0213	0.0251	0.0286	0.0244	0.0274	0.0273
	MSE	$\hat{\alpha}$	0.0032	0.0044	0.0041	0.0047	0.0044	0.0037
		$\hat{\theta}$	0.0007	0.0010	0.0014	0.0008	0.0012	0.0012
	MRE	$\hat{\alpha}$	0.2862	0.3337	0.3412	0.3642	0.3374	0.3257
		$\hat{\theta}$	0.0852	0.1006	0.1143	0.0977	0.1095	0.1094
500	|Bias|	$\hat{\alpha}$	0.0429	0.0501	0.0512	0.0546	0.0506	0.0489
		$\hat{\theta}$	0.0213	0.0251	0.0286	0.0244	0.0274	0.0273
	MSE	$\hat{\alpha}$	0.0032	0.0044	0.0041	0.0047	0.0044	0.0037
		$\hat{\theta}$	0.0007	0.0010	0.0014	0.0008	0.0012	0.0012
	MRE	$\hat{\alpha}$	0.2862	0.3337	0.3412	0.3642	0.3374	0.3257
		$\hat{\theta}$	0.0852	0.1006	0.1143	0.0977	0.1095	0.1094

$$CvM|_{III} = \frac{1}{12n} + \sum_{v=1}^{n}\left(F_{v:n}|_{X;\alpha,\theta} - \frac{2v-1}{2n}\right)\Delta_u|_{x_v;\alpha,\theta} = 0; u = 1,2.$$

4.4. *Maximum product of spacing estimation ($MPS|_{IV}$)*

If $X_v \sim$NAL-Exp $(x_v;\alpha,\theta)$ with $\alpha,\theta > 0$, then we are determined to derive the NAL-Exp distribution estimates $(\hat{\alpha},\hat{\theta})$ if we can successfully maximize the following analytical expression:

$$MPS|_{IV} = \frac{1}{n+1}\sum_{v=1}^{n+1}\left(\log F_{v:n}|_{X;\alpha,\theta} - F_{v-1:n}|_{X;\alpha,\theta}\right).$$

4.5. *Ordinary least squares estimation ($OLS|_V$)*

If $X_v \sim$NAL-Exp $(x_v;\alpha,\theta)$ with $\alpha,\theta > 0$, then we are determined to derive the NAL-Exp distribution estimates $(\hat{\alpha},\hat{\theta})$ if we can successfully minimize the following non-linear analytical expression:

$$OLS|_V = \sum_{v=1}^{n}\left(F_{v:n}|_{X;\alpha,\theta} - \frac{v}{n+1}\right)\Delta_u|_{x_v;\alpha,\theta} = 0; u = 1,2.$$

4.6. *Weighted least squares estimation ($WLS|_{VI}$)*

If $X_v \sim$NAL-Exp $(x_v;\alpha,\theta)$ with $\alpha,\theta > 0$, then we are determined to derive the NAL-Exp distribution estimates $(\hat{\alpha},\hat{\theta})$ if we can

Table 10
Simulation outcomes for ($\alpha = 0.20, \theta = 0.70$).

n	Est	Par	$ML\|_{I}$	$AD\|_{II}$	$CvM\|_{III}$	$MPS\|_{IV}$	$OLS\|_{V}$	$WLS\|_{VI}$
20	|Bias|	$\hat{\alpha}$	0.1485	0.1511	0.1649	0.2430	0.2031	0.1599
		$\hat{\theta}$	0.2730	0.2253	0.2907	0.1697	0.2395	0.1895
	MSE	$\hat{\alpha}$	0.0397	0.0422	0.0575	0.1475	0.1108	0.0648
		$\hat{\theta}$	0.1870	0.1072	0.1792	0.0399	0.1086	0.0742
	MRE	$\hat{\alpha}$	0.7427	0.7553	0.8246	1.2151	1.0156	0.7997
		$\hat{\theta}$	0.3901	0.3218	0.4152	0.2425	0.3421	0.2708
40	|Bias|	$\hat{\alpha}$	0.1019	0.1222	0.1270	0.1702	0.1401	0.1445
		$\hat{\theta}$	0.1245	0.1512	0.2054	0.1399	0.1531	0.1458
	MSE	$\hat{\alpha}$	0.0225	0.0235	0.0344	0.0663	0.0395	0.0408
		$\hat{\theta}$	0.0249	0.0348	0.1015	0.0289	0.0401	0.0300
	MRE	$\hat{\alpha}$	0.5096	0.6111	0.6350	0.8512	0.7006	0.7225
		$\hat{\theta}$	0.1778	0.2160	0.2934	0.1998	0.2186	0.2083
80	|Bias|	$\hat{\alpha}$	0.0639	0.0803	0.0848	0.0719	0.0838	0.0850
		$\hat{\theta}$	0.0765	0.1012	0.1227	0.0969	0.1209	0.0975
	MSE	$\hat{\alpha}$	0.0066	0.0120	0.0123	0.0090	0.0109	0.0121
		$\hat{\theta}$	0.0091	0.0168	0.0245	0.0144	0.0264	0.0152
	MRE	$\hat{\alpha}$	0.3195	0.4016	0.4241	0.3593	0.4190	0.4251
		$\hat{\theta}$	0.1093	0.1446	0.1753	0.1385	0.1728	0.1393
100	|Bias|	$\hat{\alpha}$	0.0655	0.0735	0.0770	0.0707	0.0787	0.0708
		$\hat{\theta}$	0.0798	0.0819	0.1013	0.0808	0.1062	0.0862
	MSE	$\hat{\alpha}$	0.0094	0.0096	0.0091	0.0099	0.0105	0.0070
		$\hat{\theta}$	0.0110	0.0116	0.0178	0.0105	0.0189	0.0113
	MRE	$\hat{\alpha}$	0.3273	0.3676	0.3852	0.3534	0.3937	0.3540
		$\hat{\theta}$	0.1141	0.1170	0.1447	0.1155	0.1517	0.1232
150	|Bias|	$\hat{\alpha}$	0.0567	0.0588	0.0623	0.0557	0.0625	0.0510
		$\hat{\theta}$	0.0645	0.0741	0.0769	0.0582	0.0806	0.0650
	MSE	$\hat{\alpha}$	0.0054	0.0057	0.0060	0.0056	0.0063	0.0041
		$\hat{\theta}$	0.0066	0.0096	0.0099	0.0053	0.0115	0.0064
	MRE	$\hat{\alpha}$	0.2836	0.2939	0.3115	0.2784	0.3127	0.2550
		$\hat{\theta}$	0.0921	0.1059	0.1098	0.0832	0.1152	0.0929
500	|Bias|	$\hat{\alpha}$	0.0567	0.0588	0.0623	0.0557	0.0625	0.0510
		$\hat{\theta}$	0.0645	0.0741	0.0769	0.0582	0.0806	0.0650
	MSE	$\hat{\alpha}$	0.0054	0.0057	0.0060	0.0056	0.0063	0.0041
		$\hat{\theta}$	0.0066	0.0096	0.0099	0.0053	0.0115	0.0064
	MRE	$\hat{\alpha}$	0.2836	0.2939	0.3115	0.2784	0.3127	0.2550
		$\hat{\theta}$	0.0921	0.1059	0.1098	0.0832	0.1152	0.0929

successfully minimize the following non-linear analytical expression:

$$WLS|_{VI} = \frac{(n+1)(n+1)^2}{v(n-v+1)} \sum_{v=1}^{n} \left(F_{v:n}|_{X;\alpha,\theta} - \frac{v}{n+1} \right) \Delta_u|_{x_v;\alpha,\theta} = 0; u = 1, 2.$$

where $\Delta_1|_{x_v;\alpha,\theta} = \frac{\partial}{\partial\alpha}\left[\frac{\log(1+\alpha-\alpha^{e^{-\theta x_v}})}{\log\alpha}\right]$ and $\Delta_2|_{x_v;\alpha,\theta} = \frac{\partial}{\partial\theta}\left[\frac{\log(1+\alpha-\alpha^{e^{-\theta x_v}})}{\log\alpha}\right]$ and its solution can be determined numerically.

5. Simulation study

This section aims to evaluate the efficacy of the previously introduced estimation techniques through a thorough simulation study. To accomplish this task, a pertinent random sample is derived from (20), with the respective sizes of 20, 40, 80, 100, 150, and 500, and subsequently replicated 500 times. Three commonly used statistical measures, namely absolute bias, mean squared error (MSE), and mean relative error (MRE), are defined as follows:

$$|\text{Bias}||_{\hat{\Gamma}} = \frac{1}{500} \sum_{i=1}^{500} (\hat{\Gamma}_i - \Gamma), \text{MSE}|_{\hat{\Gamma}} = \frac{1}{500} \sum_{i=1}^{500} (\hat{\Gamma}_i - \Gamma)^2, \text{MRE}|_{\hat{\Gamma}} = \frac{1}{500} \sum_{i=1}^{500} |\hat{\Gamma}_i - \Gamma| / \Gamma.$$

Our findings (see Tables 4–12) indicate that $\text{Bias}|_{\hat{\Gamma}}$, $\text{MSE}|_{\hat{\Gamma}}$, and $\text{MRE}|_{\hat{\Gamma}}$ for all parameter values decrease as the sample size increases. This is something that we are able to observe. Now that we are in this position, we can conclude that every approach to estimating parameters ($\hat{\Gamma} = \hat{\alpha}, \hat{\theta}$) works really well, as the characteristic of consistency can be seen throughout all of the estimators.

Table 11
Simulation outcomes for $(\alpha = 0.05, \theta = 1.75)$.

n	Est	Par	$ML\|_I$	$AD\|_{II}$	$CvM\|_{III}$	$MPS\|_{IV}$	$OLS\|_V$	$WLS\|_{VI}$
20	|Bias|	$\hat{\alpha}$	0.0570	0.0659	0.0665	0.1147	0.0857	0.0833
		$\hat{\theta}$	0.4751	0.4339	0.7424	0.5352	0.5666	0.4987
	MSE	$\hat{\alpha}$	0.0070	0.0117	0.0138	0.0387	0.0212	0.0163
		$\hat{\theta}$	0.4809	0.3766	0.9733	0.4924	0.5991	0.4408
	MRE	$\hat{\alpha}$	1.1399	1.3176	1.3305	2.2947	1.7135	1.6663
		$\hat{\theta}$	0.2715	0.2479	0.4242	0.3058	0.3238	0.2850
40	|Bias|	$\hat{\alpha}$	0.0405	0.0463	0.0421	0.0582	0.0565	0.0428
		$\hat{\theta}$	0.4553	0.3933	0.4692	0.3536	0.4201	0.3867
	MSE	$\hat{\alpha}$	0.0028	0.0048	0.0034	0.0068	0.0064	0.0031
		$\hat{\theta}$	0.4454	0.2954	0.4135	0.1698	0.3856	0.2607
	MRE	$\hat{\alpha}$	0.8102	0.9250	0.8422	1.1649	1.1290	0.8558
		$\hat{\theta}$	0.2602	0.2247	0.2681	0.2021	0.2401	0.2210
80	|Bias|	$\hat{\alpha}$	0.0261	0.0310	0.0305	0.0367	0.0387	0.0249
		$\hat{\theta}$	0.2438	0.2910	0.3321	0.2373	0.2983	0.2162
	MSE	$\hat{\alpha}$	0.0010	0.0017	0.0018	0.0026	0.0028	0.0011
		$\hat{\theta}$	0.1441	0.1635	0.2452	0.0793	0.1547	0.0780
	MRE	$\hat{\alpha}$	0.5228	0.6194	0.6094	0.7337	0.7738	0.4981
		$\hat{\theta}$	0.1393	0.1663	0.1898	0.1356	0.1704	0.1235
100	|Bias|	$\hat{\alpha}$	0.0243	0.0281	0.0310	0.0278	0.0319	0.0289
		$\hat{\theta}$	0.2169	0.2353	0.3318	0.1901	0.2591	0.2571
	MSE	$\hat{\alpha}$	0.0010	0.0013	0.0014	0.0015	0.0017	0.0016
		$\hat{\theta}$	0.0832	0.0977	0.2215	0.0522	0.1074	0.1197
	MRE	$\hat{\alpha}$	0.4859	0.5621	0.6207	0.5552	0.6386	0.5779
		$\hat{\theta}$	0.1239	0.1345	0.1896	0.1086	0.1480	0.1469
150	|Bias|	$\hat{\alpha}$	0.0226	0.0206	0.0298	0.0227	0.0285	0.0222
		$\hat{\theta}$	0.1925	0.1740	0.2523	0.1758	0.2273	0.1878
	MSE	$\hat{\alpha}$	0.0008	0.0006	0.0013	0.0008	0.0014	0.0008
		$\hat{\theta}$	0.0602	0.0526	0.1106	0.0441	0.0734	0.0539
	MRE	$\hat{\alpha}$	0.4521	0.4125	0.5965	0.4539	0.5708	0.4434
		$\hat{\theta}$	0.1100	0.0994	0.1442	0.1005	0.1299	0.1073
500	|Bias|	$\hat{\alpha}$	0.0226	0.0206	0.0298	0.0227	0.0285	0.0222
		$\hat{\theta}$	0.1925	0.1740	0.2523	0.1758	0.2273	0.1878
	MSE	$\hat{\alpha}$	0.0008	0.0006	0.0013	0.0008	0.0014	0.0008
		$\hat{\theta}$	0.0602	0.0526	0.1106	0.0441	0.0734	0.0539
	MRE	$\hat{\alpha}$	0.4521	0.4125	0.5965	0.4539	0.5708	0.4434
		$\hat{\theta}$	0.1100	0.0994	0.1442	0.1005	0.1299	0.1073

6. Application to precipitation data

This section thoroughly investigates four distinct datasets of precipitation data, with a focus on evaluating the performance of the proposed ANL-Exp distribution. In addition to the ANL-Exp distribution, the examination also encompasses its competitors in the field. The analysis aims to shed light on the effectiveness of the proposed model in comparison to its peers and provide a comprehensive understanding of its capabilities.

The first data set includes 59 years (1950–2009) of information, detailing the heaviest individual year's rain in Karachi, Pakistan (with the exception of 1987). The measurements were taken in inches. In order to properly manage water resources and construct flood barriers, accurate precipitation statistics are essential. Precipitation records can be used to forecast extreme weather events such as floods and droughts. Large hydraulic systems may also benefit from the rainfall data since it can be used to reduce potential dangers. The following list of descriptive information reported by Bhatti et al. [30] may be helpful: minimum value = 5.6, 1st quartile = 39.6, median = 92.7, mean = 118.4, 3rd quartile = 160.2, maximum value = 429.3, skewness = 0.9982, kurtosis = 3.7660, and 95% confidence interval = (94.1047, 142.6885).

The second data set reports statistics on 30 observations of the precipitation that occurred in March in Minneapolis/St. Paul (USA), measured in inches. The following list of descriptive information reported by Hinkley [31] may be helpful: minimum value = 0.320, 1st quartile = 0.915, median = 1.470, mean = 1.675, 3rd quartile = 2.087, maximum value = 4.750, skewness = 1.032, kurtosis = 4.206, and 95% confidence interval = (1.3013, 2.0486).

In the third data set, the Los Angeles (USA) Civic Centre kept a record of the total yearly rainfall that occurred during the month of January from 1880 to 1916. The measurements were taken in inches. The following list of descriptive information reported by Selim [32] may be helpful: minimum value = 0.070, 1st quartile = 1.170, median = 2.490, mean = 3.495, 3rd quartile = 5.840, maximum value = 13.300, skewness = 1.106, kurtosis = 3.890, and 95% confidence interval = (2.4397, 4.5494).

This particular fourth data set pertains to the total monthly rainfall that was recorded in the month of April in Sao Carlos, which is situated in the southeasterly region of Brazil. The study of the behavior of dry and wet periods has been shown to be geopolitical and

Table 12
Simulation outcomes for $(\alpha = 0.79, \theta = 0.31)$.

n	Est	Par	$ML|_I$	$AD|_{II}$	$CvM|_{III}$	$MPS|_{IV}$	$OLS|_V$	$WLS|_{VI}$
20	|Bias|	$\hat{\alpha}$	0.4431	1.0512	1.2502	1.6967	1.6035	1.3848
		$\hat{\theta}$	0.1142	0.1253	0.1677	0.1062	0.1309	0.1294
	MSE	$\hat{\alpha}$	0.3809	6.7835	8.7752	23.8327	20.0615	19.8796
		$\hat{\theta}$	0.0252	0.0283	0.0489	0.0173	0.0290	0.0289
	MRE	$\hat{\alpha}$	0.5609	1.3306	1.5826	2.1477	2.0298	1.7529
		$\hat{\theta}$	0.3683	0.4042	0.5411	0.3427	0.4224	0.4174
40	|Bias|	$\hat{\alpha}$	0.3850	0.3975	0.4405	0.4944	0.7100	0.4527
		$\hat{\theta}$	0.0825	0.0872	0.0982	0.0781	0.1055	0.0807
	MSE	$\hat{\alpha}$	0.4416	0.5020	0.4478	0.4897	1.8060	0.4197
		$\hat{\theta}$	0.0125	0.0140	0.0184	0.0088	0.0178	0.0114
	MRE	$\hat{\alpha}$	0.4874	0.5031	0.5576	0.6258	0.8987	0.5730
		$\hat{\theta}$	0.2661	0.2814	0.3166	0.2518	0.3404	0.2604
80	|Bias|	$\hat{\alpha}$	0.2667	0.2795	0.3131	0.2347	0.2940	0.3100
		$\hat{\theta}$	0.0551	0.0593	0.0704	0.0452	0.0594	0.0635
	MSE	$\hat{\alpha}$	0.1191	0.1223	0.1781	0.0980	0.1984	0.1821
		$\hat{\theta}$	0.0047	0.0059	0.0081	0.0032	0.0054	0.0066
	MRE	$\hat{\alpha}$	0.3376	0.3538	0.3963	0.2971	0.3721	0.3923
		$\hat{\theta}$	0.1778	0.1913	0.2271	0.1459	0.1915	0.2048
100	|Bias|	$\hat{\alpha}$	0.2106	0.2569	0.2627	0.2282	0.2482	0.2207
		$\hat{\theta}$	0.0488	0.0565	0.0541	0.0480	0.0574	0.0454
	MSE	$\hat{\alpha}$	0.0748	0.1218	0.1278	0.0932	0.1213	0.0753
		$\hat{\theta}$	0.0039	0.0051	0.0047	0.0034	0.0051	0.0035
	MRE	$\hat{\alpha}$	0.2666	0.3251	0.3325	0.2889	0.3141	0.2793
		$\hat{\theta}$	0.1574	0.1822	0.1745	0.1549	0.1853	0.1463
150	|Bias|	$\hat{\alpha}$	0.1799	0.2042	0.2189	0.2033	0.2447	0.1705
		$\hat{\theta}$	0.0377	0.0452	0.0497	0.0407	0.0475	0.0400
	MSE	$\hat{\alpha}$	0.0526	0.0684	0.0765	0.0729	0.1209	0.0456
		$\hat{\theta}$	0.0023	0.0030	0.0048	0.0026	0.0038	0.0024
	MRE	$\hat{\alpha}$	0.2278	0.2585	0.2771	0.2574	0.3097	0.2158
		$\hat{\theta}$	0.1217	0.1458	0.1602	0.1312	0.1533	0.1289
500	|Bias|	$\hat{\alpha}$	0.1799	0.2042	0.2189	0.2033	0.2447	0.1705
		$\hat{\theta}$	0.0377	0.0452	0.0497	0.0407	0.0475	0.0400
	MSE	$\hat{\alpha}$	0.0526	0.0684	0.0765	0.0729	0.1209	0.0456
		$\hat{\theta}$	0.0023	0.0030	0.0048	0.0026	0.0038	0.0024
	MRE	$\hat{\alpha}$	0.2278	0.2585	0.2771	0.2574	0.3097	0.2158
		$\hat{\theta}$	0.1217	0.1458	0.1602	0.1312	0.1533	0.1289

Table 13
Statistics for annual precipitation in Karachi (Pakistan).

Model	Estimates with SE (.)				Fit statistics	
	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\eta}$	$\hat{\varphi}$	KS	p-value
NAL-Exp	*0.3947(0.1830)*	*0.01285(0.0025)*	-	-	*0.0867*	*0.7670*
AP-Exp	–	–	4.8143 (3.6596)	0.0118 (0.0019)	0.0922	0.6972
NH-Exp	0.0040 (0.0012)	–	1.7191 (0.4213)	-	0.1000	0.5968
E-Exp	-	0.0105 (0.0016)	-	1.4471 (0.2606)	0.1034	0.5530
Exp	118.47 (15.4330)	-	-	–	0.1046	0.5390
Etr-Exp	-	0.0116 (0.0139)	1.3022 (3.2462)	–	0.1050	0.5330
L-Exp	0.0092 (0.0009)	–	2.0987 (0.3057)	–	0.1202	0.3610
LN	–	4.3853 (–)	0.9952 (–)	–	0.1218	0.3455
GEV	−0.1044 (–)	–	–	139.8824 (–)	0.3678	0.0090
G	5446.275 (–)	–	–	–	0.6035	0.0007

economically vital for the growth of such a metropolis, which plays an active industrial role and is of great agricultural significance. The following list of descriptive information may be helpful, including: minimum value = 0.50, 1st quartile = 34.90, median = 76.00, mean = 80.88, 3rd quartile = 112.40, maximum value = 204.20, skewness = 0.480, kurtosis = 2.511, and 95% confidence interval = (66.4827, 95.2832). This data set has been reported by Bakouch et al. [33].

Lastly, the fifth set of data includes the highest quantity of precipitation, measured in millimeters (mm), that was ever seen in the city of Kalat, Pakistan, over the period of time between 1981 and 2010 (30 years). The data set consists of thirty different values, each of which represents the highest amount of rainfall that occurred during a certain year. The following list of descriptive information

Table 14
Statistics for precipitation in Minneapolis/St. Paul (USA).

Model	Estimates with SE (.)				Fit statistics	
	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\eta}$	$\hat{\varphi}$	KS	p-value
NAL-Exp	*0.0505 (0.0597)*	*1.7553(0.4675)*	-	-	*0.0647*	*0.9996*
MLII-Exp	–	0.9482 (0.0717)	0.2673 (0.3122)	3.8150 (3.4229)	0.0736	0.9969
L-Exp	0.6350 (0.0689)	–	2.9218 (0.5280)	–	0.0888	0.9719
MO-Exp	3.5998 (1.0590)	–	–	–	0.1103	0.8585
NH-Exp	0.0197 (–)	–	19.7773 (–)	–	0.1600	0.4265
HL-Exp	–	–	–	0.8696 (0.1287)	0.1895	0.2314
DUS-Exp	–	0.7798 (0.1216)	–	–	0.1988	0.1868
Etr-Exp	–	0.8640 (6.6706)	1.1739 (17.2513)	–	0.2352	0.0725
Exp	1.6749 (0.3057)	–	–	–	0.2352	0.0723
LN	–	0.3373 (–)	0.6226 (–)	–	0.0912	0.9640
GEV	−0.1824 (–)	–	–	1.2910 (–)	0.3790	0.0044
G	1.0647 (–)	–	–	–	0.3879	0.0002

Table 15
Statistics for precipitation in Los Angeles (USA).

Model	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\eta}$	$\hat{\varphi}$	KS	p-value
NAL-Exp	*0.7960(0.4452)*	*0.3191(0.0952)*	–	–	*0.0851*	*0.9514*
AP-Exp	–	–	1.4861 (1.4927)	0.3143 (0.0862)	0.0858	0.9483
Etr-Exp	–	0.3985 (2.0551)	1.2654 (13.1191)	–	0.0878	0.9376
E-Exp	–	0.2943 (0.0621)	–	1.0447 (0.2248)	0.0886	0.9335
MLII-Exp	–	0.0808 (0.5063)	0.2438 (–)	1.1928 (–)	0.0887	0.9327
NH-Exp	0.1741 (0.1490)	–	1.3948 (0.8202)	–	0.0933	0.9042
HL-Exp	–	–	–	0.3877 (0.0539)	0.1283	0.5767
L-Exp	0.3234 (0.0525)	–	1.7789 (0.3548)	–	0.1495	0.3801
MO-Exp	13.2476 (4.7851)	–	–	–	0.2449	0.0237
LN	–	0.7052 (–)	1.2408 (–)	–	0.1158	0.7035
GEV	0.0334 (–)	–	–	2.6599 (–)	0.3685	0.0081
G	2.7661 (–)	–	–	–	0.3685	0.0001

Table 16
Statistics for monthly precipitation Sao Carlos (Brazil).

Model	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\eta}$	$\hat{\varphi}$	KS	p-value
NAL-Exp	*0.1496(0.0915)*	*0.0265(0.0051)*	–	–	*0.0729*	*0.9213*
L-Exp	0.01285 (0.0012)	–	2.5685 (0.3688)	–	0.0736	0.9158
AP-Exp	–	–	15.5871 (12.6061)	0.0209 (0.0029)	0.0898	0.7523
E-Exp	–	0.0157 (0.0024)	–	1.5137 (0.2835)	0.1096	0.5125
HL-Exp	–	–	–	0.0175 (0.0019)	0.1113	0.4924
NH-Exp	0.0031 (0.0008)	–	2.9111 (0.6735)	–	0.1159	0.4419
DUS-Exp	–	0.01585 (0.0018)	–	–	0.1236	0.3633
Etr-Exp	–	0.0167 (0.0237)	1.3434 (4.0384)	–	0.1604	0.1171
Exp	80.850 (11.1011)	–	–	–	0.1607	0.1159
LN	–	4.0343 (–)	1.1354 (–)	–	0.1565	0.1333
GEV	−0.3987 (–)	–	–	128.627 (–)	0.3832	0.0078
G	3472.360 (–)	–	–	–	0.6105	0.0012

reported by Ahmad et al. [34] may be helpful: minimum value = 21.7, 1st quartile = 57.8, median = 73.1, mean = 92.06, 3rd quartile = 95.8, maximum value = 546, skewness = 3.9407, kurtosis = 20.8015, and 95% confidence interval = (56.5796, 127.5423).

Several well-known alternative models have been suggested and compared to the proposed NAL-Exp distribution include: logistic exponential (defined as L-Exp) by Lan and Leemis [35], alpha power exponential (defined as AP-Exp) by Mahdavi and Kundu [36], exponentiated exponential (defined as E-Exp) by Ahuja and Nash [37], half logistic exponential (defined as HL-Exp) by Balakrishnan [38], Nadarajah-Haghighi exponential (defined as NH-Exp) by Nadarajah and Haghighi [39], DUS exponential (defined as DUS-Exp) by Kumar et al. [40], Earlang truncated exponential (defined as Etr-Exp) by El-Alosey [41], exponential (defined as Exp) by Nadarajah and Kotz [42], Marshall-Olkin exponential (defined as MO-Exp) by Salah et al. [43], modified Lehmann type II (defined as MLII-Exp) by Balogun et al. [44], generalized Pareto (defined as GP) by Pickands [45]. The LN, G, and GEV distributions are also frequently discussed as common models.

Estimates as well as standard errors (defined as SEs) along with the common fit test (defined as FT) are reported in Tables 13–17 for the contestant as well as the proposed NAL-Exp distributions. FT includes Kolmogorov-Smirnov (defined as $KS = \text{Sup}_x \left| F_{v:n}|_{X;\alpha,\theta} - F|_{X;\alpha,\theta} \right|$)

Table 17
Statistics for annual precipitation Kalat (Pakistan).

Model	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\eta}$	$\hat{\varphi}$	KS	p-value
NAL-Exp	*0.1908(0.1372)*	*0.0235(0.0062)*	–	–	*0.1616*	*0.3931*
MLII-Exp		0.9428 (0.0333)	0.2701 (0.0606)	0.0665 (0.0019)	0.1716	0.3227
GP	–	0.1221 (–)	61.0114 (–)	–	0.1939	0.1980
HL-Exp	–	–	–	0.0161 (0.0025)	0.2280	0.0832
DUS-Exp	–	0.0143 (0.0022)		–	0.2358	0.0667
NH-Exp	0.0082 (0.0037)	–	1.2114 (0.3552)	–	0.2525	0.0407
Etr-Exp	–	0.0146 (0.0325)	1.3520 (6.3968)	–	0.2694	0.0239
Exp	91.630 (16.935)	–	–	–	0.2705	0.0230
MO-Exp	8408.234 (–)	–	–	–	1.0000	0.0030
G	1659.47 (–)	–	–	–	0.5652	0.0025
GEV	0.1678 (–)	–	–	43.0425 (–)	0.4313	0.0010

Table 18
Classical estimates for annual precipitation in Karachi (Pakistan).

Estimates	$ML\|_I$	$AD\|_{II}$	$CvM\|_{III}$	$MPS\|_{IV}$	$OLS\|_V$	$WLS\|_{VI}$
$\hat{\alpha}$	0.3961	0.4662	0.5277	0.4897	*0.5796*	0.5073
$\hat{\theta}$	0.0128	0.0118	0.0110	0.0115	*0.0104*	0.0113
CVM	0.0525	0.0511	0.0505	0.0508	*0.0504*	0.0507
AD	0.3057	0.2954	0.2905	0.2931	*0.2887*	0.2918
KS	0.0859	0.0816	0.0771	0.0797	*0.0736*	0.0791
KS (p-value)	0.7768	0.8268	0.8747	0.8474	*0.9068*	0.8537

Table 19
Classical estimates for precipitation in Minneapolis/St. Paul (USA).

Estimates	$ML\|_I$	$AD\|_{II}$	$CvM\|_{III}$	$MPS\|_{IV}$	$OLS\|_V$	$WLS\|_{VI}$
$\hat{\alpha}$	0.0505	*0.0566*	0.0518	0.1073	0.0749	0.0681
$\hat{\theta}$	1.7554	*1.7333*	1.7978	1.4212	1.6211	1.6473
CVM	0.0305	*0.0299*	0.0303	0.0263	0.0283	0.0289
AD	0.2228	*0.2191*	0.2217	0.1962	0.2094	0.2129
KS	0.0647	*0.0560*	0.0622	0.0802	0.0642	0.0608
KS (p-value)	0.9996	*1.0000*	0.9998	0.9904	0.9997	0.9999

Table 20
Classical estimates for precipitation in Los Angeles (USA).

Estimates	$ML\|_I$	$AD\|_{II}$	$CvM\|_{III}$	$MPS\|_{IV}$	$OLS\|_V$	$WLS\|_{VI}$
$\hat{\alpha}$	0.7960	*1.0203*	1.1180	1.0613	1.3044	1.1170
$\hat{\theta}$	0.3191	*0.2719*	0.2547	0.2648	0.2281	0.2552
CVM	0.0459	*0.0447*	0.0448	0.0447	0.0455	0.0448
AD	0.3157	*0.3248*	0.3314	0.3274	0.3458	0.3313
KS	0.0851	*0.0811*	0.0841	0.0827	0.0896	0.0844
KS (p-value)	0.9515	*0.9682*	0.9560	0.9619	0.9279	0.9549

with a p-value. The simplest rule to consider for the best-fit model is the lowest value of FT that makes the proposed NAL-Exp distribution qualify and compete with known contestants, as outcomes are presented in Tables 13–17.

Additionally, Tables 18–22 show the outcomes of classical estimates that allow us to use $OLS|_V$ to analyze Karachi's annual precipitation (Pakistan), $AD|_{II}$ to analyze Minneapolis/St. Paul, and Los Angeles (USA), $WLS|_{VI}$ to analyze Sao Carlos (Brazil), and $OLS|_V$ to analyze annual precipitation Kalat (Pakistan) data. Furthermore, Figs. 2–7 depict some empirically fitted plots (I–V), including pdf, cdf, pp-plot, profile log-likelihood (α, θ), violin, and TTT-plots. The statistical software programme R is the tool of choice for all numerical computations, including the packages Model-Adequacy, nlmib, and Optim.

7. Novelties and contributions

- Proposed NAL-G Class: Our study introduces a novel approach to address the limitations of traditional precipitation prediction models. We propose the use of the alpha logarithmic-generated (NAL-G) class of distributions, a novel and innovative technique for modeling precipitation data. The NAL-G class offers improved accuracy and reliability compared to classic distribution models.

Table 21
Classical estimates for monthly precipitation Sao Carlos (Brazil).

Estimates	$ML\|_{I}$	$AD\|_{II}$	$CvM\|_{III}$	$MPS\|_{IV}$	$OLS\|_{V}$	$WLS\|_{VI}$
$\hat{\alpha}$	0.1504	0.1914	0.2051	0.1930	0.2312	*0.1996*
$\hat{\theta}$	0.0265	0.0242	0.0233	0.0242	0.0222	*0.0238*
CVM	0.0400	0.0432	0.0444	0.0434	0.0466	*0.0439*
AD	0.3132	0.3403	0.3504	0.3412	0.3686	*0.3460*
KS	0.0732	0.0679	0.0673	0.0695	0.0741	*0.0662*
KS (p-value)	0.9194	0.9537	0.9567	0.9443	0.9122	*0.9623*

Table 22
Classical estimates for annual precipitation Kalat (Pakistan).

Estimates	$ML\|_{I}$	$AD\|_{II}$	$CvM\|_{III}$	$MPS\|_{IV}$	$OLS\|_{V}$	$WLS\|_{VI}$
$\hat{\alpha}$	0.1906	0.0256	0.0022	0.3120	*0.0037*	0.0041
$\hat{\theta}$	0.0236	0.0432	0.0664	0.0188	*0.0616*	0.0599
CVM	0.2794	0.3102	0.3732	0.2690	*0.3584*	0.3586
AD	1.7806	1.9343	2.2468	1.7231	*2.1736*	2.1769
KS	0.1617	0.1221	0.1090	0.1818	*0.1097*	0.1199
KS (p-value)	0.3923	0.7350	0.8445	0.2599	*0.8395*	0.7544

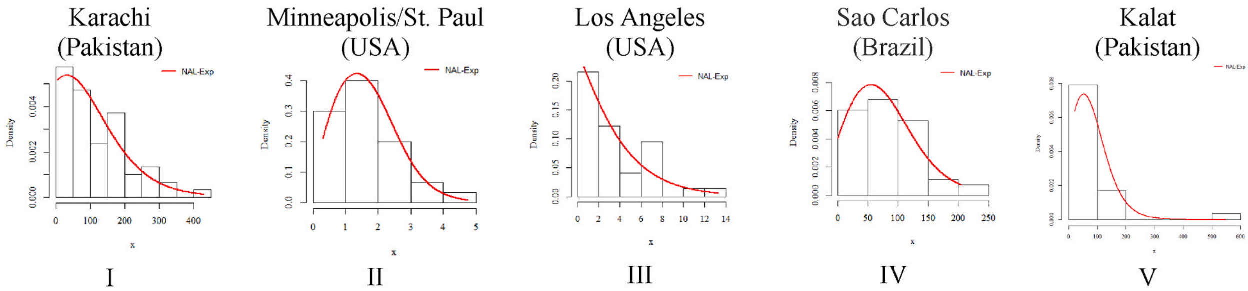

Fig. 2. Fitted density plots (I–V) for precipitation in Pakistan, USA, Brazil and Kalat.

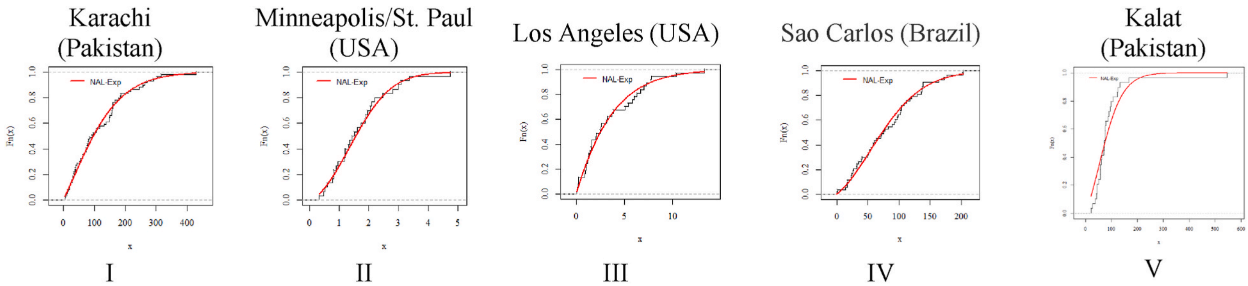

Fig. 3. Fitted distribution function plots (I–V) for precipitation in Pakistan, USA, Brazil and Kalat.

- Enhanced NAL-Exp Distribution: Within the NAL-G class, we particularly focus on the NAL-Exp distribution, which demonstrates superior performance in modeling rainfall data. Its increased flexibility and ability to capture the complex structure of precipitation make it a valuable tool in hydrological modeling and analysis.
- Comprehensive Evaluation: We conduct an exhaustive investigation into the mathematical properties of the NAL-G class and the NAL-Exp distribution. This includes moments, quantile functions, entropy measures, and order statistics. The comprehensive evaluation provides a deeper understanding of the proposed approach's capabilities and robustness.
- Superior Performance: Through extensive simulations and comparisons, our findings show that the NAL-Exp distribution outperforms other commonly used models. It demonstrates its potential as a powerful tool for accurate and reliable rainfall data modeling, which has significant implications for hydrology, climate modeling, and weather prediction.
- Alternative to Established Distributions: Our proposed approach offers an alternative to several established distributions found in the literature. The NAL-G class provides simple closed-form solutions for reliability and hazard rate functions, facilitating the analysis and prediction of future rainfall events.

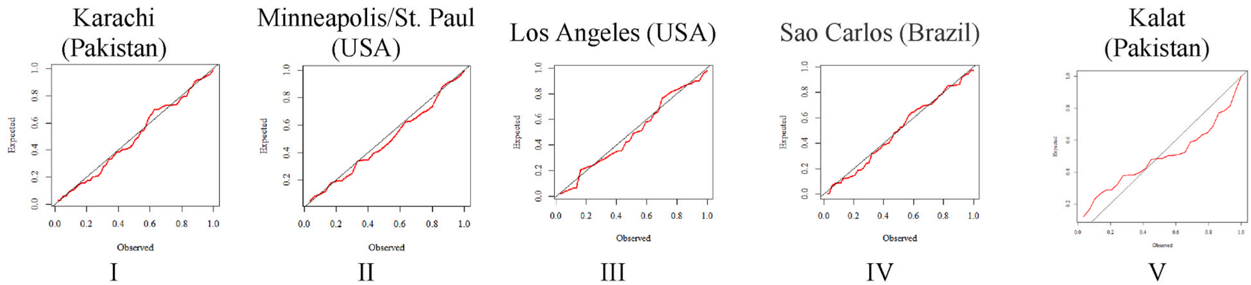

Fig. 4. Probability-Probability plots (I–V) for precipitation in Pakistan, USA, Brazil and Kalat.

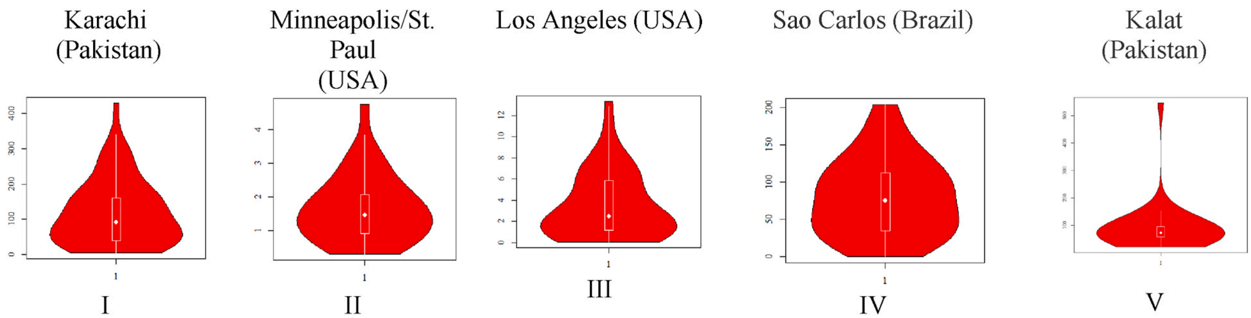

Fig. 5. Violin plots (I–V) for Precipitation in Pakistan, USA, Brazil and Kalat.

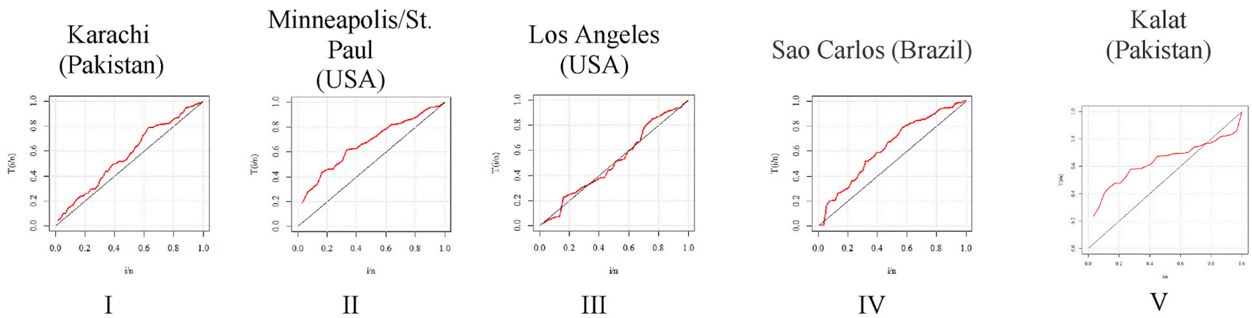

Fig. 6. TTT plots (I–V) for Precipitation in Pakistan, USA, Brazil and Kalat.

- By bolding these key contributions, we aim to clearly convey the novelty and significance of our research, emphasizing its potential impact on the field of hydrology and related areas.

8. Conclusions

In conclusion, this scientific study introduces a novel and intriguing approach to address the limitations of traditional precipitation prediction models. By leveraging the alpha logarithmic-generated (NAL-G) class of distributions, our proposed method demonstrates enhanced accuracy and reliability compared to commonly used classical distribution models for rainfall data modeling. The NAL-G class, particularly the NAL-Exp distribution, emerges as a promising and innovative technique for modeling precipitation data, especially due to its ability to capture the intricate structure of rainfall. This makes it a valuable tool in hydrological modeling and analysis, with potential practical applications in hydrology, climate modeling, and weather prediction. The authors thoroughly investigate the mathematical properties of the NAL-G class and the NAL-Exp distribution, encompassing moments, quantile functions, entropy measures, and order statistics, providing a comprehensive evaluation of the proposed approach. The results highlight the superiority of the NAL-Exp distribution over other commonly used models, emphasizing its potential for accurately representing rainfall data across various locations, as indicated by KS (p-value) for Karachi (Pakistan) as 0.7670, Minneapolis/St. Paul (USA) as 0.9960, Los Angeles (USA) as 0.9514, Sao Carlos (Brazil) as 0.9213 and Kalat (Pakistan) as 0.3925. Furthermore, the proposed method offers simple closed-form solutions for reliability and hazard rate functions, facilitating the analysis and prediction of future rainfall events. It also serves as an alternative to existing distributions found in relevant research, expanding the options available for modeling precipitation data.

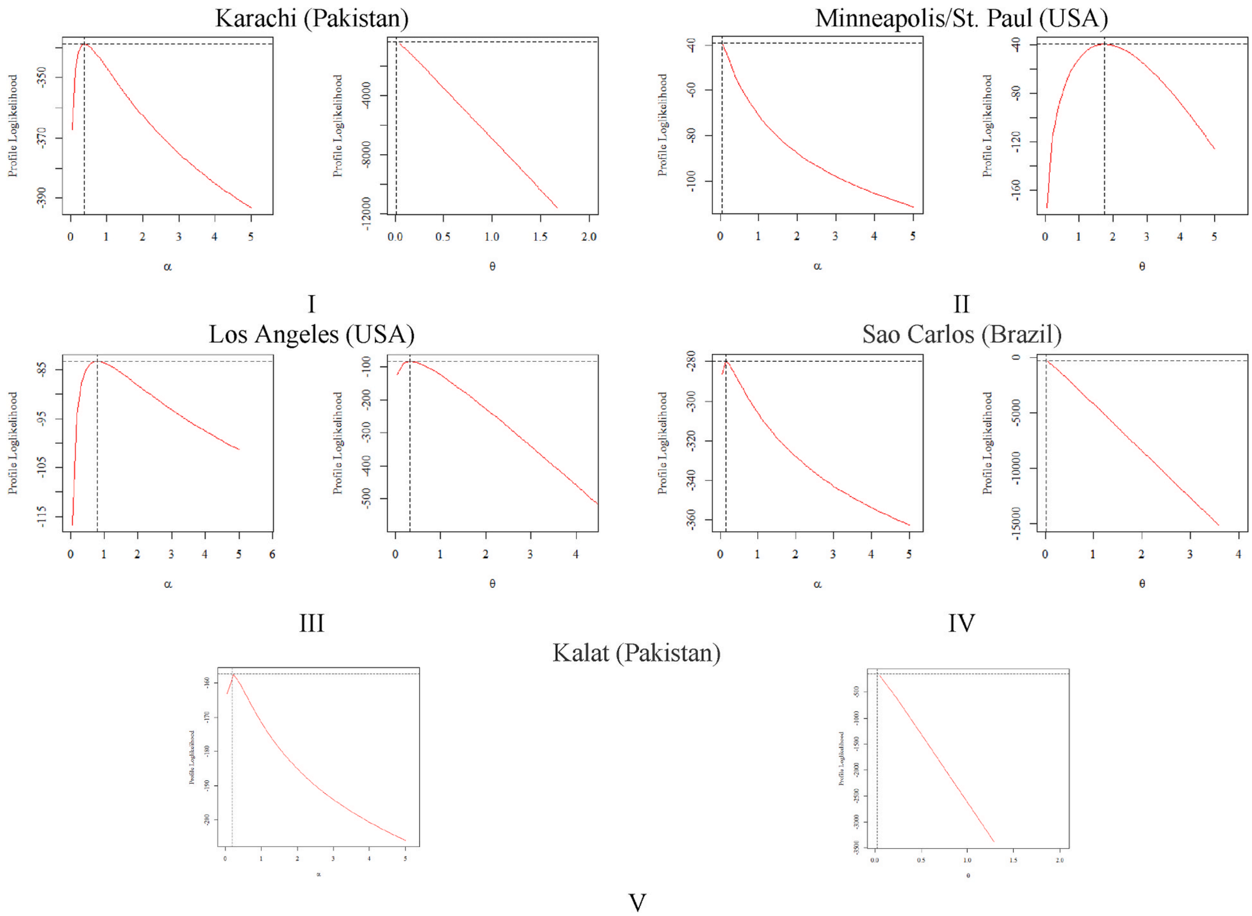

Fig. 7. Profile log-likelihood plots (I–V) for precipitation in Pakistan, USA, Brazil and Kalat.

Our findings demonstrate that the absolute bias, mean squared error (MSE), and mean relative error (MRE) consistently decrease with increasing sample size, underscoring the robustness and consistency of each estimation approach. This supports the reliability of the proposed method across diverse data scenarios. With the potential to significantly impact hydrology, climate modeling, and weather prediction, the proposed NAL-G class and NAL-Exp distribution offer practical benefits and provide valuable insights for future research and innovations in this field. The method's adaptability to various data scenarios and its consistent performance with larger datasets inspire further exploration of its applications in hydrological modeling and related domains.

9. Future work

This section discusses several possible future research directions that might expand on the findings of our study and enhance the modeling and analysis of rainfall data. Future research will address current issues, take into account more climatic variables, expand the models to multivariate analysis, and investigate practical applications in many industries. In order to better manage water resources, adapt to climate change, and prepare for disasters, we may improve our knowledge of rainfall patterns, increase forecasting accuracy, and create more reliable models by following these study approaches.

Author contribution statement

Aned Al Mutairi: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Data availability statement

All previously analyzed datasets are appropriately cited and referenced in the paper.

Additional information

No additional information is available for this paper.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

Precipitation data for Karachi (Pakistan)

117.6, 157.7, 148.6, 11.4, 5.6, 63.6, 62.4, 11.8, 6.5, 54.9,39.9, 16.8, 30.2, 38.4, 76.9, 73.4, 85, 256.3, 24.9, 148.6, 160.5, 131.3, 77, 155.2, 217.2, 105.5, 166.8, 157.9, 73.6, 291.4, 210.3, 315.7, 107.7, 33.3, 302.6, 159.1, 78.7, 33.2, 52.2, 92.7,150.4, 43.7, 68.3, 20.8, 179.4, 245.7, 19.5, 30, 270.4, 160, 96.3, 185.7, 429.3, 184.9, 262.5, 80.6, 138.2, 28, 39.3.

Precipitation data for Minneapolis/St Paul

0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47,1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

Precipitation data for Los Angeles

1.33, 1.43, 1.01, 1.62, 3.15, 1.05, 7.72, 0.2, 6.03, 0.25, 7.83, 0.25, 0.88, 6.29, 0.94, 5.84, 3.23, 3.7, 1.26, 2.64, 1.17, 2.49, 1.62, 2.1, 0.14, 2.57, 3.85, 7.02, 5.04, 7.27, 1.53, 6.7, 0.07, 2.01, 10.35, 5.42, 13.3.

Precipitation data for Sao Carlos (Month April) (Brazil)

59.00, 102.20, 17.30, 23.00, 50.60, 27.00, 203.00, 40.90, 53.00, 177.40, 94.60, 129.40, 76.00, 93.20, 22.80, 98.80, 77.70, 204.20, 16.90, 55.10, 103.90, 34.90, 39.70, 137.70, 104.20, 117.60, 17.10, 120.80, 164.90, 50.20, 172.80, 58.50, 112.40, 24.50, 32.80, 64.00, 72.10, 139.30, 0.50, 70.90, 0.80, 82.70, 108.60, 32.30, 13.60, 25.70, 135.80, 136.80, 89.70, 139.20, 102.80, 97.30, 60.60.

Precipitation data for Kalat (Pakistan)

58.65, 60.71, 64.01, 42.6, 75.8, 88.6, 90.1, 97.9, 105.6, 73.1, 76.6, 78.5, 58.3, 122.5, 57.8, 546, 125.8, 50.5, 45.9, 21.7, 45.5, 38, 75.4, 168.2, 72.9, 95.8, 133.4, 71.9, 28.

The partial derivatives of the MLE shown in Section 4.2, denoted by $\boldsymbol{ML}|_{\boldsymbol{I}}$ with respect to $\boldsymbol{\alpha}$ and $\boldsymbol{\theta}$, are presented below.

$$\frac{\partial ML|_I}{\partial \alpha} = -\frac{e^{-\left(\theta \sum_{v=1}^{n} x_v\right)}}{\alpha} - \sum_{v=1}^{n} \left[\frac{1 + \theta x_v \alpha^{e^{-\theta x_v} - 1}}{1 + \alpha - \alpha^{e^{-\theta x_v}}}\right].$$

$$\frac{\partial ML|_I}{\partial \theta} = \frac{n}{\theta} - \sum_{v=1}^{n} x_v + e^{-\left(\theta \sum_{v=1}^{n} x_v\right)} log\alpha + \sum_{v=1}^{n} \left[\frac{x_v e^{-\theta x_v} \alpha^{e^{-\theta x_v}}}{1 + \alpha - \alpha^{e^{-\theta x_v}}}\right].$$

R code

```
cdf_new = function (par,x)
{
  a = par [1]
  b = par [2]
  log (1 + a − a^(1 − (1 − exp (−b*x))))/log(a)
  }
pdf_new = function (par,x)
{
  a = par [1]
  b = par [2]
  a^(1 − (1 − exp (−b * x))) * (log(a) * (exp (−b * x) * b))/(1 +
  a − a^(1 − (1 − exp (−b * x))))/log(a)
  }
result_new = goodness. fit (pdf = pdf_new, cdf = cdf_new, starts = c (1.1,.1), data = x, method =
"B", domain = c (0,Inf))
```

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