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Research article

A statistical framework for a new Kavya-Manoharan Bilal distribution using ranked set sampling and simple random sampling

Anum Shafiq [a,b,*], Tabassum Naz Sindhu [c], Muhammad Bilal Riaz [b,d,**], Marwa K. H. Hassan [e], Tahani A. Abushal [f]

[a] School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing, 210044, China
[b] IT4Innovations, VSB -Technical University of Ostrava, Ostrava, Czech Republic
[c] Department of Statistics, Quaid-i-Azam University, Islamabad, 44000, Pakistan
[d] Department of Computer Science and Mathematics, Lebanese American University, Byblos, Lebanon
[e] Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, 11566, Egypt
[f] Dept. of Mathematical Sciences, Umm Al-Qura University, Makkah, 24382, Saudi Arabia

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ABSTRACT

In survival and stochastic lifespan modeling, numerous families of distributions are sometimes considered unnatural, unjustifiable theoretically, and occasionally superfluous. Here, a novel parsimonious survival model is developed using the Bilal distribution (BD) and the Kavya-Manoharan (KM) parsimonious transformation family. In addition to other analytical properties, the forms of probability density function (PDF) and behavior of the distributions' hazard rates are analyzed. The insights are theoretical as well as practical. Theoretically, we offer explicit equations for the single and product moments of order statistics from Kavya-Manoharan Bilal Distribution. Practically, maximum likelihood (ML) technique, which is based on simple random sampling (SRS) and ranked set sampling (RSS) sample schemes, is employed to estimate the parameters. Numerical simulations are used as the primary methodology to compare the various sampling techniques.

1. Introduction

To improve survival data modelling, statisticians and applied researchers are becoming more motivated to establish adaptive lifetime models. As a result, substantial progress has been made in the generalization and application of several well-known lifetime models. The Bilal distribution (BD) (ξ), a novel one-parameter lifespan distribution, was first presented by Abd-Elrahman [1]. The BD (ξ), has been demonstrated to fall under the category of new renewal failure rates that are better than average. The underlying PDF and cumulative distribution function (CDF) are specified as follows in the description:

* Corresponding author. School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing, 210044, China.
** Corresponding author. IT4Innovations, VSB -Technical University of Ostrava, Ostrava, Poruba, 70800, Czech Republic.
E-mail addresses: anumshafiq@ymail.com (A. Shafiq), muhammad.bilal.riaz@vsb.cz (M.B. Riaz).

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$$f(w;\xi)=\frac{6}{\xi}e^{\frac{-2w}{\xi}}\left(1-e^{\frac{-w}{\xi}}\right); w,\xi>0, \tag{1}$$

and

$$F(w;\xi)=1-e^{\frac{-2w}{\xi}}\left(3-2e^{\frac{-w}{\xi}}\right), \tag{2}$$

respectively. The parameter ξ is the scale parameter. Now W will be referred to as a random variable with PDF (1). Below is a description of the main attribute of the BD. The hazard rate function (hrf) of the BD can only be increasing. As a result, it is useless for modelling non-monotonic human mortality or mechanical lifetimes. The BD may have received less consideration because of the prominence of exponential model, but interest in its extensions, refinements, and associated applications has recently surged. Altun et al. [2], proposed a new one-parameter discrete distribution, called a discrete BD. Maya et al. [3], addressed the use of U-statistics in the estimation of the scale parameter of the BD. The log-BD and associated regression, which Altun et al. [4], introduced, enhance modelling of the highly skewed dependent variables with related covariates. The BD has been generalized in a number of ways. Abd-Elrahman [5], suggested a novel two-parameter lifespan model as a generalized version of the BD (ξ). He examined at the characteristics of this distribution's failure rate function and probability density. The maximum likelihood estimates (MLEs) of uncertain parameters were computed under the entire sample and a thorough mathematical study of the GB distribution was offered. A Type-II censored sample was used by Abd-Elrahman [6], to provide MLEs, Bayesian predictions of the unknown parameters, and reliability functions. Shi et al. [7], examined entropy and parameter estimation using the generalized Bilal (GB) model centered on adaptive Type-II progressive hybrid (A-T–II–PH) censored data.

Greater flexibility is provided by additional parameters, but the complexity of the estimation also rises. The Dinesh-Umesh-Sanjay (DUS) transformation was suggested by Kumar et al. [8], as a solution to overcome this to create new parsimonious classes of distributions. This is how it goes. The DUS transformation creates a new cdf $G(w)$ expressed as $G(w)=\frac{e^{F(w)}-1}{e-1}$ if $F(.)$ is the baseline CDF. The advantage of using such transformation is that no more parameters are added, resulting in a parameter-precise distribution. A novel class of distributions with several variable hazard rates was thus proposed by Maurya et al. [9], in this fashion. They experimented with the generalized DUS (GDUS) transformation by applying the exponentiated cdf to the DUS transformation. A generalized lifetime model based on the DUS transformation was proposed by Kavya & Manoharan [10], with the cdf of the GDUS transformation provided by $G(w;\kappa)=\frac{e^{F^{\kappa}(w)}-1}{e-1}$, where $\kappa>0$ and $F(.)$ is the baseline CDF. The KM transformation family of distributions is a new transformation that was just developed by Kavya & Manoharan [11]. Both the cdf and the pdf are

$$G_{KM}(w)=\frac{e}{e-1}\left(1-e^{-F(w)}\right), w>0, \tag{3}$$

and

$$g_{KM}(w)=\frac{e}{e-1}f(w)e^{-F(w)}, \tag{4}$$

The HRF is acquired by

$$h_{KM}(w)=\frac{f(w)e^{1-F(w)}}{e^{1-F(w)}-1}. \tag{5}$$

This family develops updated lifetime models or distributions from a given benchmark distribution. The exponential and Weibull distributions were employed as foundation distributions by (Kavya & Manoharan [11]) because reliability theory and survival analysis frequently employ them. The field of statistics involves the gathering, structuring, arrangement, and analysis of data, as well as drawing conclusions from samples of the complete population. To choose a good sample from the population, there are a number of sampling techniques available. The simplest technique is the simple random sampling (SRS) mechanism. Accordingly, a population of N size is used to select a sample of n size, giving each class of n items an equal opportunity to be represented in the sample. McIntyre [12] developed the RSS (ranked set sampling) design as an effective alternative to SRS design for enhancing precision and estimation efficiency when the measurements of relevant variables are costly to measure or hard to attain, nevertheless easy to rank. The authors (Dell & Clutter [13] and (Takahasi & Wakimoto [14]), were the first to design the mathematical underpinnings of the RSS technique. Bhushan et al. [15], introduced some novel class estimators using ranked set sampling to evaluate the population mean utilizing additional information on an auxiliary variable. Bhushan & Kumar [16], proposed an efficient class of estimators for population mean in a ranked set sampling framework. Mahdizadeh & Zamanzade [17], studied the estimation of a symmetric distribution function under multistage ranked set sampling. They developed a nonparametric estimator and explored its theoretical properties with numerical studies. Mahdizadeh and Zamanzade [18] solved the issue of using ranked set sampling to generate a confidence interval with the reliability parameter. They compared their suggested intervals to others based on everything from basic random sampling to Monte Carlo simulations and advocated for both asymptotic and resampling-based intervals. Mahdizadeh & Zamanzade [19], developed a nonparametric reliability estimator based on multistage ranked set sampling and studied its efficiency relative to simple random sampling. One might see Chen et al. [20], for an extensive study of the idea, procedures, and usages of RSS scheme.

Numerous studies have recently concentrated on RSS-centered parameters of estimation for a spectrum of important real-life models. The RSS stratagem is more effective than the SRS technique and other conventional sampling procedures, as these investigations have repeatedly shown (Abu-Dayyeh et al. [21]; He et al. [22]; Sabry et al. [23]). Many studies have developed and used

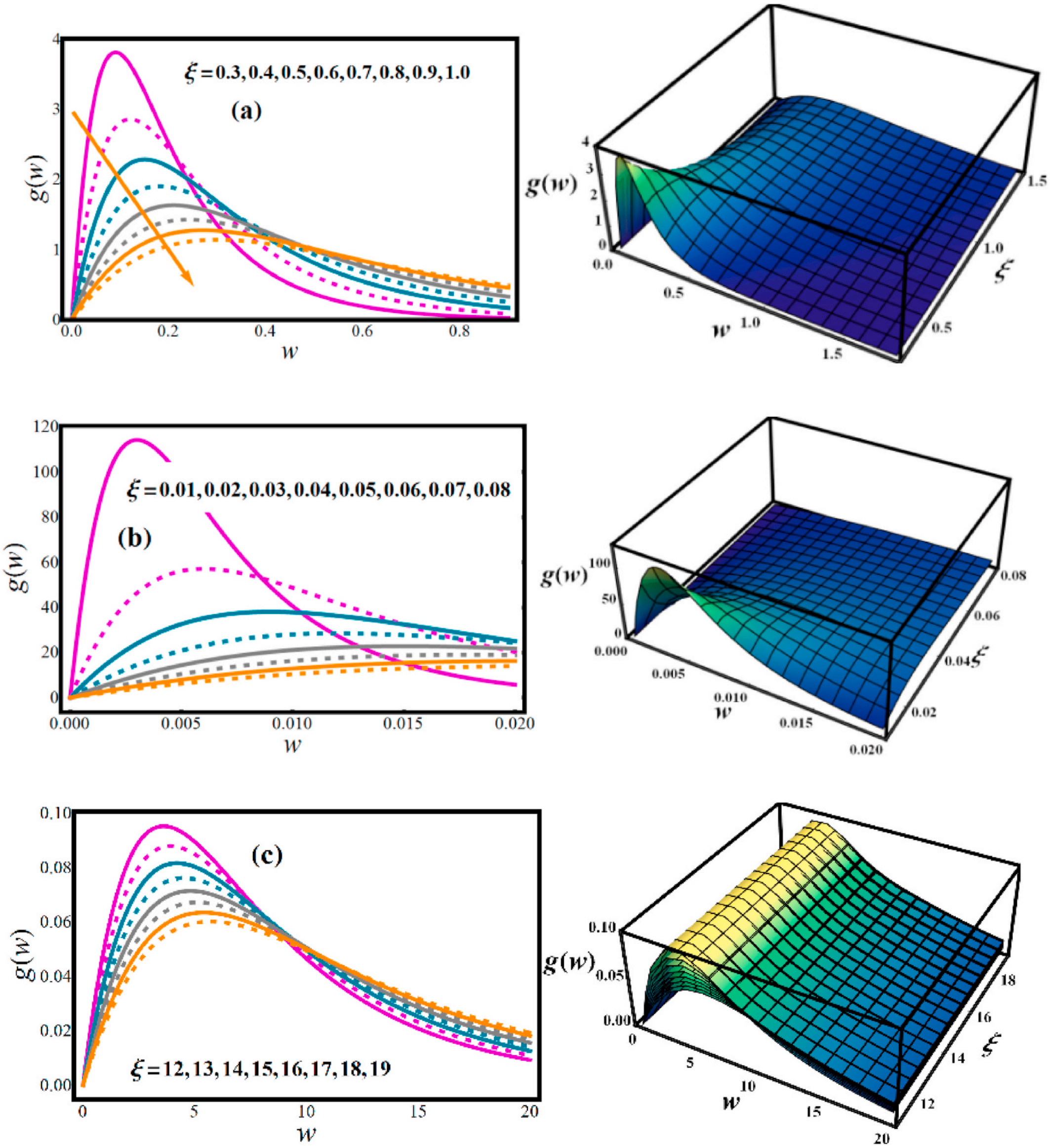

Fig. 1. Different layouts of pdf for KMBD.

the performance of ML estimation under the RSS method. Such as authors (Pedroso et al. [24]; Taconeli et al. [25]) describe how the parameter estimation for the two-parameter Birnbaum Saunders and Lindley distribution is derived using simple random sampling and ranked set sampling. For a thorough analysis and information on the RSS system and its uses consider, for instance (Ahmed & Shabbir [26]; Dorniani et al. [27]; Al-Omari & Bouza [28]), and the sources therein.

In this article, we extend the study by introducing a novel extension of the BD, referred to as the Kavya-Manoharan Bilal Distribution (KMBD). Our investigation into the KMBD is driven by several key motivations: (i) the desire to develop distributions characterized by diverse shapes, (ii) the need for distributions with both monotone and non-monotone failure rate functions, (iii) the exploration of analytic measures and reliability properties for the KMBD, (iv) the examination of parameter estimation techniques focusing on both SRS and RSS for the KMBD, and (v) the empirical inference from goodness-of-fit statistics and graphical tools.

2. KM-Bilal model

We construct a novel flexible model called Kavya–Manoharan transformation Bilal distribution (KMBD) by using Equation (1) and Equation (2) in Equation (3) and Equation (4), we get a new distribution, which has CDF as below

$$\mathrm{G}_{KMB}(w) = \frac{e}{e-1}\left(1 - \exp\left(-\left\{1 - e^{\frac{-2w}{\xi}}\left(3 - 2e^{\frac{-w}{\xi}}\right)\right\}\right)\right), w, \xi > 0, \tag{6}$$

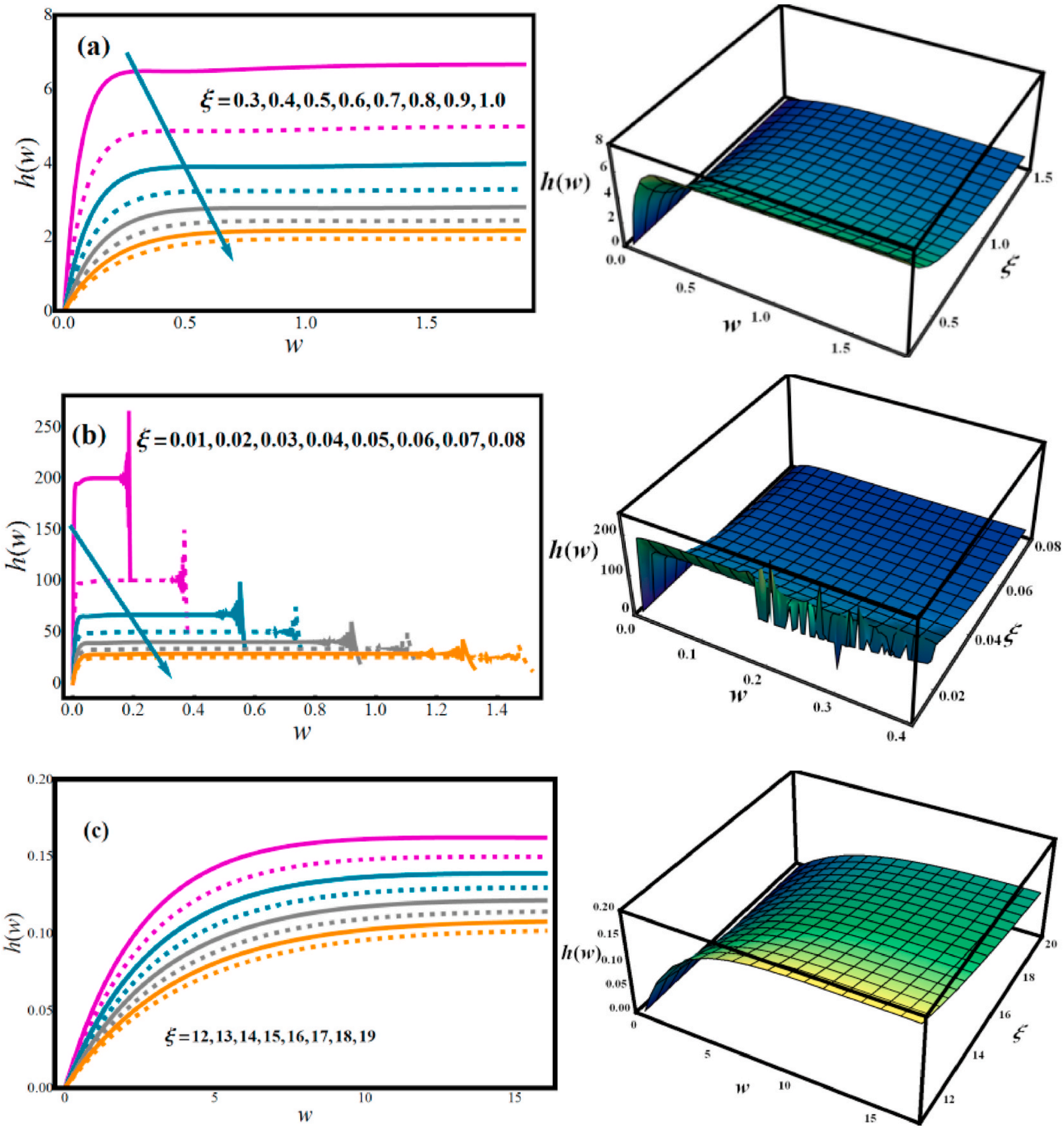

Fig. 2. Different layouts of hrf for KMBD.

And the PDF of W is

$$g_{KMB}(w|\,\xi)=\frac{6\left(1-e^{\frac{-w}{\xi}}\right)}{\xi(e-1)}\exp\left(-\left\{\frac{2w}{\xi}-e^{\frac{-2w}{\xi}}\left(3-2e^{\frac{-w}{\xi}}\right)\right\}\right),w,\xi>0, \tag{7}$$

The survivor or reliability function is another name for this survival function. It is defined as the probability of a system failing or having a mortality rate (Gupta & Nanda [29]; Lyu [30]). The KMBD's survival function and HRF using in Equation (5) are given in Equation (8) and Equation [9] respectively.

$$S_{KMB}(w)=1-\frac{e}{e-1}\left(1-\exp\left(-\left\{1-e^{\frac{-2w}{\xi}}\left(3-2e^{\frac{-w}{\xi}}\right)\right\}\right)\right),w,\xi>0, \tag{8}$$

$$h_{KMB}(w)=\frac{6\left(1-e^{\frac{-w}{\xi}}\right)\exp\left(-\left\{\frac{2w}{\xi}-e^{\frac{-2w}{\xi}}\left(3-2e^{\frac{-w}{\xi}}\right)\right\}\right)}{\xi\left\{(e-1)-e\left(1-\exp\left(-\left\{1-e^{\frac{-2w}{\xi}}\left(3-2e^{\frac{-w}{\xi}}\right)\right\}\right)\right)\right\}}. \tag{9}$$

The formulation for the cumulative risk exposure function ($C^{'}(t|\,\xi)$) is established in Equation (10) as

$$C'(t|\ \xi) = \int_0^t h(w|\ \xi)dw = -\log(S(t|\ \xi)). \tag{10}$$

Hence $(C'(t|\ \xi))$ of the KMB model is given in Equation (11) as

$$C'(t|\ \xi) = \int_0^t h(w|\ \xi)dw = -\log\left[1 - \frac{e}{e-1}\left(1 - \exp\left(-\left\{1 - e^{\frac{-2t}{\xi}}\left(3 - 2e^{\frac{-t}{\xi}}\right)\right\}\right)\right)\right]. \tag{11}$$

The RHRF (reversed hazard rate function) is described as the ratio of the PDF and the relevant CDF. The RHRF $h'(w|\xi) = f(w|\xi)[F(w|\xi)]^{-1}$ have recently piqued the interest of experts (refer to Ref. Chandra & Roy [31] for definitions, characterizations and other information). The RHRF of the KMB model is given in Equation (12) as

$$h'_{KMB}(w|\ \xi) = \frac{6\left(1 - e^{\frac{-w}{\xi}}\right)\exp\left(-\left\{\frac{2w}{\xi} - e^{\frac{-2w}{\xi}}\left(3 - 2e^{\frac{-w}{\xi}}\right)\right\}\right)}{\xi e\left(1 - \exp\left(-\left\{1 - e^{\frac{-2w}{\xi}}\left(3 - 2e^{\frac{-w}{\xi}}\right)\right\}\right)\right)} \tag{12}$$

Due to its connection to HRF, Mills Ratio $\{S(w|\ \xi)[f(w|\ \xi)]^{-1}\}$ is an exceptional approach for assessing reliability. The Mills Ratio of the KMB model is given in Equation (13) as

$$MR_{KMB}(w|\xi) = \frac{\xi\left\{(e-1) - e\left(1 - \exp\left(-\left\{1 - e^{\frac{-2w}{\xi}}\left(3 - 2e^{\frac{-w}{\xi}}\right)\right\}\right)\right)\right\}}{6\left(1 - e^{\frac{-w}{\xi}}\right)\exp\left(-\left\{\frac{2w}{\xi} - e^{\frac{-2w}{\xi}}\left(3 - 2e^{\frac{-w}{\xi}}\right)\right\}\right)}. \tag{13}$$

The PDF and HRF of the KMBD is graphically depicted in Fig. 1(a–c) and 2 (a-c) with different parameter values. Fig. 1(a–c) illustrates the asymmetric and unimodal configurations of the PDF, providing a visual representation of the distribution's shape. This graphical representation serves as a valuable tool for better understanding the characteristics of the proposed KMBD. Fig. 2(a–c) displays the growing, and upside-down bathtub (UBT) forms of the HRF. Engineers and analysts can evaluate the risk of failure at various phases of a system's life cycle by knowing the shape of the hazard function. This data is essential for determining probable failure modes, assessing their effects, and putting suitable risk-reduction plans into action. The HRF has a monotonic growing and upside-down trend when parametric values are considered. These versatile HRF forms are ideally suited for real-time applications, where the necessity for hazard rate trends to display both monotonic and non-monotonic characteristics is prevalent. For instance, the data of 3878 cases of breast cancer that were first identified in Edinburgh between 1954 and 1964 has been analyzed by the researchers. They discovered that the death rate peaked after the first year of follow-up, declined progressively after that, and was low in the subsequent years [32]. UBT hazard rate is connected in this instance.

3. Analytical properties

Here, we analyze several analytical features of the KMBD.

3.1. Moment properties

The rth moment of a r.v. W with the KMBD can be expressed explicitly in the following way given in Equation (14) for any integer r as

$$\mu'_r = \mathrm{E}(W^r) = \int_0^{+\infty} w^r f(w)dw, \tag{14}$$

$$\mu'_r = \int_0^{+\infty} w^r \frac{6\left(1 - e^{\frac{-w}{\xi}}\right)}{\xi(e-1)}\exp\left(-\left\{\frac{2w}{\xi} - e^{\frac{-2w}{\xi}}\left(3 - 2e^{\frac{-w}{\xi}}\right)\right\}\right)dw, \tag{15}$$

We know $e^{\varsigma} = \sum_{i=0}^{+\infty}\frac{\varsigma^i}{i!}$ and for $|z| < 1, s \in R^+$, we have $(1-z)^s = \sum_{j=0}^{+\infty}(-1)^j\frac{\Gamma(s+1)}{j!\Gamma(s+1-j)}z^j$, using these in Equation (15), which is equivalent to,

$$\mu'_r = \frac{6\xi^r}{(e-1)}\sum_{k,m=0}^{+\infty}\frac{(-1)^k 3^{m-k}2^k\Gamma(m+1)\Gamma(1+r)}{k!m!\Gamma(m+1-k)} \times \left[\frac{1}{(2+2m+k)^{r+1}} - \frac{1}{(3+2m+k)^{r+1}}\right], r = 1, 2, \dots \tag{16}$$

The average given in Equation [17] of the PDF of the KMBD is

$$\mu' = \frac{6\xi}{(e-1)} \sum_{k,m=0}^{+\infty} \frac{(-1)^k 3^{m-k} 2^k \Gamma(m+1)}{k!m!\Gamma(m+1-k)} \times \left[\frac{1}{(2+2m+k)^2} - \frac{1}{(3+2m+k)^2}\right] = \mu \tag{17}$$

Whereas the dispersion given in Equation (18) is determined by

$$\varsigma^2 = \left\{\frac{12\xi}{(e-1)} \sum_{k,m=0}^{+\infty} \frac{(-1)^k 3^{m-k} 2^k \Gamma(m+1)}{k!m!\Gamma(m+1-k)} \times \left[\frac{1}{(2+2m+k)^3} - \frac{1}{(3+2m+k)^3}\right]\right\} - \mu^2. \tag{18}$$

The square root is used to determine the standard deviation ς. The rth central moment using Equation (19) of a *r.v.* W with the KMB model can be expressed explicitly in Equation (20) as

$$\mu_r = E(w-\mu)^r = \int_0^{+\infty} (w-\mu)^r f(w|\ \xi) dw, \tag{19}$$

We can use the conventional binomial formula to expand the expressions of these measures, and we end up with

$$\mu_r = \sum_{s=0}^{r} (-1)^s \binom{r}{s} \mu^s \frac{6\xi^{r-s}}{(e-1)} \left\{ \sum_{k,m=0}^{+\infty} \frac{(-1)^k 3^{m-k} 2^k \Gamma(m+1)\Gamma(1+r-s)}{k!m!\Gamma(m+1-k)} \times \left[\frac{1}{(2+2m+k)^{r-s+1}} - \frac{1}{(3+2m+k)^{r-s+1}}\right] \right\}. \tag{20}$$

Notably, the very first four moments about origin can be obtained by putting $r = 1, 2, 3, 4$ in Equation (16). The moment skewness of *r.v.* W is calculated using the following equation, $\breve{S}_k = E(w-\mu)^3/\varsigma^3$ and the moment kurtosis of *r.v.* W can be described as $\breve{K}_u = E(w-\mu)^4/\varsigma^4$.

3.2. Moment generating function (MGF)

The MGF given in Equation (21) of the KMBD is:

$$\widetilde{M}_W(d) = E(e^{wd}) = \frac{6}{\xi(e-1)} \int_0^{\infty} e^{wd}\left(1 - e^{\frac{-w}{\xi}}\right) \exp\left(-\left\{\frac{2w}{\xi} - e^{\frac{-2w}{\xi}}\left(3 - 2e^{\frac{-w}{\xi}}\right)\right\}\right) dw, \tag{21}$$

Hence

$$\widetilde{M}_W(d) = \frac{6}{(e-1)} \sum_{j=0}^{+\infty} \frac{d^j}{j!} \xi^j \left\{ \sum_{k,m=0}^{+\infty} \frac{(-1)^k 3^{m-k} 2^k \Gamma(m+1)\Gamma(1+j)}{k!m!\Gamma(m+1-k)} \times \left[\frac{1}{(2+2m+k)^{j+1}} - \frac{1}{(3+2m+k)^{j+1}}\right]\right\}. \tag{22}$$

Additionally, using the Equation (22), we may derive the rth raw moment from MGF

$$\mu_r' = \left.\frac{d^r \widetilde{M}_W(d)}{dw^r}\right|_{d=0}.$$

3.3. Cumulants

The characteristic function (CF), $\breve{\mathbb{C}}_t(d) = E[e^{itd}]$ of KMBD is obtained by substituting d with 'id' in Equation (20), the final expression of CF given in Equation (23) as

$$\breve{\mathbb{C}}_W(d) = E(e^{iwd}) = \frac{6}{(e-1)} \sum_{j=0}^{+\infty} \frac{(id)^j}{j!} \xi^j \left\{ \sum_{k,m=0}^{+\infty} \frac{(-1)^k 3^{m-k} 2^k \Gamma(m+1)\Gamma(1+j)}{k!m!\Gamma(m+1-k)} \times \left[\frac{1}{(2+2m+k)^{j+1}} - \frac{1}{(3+2m+k)^{j+1}}\right]\right\}. \tag{23}$$

where $i = \sqrt{-1}$ is complex number.

3.4. Cumulant generating function (CGF)

The CGF is $\log\{\breve{\mathbb{C}}_W(d)\}$ where $\breve{\mathbb{C}}_W(d)$ is given in (23).

3.5. Probability generating function (PGF)

In (20), the PGF is obtained by substituting d with $\ln(d)$ as follows:

$$G_W(d) = E(d^W) = E(e^{w \ln d})$$

The PGF given in Equation (24) of the KMBD is:

$$G_W(d) = \frac{6}{(e-1)} \sum_{j=0}^{+\infty} \frac{(\ln d)^j \xi^j}{j!} \left\{ \sum_{k,m=0}^{+\infty} \frac{(-1)^k 3^{m-k} 2^k \Gamma(m+1)\Gamma(1+j)}{k! m! \Gamma(m+1-k)} \times \left[\frac{1}{(2+2m+k)^{j+1}} - \frac{1}{(3+2m+k)^{j+1}} \right] \right\}. \tag{24}$$

3.6. Incomplete moments

The incomplete moments given in Equation (25) would be:

$$\psi_s(t) = E(W^s | W < t) = \int_0^t w^s g(w) dw. \tag{25}$$

The incomplete moments of the KMBD given in Equation (26) is

$$\psi_s(t) = \frac{6\xi^s}{(e-1)} \sum_{k,m=0}^{+\infty} \frac{(-1)^k 3^{m-k} 2^k \Gamma(m+1)}{k! m! \Gamma(m+1-k)} \times \left[\frac{\gamma\left(1+s, \frac{t(2+2m+k)}{\xi}\right)}{(2+2m+k)^{s+1}} - \frac{\gamma\left(1+s, \frac{t(3+2m+k)}{\xi}\right)}{(3+2m+k)^{s+1}} \right], \tag{26}$$

where $\gamma(\varsigma, t) = \int_0^t x^{\varsigma-1} e^{-x} dx.$

4. Some measures of reliability and entropy

We explore certain reliability indicators, such as the mean residual function for the KMBD, the vitality function, conditional survival function (CSF) and failure rate average (FRA) function. In this part, we also extract the expressions for stress strength reliability and entropy.

4.1. Vitality function and mean residual life (MRL)

A continuous random variable W with PDF $f(w)$ is said to have the following vitality function:

$$\breve{v}(w) = E(W | W \geq x). \tag{27}$$

In the framework of reliability, Equation (27) can be defined as the mean longevity of items with an age more than x. Another way to express vitality function given in Equation [28] is as follows:

$$\breve{v}(x) = \frac{1}{S(x)} \int_x^{+\infty} w g(w) dw, \tag{28}$$

where $f(w)$ and $S(w)$ are the PDF and survival function of the model.

Applying the results $e^{\varsigma} = \sum_{i=0}^{+\infty} \frac{\varsigma^i}{i!}$ and for $|z| < 1, s \in R^+$, $(1-z)^s = \sum_{j=0}^{+\infty} (-1)^j \frac{\Gamma(s+1)}{j!\Gamma(s+1-j)} z^j$, and $\Gamma(\varsigma, w) = \int_w^{+\infty} x^{\varsigma-1} e^{-x} dx$, we get,

$$\int_x^{+\infty} w g(w) dw = \frac{6\xi^2}{(e-1)} \sum_{k,m=0}^{+\infty} \frac{(-1)^k 3^{m-k} 2^k \Gamma(m+1)}{k! m! \Gamma(m+1-k)} \times \left[\frac{\Gamma\left(2, \frac{x}{\xi}(2+2m+k)\right)}{(2+2m+k)^2} - \frac{\Gamma\left(2, \frac{x}{\xi}(3+2m+k)\right)}{(3+2m+k)^2} \right]. \tag{29}$$

Using the result of Equation (29) in Equation (28) to obtain the final version of vitality function. The expected lifetime in reliability problems is called as the mean time to failure, and the anticipated future lifespan is defined as the mean residual lifetime. For a continuous random variable W, with $E(W) \leq \infty$, subsequently, the Borel measurable function (MRLF) is described in Equation (30) as the mean residual life function,

$$\widehat{m}(x) = E\{X - x | X \geq x\} \tag{30}$$

For KMBD's MRLF in Equation (31) is

$$\widehat{m}_{KMB}(x|\ \xi)=\frac{1}{S(x|\ \xi)}\int_{x}^{+\infty}S(w|\ \xi)dw, \tag{31}$$

The vitality function can also be used to express the MRLF. In other words, MRLF $\widehat{m}_{KMB}(x|\ \xi)$ can be expressed as,

$$\widehat{m}_{KMB}(x|\ \xi)=\breve{v}(x)-x. \tag{32}$$

As a result, using Equation (32) the MRLF of the KMBD is determined as

$$\widehat{m}_{KMB}(x|\ \xi)=\frac{\frac{6\xi^2}{(e-1)}\sum_{k,m=0}^{+\infty}\frac{(-1)^k3^{m-k}2^k\Gamma(m+1)}{k!m!\Gamma(m+1-k)}\times\left[\frac{\Gamma\left(2,\frac{x}{\xi}(2+2m+k)\right)}{(2+2m+k)^2}-\frac{\Gamma\left(2,\frac{x}{\xi}(3+2m+k)\right)}{(3+2m+k)^2}\right]}{1-\frac{e}{e-1}\left(1-\exp\left(-\left\{1-e^{\frac{-2x}{\xi}}\left(3-2e^{\frac{-x}{\xi}}\right)\right\}\right)\right)}-x. \tag{33}$$

4.2. Mean inactivity time function

The MIT in Equation (34), also recognized as the mean past lifespan function, is a well reputed integrity indicator with uses in different fields like survival investigation, reliability theory, and actuarial research. Consider W be $r.v$ with a life span and having CDF given in Equation (6)

$$\mu_W(t)=\begin{cases}\int_0^t\frac{G(w|\ .)dw,}{G(t|\ .)} & t>0\\ 0 & t\le 0\end{cases} \tag{34}$$

$$\mu_W(t)=\frac{\int_0^t\left(\frac{e}{e-1}\left(1-\exp\left(-\left\{1-e^{\frac{-2w}{\xi}}\left(3-2e^{\frac{-w}{\xi}}\right)\right\}\right)\right)\right)dw}{\left(\frac{e}{e-1}\left(1-\exp\left(-\left\{1-e^{\frac{-2t}{\xi}}\left(3-2e^{\frac{-t}{\xi}}\right)\right\}\right)\right)\right)}, \tag{35}$$

After some mathematics applied in Equation (35), we come to the following explicit expression given in Equation (36) as

$$\mu_W(t)=\frac{\left[t-e^{-1}\sum_{k,m=0}^{+\infty}\frac{(-1)^k3^{m-k}2^k\Gamma(m+1)}{k!m!\Gamma(m+1-k)}\left\{\frac{\xi}{2m-k}\left(e^{(2m-k)\frac{t}{\xi}}-1\right)\right\}\right]}{\left(1-\exp\left(-\left\{1-e^{\frac{-2t}{\xi}}\left(3-2e^{\frac{-t}{\xi}}\right)\right\}\right)\right)}. \tag{36}$$

4.3. Strong mean inactivity time (SMIT) function

$$S\mu_W(t)=\begin{cases}\int_0^t\frac{2wG(w|\ .)dw,}{G(t|\ .)} & t>0\\ 0 & t\le 0\end{cases} \tag{37}$$

Some mathematics shows that the Equation [33] is parallel to Equation [34],

$$S\mu_W(t)=\frac{2\left[\frac{t^2}{2}-e^{-1}\sum_{k,m=0}^{+\infty}\frac{(-1)^k3^{m-k}2^k\Gamma(m+1)}{k!m!\Gamma(m+1-k)}\frac{\xi}{2m-k}\left\{te^{(2m-k)\frac{t}{\xi}}-\frac{\xi}{2m-k}\left(e^{(2m-k)\frac{t}{\xi}}-1\right)\right\}\right]}{1-\exp\left(-\left\{1-e^{\frac{-2t}{\xi}}\left(3-2e^{\frac{-t}{\xi}}\right)\right\}\right)}. \tag{38}$$

4.4. Conditional survival function (CSF) and failure rate average (FRA)

FRA and CSF given in Equations (39-40) are two additional helpful reliability functions (DNP, [35]). We may find IFRA (increasing failure rate average) and DFRA (decreasing failure rate average) by assessing FRA on t.

The FRA of T is;

$$FRA(w|\xi)=\frac{C^{'}(w|\xi)}{w}, w>0, \tag{39}$$

where $C^{'}(.)$ is CHR, and given in (11). The conditional survival of W is described accordingly

$$P(W>w+t|W>t)=S(w|t)=\frac{S(w+t)}{S(w)},t>0,w>0,S(.)>0, \tag{40}$$

where $S(.)$ is SF and given in (8).

4.5. Entropy measures

Entropy is an indicator of the variability of uncertainty for a random variable W. Entropy is a measure of uncertainty that is utilized in fields like engineering and the natural sciences, and it is discussed by (Nasiru et al. [36]; Oguntunde et al. [37]).

The entropy of $r.v.$ W is a measure of risk. The Rényi entropy given Equations (41) of W is defined as:

$$\breve{\mathbb{R}}_\delta(W)=\frac{1}{1-\delta}\log\int_0^{+\infty} g^\delta(w)dw,\delta>0 \text{ and } \delta\neq 1. \tag{41}$$

First, $g(w)$ is simplified in terms of $g^\delta(w)$ by considering Equation [7] and applying the binomial expansion as:

$$g^\delta(w)=\left(\frac{6}{\xi(e-1)}\right)^\delta\sum_{i,k,m=0}^{+\infty}\binom{\delta}{i}\frac{(-1)^{i+k}3^{m-k}2^k\delta^m\Gamma(m+1)}{k!m!\Gamma(m+1-k)}e^{-(i+k+2m+2\delta)\frac{w}{\xi}}. \tag{42}$$

and substituting Equation (42) into Equation (41) gives the Rényi entropy of W given in Equation (43) as:

$$\breve{\mathbb{R}}_\delta(W)=\frac{\delta^m}{1-\delta}\left(\frac{6}{\xi(e-1)}\right)^\delta\sum_{i,k,m=0}^{+\infty}\binom{\delta}{i}\frac{(-1)^{i+k}3^{m-k}2^k\Gamma(m+1)\xi}{k!m!\Gamma(m+1-k)(i+k+2m+2\delta)}. \tag{43}$$

Tsallis entropy defined in Equation (44) is

$$\Omega_\delta(W)=\frac{1}{\delta-1}\left(1-\int_0^{+\infty}g^\delta(w)dw\right),\delta>0,\delta\neq 1. \tag{44}$$

Finally, the reduced version of the Tsallis entropy given in Equation (45) is obtained after some simplification.

$$\Omega_\delta(W)=\frac{1}{\delta-1}\left(1-\left(\frac{6}{\xi(e-1)}\right)^\delta\sum_{i,k,m=0}^{+\infty}\binom{\delta}{i}\frac{(-1)^{i+k}3^{m-k}2^k\delta^m\Gamma(m+1)\xi}{k!m!\Gamma(m+1-k)(i+k+2m+2\delta)}\right). \tag{45}$$

Mathai and Haubold [38], generalized the classical Shannon entropy is defined by Equation (46)

$$\mathrm{M_H}(W)=\frac{1}{\delta-1}\left(\int_0^\infty g^{2-\delta}(w)dw-1\right),\delta>0,\delta\neq 1. \tag{46}$$

After some simplification, the reduced version of the Mathai & Haubold entropy of W is obtained and given in Equation (47)

$$\mathrm{M_H}(W)=\frac{1}{\delta-1}\left\{\sum_{i,k,m=0}^{+\infty}\binom{\delta}{i}\left(\frac{6}{\xi(e-1)}\right)^{2-\delta}\frac{(-1)^{i+k}3^{m-k}2^k(2-\delta)^m\Gamma(m+1)\xi}{k!m!\Gamma(m+1-k)(i+k+2m+2(2-\delta))}\right\}. \tag{47}$$

The Havrda and Charvat introduced $\delta-$ entropy measure. It is defined by Equation (48)

$$\mathrm{H_{HC}}(W)=\frac{1}{2^{1-\delta}-1}\left(\int_0^\infty g^\delta(w)dw-1\right),\delta>0,\delta\neq 1, \tag{48}$$

and the reduced version of the Mathai & Haubold is given in Equation (49)

$$\mathrm{H_{HC}}(W)=\frac{1}{2^{1-\delta}-1}\left(\sum_{i,k,m=0}^{+\infty}\binom{\delta}{i}\left(\frac{6}{\xi(e-1)}\right)^\delta\frac{(-1)^{i+k}3^{m-k}2^k\delta^m\Gamma(m+1)\xi}{k!m!\Gamma(m+1-k)(i+k+2m+2\delta)}-1\right). \tag{49}$$

Boekee and Lubba entropy is defined in Equation (50) of W is:

$$BL_\nu(W)=\frac{\delta}{\delta-1}\left(1-\left(\int_0^\infty g^\delta(w)dw\right)^{\frac{1}{\delta}}\right),\delta>0,\delta\neq 1. \tag{50}$$

This entropy has been reduced to its simplest form and given in Equation (51).

$$BL_\delta(W)=\frac{\delta}{\delta-1}\left(1-\left(\sum_{i,k,m=0}^{+\infty}\binom{\delta}{i}\left(\frac{6}{\xi(e-1)}\right)^\delta\frac{(-1)^{i+k}3^{m-k}2^k\delta^m\Gamma(m+1)\xi}{k!m!\Gamma(m+1-k)(i+k+2m+2\delta)}\right)^{\frac{1}{\delta}}\right). \tag{51}$$

When researching the estimate of limit parameters, Arimoto [37] developed a type of generalized entropy function defined in Equation

(52) called the "Arimoto entropy," which is highly good at addressing the likelihood of decision-making.

$$\widetilde{A}(W) = \frac{\delta}{2^{1-\delta}-1}\left(\left(\int_0^\infty g^\delta(w)dw\right)^{\frac{1}{\delta}} - 1\right), \delta > 0. \tag{52}$$

This Arimoto entropy has been reduced to its simplest form and given in Equation (53).

$$\widetilde{A}(W) = \frac{\delta}{2^{1-\delta}-1}\left(\left(\sum_{i,k,m=0}^{+\infty}\binom{\delta}{i}\left(\frac{6}{\xi(e-1)}\right)^\delta \frac{(-1)^{i+k}3^{m-k}2^k\delta^m\Gamma(m+1)\xi}{k!m!\Gamma(m+1-k)(i+k+2m+2\delta)}\right)^{\frac{1}{\delta}} - 1\right). \tag{53}$$

5. Inference of the parameters

Some useful insights on the KMBD's applied study are provided in this part. It covers utilizing the ML approach with the SRS and RSS schemes to estimate the parameter ξ of the KMBD. In addition, we reveal the behavior of the guesstimates using a simulation experiment.

5.1. Inference under the SRS scheme

Suppose n be an integer and $W_1, W_2, ..., W_n$ be an SRS scheme (of size n) from the KMBD(ξ) with the PDF and CDF specified in Equations (6) and (7), respectively and $w_1, w_2, ..., w_n$ be observations of $W_1, W_2, ..., W_n$. Here $\mathbf{w} = \{w_1, w_2, ..., w_n\}$. Then log-likelihood ($l(\mathbf{w}|\ \xi)$) is generally easier to maximize. Hence the $l(\mathbf{w}|\ \xi)$ of the joint probability function of $W_1, W_2, ..., W_n$ given in Equation (54) is

$$\mathrm{l}(\mathbf{w}|\ \xi) = \mathrm{n}\log(6) + \sum_{\mathrm{i}=1}^{\mathrm{n}}\log\left(1\text{-}\mathrm{e}^{\frac{-\mathrm{w_i}}{\xi}}\right)\text{-n}\log\xi\text{-n}\log(\mathrm{e}\text{-}1)\text{-}\frac{2}{\xi}\sum_{\mathrm{i}=1}^{\mathrm{n}}\mathrm{w_i} + \sum_{\mathrm{i}=1}^{\mathrm{n}}\mathrm{e}^{\frac{-2\mathrm{w_i}}{\xi}}\left(3\text{-}2\mathrm{e}^{\frac{-\mathrm{w_i}}{\xi}}\right). \tag{54}$$

The ML estimates (MLEs) of ξ that are only based on $\mathbf{w}$ can be characterized as $\widehat{\xi}$, where $\widehat{\xi} = \mathrm{argmax}\, l(\mathbf{w}|\ \xi)$. The maximizing is applied to all potential parameter values. These MLEs also confirm the partial derivative equation: $\frac{\partial l(\mathbf{w}|\ \xi)}{\partial \xi} = 0$.

With respect to ξ, the partial derivatives of $l(\mathbf{w}; \xi)$ given in Equation (55)

Where

$$\frac{\partial l(\mathbf{w}|\ \xi)}{\partial \xi} = -\frac{n}{\xi} + \frac{2}{\xi^2}\sum_{i=1}^n w_i - \sum_{i=1}^n \frac{w_i e^{\frac{-w_i}{\xi}}}{\xi^2\left(1 - e^{\frac{-w_i}{\xi}}\right)} - \frac{1}{\xi^2}\sum_{i=1}^n w_i\left(2\,e^{\frac{-2w_i}{\xi}} - e^{\frac{-w_i}{\xi}}\left(3 - 2e^{\frac{-w_i}{\xi}}\right)\right). \tag{55}$$

With respect to ξ, the second partial derivatives of $l(\mathbf{w}; \xi)$ given in Equation (56) can be represented as

$$\frac{\partial^2 l(\mathbf{w}|\ \xi)}{\partial \xi^2} = \frac{n}{\xi^2} - \frac{4}{\xi^3}\sum_{i=1}^n w_i + 2\sum_{i=1}^n w_i\left(\frac{2\,e^{\frac{-2w_i}{\xi}} - e^{\frac{-w_i}{\xi}}\left(3 - 2e^{\frac{-w_i}{\xi}}\right)}{\xi^3}\right) - \frac{1}{\xi^2}\sum_{i=1}^n e^{\frac{-2w_i}{\xi}} w_i\left(-8 + 3e^{\frac{w_i}{\xi}}\right)$$

$$+\sum_{i=1}^n w_i\left(\frac{2\left(-1 + e^{\frac{-2w_i}{\xi}}\right)\xi - e^{\frac{-w_i}{\xi}} w_i}{\left(-1 + e^{\frac{w_i}{\xi}}\right)^2 \xi^4}\right). \tag{56}$$

Equation (55) is non-linear equations does not yield perfect solutions or the best value for MLEs. There are limited chances to obtain closed-form formulas for $\widehat{\xi}$ due to the interdependencies of these partial derivatives. Numerical techniques such as the (quasi) Newton–Raphson method can be helpful to find accurate numerical solutions.

5.2. Inference under the RSS scheme

The RSS plan can be summed up as follows:

- Let c be the overall cycle count and τ be the number of sample units selected for each cycle (fixed size).
- The stages listed below can be utilized to generate a ranked set sample of size $n = c\tau$.
- Choose τ^2 units at random from the population, then split them into τ groupings of size τ.
- Assign the units in each set a rank based on a simple, affordable ordering mechanism.
- Add the lowest ranked unit from the first set, the next lowest ranked unit from the second set, and so on to create a single quantification sample.
- Repeat steps 1 through 3, c times to get an ultimate sample with size $n = c\tau$.

RSS only utilizes one observation, $W_{(11)c}$, which is the lowest observation in the c th cycle from this set, followed by $W_{(22)c}$, which is the second-lowest observation from a separate batch of τ observations, and $W_{(\widehat{\tau}\widehat{\tau})c}$, which is the highest observation from the last group of τ occurrences.

Now, to make notations simpler, let $\{W_{(ii)k} : i = 1, 2, ..., \tau, k = 1, 2, ..., c\}$ be a RSS procedures sketched from the MKB(ξ) distribution with sample of size $n = c\tau$, where τ is the set size and c is the number of cycles or cycle size, and $\{w_{(ii)k} : i = 1, 2, ..., \tau, k = 1, 2, ..., c\}$ be the corresponding observations. Then CDF given in Equation (57) and PDF given in Equation (58) of $W_{(ii)k}$ are given by:

$$G_{i:\tau}\left(w_{(ii)k};\xi\right) = \sum_{\widehat{q}=1}^{\tau}\binom{\tau}{\widehat{q}}\left\{G_{i:\tau}\left(w_{(ii)k};\xi\right)\right\}^{\widehat{q}}\left\{1-G_{i:\tau}\left(w_{(ii)k};\xi\right)\right\}^{\tau-\widehat{q}}, \tag{57}$$

$$g_{i:\tau}\left(w_{(ii)k};\xi\right) = \Upsilon_{i:\tau}\left\{G_{i:\tau}\left(w_{(ii)k};\xi\right)\right\}^{i-1}\left\{g_{i:\tau}\left(w_{(ii)k};\xi\right)\right\}\left\{1-G_{i:\tau}\left(w_{(ii)k};\xi\right)\right\}^{\tau-i}, \tag{58}$$

$\Upsilon_{i:\tau} = \frac{\tau!}{(i-1)!(\tau-i)!}$. Observing Equation (57), we can formulate the likelihood function using Equation (59) given in Equation (60) of KMBD as

$$L(\mathbf{w};\xi) = \prod_{i=1}^{\tau}\prod_{k=1}^{c} g_{i:\widehat{\tau}}\left(w_{(ii)k};\xi\right), \tag{59}$$

$$L(\boldsymbol{w};\xi)=\prod_{i=1}^{\tau}\prod_{k=1}^{c}\Upsilon_{i:\tau}\frac{6\left(1-e^{\frac{-w_{(ii)k}}{\xi}}\right)}{\xi(e-1)}\exp\left(-\left\{\frac{2w_{(ii)k}}{\xi}-e^{\frac{-2w_{(ii)k}}{\xi}}\left(3-2e^{\frac{-w_{(ii)k}}{\xi}}\right)\right\}\right)\left\{\frac{e}{e-1}\left(1-\exp\left(-\left\{1-e^{\frac{-2w_{(ii)k}}{\xi}}\left(3-2e^{\frac{-w_{(ii)k}}{\xi}}\right)\right\}\right)\right)\right\}^{i-1}$$
$$\times\left\{1-\frac{e}{e-1}\left(1-\exp\left(-\left\{1-e^{\frac{-2w_{(ii)k}}{\xi}}\left(3-2e^{\frac{-w_{(ii)k}}{\xi}}\right)\right\}\right)\right)\right\}^{\tau-i}. \tag{60}$$

The log-likelihood function given in Equation (61) as

$$l(\mathbf{w};\xi) = \widehat{C} + n\ln 6 - n\ln\xi - n\ln(e-1) + \sum_{i=1}^{\tau}\sum_{k=1}^{c}\ln\left(1-e^{\frac{-w_{(ii)k}}{\xi}}\right) - \sum_{i=1}^{\tau}\sum_{k=1}^{c}\left(\left\{\frac{2w_{(ii)k}}{\xi}-e^{\frac{-2w_{(ii)k}}{\xi}}\left(3-2e^{\frac{-w_{(ii)k}}{\xi}}\right)\right\}\right)$$
$$+\sum_{i=1}^{\tau}\sum_{k=1}^{c}(i-1)\ln\left\{\frac{e}{e-1}\left(1-\exp\left(-\left\{1-e^{\frac{-2w_{(ii)k}}{\xi}}\left(3-2e^{\frac{-w_{(ii)k}}{\xi}}\right)\right\}\right)\right)\right\}$$
$$+\sum_{i=1}^{\tau}\sum_{k=1}^{c}(\tau-i)\ln\left\{1-\frac{e}{e-1}\left(1-\exp\left(-\left\{1-e^{\frac{-2w_{(ii)k}}{\xi}}\left(3-2e^{\frac{-w_{(ii)k}}{\xi}}\right)\right\}\right)\right)\right\},$$
$$\widehat{C} = \ln\Upsilon_{i:\widehat{\tau}}. \tag{61}$$

The estimation result represents the evaluation of survival analysis for KMBD, and it has been presented, compared graphically, and numerically discussed in Section 6. The MLE of ξ is defined by $\widehat{\xi} = \underset{\xi>0}{\operatorname{argmax}}\, l(\mathbf{w};\xi)$. There are no closed forms for $\widehat{\xi}$, however, a numerical solution can be produced by using the partial derivatives of $l(\mathbf{w};\xi)$. With respect to ξ, the partial derivatives of $l(\mathbf{w};\xi)$ given in Equation (62) can be represented as

$$\frac{\partial l(\mathbf{w};\xi)}{\partial\xi} = \frac{-n}{\xi} - \sum_{i=1}^{\tau}\sum_{k=1}^{c}\frac{w_{(ii)k}e^{\frac{-w_{(ii)k}}{\xi}}}{\left(1-e^{\frac{-w_{(ii)k}}{\xi}}\right)\xi^2} + 12\sum_{i=1}^{\tau}\sum_{k=1}^{c}\frac{(i-1)w_{(ii)k}\left\{\Theta_1\left(w_{(ii)k}\right)-\Theta_2\left(w_{(ii)k}\right)\right\}}{\left(1-e^{-1+2e^{\frac{w_{(ii)k}}{\xi}}\left(3-2e^{\frac{-w_{(ii)k}}{\xi}}\right)}\right)\xi^2}$$
$$-12\frac{e}{e-1}\sum_{i=1}^{\tau}\sum_{k=1}^{c}\frac{(\tau-i)w_{(ii)k}\left\{\Theta_1\left(w_{(ii)k}\right)-\Theta_2\left(w_{(ii)k}\right)\right\}}{\left\{1-\frac{e}{e-1}\left(1-\exp\left(-\left\{1-e^{\frac{-2w_{(ii)k}}{\xi}}\left(3-2e^{\frac{-w_{(ii)k}}{\xi}}\right)\right\}\right)\right)\right\}\xi^2}$$
$$+\sum_{i=1}^{\tau}\sum_{k=1}^{c}\left\{\frac{2w_{(ii)k}}{\xi^2}\left(1-e^{\frac{-3w_{(ii)k}}{\xi}}+e^{\frac{-2w_{(ii)k}}{\xi}}\left(3-2e^{\frac{-w_{(ii)k}}{\xi}}\right)\right)\right\}, \tag{62}$$

With respect to ξ, the second derivatives of $l(\mathbf{w};\xi)$ given in Equation (63) is

Table 1
Bias, MRE, and MSE for the KMBD(ξ) distribution under the RSS and SRS schemes.

	SRS				RSS $c = 1$			RSS $c = 2$		
ξ	n	Bias	MRE	MSE	Bias	MRE	MSE	Bias	MRE	MSE
0.75	4	0.017	1.093	0.0015	0.015	1.092	0.0014	0.013	1.090	0.0011
	10	0.016	1.021	0.0013	0.014	1.021	0.0011	0.012	1.019	0.0010
	14	0.011	1.015	0.0011	0.011	1.015	0.0009	0.011	1.013	0.0005
1.25	4	0.036	1.029	0.0013	0.035	1.028	0.0011	0.035	1.021	0.0010
	10	0.026	1.021	0.0007	0.025	1.020	0.0006	0.024	1.019	0.0005
	14	0.022	1.018	0.0005	0.022	1.017	0.0004	0.021	1.016	0.0003
1.9	4	0.047	1.025	0.0022	0.047	1.024	0.0020	0.047	1.022	0.0017
	10	0.036	1.019	0.0019	0.036	1.016	0.0015	0.035	1.014	0.0013
	14	0.023	1.012	0.0015	0.022	1.012	0.0013	0.022	1.011	0.0011

$$\frac{\partial^2 l(\mathbf{w};\xi)}{\partial\xi^2}=\frac{n}{\xi^2}+6\sum_{i=1}^{\tau}\sum_{k=1}^{c}e^{\frac{-3w_{(ii)k}}{\xi}}\frac{w_{(ii)k}\left\{2\left(e^{\frac{3w_{(ii)k}}{\xi}}-\xi+e^{\frac{w_{(ii)k}}{\xi}}\xi\right)+\left(3-2e^{\frac{-w_{(ii)k}}{\xi}}\right)w_{(ii)k}\right\}}{\xi^4}+\sum_{i=1}^{\tau}\sum_{k=1}^{c}\frac{w_{(ii)k}\left\{2\left(-1+e^{\frac{w_{(ii)k}}{\xi}}\right)\xi-e^{\frac{w_{(ii)k}}{\xi}}w_{(ii)k}\right\}}{\left(-1+e^{\frac{w_{(ii)k}}{\xi}}\right)^2\xi^4}$$

$$+12\sum_{i=1}^{\tau}\sum_{k=1}^{c}\frac{(i-1)w_{(ii)k}\left\{\Theta_1\left(w_{(ii)k}\right)-\Theta_2\left(w_{(ii)k}\right)\right\}}{\xi^2}\frac{\left\{-\Theta_1\left(w_{(ii)k}\right)+\Theta_2\left(w_{(ii)k}\right)\right\}}{\left(1-e^{-1+2e^{\frac{-2w_{(ii)k}}{\xi}}\left(3-2e^{\frac{-w_{(ii)k}}{\xi}}\right)}\right)\xi^2}-\sum_{i=1}^{\tau}$$

$$\times\sum_{k=1}^{c}\frac{12e^{-1+e^{\frac{-3w_{(ii)k}}{\xi}}\left(-4+6e^{\frac{w_{(ii)k}}{\xi}}\right)-\frac{6w_{(ii)k}}{\xi}}(i-1)w_{(ii)k}}{\xi^4\left(1-e^{-1+2e^{\frac{-2w_{(ii)k}}{\xi}}\left(3-2e^{\frac{-w_{(ii)k}}{\xi}}\right)}\right)}\left(e^{\frac{3w_{(ii)k}}{\xi}}\left(2\xi-3w_{(ii)k}\right)-2e^{\frac{4w_{(ii)k}}{\xi}}\left(\xi-w_{(ii)k}\right)+12w_{(ii)k}-24w_{(ii)k}e^{\frac{w_{(ii)k}}{\xi}}+12w_{(ii)k}e^{\frac{2w_{(ii)k}}{\xi}}\right)$$

$$-144\frac{e}{e-1}\sum_{i=1}^{\tau}\sum_{k=1}^{c}\frac{(\tau-i)(e-1)^2e^{12e^{\frac{-2w_{(ii)k}}{\xi}}-\frac{6w_{(ii)k}}{\xi}}\left(1-e^{\frac{w_{(ii)k}}{\xi}}\right)^2w_{(ii)k}^2}{\left(-e^{1+6e^{\frac{-2w_{(ii)k}}{\xi}}}+e^{1+4e^{\frac{-3w_{(ii)k}}{\xi}}}\right)\xi^4}-12\sum_{i=1}^{\tau}$$

$$\times\sum_{k=1}^{c}\frac{(\tau-i)e^{1+6e^{\frac{-2w_{(ii)k}}{\xi}}-\frac{6w_{(ii)k}}{\xi}}w_{(ii)k}\left(e^{\frac{3w_{(ii)k}}{\xi}}\left(2\xi-3w_{(ii)k}\right)-2e^{\frac{4w_{(ii)k}}{\xi}}\left(\xi-w_{(ii)k}+12w_{(ii)k}-24w_{(ii)k}e^{\frac{w_{(ii)k}}{\xi}}+12w_{(ii)k}e^{\frac{2w_{(ii)k}}{\xi}}\right)\right)}{\left(e^{1+6e^{\frac{-2w_{(ii)k}}{\xi}}}-e^{1+4e^{\frac{-3w_{(ii)k}}{\xi}}}\right)\xi^4}, \tag{63}$$

where

$$\Theta_1\left(w_{(ii)k}\right)=\exp\left(-\left\{1-2e^{\frac{-2w_{(ii)k}}{\xi}}\left(3-2e^{\frac{-w_{(ii)k}}{\xi}}\right)+\frac{3w_{(ii)k}}{\xi}\right\}\right),$$

$$\Theta_2\left(w_{(ii)k}\right)=\exp\left(-\left\{1-2e^{\frac{-2w_{(ii)k}}{\xi}}\left(3-2e^{\frac{-w_{(ii)k}}{\xi}}\right)+\frac{2w_{(ii)k}}{\xi}\right\}\right).$$

We employed the Mathematica 12 software to gain numerical answers due to the frequent complexity of the theoretical solutions. The estimated confidence intervals for the parameters ξ can be derived as follows because the MLE is asymptotically normal $\widehat{\xi}\pm\varsigma_{\frac{\tau}{2}}\sqrt{\widehat{\sigma}_{\widehat{\xi}}{}^2}$, here $\widehat{\sigma}^2_{(.)}$ is the variance of the respective parameter ξ and using Eqs. (56) and (63), $\widehat{\sigma}_{\widehat{\xi}}{}^2=\frac{-1}{\frac{\partial^2 l(\mathbf{w}|\ \xi)}{\partial\xi^2}}$, where the SDN curve's value is $\varsigma_{\frac{\tau}{2}}$ and τ is the level of significance.

6. Simulation experiment

This part compares the ML estimators of the unknown parameter ξ for KMBD based on RSS and SRS using a numerical investigation. Based on the biases, mean square errors (MSEs), and mean relative errors (MREs), a validation study is conducted. Various set sizes, different cycles, and distinct parameters levels are all considered when utilizing Monte Carlo simulation. The MLEs and recommended criterion measures are obtained utilizing the below algorithm.

Step 1. Generate a random sample from the KMBD with sizes $n = c\tau$ and set sizes $\tau = 4$, 10, 14, and $c = 1$, and $\tau = 2$, 5, 7, and $c = 2$. Step 2: For the estimation technique, the parameter values are chosen as $\xi = (0.75, 1.25.1.9, 3, 7, 15)$. Step 3: The estimators ξ_{SRS} and ξ_{RSS} are computed under SRS and RSS for the selected parameter's set and each sample of n size. Step 4: Perform steps 1 through 3, N times to represent various samples, where N is equal to 5000. The bias, MSEs and mean relative errors (MREs), of the estimations are

Table 2
Bias, MRE, and MSE for the KMBD(ξ) distribution under the RSS and SRS schemes.

	SRS				RSS $c = 1$			RSS $c = 2$		
ξ	n	Bias	MRE	MSE	Bias	MRE	MSE	Bias	MRE	MSE
3.0	4	−0.032	0.990	0.0010	−0.031	0.990	0.0009	−0.031	0.990	0.0006
	10	−0.019	0.994	0.0004	−0.019	0.995	0.0003	−0.018	0.996	0.0001
	14	−0.017	0.995	0.0003	−0.016	0.996	0.0002	−0.016	0.997	0.0001
7.0	4	−0.133	0.981	0.0178	−0.133	0.981	0.0171	−0.133	0.982	0.0169
	10	−0.074	0.989	0.0055	−0.071	0.990	0.0049	−0.071	0.990	0.0041
	14	−0.058	0.992	0.0034	−0.058	0.995	0.0030	−0.057	0.995	0.0027
15	4	0.632	1.042	0.3990	0.631	1.035	0.3961	0.630	1.029	0.3912
	10	0.121	1.008	0.0146	0.121	1.008	0.0129	0.120	1.006	0.0117
	14	0.109	1.007	0.1178	0.108	1.006	0.1143	0.108	1.004	0.1122

Table 3
Lower, upper bounds of CP and CI for KMBD(ξ) model under RSS and SRS schemes.

	SRS				RSS $c = 1$			RSS $c = 2$		
ξ	n	LB	UB	CP%	LB	UB	CP%	LB	UB	CP%
0.75	4	−2.727	4.262	86.141	−2.268	4.432	86.717	−2.100	4.564	86.794
	10	−2.730	4.162	86.329	−2.530	4.102	86.841	−2.510	4.060	86.962
	14	−2.766	3.989	86.596	−2.610	3.961	86.958	−2.694	3.861	86.988
1.25	4	−0.321	2.892	93.611	−0.328	2.749	93.879	−0.335	2.723	93.917
	10	−0.352	2.803	93.725	−0.357	2.713	93.892	−0.359	2.692	93.930
	14	−0.362	2.695	93.918	−0.365	2.672	93.958	−0.368	2.661	93.975
1.9	4	1.084	2.810	96.574	1.046	2.834	96.450	1.027	2.870	96.342
	10	1.066	2.806	96.547	1.029	2.830	96.425	1.012	2.861	96.330
	14	1.043	2.802	96.509	0.978	2.813	96.359	0.978	2.851	96.282

Table 4
Lower, upper bounds of CP and CI and KMBD(ξ) model under RSS and SRS schemes.

	SRS				RSS $c = 1$			RSS $c = 2$		
ξ	n	LB	UB	CP%	LB	UB	CP%	LB	UB	CP%
3.0	4	2.510	3.427	98.193	2.460	3.466	98.017	2.4153	3.5825	97.698
	10	2.526	3.436	99.453	2.405	3.594	97.656	2.3956	3.5999	97.625
	14	2.529	3.438	98.206	2.300	3.519	97.594	2.1648	3.5789	97.209
7.0	4	6.136	7.017	98.336	6.365	7.317	98.336	6.4327	7.5648	97.876
	10	6.097	7.594	97.184	6.017	7.521	97.167	5.9887	7.5894	96.987
	14	6.014	7.699	96.831	5.862	7.645	96.643	5.734	7.699	96.297
15	4	15.094	15.670	99.108	15.00	15.640	99.049	15.010	15.750	98.902
	10	15.081	15.161	99.899	14.100	15.745	98.888	14.616	15.846	98.166
	14	15.069	15.149	99.880	14.347	15.777	98.850	14.616	15.846	98.030

Table 5
REs for the KMB(ξ) distribution under the SRS and RSS schemes.

ξ	n	RE1	RE2	RE3	ξ	RE1	RE2	RE3
0.75	4	1.071	1.364	1.273	3.0	1.111	1.667	1.500
	10	1.182	1.300	1.100		1.333	4.00	3.000
	14	1.222	2.200	2.200		1.500	3.00	2.000
1.25	4	0.182	1.300	1.100	7.0	1.041	1.053	1.011
	10	1.167	1.400	1.200		1.122	1.342	1.195
	14	1.250	1.667	1.333		1.133	1.260	1.111
1.9	4	1.100	1.294	1.176	15	1.007	1.020	1.013
	10	1.267	1.462	1.154		1.132	1.248	1.103
	14	1.154	1.364	1.182		1.031	1.049	1.019

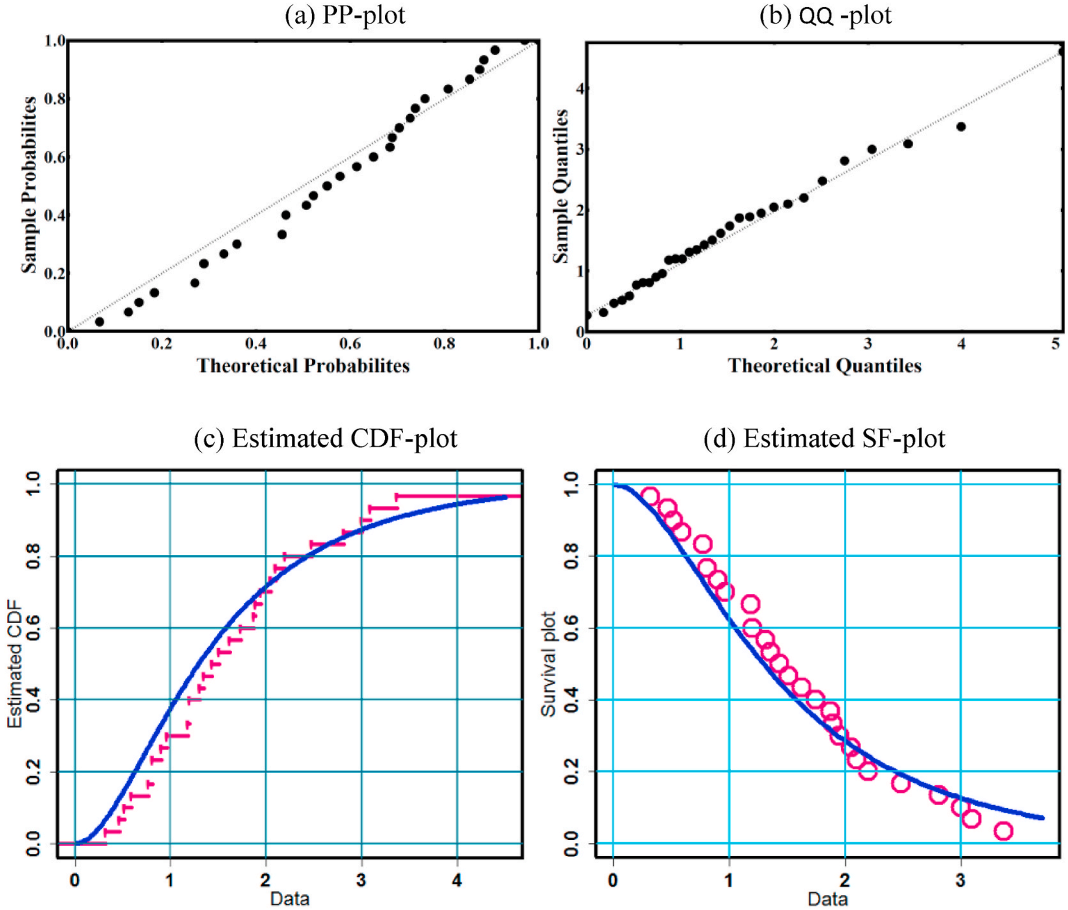

Fig. 3. PP-plot, QQ-plot, estimated CDF, SF of KMBD(ξ).

Table 6
Cycle 1.

0.52	0.77	1.2	1.87	2.48
0.77	**0.81**	1.31	1.74	1.95
0.32	0.59	**0.96**	1.51	1.62
0.81	0.81	0.96	**1.20**	2.05
0.81	1.87	1.87	1.95	**3.00**

then determined.

$$Bias_{\Theta}(n) = \frac{1}{N}\sum_{j=1}^{N}\left(\widetilde{\Theta}_j - \Theta\right),\ MSE_{\Theta}(n) = \frac{1}{N}\sum_{j=1}^{N}\left(\widetilde{\Theta}_j - \Theta\right)^2,\ MRE_{\Theta}(n) = \frac{1}{N}\sum_{j=1}^{N}\left(\widetilde{\Theta}_j / \Theta\right).$$

We also report the relative efficiencies (REs) of the estimators of R based on MSE defined by the following

$$RE1 = \frac{MSE\left(\widetilde{\Theta}_{ML,SRS}\right)}{MSE\left(\widetilde{\Theta}_{ML,RSS}^{\ c=1}\right)}, RE2 = \frac{MSE\left(\widetilde{\Theta}_{ML,SRS}\right)}{MSE\left(\widetilde{\Theta}_{ML,RSS}^{\ c=2}\right)}, RE3 = \frac{MSE\left(\widetilde{\Theta}_{ML,RSS}^{\ c=1}\right)}{MSE\left(\widetilde{\Theta}_{ML,RSS}^{\ c=2}\right)}$$

Larger values (>1) of RE indicate that the efficiency of the estimator given in the denominator outperforms the estimator given in the numerator. Tables 1- 5 present the results of the simulation investigation. Tables 1 and 2 give the outcomes of simulation study for bias, MSE and MRE. The results of the simulation studies regarding coverage probabilities (CPs) and confidence intervals are displayed in Tables 3 and 4. Since intervals get shorter as sample size increases, so does confidence interval accuracy. It is consistently noted that bias and MSE values derived from RSS consistently exhibit lower values compared to those relying on SRS. In addition, for both of the SRS and RSS techniques, the MSE levels decrease as the sample size rises. Additionally, it is evident from Tables 1 and 2 that all the estimators exhibit negligible bias in terms of the bias criterion. In comparison to the SRS scheme, the CPs for the RSS scheme is

Table 7
Cycle 2.

0.9	0.96	1.2	2.2	2.81
0.9	**0.96**	1.2	2.05	4.75
0.96	1.87	**2.05**	2.48	3.00
0.81	1.31	1.43	**1.87**	1.89
0.47	0.77	0.96	1.95	**3.37**

Table 8
Cycle 3.

0.81	1.51	2.10	2.81	3.00
0.77	**1.35**	2.20	3.09	4.75
0.47	0.90	**1.87**	2.10	2.20
0.81	0.96	1.74	**1.89**	3.37
0.81	1.20	1.43	1.89	**3.37**

Table 9
Cycle 4.

0.77	0.81	1.95	2.81	3.37
1.43	**1.74**	1.89	2.81	3.09
0.81	1.20	**1.51**	1.62	1.89
0.32	0.52	1.20	**1.89**	2.20
0.77	0.90	1.43	1.87	**2.05**

Table 10
Parameter Estimation and measure for KMBD using SRS and RSS.

	SRS			RSS		
Est. par	Estimate	LB	UB	Estimate	LB	UB
$\widehat{\xi}$	2.2025	2.1422	2.2628	2.7267	2.6855	2.7679
Anderson-Darling(Statistic)	1.3967	*p-value*	0.2017	0.9297	*p-value*	0.3940
Cramer-von Mises (Statistic)	0.2722		0.1622	0.1333		0.4447
Pearson χ^2 (Statistic)	8.0		0.2381	9.35		0.1523

superior. In most scenarios, the bias and MSE values decrease as the number of cycles rises. Based on the efficiencies observed, the following conclusions can be drawn from Table 5. The estimators of the ξ, based on RSS are more efficient than the corresponding estimators based on SRS for all sample sizes and the all parameter values, see the columns for RE1 and RE2. As the number of cycles increases, the REs rises as well; observe the columns RE3.

7. Data investigation

Here, actual data analysis using Hinkley's [34] original data is done. The March precipitation data set for Minneapolis/St Paul includes 30 observations (in inches). The proposed approach, as presented in the investigations by Sindhu et al. [[39,40]] and Shafiq et al. [41], demonstrates a good fit of the data to the KMB distribution. The MLE of the parameter ξ for the entire sample is 2.38. With a p-value of 0.4660, the Anderson Darling is 0.8188 and value of Pearson χ^2 is 4.6667 with p-value 0.70057. We can corroborate these findings by referring to Fig. 3 with two alternative sampling techniques, SRS and RSS, a random sample of 20 people is chosen for analysis.

In SRS, the sample is {2.05, 0.81, 1.87, 2.2, 1.43, 1.89, 1.74, 1.31, 0.81, 1.51, 1.95, 1.18, 0.47, 3.37, 0.32, 2.48, 1.35, 1.62, 0.9, and 1.2}. With two alternative sampling techniques, SRS and RSS, a random sample of 20 values is chosen for analysis. In the RSS, we defined $\widehat{c} = 4$ and $\widehat{\tau} = 5$; see Tables 5–9. The analysis's findings are provided in Table 10.

8. Conclusions

In this investigation, we utilized ranked set and simple random sample strategies for parameter estimation of the Kavya-Manoharan Bilal Distribution (KMBD) through maximum likelihood estimation. Our comparative analysis considered various scenarios, sample sizes, the quantity of sample units per iteration, and cycle counts, leading to insightful conclusions. Notably, bias and mean squared error (MSE) decreased with increasing sample size, the number of sample units per cycle, and the number of cycles.

To assess the estimation method's effectiveness and compare different sets and cycle lengths, we applied it to real data, estimating parameters and determining the best-fit measurement criteria using the KMBD under both SRS and RSS. Additionally, we established 95 % confidence intervals. Theoretical findings and numerical results underscore the superior efficiency of ML estimates derived from the RSS procedure compared to the SRS strategy.

In future endeavors, we plan to extend the application of the proposed distribution to censored sample methods. Exploring various censoring strategies, including type-I and type-II censored samples, and generating random censored samples based on the new sample methods will be a focal point. Furthermore, our work may evolve to encompass the application of the suggested model in different types of accelerated life testing, such as constant and partially constant tests, and potentially progressive stress accelerated life tests.

Data availability statement

The data that supports the findings of this study are available within the article.

CRediT authorship contribution statement

Anum Shafiq: Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization. **Tabassum Naz Sindhu:** Writing – review & editing, Writing – original draft, Visualization, Software, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Muhammad Bilal Riaz:** Writing – review & editing, Writing – original draft, Validation, Supervision, Investigation, Funding acquisition, Formal analysis. **Marwa K.H. Hassan:** Writing – review & editing, Writing – original draft, Visualization, Validation, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Tahani A. Abushal:** Writing – review & editing, Writing – original draft, Visualization, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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