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British Journal of Educational Psychology (2022), 92, 688–706

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Predicting adaptive expertise with rational number arithmetic

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Background. Adaptive expertise is a highly valued outcome of mathematics curricula. One aspect of adaptive expertise with rational numbers is adaptive rational number knowledge, which refers to the ability to integrate knowledge of numerical characteristics and relations in solving novel tasks. Even among students with strong conceptual and procedural knowledge of rational numbers, there are substantial individual differences in adaptive rational number knowledge.

Aims. We aimed to examine how a wide range of domain-general and mathematically specific skills and knowledge predicted different aspects of rational number knowledge, including procedural, conceptual, and adaptive rational number knowledge.

Sample. 173 6th and 7th grade students from a school in the southeastern US (51% female) participated in the study.

Methods. At three time points across 1.5 years, we measured students' domain-general and domain-specific skills and knowledge. We used multiple hierarchal regression analysis to examine how these predictors related to rational number knowledge at the third time point.

Result. Prior knowledge of rational numbers, general mathematical calculation knowledge, and spontaneous focusing on multiplicative relations (SFOR) tendency uniquely predicted adaptive rational number knowledge, after taking into account domain-general and mathematically specific skills and knowledge. Although conceptual knowledge of rational numbers and general mathematical achievement also predicted later conceptual and procedural knowledge of rational numbers, SFOR tendency did not.

Conclusion. Results suggest expanding investigations of mathematical development to also explore different features of adaptive expertise as well as spontaneous mathematical focusing tendencies.

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DOI:10.1111/bjep.12471

Rational number knowledge is crucial for work-life mathematics (Acme, 2011) and a strong predictor of more advanced mathematical learning (Siegler et al., 2012). While many previous studies have examined the developmental precursors of rational number knowledge (Bailey, Siegler, & Geary, 2014; Bailey, Watts, Littlefield, & Geary, 2014; Hansen, Jordan, & Carrique, 2015; McMullen, Brezovszky, et al., 2016; McMullen, Hannula-Sormunen, & Lehtinen, 2014; Vukovic et al., 2014), these studies have mainly had two main aims: (1) to determine which mathematical skills support learning rational numbers (Vukovic et al., 2014) and (2) to identify warning signs for difficulties in learning rational numbers (Hansen et al., 2015). However, little is known about the predictors of advanced rational number knowledge, such as adaptive rational number knowledge.

Adaptive expertise in mathematics

Adaptive expertise, in contrast with routine expertise, is one feature of exceptional mathematical thinking (Baroody, 2003; Hatano & Oura, 2003; Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009) and is a highly valued outcome of mathematics curricula (Common Core State Standards Initiative, 2010; National Core Curriculum for Basic Education, 2014). In contrast to routine expertise, adaptive expertise is typified by richly connected procedural and conceptual knowledge that can be flexibly applied in novel contexts (Baroody, 2003). The notion of adaptive expertise in mathematics education has motivated research on, among other topics, flexibility with problem-solving strategies (Baroody & Rosu, 2004; Verschaffel et al., 2009) and advanced arithmetic and numerical knowledge (Markovits & Sowder, 1994; McMullen, Brezovszky, et al., 2016). Most notably, procedural flexibility with whole number arithmetic (Siegler & Lemaire, 1997), fractions (Fazio, DeWolf, & Siegler, 2016), and linear algebra (Schneider, Rittle-Johnson, & Star, 2011) have been identified as behavioural manifestations of adaptive expertise. Procedural flexibility is defined by an ability to switch between multiple strategies for solving a particular task (Verschaffel et al., 2009).

Another behavioural manifestation of adaptive expertise, adaptive number knowledge, is defined by the ability to integrate conceptual and procedural knowledge of numerical characteristics and relations into solutions for novel tasks (McMullen, Brezovszky, et al., 2016; McMullen, Hannula-Sormunen, Lehtinen, & Siegler, 2020). Adaptive number knowledge describes the ability of students to recognise relevant arithmetic and numerical relations between a given set of numbers (e.g., ½, 0.5, ¼, 0.25, and 4) and integrate this knowledge in creating unique arithmetic sentences that equal a target number (e.g., 1). For example, high levels of adaptive number knowledge are supported by a rich understanding of the link between magnitudes and the effects of each arithmetic operation. This could take the form of recognising it is possible to calculate 1/4 using 1/2 (i.e., $½ \times n = 1/4$ is possible), if one multiplies 1/2 by a number less than1, even if 1/2 is larger than 1/4. Additionally, having strong knowledge of the relations between fractions and decimals supported the creation of more solutions, especially those that were mathematically equivalent—for example, knowing that since 0.5 is the same magnitude at ½ both $1/2 + 1/2$ and $0.5 + 0.5$ are equal to 1.

Recent evidence suggests the arithmetic sentence production task distinguishes between (1) those students who can adapt their numerical knowledge to find success on a novel task, indicating more adaptive expertise, (2) those who mostly rely on memorised facts and procedures, indicating more routine expertise, and (3) those who have little knowledge whatsoever (McMullen, Brezovszky, et al., 2017; McMullen et al., 2020). Importantly, levels of procedural and conceptual knowledge were equally high among those with adaptive and routine expertise, indicating that adaptive number knowledge is distinct from routine procedural and conceptual knowledge.

Adaptive expertise should be valuable for future learning (Baroody, 2003). Adaptive number knowledge appears to be so, predicting algebra knowledge above and beyond measures of procedural and conceptual knowledge (McMullen, Brezovszky, et al., 2017; McMullen et al., 2020). Adaptive number knowledge may be supportive in recognising crucial numerical relations within linear equations. For instance, in solving $3(x + 1) = 15$, recognising that dividing both sides of the equation by 3 is an advantageous first step (as opposed to first distributing the three across $x + 1$).

Although there has been a serious effort to examine the nature of adaptive expertise in school mathematics (Verschaffel et al., 2009), little is known about its developmental foundations. One major distinction between developing adaptive versus routine expertise is a diversity of experiences in applying knowledge and skills (Feltovich, Spiro, & Coulson, 1997). Thus, it is likely that specific conditions for developing adaptive expertise are needed. Echoing these calls for dynamic learning environments, Hatano (2003, p. xi) argued that adaptive expertise requires an educational environment in which an individual needs to meet 'varied and changing demands' rather than an environment, that is, 'oriented towards solving a fixed class of problems skillfully'. In the typical mathematics classroom, it is not clear if many students will have the opportunity to be challenged with varied and changing demands (Sievert, van den Ham, Niedermeyer, & Heinze, 2019). Despite this, some students seem to develop adaptive expertise with rational numbers (McMullen et al., 2020).

Developmental predictors of rational number knowledge

The developmental predictors of rational number knowledge have been extensively explored in the past 10 years. However, the same predictors may not remain equally important when examining adaptive expertise with rational numbers. Thus, in this study, we aim to examine the predictors of adaptive, conceptual, and procedural rational number knowledge.

Adaptive rational number knowledge was measured with the arithmetic sentence production task with rational numbers. Conceptual and procedural knowledge were measured with a selection of tasks that did not require the integration of multiple features of rational number knowledge in solving a novel task. Instead, these tasks required the typical application of isolated features of rational number knowledge, including estimating the value of a given rational number, matching equivalent representations, and estimating or calculating the outcome of a given arithmetic task. Previous evidence suggests that adaptive rational number knowledge was distinct from students' conceptual and procedural knowledge when measured by these tasks (McMullen et al., 2020).

Previous evidence suggests students with strong overall mathematical knowledge are well poised to learn about rational numbers (Rinne, Ye, & Jordan, 2017; Van Hoof, 2015; Van Hoof et al., 2016). More specifically, knowledge of whole numbers and prior rational number knowledge predict later rational number knowledge (Bailey, Hansen, & Jordan, 2017; McMullen, Brezovszky, et al., 2017). Both routine procedural and conceptual knowledge are expected to support adaptive expertise (Baroody, 2003). Thus, we expect that general mathematical knowledge, especially prior rational number knowledge, will have a strong predictive value for later rational number knowledge, including adaptive rational number knowledge.

Other domain-specific features of students may also play a role in the development of rational number knowledge. Reasoning about spatial proportional relations (Möhring, Newcombe, Levine, & Frick, 2016) and multiplicative relations (McMullen, Brezovszky, et al., 2016) have also been shown to be related to rational number knowledge. Spatial reasoning is related to high-level performance in mathematics and other STEM domains (Wai, Lubinski, & Benbow, 2009). Further, interest in mathematics is an important contributor to success in maths (Eccles & Wigfield, 2002) and may be particularly important for dispositional factors involved with adaptive expertise (Verschaffel et al., 2009). Finally, non-verbal intelligence is also related to rational number knowledge and its development (McMullen, Brezovszky, et al., 2016; Seethaler, Fuchs, Star, & Bryant, 2011).

Recent research on spontaneous mathematical focussing tendencies has revealed their relevance for the development of formal mathematical skills (McMullen, Hannula-Sormunen, Kainulainen, Kiili, & Lehtinen, 2019; Verschaffel et al., 2020; Wijns et al., 2019). Previous studies have shown that students who have a higher tendency of Spontaneous Focussing On multiplicative Relations (SFOR) in situations that are not explicitly mathematical, such as *I drank half the milk* and *I have three times as much juice in my glass*, show greater improvement in rational number development than their peers, even after taking into account non-verbal intelligence, mathematical achievement, whole number magnitude knowledge and skills, and the ability to describe multiplicative relations when explicitly guided to do so (McMullen, Brezovszky, et al., 2016, 2017; McMullen, Hannula-Sormunen, & Lehtinen, 2017; Van Hoof et al., 2016). Dispositional factors akin to SFOR tendency may be related to the development of adaptive expertise (Torbeyns et al., 2009). For example, everyday situations may offer the varying demands needed to develop adaptive expertise with mathematical knowledge (Feltovich et al., 1997). However, it is necessary to first explicitly focus ones' attention on the mathematical features of everyday life to use them (Hannula & Lehtinen, 2005). Thus, those students with a higher SFOR tendency may gain more experience applying their knowledge of multiplicative relations, including rational numbers, in novel situations. This increased exposure to mathematics in novel situations may support the development of adaptive expertise with rational numbers.

Methods

Participants

Students from the sixth (Mean age = 12.7 years; Range = 11.8–13.8) and seventh (Mean age = 13.7 years; Range = 13.1–14.9) grades of a school in the southeastern United States (N = 173; 51% female) participated in the study. The population of the school was fairly diverse; at the last measurement point 51% of the students were White, 28% African-American, 11% Hispanic, and 5% Asian. As well, 43% of students were eligible for free or reduced lunch. Students were enrolled in different mathematics classes ranging from Maths 1 to Algebra and Geometry taught by nine different teachers. We accounted for these variations in students enrollment by including a measure of overall mathematics achievement. In order to mask the mathematical nature of the tasks (see below), students were recruited and tested in their science classrooms and not in their maths classes. All participants had parental permission to participate in the study and gave their assent before participating. The ethical board of the first authors' institution approved the study. At the first measurement point (see Figure 1 for an overview), participants completed measures of rational number conceptual knowledge, SFOR tendency, and multiplicative

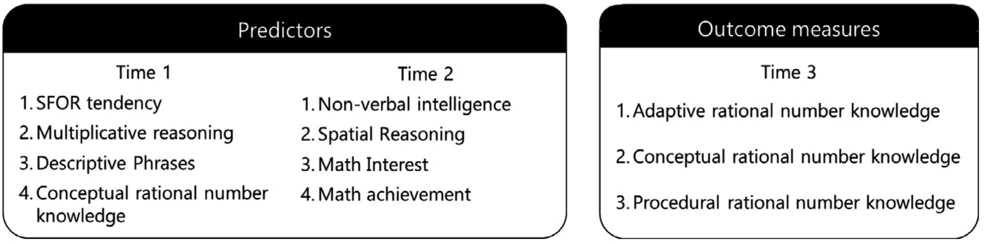

Figure 1. Overview of the testing procedure.

reasoning. At the second measurement point, participants completed measures of non-verbal intelligence, spatial reasoning, general mathematical achievement, and mathematical interest. One year after this, at the last time point, participants completed measures of rational number conceptual and procedural knowledge and a measure of adaptive rational number knowledge. Anonymized data used in this study is available at https://osf.io/m3uk9/

Predictor measures

SFOR tendency

At the first measurement point, SFOR tendency was measured with two picture description tasks, each involving four items (McMullen, Brezovszky, et al., 2016). These tasks were presented before any mention of mathematics and completed during science class. Students were asked to describe situations that did not explicitly call for mathematics to be used, but where it could be. Since students were not instructed to pay attention to mathematical relations in these tasks, any use of exact multiplicative relations first required the students to spontaneously focus on multiplicative relations. All items were presented in colour booklets, one item per page, with the pictures also projected onto a screen at the front of the room. SFOR measures can be accessed at https://osf.io/rnu2y/.

The *teleportation task* has been used previously as a measure of multiplicative SFOR tendency (McMullen, Brezovszky, et al., 2016).[1] Students were told that a teleportation device was sending items to space colonies, but that the material changed en route. They were first asked to: 'describe in as many ways as possible', how three sets of material changed. For example, three blue cans, four blue milk cartons, and one basket of blueberries changed into 9 red-orange cans, 12 red juice boxes, and 3 baskets of red apples. There were many different ways to describe the changes (e.g., colour, shape, exact number of objects), including by a common multiplicative factor (e.g., multiplied by three, divided by two). Next, a new set of the same items as in the previous trial, but in different quantities, was presented. Students were asked to draw what they expect would arrive based on what happened 'last time'. Students were told they could look back to the previous page, and coloured pencils were available if they chose to use them. They completed two sets of writing and drawing items on the teleportation task.

On the *lunch task* (Figure 2), students were shown images of two lunches side by side on separate plates and asked to 'describe in as many ways as possible how the lunches are

[1] *Degrande, Verschaffel, & Van Dooren (2017) have applied the same task design with additive, multiplicative, and ambiguous relations (i.e. either additive or multiplicative). Given the strong relation between multiplicative and fractional relations (Möhring et al., 2016), we examine only multiplicative relations.*

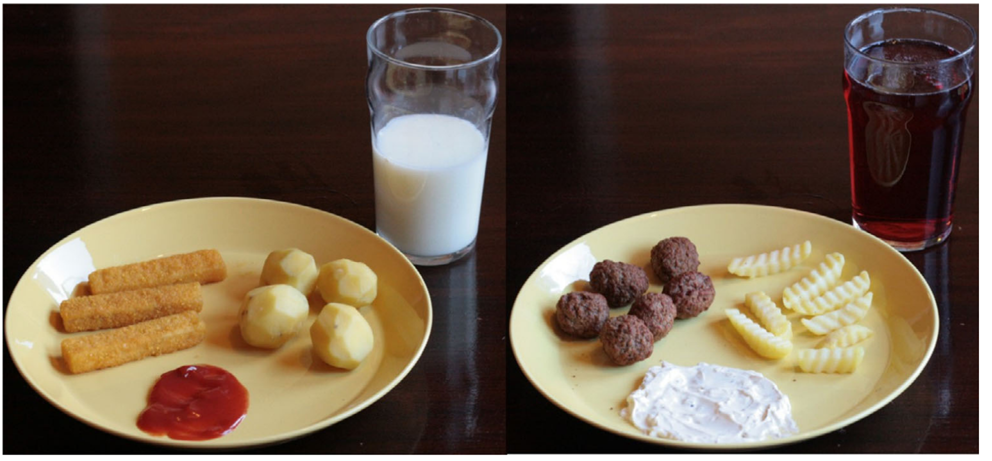

Figure 2. Example of Lunch task stimuli. Copyright CC-By Attribution 4.0 International. Originally published at https://osf.io/ghx74/

different from each other'. The lunches varied in the types and amounts of food on each plate, including a common multiplicative relation. On a subsequent trial with the same types of food in different amounts, students were asked to draw what they expected the second plate to include based on the previous item. In total, students completed two trials of the writing item and two trials of the drawing item on the lunch task.

SFOR tendency was calculated by combining scores from the teleportation and lunch tasks. For the written description items, the total number of multiplicative relation descriptions was calculated (e.g., 'three times as many potatoes' would be given one point). For the drawn items, the number of items that were drawn using correct multiplicative relations was calculated. Inter-rater reliability was high for two independent coders' scorings of 35 participants' responses (Intraclass Correlation = .96). Separate standardised scores for the written and drawn sub-scales were calculated and added together to create an overall SFOR tendency score. Test–retest reliability, using novel SFOR tests and testers at each time point, is acceptable (Spearman's ρ = .80) for this composite measure in previous studies (McMullen et al., 2019), and inter-item reliability was acceptable for the eight items in this study (Cronbach's α = .70).

Descriptive phrases

To account for differences in students' overall descriptive fluency, the total number of descriptive phrases, including all descriptions of colour, content, numbers, and non-specific quantitative relations (e.g., more/less), were calculated for the four SFOR description items. Reliability was acceptable across the four items (Cronbach's α = .71).

Multiplicative reasoning

To determine the influence of the ability to recognise and describe multiplicative relations when explicitly guided to do so (as opposed to spontaneously; Hannula & Lehtinen, 2005), after students completed all SFOR tasks at the first time point, they were presented the same items again with new instructions. They were given instructions to describe or draw the multiplicative relations for each item (e.g., 'describe how the packages were divided'). One point was given for each correct multiplicative relation that was described or drawn. Reliability of the number of multiplicative responses was acceptable for all eight items (Cronbach's α = .71).

Rational number conceptual knowledge

At the first measurement point, we measured two aspects of participants' rational number conceptual knowledge, knowledge of magnitudes of rational numbers, and operations with rational numbers.

Rational number magnitude knowledge was assessed by a magnitude comparison task (Stafylidou & Vosniadou, 2004; Van Hoof, Janssen, Verschaffel, & Van Dooren, 2015). Participants were asked to 'Circle the larger [fraction/decimal]. If the numbers are equal, circle both'. There were six fraction comparison items (5/9 vs. 5/7; 2/3 vs. 3/5; 7/15 vs. 3/4; 2/3 vs. 6/9; 2/5 vs. 2/7; 5/4 vs. 6/8) and six decimal comparison items (3.5407 vs. 3.65; 0.4 vs. 0.40; 0.38 vs. 0.6; 1.43 vs. 1.253; 7.08 vs. 7.7; 0.1 vs. 0.02). Answers were scored as correct or incorrect, with the maximum score for the test being 12. Reliability was high across the 12 items (Guttman's Lambda = .74).

Rational number arithmetic operations knowledge was measured using six items adapted from Van Hoof et al. (2015). Items tested students' knowledge of the effects of arithmetic operations with fractions and decimals (e.g., 'Is the outcome of $50 \times 1/2$ smaller or larger than 50?'; 'What is half of 1/8?'). All items were incongruent, such that reasoning based on features of arithmetic with whole numbers would lead to incorrect answers (e.g., Multiplication always makes a number bigger). Reliability was acceptable for these items (Guttman's Lambda = .74).

Non-verbal intelligence

At the second time point, Raven's coloured Progressive Matrices (Raven, 1976) were used as a measure of non-verbal intelligence. A total of 12 items (D1–D12) were used. Items were scored as correct or incorrect, with a maximum score of 12. Reliability was good for these items (Guttman's Lambda = .77).

General mathematics achievement

At the second time point, students completed the Woodcock-Johnson III Calculation Fluency sub-test (Woodcock et al., 2001), which included arithmetic and algebraic computation items, as a general measure of their mathematical achievement. Only items that did not include rational numbers were used as a measure of general mathematical ability outside of rational number knowledge. In total, there were 23 items, for which students were given one point per correct answer. Reliability for these items was acceptable (Guttman's Lambda = .68).

Spatial reasoning

At the second time point, the first twelve items of the Mental Rotations Test (Peters et al., 1995) were used to assess students' spatial reasoning. On this test, students are asked to identify matching objects that have been rotated in space. Each item had two correct answers. Students were given one point for each correct answer, for a total of 24 possible points. Reliability was acceptable for these items (Guttman's Lambda = .62).

Interest in mathematics

At the second time point, students' motivation to learn mathematics was measured through three items formulated by Berger and Karabenick (2011) ('I like math', 'I enjoy doing math', 'Math is exciting to me'). Students rated themselves on 5-point Likert scales ranging from 'completely disagree' (1 point) to 'completely agree' (5 points). Reliability was excellent for these three items (Cronbach's α = .92).

Outcome measures

At the third time point, we measured participants' adaptive rational number knowledge, rational number procedural knowledge, and rational number conceptual knowledge. Outcome measures can be accessed at https://osf.io/rnu2y/.

Adaptive rational number knowledge

The arithmetic sentence production task with rational numbers was used to measure adaptive rational number knowledge. The arithmetic sentence production task has been shown to capture differences between routine and adaptive expertise with arithmetic (McMullen et al., 2020). High-level performance on the task requires the integration of conceptual and procedural knowledge in solving a novel task. In the task, students must integrate their knowledge of rational number representations, magnitudes, and effects of operations in coming up with procedural solutions to the novel task. As with the whole number version of the task (McMullen, Brezovszky, et al., 2017), students were given 90 seconds to generate and write down as many mathematically correct arithmetic sentences as possible that used subsets of five numbers and the four arithmetic operations to produce a target number. Each of the four items included two pairs of equivalent fractions and decimals (e.g., ½ and 0.5, ¼ and 0.25) and a single whole number (e.g., 4) as the numbers the students should make the target number (e.g., 1). Students could use each number repeatedly; answers were counted as correct if they were mathematically correct, only used the given numbers, and were not literal repetitions of a previous answer (e.g., ½ + ½ was counted as correct even if the student previously answered 0.5 + 0.5). Students were given one point for each correct arithmetic sentence they wrote. Reliability was good (Cronbach's α = .86).

Rational number conceptual knowledge

Three aspects of rational number conceptual knowledge were measured at the third time point: knowledge of magnitudes of rational numbers, operations with rational numbers, and representations of rational numbers.

Rational number magnitude knowledge was assessed through two tasks: an ordering task and a number line estimation task (Schneider & Siegler, 2010; Stafylidou & Vosniadou, 2004; Van Hoof et al., 2015). The ordering task included three fraction items (e.g., 'Put the numbers in order from smallest to largest': 6/12; 5/7; 2/6) and three decimal items (e.g., 'Put the numbers in order from smallest to largest': 5.89; 5.886; 6.5). Each item was scored as correct or incorrect with a maximum score of 6 for the test. Reliability was good (Guttman's Lambda = .72).

Number line estimation was assessed on a 0–1 number line with four items (0.6, 1/5, 3/7, and 0.42), and on a 0–5 number line with four other items (11/7, 3.7, 9/2, and 0.83). Percent absolute error was used to measure accuracy on both number lines (Siegler et al., 2009). Reliability was acceptable for these items (Cronbach's $\alpha = .70$). Scores on the ordering and number line estimation tasks were independently standardised and summed to create an overall measure of magnitude knowledge.

Rational number arithmetic operations knowledge was measured using a parallel version of the six items adapted from Van Hoof et al. (2015) used at the first measurement point (e.g., 'Is the outcome of 40 × 1/3 smaller or larger than 40?'; 'What is half of 1/6?'). Reliability was good for these items (Guttman's Lambda = .79).

Rational number representation knowledge was examined via an adapted version of the Number Sets Test (Geary et al., 2009; 2010). Students had 1 min to identify as many symbolic and non-symbolic representations as possible that equalled first ½ and then 0.9. Each item had 15 alternative answers, with nine and eight correct matches per item. Correct answers added a point; incorrect answers deleted a point. Reliability was good for these items (Cronbach's $\alpha = .82$).

Rational number procedural knowledge

Participants were asked to solve 12 fraction arithmetic problems (2/3 − 1/3; 4/7 ÷ 1/2; ¾ × 1/5; 8 ½ ÷ 4 1/8; 5/7 − 1/2; 1/5 + 2/3; 7/8 + 2/8; 2 ¾ + 4 1/8; 2 6/7 + 5 1/2; 5/8 ÷ 3/8; 3 2/3 − 3/4; 3/5 × 1/5) and 12 decimal arithmetic problems (1.05 × 0.2; 0.71 − 0.4; 0.11 + 0.7; 5.29 − 4.2; 3.4 + 1.02; 0.38 − 0.14; 0.4 + 0.2; 0.9 ÷ 0.3; 0.4 × 0.52; 0.111 × 0.097; 3.06 × 5.3; 0.84 ÷ 0.4). Answers were scored as correct or incorrect, with the maximum score for the test being 24. Reliability was good for these items (Guttman's Lambda = .86).

Analysis

At the first measurement point, an equally weighted composite score of rational number conceptual knowledge was created by separately standardising scores from the comparison and operations tasks and calculating the average scores of these standardised values. At the third measurement point, ordering, number line estimation, representations, and operation tasks were combined to create a composite rational number conceptual knowledge score using the same procedure.

To examine the predictors of rational number knowledge, a series of linear regressions were estimated with adaptive, conceptual, and procedural rational number knowledge used as the dependent variables. Multi-collinearity indices were acceptable for all independent variables (tolerances > 0.44, VIFs < 2.2). Explanatory power was examined by inputting each predictor or set of predictors as the last step to calculate the change in R^2 that was unique to that (set of) predictor(s). The domain general abilities of descriptive phrases, non-verbal intelligence, and spatial reasoning were grouped as a single step. The general mathematical predictor variables of mathematical achievement, maths interest, and multiplicative reasoning were grouped. Rational number conceptual knowledge at Time 1 and SFOR tendency were separately added as their own steps.

Results

Predicting rational number knowledge

Table 1 reports the means, standard deviations, and correlation matrix for all measures. All predictor variables were at least moderately related to an aspect of rational number knowledge and were therefore included in the regression models.

Overall, 47% of the variance in adaptive rational number knowledge was explained by the predictors included in the regression model, with 64% and 52% of conceptual and procedural knowledge explained, respectively (Table 2). The domain-general predictors of the number of descriptive phrases, non-verbal intelligence, and spatial reasoning did not explain any unique variance in adaptive, conceptual, or procedural rational number knowledge, after controlling for domain-specific skills and abilities. These results further previous evidence that domain-specific prior knowledge is an important feature of mathematical development.

After taking into account domain general predictors, rational number knowledge, and SFOR tendency, the general mathematical predictors of maths interest, mathematical achievement, and multiplicative reasoning collectively explained between 5% of variance in adaptive and conceptual rational number knowledge and 11% of variance in procedural rational number knowledge. Mathematical achievement one year prior was a unique predictor of adaptive, conceptual, and procedural rational number knowledge one year later. Multiplicative reasoning was a unique predictor of conceptual and procedural rational number knowledge one year later, but not adaptive rational number knowledge. Maths interest did not uniquely predict rational number knowledge one year later. Thus, in total, prior mathematical achievement is unsurprisingly revealed as an important predictor of later rational number knowledge. As well, multiplicative reasoning appears to be somewhat important for later routine rational number knowledge of concepts and procedures.

Prior rational number conceptual knowledge measured at Time 1 uniquely explained 3% of variance in adaptive and procedural rational number knowledge and 11% of variance in conceptual knowledge measured at Time 3. While the unique predictive power of prior rational number knowledge was fairly limited for adaptive and procedural knowledge, these results further confirm the importance of prior knowledge in mathematical development.

Finally, after taking into account domain-general, general maths predictors, and rational number conceptual knowledge, SFOR tendency one year prior explained an additional 3% variance in adaptive rational number knowledge, but no additional variance in conceptual and procedural knowledge of rational numbers. These results reveal, for the first time, how SFOR tendency is a unique predictor of adaptive rational number knowledge.

Discussion

This study highlights a less common aim in examining individual differences in mathematical development: the development of advanced mathematical skills. We present the first evidence of what predicts adaptive rational number knowledge across a year and a half period. Alongside general mathematical ability and prior rational number knowledge, a disposition to spontaneously recognise and use multiplicative relations, SFOR tendency, appears to support features of adaptive expertise with rational numbers. However, unlike in previous studies (Van Hoof et al., 2016), this dispositional factor did

Table 1. Means, standard deviations, and correlation matrix for all measures ($N = 173$)

	Mean	*SD*	SFOR	Descript phrases	Mult reason	Comp (time 1)	Oper (time 1)	Non-verbal intell	Maths Int	Spatial reason	Maths achieve	Adaptive	Ord (time 3)	Num line Est	Oper (time 3)	Rep
SFOR	10.69	6.23														
Descriptive phrases	29.57	10.57	.14													
Multiplicative reasoning	15.27	6.12	.55***	.30***												
Rational number comparison (Time 1)	9.20	3.68	.45***	.20**	.53***											
Rational number operations (Time 1)	2.53	1.95	.47***	.17*	.47***	.55***										
Non-verbal Intelligence	9.26	2.04	.22**	.01	.30***	.27***	.23**									
Maths interest	9.41	3.75	.06	.05	.10	.16*	.22**	.03								
Spatial reasoning	12.99	4.90	.26***	.01	.10	.26***	.30***	-.05	.14							
Maths achievement	13.58	2.95	.40***	.14	.39***	.52***	.59***	.26***	.24**	.28***						
Adaptive rational number knowledge	9.05	7.21	.54***	.20**	.49***	.57***	.59***	.24**	.19*	.28***	.57***					
Rational number ordering (time 3)	2.78	2.00	.43***	.17**	.50***	.64***	.60***	.35***	.07	.23**	.57***	.67***				
Rational number line estimation (time 3)	16.61	10.34	.40***	.08	.46***	.62***	.51***	.26***	.11	.21**	.53***	.62***	.64***			
Rational number operations (time 3)	2.73	2.07	.45***	.17*	.54***	.61***	.70***	.26***	.14	.25***	.52***	.70***	.70***	.69***		
Rational number representations	11.26	4.45	.45***	.27***	.55***	.66***	.62***	.26***	.15	.32***	.56***	.70***	.70***	.66***	.71***	
Rational number arithmetic	10.19	5.62	.46***	.18*	.55***	.54***	.59***	.26***	.18*	.21**	.63***	.67***	.67***	.61***	.69***	.67***

Note. *p < .05; **p < .01; ***p < .001.

Table 2. Hierarchical linear regression analyses: predicting rational number knowledge at time 3 ($N = 173$)

Step		Adaptive rational number knowledge		Rational number conceptual knowledge		Rational number procedural knowledge	
		b (SE)	ΔR^2 (as last step)	b (SE)	ΔR^2 (as last step)	b (SE)	ΔR^2 (as last step)
1	Domain-general predictors		.003		.00		.00
	Descriptive phrases	.05		−.00		−.02	
	Non-verbal intelligence	.02		.04		.01	
	Spatial reasoning	.05		.04		−.01	
2	Maths-specific predictors		.05***		.05***		.11***
	Maths interest	.06		−.04		.00	
	Maths achievement	.26***		.23***		.37***	
	Multiplicative reasoning	.08		.17*		.23**	
3	Rational number magnitude and operations knowledge	.26**	.03**	.50***	.11***	.24**	.03**
4	SFOR Tendency	.23**	.03**	.02	.00	.05	.00
	Intercept (unstandardised)	−10.38 (2.86)		−2.37 (0.27)		−5.98 (2.14)	
	ANOVA	$F(8, 165) = 20.52, p < .001$		$F(8, 165) = 39.05, p < .001$		$F(8, 165) = 24.16, p < .001$	
	Adj. R^2		.47		.64		.52

Note. $*p < .05$; $**p < .01$; $***p < .001$.

not predict the routine features of rational number conceptual and procedural knowledge. We discuss the implications of these findings for the development of mathematical knowledge and mathematical instruction.

SFOR tendency and the development of adaptive expertise

These results extend our understanding of the development of adaptive expertise in mathematics by providing the first evidence that spontaneous mathematical focussing tendencies may be beneficial for the development of this highly valued, but rarely achieved, outcome of mathematics instruction. In contrast with previous research that found primary students' SFOR tendency predicted their rational number conceptual development (McMullen, Brezovszky, et al., 2016; Van Hoof et al., 2016), this study found that, in lower secondary students, SFOR tendency predicted adaptive rational number knowledge, but not rational number conceptual knowledge. This may be explained by rational numbers not being as large a portion of the middle school curriculum as in primary school, leading to different developmental patterns with routine knowledge, which should be more closely tied to classroom instruction.

This study cannot explicitly determine if or how SFOR tendency may contribute to the development of adaptive rational number knowledge. However, previous research has suggested that a higher SFOR tendency is presumed to contribute to more self-initiated practice with mathematical, particularly explicit multiplicative, relations in everyday situations (Lehtinen, Hannula-Sormunen, McMullen, & Gruber, 2017; McMullen et al., 2019). More self-initiated practice with multiplicative reasoning in everyday situations may have multiple benefits for learning about formal mathematics. Spontaneous mathematical focussing tendencies, such as SFOR, have been found to influence and predict formal mathematical learning over extended periods throughout schooling ages (see McMullen et al., 2020 and Verschaffel et al., 2020 for recent reviews). Initially, self-initiated practice may allow students to more easily consolidate various concrete examples of mathematical objects into a more coherent and consistent concept (Schwartz, Bransford, & Sears, 2005). For example, the benefit of SFOR tendency in late primary school in the development of rational number conceptual knowledge (McMullen, Hannula-Sormunen, Hannula-Sormunen, Laakkonen, & Lehtinen, 2016; Van Hoof et al., 2016) may have been driven by those students with a higher SFOR tendency getting more experience reasoning about the concept of half in their everyday lives. This extra practice may provide benefits beyond their formal fraction instruction for their understanding of rational number magnitudes.

At later stages, advanced knowledge of rational numbers may be better supported by reasoning about mathematics in everyday experiences. The complexity and novelty of reasoning about mathematics in everyday life are desirable features of learning environments that are expected to support the development of high-level conceptual knowledge and adaptive expertise (Feltovich et al., 1997; Hatano & Oura, 2003). Mathematical features of everyday situations are inherently ill-defined and complex. There is rarely one simple way to mathematically model the relations embedded in everyday situations; the web of inputs, representations, and estimations required to reason mathematically is complex. This is in direct contrast to the typical tasks required of students in the mathematics classroom, which are often concise, well-structured, and require little, if any, mathematical modelling, even when purportedly about everyday life (Pongsakdi, 2017; Verschaffel, De Corte, & Lasure, 1994).

In line with the theory of adaptive expertise (Hatano & Oura, 2003), this learning process does not result in simply gaining more knowledge from existing instruction; instead, it may reflect a qualitatively different disposition to explore mathematical content and make explicit connections across topics. According to our results, the connection between SFOR tendency and adaptive rational number knowledge is not explained by, among other things, an interest in mathematics, overall mathematical achievement, prior rational number knowledge, or non-verbal intelligence. This suggests that SFOR tendency is not simply reflective of stronger skills or even a general disposition towards mathematics. SFOR requires recognition that there is mathematics embedded in a situation, that the mathematics is relational in nature, and that these relations are multiplicative. This is certainly not a simple process, and it requires strong relational reasoning and mathematical knowledge (McMullen, Brezovszky, et al., 2016). Nonetheless, the relation between SFOR tendency and adaptive rational number knowledge is not entirely explained by the ability to reason with relational features, as indexed by performance on the multiplicative reasoning tasks.

Implications for instruction

SFOR tendency itself does not appear to be a fixed trait, that is, innate to an individual. Instead, it appears possible to increase students' SFOR tendency by modelling the recognition and use of multiplicative relations in everyday situations (McMullen et al., 2019). These activities have been found to improve primary school students' multiplicative reasoning and lead to long-term gains in fraction conceptual knowledge (Määttä, Hannula-Sormunen, Halme, & McMullen, in press). However, it is not clear if enhancing students' tendency to recognise and use multiplicative relations in their everyday lives would improve their adaptive expertise with rational numbers.

Nonetheless, efforts to support the development of adaptive expertise with rational numbers should be encouraged and this study provides some evidence that improving spontaneous mathematical focussing tendencies may be one way to do so. Traditional instruction may not be enough to support the widespread development of adaptive expertise (Feltovich et al., 1997). Even those students with high prior knowledge rarely develop features of adaptive expertise without explicit opportunities to do so (Sievert et al., 2019). Previous studies have shown that comparisons between multiple problem solving strategies are one important key to the development of adaptive expertise (Rittle-Johnson & Star, 2009). More recently, evidence from game-based learning environments suggests that giving students the opportunity for playful exploration of the relations between numerical characteristics and arithmetic operations is a promising avenue for promoting adaptive expertise (Brezovszky et al., 2019; Kärki et al., in press; Yu & Denham, 2021). Expanding these activities to involve also everyday features of students' surroundings using realistic scenarios and everyday objects may promote their spontaneous use of multiplicative reasoning also outside of the training tasks. If this leads to increased self-initiated practice with multiplicative reasoning, as has been found possible with young children's tendency of spontaneous focussing on numerosity (Hannula-Sormunen, Lehtinen, & Räsänen, 2020), similar gains in rational number knowledge, including adaptive expertise, may be possible.

Limitations and future directions

The main limitation of this study is both SFOR tendency and adaptive rational number knowledge are only measured with a single task type. SFOR tendency is mainly assessed via picture description and drawing tasks, without much evidence of actual spontaneous focussing in real-world situations. The contrived scenarios presented to the students in the teleportation and lunch tasks are not direct measures of what they may do in everyday situations. However, this study does improve on previous SFOR measures by including the lunch task, with more authentic real-world stimuli that are rich in non-mathematical features and are not as highly structured (cf. Degrande, Verschaffel, & Van Dooren, 2017). Although this limitation limits the ability to directly test the hypothesis that it is self-initiated practice that drives the relation between SFOR tendency and adaptive expertise with rational numbers, the recognition that this relation remains even after controlling for a wealth of potential confounds suggests that a causal relation is plausible (Bailey, Duncan, Watts, Clements, & Sarama, 2018). Nonetheless, expanding the measures of both SFOR tendency and adaptive rational number knowledge would be crucial for future studies.

As well, the lack of direct evidence of the causality of the relation between SFOR tendency and adaptive expertise with rational numbers requires temperance in drawing conclusions from the present study. In particular, there is no evidence of the proposed causal mechanism, an increase in self-initiated practice supported by SFOR. However, the underlying relation appears unlikely to be caused by a confounding common cause, such as mathematical interest or achievement or underlying cognitive abilities such as non-verbal intelligence. Nonetheless, an RCT that could directly examine the causal relation between SFOR tendency and rational number development would still be valuable, especially in determining any potential causal mechanism.

Finally, this study included participants from a single school. While there was a large amount of diversity in the mathematics courses the students took during the study period, it is possible that a single educational setting does not capture an appropriate level of variation in the features of learning environments that influence the development of adaptive expertise. Future studies should increase the number of schools included and also aim to take into account teacher and curricular influences on the development of adaptive expertise (Sievert et al., 2019) through, for example, multi-level modelling.

Conclusions

These findings further our understanding of the development of adaptive expertise and the role spontaneous mathematical focussing tendencies may play in this development. The notion of adaptive expertise suggests that students should be able to flexibly and adaptively apply their knowledge in novel situations. Students with a higher tendency to recognise the relevance of mathematical features of their environment may have an advantage over their peers in developing adaptive expertise. Analyses of rational numbers should be broadened to include tasks, such as the ones used to measure SFOR and adaptive number knowledge in this study, which are not usually part of classroom instruction.

Acknowledgements

We gratefully acknowledge the support of the Academy of Finland (Grant 310388) and its Strategic Research Council (Grant 312528), Cassondra Griger, Elena Leib, and all participants and teachers.

Conflicts of interest

All authors declare no conflict of interest.

Author contribution

Jake McMullen: Conceptualization (equal); Data curation; Formal analysis; Funding acquisition; Investigation; Methodology (equal); Writing – original draft; Writing – review & editing (equal). **Minna M Hannula-Sormunen:** Conceptualization (equal); Methodology (equal); Writing – review & editing (equal). **Erno Lehtinen:** Conceptualization (equal); Methodology (equal); Writing – review & editing (equal). **Robert S. Siegler:** Conceptualization (equal); Methodology (equal); Writing – review & editing (equal).

Data availability statement

Data associated with this study is available at https://osf.io/m3uk9/

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Received 30 April 2021