Research article

Implementing GeoGebra integrated with multi-teaching approaches guided by the APOS theory to enhance students' conceptual understanding of limit in Ethiopian Universities

Mulat Gebeyehu Baye [a,*], Mulugeta Atnafu Ayele [b], Tadele Ejigu Wondimuneh [a]

[a] Department of Mathematics Bahir Dar University, Ethiopia
[b] Department of Science & Mathematics Education, College of Education & Behavioral Studies, Addis Ababa University, Ethiopia

ARTICLE INFO

ABSTRACT

The notion of limit is one of the fundamental concepts which underpins advanced calculus of one or more variables in the field of analysis. However, understanding the concept of limit has been an impenetrable problem for many students in Ethiopian Universities. Only very few literatures were documented focusing on overcoming the difficulty of learning the concept of limit. For this reason, the overarching aim of the present study is to enhance students' conceptual understanding of limit by empowering their visualization skills using GeoGebra integrated with multi-teaching approaches. The study employed mixed methods experimental (intervention) design within an APOS paradigm. Both qualitative and quantitative data were collected. Qualitative data was collected using students' reflections and interviews, whereas quantitative data was collected through pretest and posttest using diagnostic tests. The results of the qualitative data analysis revealed that the learning milieu created a positive impact on students' understanding of the concept of limit. Additionally, students provided coherent and viable reasons while making mental constructions and their coordination in the learning process based on the genetic decomposition grounded in APOS theory. Furthermore, the results of the quantitative (posttest) data analysis proved that students' mean scores on conceptual understanding of limit in the experimental group was significantly better than those in the control group. Thus, it could be possible to conclude that students' conceptual understanding of limit is improved using GeoGebra integrated with multi-teaching approaches within an APOS paradigm. The findings open a great opportunity to suggest technology integrated mathematics curriculums for the teaching and learning of mathematics.

1. Introduction

Calculus is an increasingly important area in the field of analysis and applied mathematics [1]. It entails a comprehensive account of concepts such as limit, continuity, derivative and integrals of functions among other things. Limit is one of the fundamental concepts which underpins advanced calculus of one or more variables in the field of analysis [2, 3]. For example, concepts such as continuity, derivative, integrals of functions and convergence theories have been defined using the notion of limit. The concept of limit is an abstract and complex idea; as a result, it demands and develops students' advanced mathematical thinking [3].

The notion of limit has been given to students in different disciplines in most of the Universities in Ethiopia. However, students are facing difficulties in learning the concept [4]. Most of the students do not have conceptual understanding of limit. Specially, understanding the formal conception of limit is found to be an impenetrable problem for many students. There has been a universal agreement amongst mathematics teachers and researchers that students learn the concept of limit amid major challenges in conceptual understanding [5, 6]. This problem triggered us to be part of a solution and consequently add new points to the existing literatures. In fact, in resolving the aforementioned problems, different endeavors were made by researchers and mathematics teachers using pedagogical strategies, technology integrated with pedagogical strategies and theoretical models. For example, Cottrill et al. [7] did a strong attempt to improve students' conceptual understanding of limit using technology (Computer programing) integrated with pedagogical strategies and a theoretical model.

Apart from this, mathematics software such as Geometer's Sketchpad, CAS, Spreadsheet and GeoGebra are some of the commonly used technologies that support mathematics teaching and learning. GeoGebra is one of the newly emerging technologies in mathematics education to support the teaching-learning process of mathematics from primary school up to university level [8]. It is a freely available multi-platform dynamic mathematics software which brings geometry, algebra and calculus together. The dynamic nature, multi-platform, friendly use, multiple representation and open accessibility are among others that make GeoGebra be a powerful and preferred tool for students' learning [9]. However, the use of GeoGebra alone is unlikely to bring a significant change on learners conceptual understanding. For example, a study documented that students' posttest achievement was significantly lower than their pretest achievement after the treatment using GeoGebra [10]. Therefore, it is demanding to integrate it with worthwhile teaching approaches.

On the other hand, considering a theoretical model of understanding mathematical concepts plays a significant role to enhance students' learning [11]. In line with this, APOS theory is a powerful theory for modeling students' understanding of mathematical concepts, its nature and development [12]. The theory is an extension of Piaget's idea of reflective abstraction [13]. It was originally created to apply Piaget's concepts of reflective abstraction to postsecondary mathematics education, but some works have also been done in the context of primary and secondary school mathematics [12]. The main points of the theory were introduced by Dubinsky [14]. The acronym APOS refers to actions, processes, objects and schemas, which was first introduced by Cottrill et al. [7]. According to the APOS theory, mathematical knowledge arises from an individual's proclivity to respond to a perceived problem situation through reflection about the problem and its solution in a social context and by constructing or reconstructing actions, processes and objects, and organizing these in schemas so as to deal with the problem situation [12]. Cottrill et al. [7] used APOS paradigm to investigate students' conceptual understanding of limit. It involves interconnections of theoretical analysis, design and implement instruction, and empirical data. In their study, a genetic decomposition model was employed. Genetic decomposition is a description of mental constructs that learners might experience while understanding a mathematical concept. However, students were not managed activities on the concept of limit, which demanded higher level mental structures based on the genetic decomposition.

In sum, students' mental constructions may be elicited based on successful enactment of physical operations using computer activities, and paper and pencil work. In this regard, GeoGebra may be considered as a powerful cognitive tool to link students' physical activities on computers, and paper and pencil work with mental operations in a better way as compared to other mathematical software [9, 15]. In addition to this, for understanding abstract concepts like limit and some other mathematical concepts in a better way, integrating teaching approaches with worthwhile technology might be inevitable [16]. These were some of the rationales or reasons why GeoGebra, multi-teaching and APOS theory are used in this study. Details are available in the literature review sections.

The purpose of the present study is to enhance students' conceptual understanding of the notion of limit by boosting students' visualization skills using GeoGebra integrated with multi-teaching approaches. Experimental (intervention) design coupled with the APOS paradigm is used to explore students' conceptual understanding of limit. The findings of the present study might have invaluable contribution to teaching and learning of mathematics for University mathematics students.

To achieve this, the following research questions were framed.

1. What were the views of students about learning the limit concept using GeoGebra integrated with multi-teaching approaches?
2. What is the nature of students' reasoning while working on carefully designed limit activities based on the genetic decomposition model?
3. Is there a statistically significant difference in students' mean scores on conceptual understanding of the notion of limit between the experimental group and the control group?

2. Literature review

2.1. The concept of limit

The notion of limit has been a controversial and much debated concept in the area of calculus. Güçler [6] asserted that students' misconceptions of limit were mainly associated with the phrases "approaches", "tends to" and "gets close to" in expressing the symbol "$\lim_{x \to a} f(x)$" in the informal conception. Another researcher also argued that the phrases "approaches", "tends to" and "gets close to" entail dynamic or process conception, whereas the conventional ε-δ form shows a static conception; as a result of this, the dynamic or process conception interferes with the formal conception of limit and hinders students' learning [17].

In contrast to the aforementioned studies, however, Cottrill et al. [7] contended that the dynamic conception of limit was a foundation for the formal conception of limit and did not hinder students' learning. They believed that the quantification associated with the ε-δ form in the formal conception makes it more complex and abstract for students' learning. In addition to this, in our teaching experience of mathematics, we observed that mathematics teachers oftentimes introduced the conventional ε-δ definition without establishing sound connections with the intuitive notion and proved limit problems using ε-δ form irrespective of students' understanding. Apart from this, research outputs on overcoming such obstacles were very scant, if any, and were only theoretical considerations driven mainly by cognitive constructivism. For instance, Cottrill et al. [7] utilized "APOS theory" coupled with genetic decomposition (a seven step model) to investigate the conception of limit; Bagni [18] explored the notion of limit using "Register semiotic representations"; and Juter [19] explained students' concept growth using "Three worlds of mathematics" coupled with "Concept image and concept definition". Nevertheless, endeavors to understand epsilon-delta form of defining and proving the limit concept did not make a significant contribution to alleviating the perennial problems in mathematics education, and students are still struggling to learn abstract mathematical concepts. Moreover, difficulty of learning the limit concept has brought about considerable challenges in learning subsequent mathematical concepts [20] for many years to come. As a result, the quest for conceptual understanding of limit becomes more intensified in mathematics education. The commonly used intuitive and formal definition of limit are briefly indicated below.

Intuitive (informal) definition of limit: Let f be a function defined at each point of some open interval containing α, except possibly at α itself. Then a number L is the limit of $f(x)$ as x approaches α, written $\lim_{x \to a} f(x) = L$ iff $\lim_{x \to a} - f(x) = L = \lim_{x \to a} + f(x)$.

Formal definition of limit: Let f be a function defined at each point of some open interval containing α, except possibly at α itself. Then a number L is the limit of $f(x)$ as x approaches α, iff for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. In this case it is written as $\lim_{x \to a} f(x) = L$.

A recent study by Swinyard [21] disclosed a reinvention of the formal conception of limit using the radical constructivism theoretical framework. In his study, Swinyard [21] probed the existence of a coherent definition of limit, wherein students constructed nine steps in the reinvention process. In the same vein, Swinyard and Larsen [22] extended the conceptual framework of Cottrill et al. [7] so as to explore students' reinvention of the formal definition based on Swinyard's [21] descriptive approach. They found an encouraging result although it was not validated in a normal whole class context. However, they utilized the concept of limit at infinity to establish limit at a point during the formal

conception of limit, which might be against students' concept image of limit [23], contradicting the natural way of learning mathematics. We argued that students must learn first simple and concrete concepts based on their cognitive development to ascertain conceptual understanding, but the concept of limit at infinity might be a more abstract concept than limit at a point with regard to formal conception of limit. The debates still continue unabated. Thus, we strongly believe that a body of research focusing on such abstract and complex concepts may trigger pedagogical suggestions which could help to alleviate the existing situations. As we have worked with the concept of limit for several years, we have developed personal interests and enthusiasm to make a substantial contribution to the existing literature, and also solve the problem of teaching and learning of the concept of limit in Ethiopian Universities. With this back drop of the concept of limit, let us discuss in brief teaching approaches, the role of GeoGebra and the APOS theory to substantiate one's understanding of our endeavor to synthesize the present approach for addressing the aforementioned problem.

2.2. Multi-teaching approaches

Flexible use of pair, group and whole class discussion which involves teacher's questioning, and variety of compositions in lessons are among other things that account for effective teaching [24]. Accordingly, the present study employed multi-teaching approaches incorporating brainstorming, individual learning, group discussions and interactive lecture to enhance students' learning. Brainstorming encourages students' engagement and increases their achievement [25]. Providing students opportunities to attempt some activities individually might help to nurture their creativity and critical thinking skills. Moreover, learners actively construct, elucidate and restructure knowledge in an individual manner [26]. Furthermore, integrative implementation of group discussion and interactive lecture could engender students' learning [27].

2.3. GeoGebra and mathematics learning

GeoGebra has different interfaces that serve for teaching and learning of mathematical concepts. Algebraic view, graphics view and spreadsheet view are some of the commonly used GeoGebra interfaces in mathematics education [8]. Mathematical objects such as points, lines, planes and space figures could be sketched and also dynamically described with ease using GeoGebra. GeoGebra played a predominant role in mediating the physical operations of students' paper and pencil work, computer activity, and mental process involved in learning new concepts. Indeed, students could stimulate the data on the GeoGebra window by manipulating the slider and information may evolve out of the data [15]. Consequently, the working memory tracks the information and executes processing henceforth to construct knowledge. In addition, GeoGebra could empower learners to articulate their visual and analytic thinking in the learning process. It also encourages students' visualization and understanding while learning [28]. Different research studies revealed that empowering students' visualization skills of mathematical concepts certainly improves their learning [29, 30]. Arcavi [30] pointed out that visualization serves not only as an illustrative purpose, but also as a fundamental component of reasoning in conceptual learning. However, he suggested managing visualization tools carefully as it might not be a panacea for every problem in mathematics. Moreover, Yung and Paas [29] reported that computer assisted visualization could enhance deep learning and minimize cognitive load problems. For example, students develop positive opinions and improve their achievement in understanding statistics concepts using GeoGebra [10]. Nobre et al. [31] also documented that GeoGebra software increases students' motivation for learning calculus contents. Furthermore, integrating GeoGebra with constructivist instruction enhances students' learning better than constructivist instruction alone [16, 32]. In general, integrating GeoGebra with flexible teaching approaches could provide learners with myriad options to articulate and figure out their learning.

2.4. Theoretical framework

2.4.1. The APOS theory

APOS theory has tremendous benefits for instructional design associated with how students might make mental constructions in the learning process based on a genetic decomposition [11]. Mental mechanisms such as interiorization, encapsulation and thematization are reflective abstractions, which lead to constructions of mental structures like actions, objects, processes and schemas [11]. Action is the first level in concept formation which is governed by external transformation of a formerly perceived object [33]. In this case, externally perceived actions could spur construction of internal mental actions in an individual's mind. Such an internal mental action is said to be process. It is noteworthy to note that process is the second level in concept construction, which is realized because of interiorization of an action. Students accomplish the transformation totally in their mind devoid of explicitly executing each step during processing [33, 34]. Students' mental construction of process could be conceived through reflections on repeated actions and contemplation based on their prior and tacit knowledge. Encapsulation of two or more processes forms cognitive object, which is the third level in concept construction. Cognitive objects are mental structures created through students' cognitive imagination [33]. The collective and synchronized organization of actions, processes and objects provides a coherent cognitive structure called schemas, which espoused conceiving the situation and solving the problem. Schemas entail "logico-mathematical structures", which build individual's analytic thinking; and thematization of schemas may create cognitive object [11]. The construction of "logico-mathematical structures" of an individual is explained based on Piaget's concept of reflective abstraction during cognitive development [35]. A growing number of research studies used the APOS theory to examine students' mathematics learning [11]. In the present study, we employed the APOS theory to guide instruction and analyze data based on a genetic decomposition. The framework of the present study is adapted from Cottrill et al. [7] indicated in Figure 1 below.

In this framework, theoretical analysis comes first to model cognition of the limit concept, which is the basis for establishing the genetic decomposition. The theoretical analysis is established based on conceptualization of the problem. In fact, our conceptualization of the problem arises from the theory of learning, different literatures, our own understanding of the concept, and our experience of teaching and learning the concept of limit. The theoretical analysis calls for instructional design and implementation, and this in turn provides opportunities to collect data and reconsider the theoretical analysis. Moreover, based on observation results and students' feedback, a revised genetic decomposition
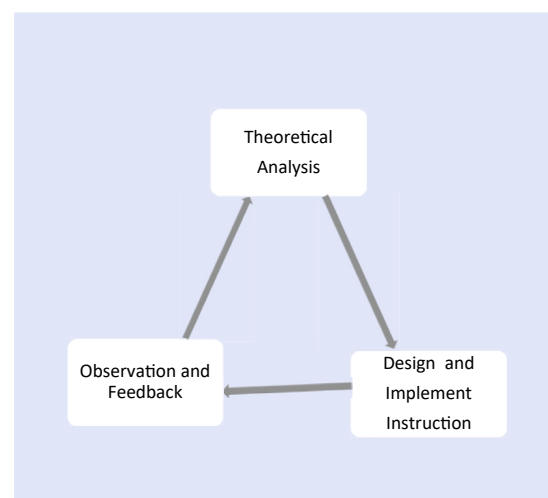

Figure 1. A theoretical perspective (paradigm).

might be realigned to construct and reconstruct epistemology of the concept. While observation was taking place, we focused on whether or not students made the required specific construction. While the analysis of data was guided by the theoretical perspective, our genetic decomposition was influenced by the data. This cyclical nature continuous until students successfully construct epistemology of the concept, which calls for by the genetic decomposition.

3. Methodology

3.1. Research design and methods of data collection

There is no single best approach in doing research [36]. Neither a quantitative nor qualitative approach provides an inevitable evidence of truth. Therefore, both stories and numbers inevitably provide ample information to properly articulate the problem in a better way and to suggest a practical solution [36]. Researchers of the present study also believe that to point out the underpinning epistemology of an abstract and difficult concept, for example the notion of limit, employing mixed methods research is extremely appropriate. Hence, the study employed mixed methods experimental (intervention) design within an APOS paradigm. Both quantitative and qualitative methods were used in the present study. Quantitative data was collected using diagnostic tests, whereas qualitative data were collected through students' reflection (view) and interview.

Data was collected after obtaining approval from the Research Ethics Committee of College of Science, Bahir Dar University (PRCSVD/010/ 2011) and securing students' written informed consent. The entire research design of the present study is adapted from Creswell and Creswell [36] illustrated in Figure 2 below. The design could help to explore how students experience the treatment while learning the concept of limit. It is also helpful to convey fidelity of implemented procedures and to identify students' barriers while learning the concept of limit. The design provides triangulated trustworthy information about the study.

3.2. Participants of the study

First year mathematics students in science stream at Bahir Dar and Wollo Universities in 2018 participated (single section in each university) in the present study. First year mathematics students in 2018 in Ethiopian Universities were considered as population of the present study. One section of students at Bahir Dar Univeristy was used as an experimental group and another section at Wollo University was used as a control group using simple random sampling technique. The students took introductory calculus course before a year while they were pre-college students. In 2018 a pretest to assess the students' background knowledge about the concept of limit was administered to both groups. Next, the limit concept was delivered using GeoGebra integrated with multi-teaching approaches based on the APOS paradigm for the experimental group, whereas the same content was given to the control group using the traditional (lecture dominated) method. A teacher with 3 years of teaching experience was assigned to teach the experimental group, and another teacher with 5 years of teaching experience was assigned to teach the control group. Both teachers had master's degrees in teaching mathematics.

3.3. Instruments for data collection

3.3.1. Two-tier diagnostic tests

The current investigators did literature reviews [20, 37] for developing two-tier diagnostic tests. The primary purpose of the diagnostic tests was to gauge students' higher cognitive thinking in relation to their conceptual understanding of limit. The first tier of each Item was a multiple choice question having four choices focusing on content, while the second tier was also a multiple choice question with equal number of choices addressing students' possible justifications or reasons for the answer given in the first tier. Each diagnostic test incorporated six Items each involving two questions, some Items from informal conception of limit, some Items requiring coordinated schema and some Items from formal schema. An example of two tier diagnostic test is indicated as shown below.

Item 1:

FT: If $|x-3| < a$ for all x then $|x^2 - 9| < 1$, which one is a possible value of a,

A. 1/7,　　B. 1/5　　C. 7　　D. 1

ST: Which one could be a possible reason in the above problem?

A. Because if $|x - 3| < 1$, then $|x^2 - 9| \leq \frac{1}{7}|x - 3|$

B. Because if $|x - 3| < 1$, then $|x^2 - 9| \leq 5|x - 3|$

C. Because if $|x - 3| < 1$, then $|x^2 - 9| \leq 7|x - 3|$

D. Because if $|x - 3| < 1$, then $|x^2 - 9| \leq |x - 3|$

Answer: Students first simplify $|x^2 - 9|$ in order to express it in terms of $|x - 3|$ and then apply comparison to decide the value of a.
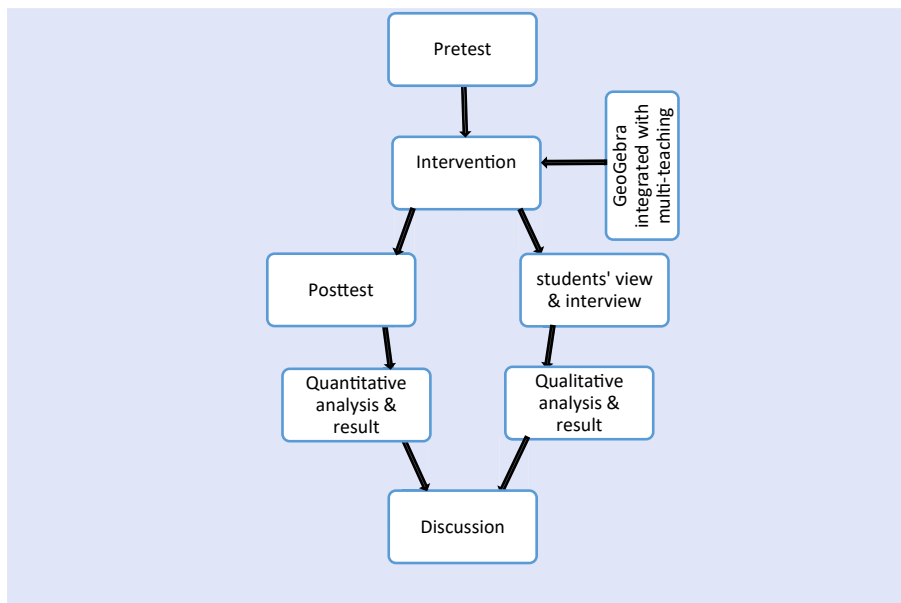

Figure 2. Mixed methods experimental (intervention) design.

Now we have $|x^2 - 9| = |(x-3)(x+3)| = |x-3||x+3|$, but for x sufficiently closer to 3, it is possible to consider an assumption $|x-3| < 1$, as a result we get $|x^2 - 9| = |(x-3)(x+3)| = |x-3||x+3| \leq 7|x-3| < 1$, which implies $|x-3| < 1/7$ and hence, $a = 1/7$, since $\frac{1}{7} < 1$. Therefore, A is the correct choice for the first tier FT and C is the correct choice for the second tier ST. Students who might have conceptual understanding of FT, could answer ST with ease. As it is shown above the second tier is an intermediary step while finding the value of a.

3.3.2. Students' reflections and interview

The investigators of the present study continuously observed the experimental group and collected feedback from students so as to improve their learning using open-ended questionnaires. However, their final reflections were collected and incorporated in this study. The following open ended questionnaires were used to collect students' reflections: "What is your comment about the impact of GeoGebra integrated with multi-teaching approaches in learning the concept of limit?" and "What were the difficulties you encounter in the learning environment?"

The investigators also conducted interviews with randomly selected students while they were learning the notion of limit. Semi-structured interviews were conducted to collect data so as to explore the participants' conceptual understanding based on a genetic decomposition. The interview data was collected using field notes and audio tape recording.

3.4. Validity and reliability

Validity and reliability tests ensure how well research instruments indicate the appropriateness and trustworthiness of the instruments. A two tier diagnostic test was among the instruments employed in the present study to measure students' conceptual understanding of the notion of limit. Both pretest and posttest were adapted from different calculus exams prepared by University mathematics teachers. Furthermore, the validity of the tests was verified using five expert University mathematics teachers, whereas the reliability of the instruments was checked using data from the pilot study. Indeed, checking the validity of the tests entailed test contents representativeness and appropriateness so as to measure students' conceptual understanding, whereas ensuring the reliability of tests addressed internal consistency of the test Items to meet its objective. Cronbach's alpha coefficient of reliability of 0.71 showed that the diagnostic tests had high internal consistency [38].

3.5. Procedures

Prior to the intervention, lesson activities integrated with GeoGebra were prepared based on the APOS theory in consultation with the teacher assigned to the experimental group. Next, the GeoGebra software was installed in the mathematics laboratory at Bahir Dar University. Moreover, appropriate training on GeoGebra software and multi-teaching approaches was given to the teacher. Students were also given 4 hours of training in how to manage the GeoGebra software during mathematics learning. Then, using theoretical analysis, an initial genetic decomposition model was developed. The theoretical analysis was carried out based on our understanding of limit concept, different literatures, theories of learning and our experience of learning and teaching calculus. After that the teaching-learning process was started in both the experimental group as well as the control group at Bahir Dar University and Wollo University, respectively. As of the first class, the investigators of this study discussed with the teacher while the teaching-learning process was going on so as to change the strategy and approach based on observed data and students' reflections in response to the given genetic decomposition for the sake of maximizing their learning. Accordingly, the teacher continuously redesigned the procedures in response to the students' learning. Along the way, the researchers of this study collected relevant data using interviews. During the instructions in the experimental group, a single

activity could be presented first using brainstorming, next individual learning followed by group discussion including pair, triple and quadruple discussions with the use of GeoGebra and without it. In addition, interactive lecture might be implemented after group discussion, which could be realized using whole class discussion led by the teacher using GeoGebra as well as without GeoGebra simply using paper and pencil work. However, the teacher could vary the sequence of approaches depending on students' prior knowledge and the type of mental constructions needed in alignment with the genetic decomposition. This continued for three weeks having two days per week for three hours each day. The approach may provide a flexible learning milieu which accommodates students' natural way of learning, and teacher's collective endeavor of leading, facilitating and teaching [24]. At the end of the limit content, students' reflections were collected from the experimental group and a posttest was administered to both groups to probe students' conceptual understanding of the limit concept.

3.6. Genetic decomposition

In the present study, a four level genetic decomposition model, which was adapted from Cottrill et al. [7], used to analyze students' conceptual understanding of limit.

The genetic decomposition model, which is driven by the underlying APOS theory, could illustrate what is going on in the minds of students while they are learning a mathematical concept based on their success or failure in executing each activity of the model. While a genetic decomposition guides instruction, its description could portray how a concept might be constructed and reconstructed in the mind of a learner and could unearth how their mental mechanisms might be realized in a specific problem context. Data obtained from students' interview is analyzed using the model. In addition, the model plays a fundamental role in instruction design for learning the concept of limit.

3.7. Methods of data analysis

The qualitative data was analyzed (interpreted) using thematic analysis by coding students' reflections. Moreover, a genetic decomposition model was applied to analyze the nature of students' reasoning using data collected from students' interviews. The model portrayed students' conceptual understanding of limit based on their action, process and encapsulation of the cognitive object inductively by employing particular instances to synthesize the formal schema in a coherent and pragmatic way. The quantitative data was analyzed using descriptive statistics and independent samples t-test. The quantitative analysis was conducted using SPSS statistical software version 21.

3.8. Data analysis and results

3.8.1. Qualitative data analysis(interpretation)

3.8.1.1. Analysis of students' reflections. In each session students were requested to reflect on a sticky note regarding the following open-ended questions a) what is your comment about the impact of GeoGebra integrated with multi-teaching approaches in learning the concept of limit? b) what were your learning difficulties in the learning environment?

Based on students' reflections, actions were taken to improve their learning. However, their last reflections were summarized and analyzed using thematic analysis. The following categories were identified during the analysis: The benefits of GeoGebra integrated with multi-teaching approaches, suggestions for other courses and learning difficulties. The thematically analyzed data was coded using different categories for in-depth analysis. Students were coded as S1, S2, S3, etc. The number of students involved in a given code was considered as frequency (f) (Table 1).

Table 1. Codes for different categories of students' view.

Category	Code	f
Benefits of GeoGebra integrated with multi-teaching approaches	Provides better visualization	13
	Increases understanding	10
	Encourages participation	10
	Promotes individual and team work	8
	Creates enjoyable learning environment	8
	Increases interest and motivation	8
	Boosts imagination	5
Suggestion for other courses	Implementing for other concepts and courses	3
Learning difficulties	Shortage of time	3
	Lack of skills to manipulate GeoGebra	3

Table 1 reveals that students conveyed positive views regarding the importance of GeoGebra and multi-teaching approaches. Students stated that GeoGebra integrated with multi-teaching approaches provided better visualization, improved understanding, encouraged participation, promoted their team work and individual work, created enjoyable learning environment, boosted their interest, motivation and imagination. Moreover, some students suggested using GeoGebra integrated with multi-teaching approaches for some other concepts in calculus and in some other courses. In addition, the majority of the students did not face learning difficulties in the learning process. Only few students faced shortage of time during brainstorming and lack of computer skills to manipulate GeoGebra software.

Thirteen students reported that GeoGebra integrated with multi-teaching approaches provided better visualization. Two students' comments were as follows:

"I have got better visualization of the relation between |x-a| and |f(x)-L| in a dynamic manner to validate L is the limit of f(x) using a slider." (S3)

"GeoGebra technology integrated with multi-teaching approaches provided us a better visualization in the learning process." (S2)

Ten students stated that GeoGebra integrated with multi-teaching approaches improved their understanding. Two students' reflections were as follows:

"Although limit was a difficult concept, employing GeoGebra increased my understanding." (S23)

"I believe that the learning environment improved my understanding of the limit concept." (S5)

Ten students stated that GeoGebra integrated with multi-teaching approaches enhanced their participation. Two students commented as follows:

"GeoGebra integrated with multi-teaching approaches encouraged me to participate in each activity." (S7)

"I have been actively participating during group discussion using GeoGebra integrated with multi-teaching." (S8)

Eight students reflected that GeoGebra integrated with multi-teaching approaches promoted individual and team learning. Two students reflected as follows:

"The learning environment encouraged us to use individual learning and team learning." (S16)

"I benefited from GeoGebra integrated with multi-teaching through individual and team learning." (S9)

Eight students reported that GeoGebra integrated with multi-teaching approaches created enjoyable learning environment. Two students' comments were as follows:

"We practiced an enjoyable teaching-learning process." (S17)

"I enjoyed the learning of limit through GeoGebra integrated with multi-teaching." (S13)

Eight students reflected that GeoGebra integrated with multi-teaching approaches increased their interest and motivation. Two students' comments were as follows:

"I have learned the relation between the informal and formal limit concept in an interesting and motivating manner." (S24)

"In my view, GeoGebra integrated with multi-teaching approaches increased my interest to learn the concept of limit." (S18)

Five students reported that GeoGebra integrated with multi-teaching approaches increased their imagination. Two students said the followings:

"The learning environment expanded my imagination of the concept of limit." (S15)

"I found GeoGebra integrated with multi-teaching approaches are powerful to increase my imagination of the relationship between δ and ε." (S11)

Three students reported that some other concepts in calculus and other mathematics courses are also abstract; hence, implementing GeoGebra might solve the problem. Two students suggested followings:

"I suggest GeoGebra to be used for learning abstract concepts in calculus and in other mathematics courses." (S24)

"I believe that GeoGebra integrated with multi-teaching approaches could be helpful for learning other abstract concepts in mathematics." (S10)

Three students reported that they faced shortage of time in the learning process. Two students complained as follows:

"There is shortage of time during the brainstorming stage in the learning process." (S22)

"I believe that learning the concept of limit using GeoGebra demands sufficient time for practice. However, I faced shortage of time during brainstorming and individual learning." (S19)

Three students stated that they faced lack of computer skills in the learning process. Two students' comments were as follows:

"Truly speaking, I did not properly utilize the key board while entering functions." (S6)

"Lack of computer skills influenced my learning of limit using GeoGebra." (S4)

From the reflections provided above, it was possible to learn that the participants were able to have better visualization and understanding of the concept of limit using GeoGebra integrated with multi-teaching approaches. Their views indicated that they were actively participating in individual as well as group learning. Moreover, many participants reported that they learned the concept of limit in an interesting, motivating and enjoyable manner. In addition, students stated that GeoGebra integrated with multi-teaching approaches increased their mental imagination. Generally, the participants suggested using GeoGebra integrated with multi-teaching approaches for learning some other abstract concepts in mathematics. On the other hand, some students reported that they faced shortage of time and computer skills while learning the concept of limit using GeoGebra integrated with multi-teaching approaches. Overall, the participants forwarded positive reflections regarding their learning of limit using GeoGebra integrated with multi-teaching approaches.

3.8.1.2. Analysis of students' interviews. The analysis of the activity below was among the different activities on the contents of limit covered in the experimental group.

Activity: Exploring and proving that $\lim_{x\to 2} f(x) = 7$, where $f(x) = 3x + 1$.

First, students were requested to enter the function $f(x) = 3x+ 1$ in the input field of GeoGebra and create a slider for the sake of manipulation by tracing points A and B along the x-axis closer to 2 and $y-$ axis closer to 7 respectively (Figure 4). In fact, most students identified the behavior of $f(x) = 3x + 1$ near 7 associated with the behavior of x near 2 (Figure 4) exhibiting step 1(a) and (b) in the genetic decomposition (Figure 3).

Consider the following excerpt taken from students' interview by researchers (R) **for determining limit candidates**

1. R: What do you observe closer to 2 on the x − axis and closer to 7 on the y − axis?
2. Taye: You know, if x is 1.9, then f(x) becomes 6.7, if x is 1.99, then f(x) becomes 6.97 and, if x is 1.999, then f(x) becomes 6.997 and so on. Besides, if x is 2.1, then $f(x)$ becomes 7.3, if x is 2.01, then f(x) becomes 7.03 and if x is 2.001, then f(x) becomes 7.003 and so on.
3. R: What is the relation between the two changes?
4. Taye: Ok, if x is within 0.1 of 2, then f(x) becomes within 0.3 of 7; if x is within 0.01 of 2, then f(x) becomes within 0.03 of 7; if x is within 0.001 of 2, then f(x) becomes within 0.003 of 7 and so on. So, in general using the slider I noticed that whenever x is closer and closer to 2, f(x) becomes also closer and closer to 7 from the algebraic and graphics view of GeoGebra. Hence, $\lim_{x\to 2} f(x) = 7$.
5. R: Why?
6. Taye: You know, f(x) approaches to 7 whenever x is gets close to 2, from either side of 2.
7. R: Ok, given $s(x) = \begin{cases} 2x, for\ x < 2 \\ 6,\ for\ x = 2 \\ x + 1, for\ x > 2 \end{cases}$, and $r(x) = \begin{cases} 2x,\ for x < 2 \\ 6,\ for x = 2 \\ x + 2, for x > 2 \end{cases}$ what is the limit of s(x) at 2?
8. Taye: Does not exist

9. R: Why?
10. Taye: Because s(x) approaches 4 as x approaches 2 from the left and s(x) approaches 3 as x approaches 2 from the right, hence, s(x) does not approach to a unique value whenever x approaches 2.
11. R: Ok, what is the limit of r(x) at 2?
12. Taye: 4
13. R: why?
14. Taye: You know, the rule of the game is closer to 2, not necessarily exactly at point 2.

From the excerpts, the action of describing the behavior of $f(x)$ near 7 at different points closer to point 2 was exhibited (2). This qualified Taye's successful actions at successive points closer to 2 on the $x-$ axis which induced changes at successive points closer to 7 on the $y-$ axis. Besides, Taye successfully coordinated the process conception of $f(x)$ near 7 (4 & 6) and a discontinuous function $r(x)$ near 4 together with the process of x near 2 (12 &14). In addition, Taye also characterized the behavior of s(x) as x gets close to 2 (8 & 10). Alex, Biru, Hana and Sara also elicited similar attributions. Hence, it could be argued that students successfully developed the process conception of limit; because, the reason that $\lim_{x\to a} f(x) = L$ might be regarded as the point-wise dependency of the behavior of $f(x)$ "near" L on the behavior of x "near" a, warrants the process conception of $f(x)$ to be sufficient and necessary condition for understanding the intuitive notion of limit [39]. This uncovered that students' coordination of processing mental images regarding the behavior of $f(x)$ closer to 7 as x close enough to 2 figure out the construction of intuitive (informal) cognitive schema within their mind. Moreover, researchers' classroom observations also confirmed a high prevalence of manifestation of similar attributions in the majority of the students in the experimental group.

Next, students were requested to enter $|x - 2|$ and $|3x + 1 - 7|$ in the input field of GeoGebra so as to notice their relations using a slider (Figure 4). Specially, students were critically observing points A, B, C and D, and the graphs of $g(x)$, $h(x)$ and $f(x)$ on the graphics by manipulating the slider (Figure 4). They were requested to describe the relationship
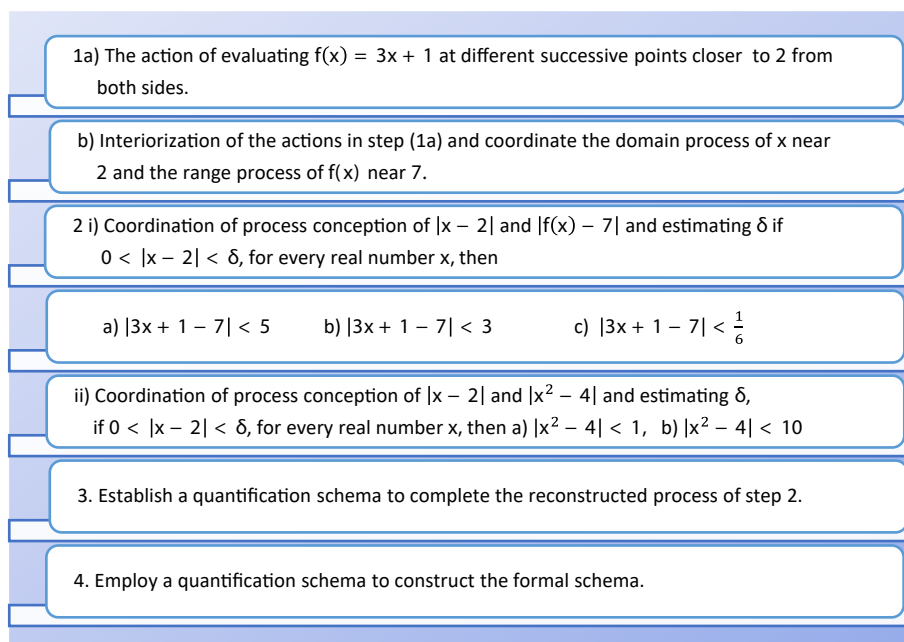

1a) The action of evaluating f(x) = 3x + 1 at different successive points closer to 2 from both sides.

b) Interiorization of the actions in step (1a) and coordinate the domain process of x near 2 and the range process of f(x) near 7.

2 i) Coordination of process conception of |x − 2| and |f(x) − 7| and estimating δ if $0 < |x − 2| < δ$, for every real number x, then

a) $|3x + 1 − 7| < 5$ b) $|3x + 1 − 7| < 3$ c) $|3x + 1 − 7| < \frac{1}{6}$

ii) Coordination of process conception of |x − 2| and $|x^2 − 4|$ and estimating δ, if $0 < |x − 2| < δ$, for every real number x, then a) $|x^2 − 4| < 1$, b) $|x^2 − 4| < 10$

3. Establish a quantification schema to complete the reconstructed process of step 2.

4. Employ a quantification schema to construct the formal schema.

Figure 3. Genetic decomposition model.

between $3x + 1$, $|x - 2|$ and $|3x + 1 - 7|$ near point 2 using the slider (Figure 4).

Consider the following excerpt taken from students' response.

15. Alex: You know, $|3x + 1 - 7| = 3|x - 2|$ using simple algebra and from the algebraic and graphics view of GeoGebra. Moreover, $|x - 2|$ and $|3x + 1 - 7|$ becomes closer to zero and $f(x) = 3x + 1$ tends to 7 as x gets close to 2, which is obvious with the use of slider.

16. Biru: Ok, from the algebraic and graphics view of GeoGebra, $|x - 2|$ is one third of $|3x + 1 - 7|$ and both approach to zero, while $f(x) = 3x + 1$ approaches to 7, whenever x gets close to 2.

17. Sara: It is quite obvious from the GeoGebra window that $|3x + 1 - 7| = 3|x - 2|$ and both converge to zero, but $f(x) = 3x + 1$ approaches to 7, whenever x approaches to 2.

From the excerpts, Alex, Biru and Sara successfully identified and coordinated the connections among the process (15, 16 & 17). The majority of the students also elicited similar attributes. This established students' concept map as a dynamic process which substantiate the creation of cognitive structures for quantification schema.

Students also explored the detail relationship between $|x - 2|$ and $|3x + 1 - 7|$ indicated as in the following excerpts **for determining a single quantification schema.**

18. R: If $|x - 2| < \delta$, for every real number x, then $|3x + 1 - 7| < 5$, how do you get the value of δ?

19. Biru: Ok, since, $|3x + 1 - 7| = 3|x - 2| < 5$ is evident using slider in the GeoGebra window or applying simple algebra. Therefore, $|x - 2| < \frac{5}{3}$, as a result, $\delta = \frac{5}{3}$.

20. R: Ok, do you think that δ is unique?

21. Biru: Um, ok, no, it is possible to take $\delta \leq \frac{5}{3}$, since if $\delta < \frac{5}{3}$, then "the distance between x and 2 " becomes very Small, as a result $|3x + 1 - 7|$ could be as small as we wish.

22. R: Ok, what if $\delta = 0$?

23. Biru: Um, uh, ok, $\delta \neq 0$, since $|x - 2| < 0$ is impossible.

24. R: Why?

25. Biru: You know, $|x - 2| = 0$ or $|x - 2| > 0$, cannot be negative. So, in both case $|x - 2| < 0$ is impossible.

26. R: Ok, good, what is the value of δ, if $|x - 2| < \delta$, for every real number x, then $|3x + 1 - 7| < 3$?; what if $|3x + 1 - 7| < \frac{1}{2}$?

27. Biru: $\delta = \frac{3}{3} = 1$ and $\delta = \frac{1}{6}$ respectively.

The excerpts uncovered that Biru subsumed the process conception and coordinated schema based on the pattern of distances of $|x - 2|$ and $|3x + 1 - 7|$ using the slider as x gets close 2 (Figure 4) to construct cognitive structures to establish a single quantification schema (19, 21, 23, 25 & 27) exhibiting step 2(i) and (ii) in the genetic decomposition (Figure 3). Moreover, Alex, Hana, Sara and Taye elicited manifestation of similar attributes in the classroom. Consider also the following excerpts for validating the limit candidate.

28. R: If $0 < |x - 2| < \delta$, for every real number x, then $|3x + 1 - 7| < \varepsilon$, where δ and ε were positive numbers. What is the relation between δ and ε?

29. Hana: Ok, $\delta = \frac{\varepsilon}{3}$

30. R: Have you noticed what δ and ε represent?
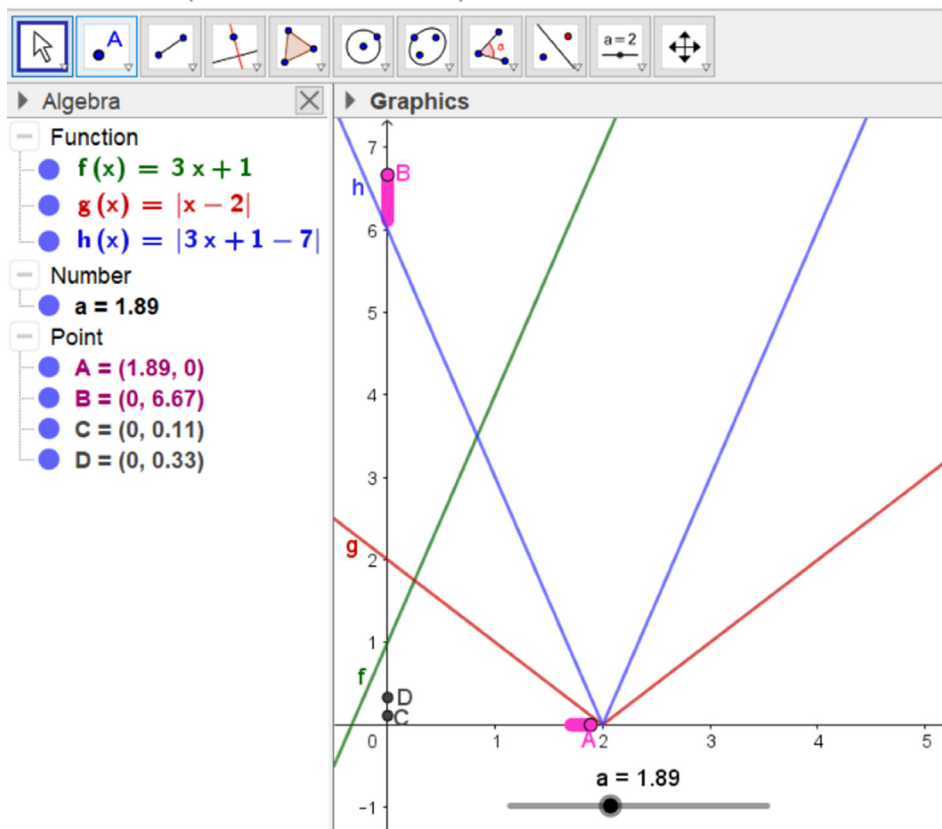

Figure 4. Algebraic and graphics representation of $f(x) = 3x + 1$, $|3x + 1 - 7|$ and $|x - 2|$ together with their property closer to 2 manipulated using slider in a dynamic manner with GeoGebra.

31. Hana: Yes, I noticed that δ represented "distance between x and 2", whereas, ε represented "the distance between 3x + 1 and 7".

32. R: Ok, if $0 < |x - 2| < \delta$ for the aforesaid δ value, can you establish the relation $|3x + 1 - 7| < \varepsilon$?

33. Hana: Yes.

34. R: H*ow?*

35. Hana: You know, $0 < |x - 2| < \delta = \frac{\varepsilon}{3}$, implies $3|x - 2| = |3x + 1 - 7| < \varepsilon$.

36. R: What do you think about the relation between $|x - 2|$ and $|3x + 1 - 7|$ if ε is arbitrary small positive number?

37. Hana: Ok, I think that $|x - 2|$ will be almost zero and this again induced $|3x + 1 - 7|$ to be almost zero.

38. R: Can you describe the relation between δ and ε?

39. Hana: Yes, I can, δ could be dependent on ε.

40. R: So, can you guess the end result warranted by this process?

41. Hana: I think that $\lim_{x \to 2}(3x + 1) = 7$.

The excerpts revealed that Hana established quantification schema based on the notion of "distance" (29, 31, 33 & 35). The thematization of the quantification schema created a coherent cognitive object in the form of limit (37, 39 & 41). Alex, Biru, Sara, and Taye also established a similar cognitive structure which was germane to convey the formal conception of limit exhibiting step 3 in the genetic decomposition (Figure 3). Consider the excerpts taken from Taye' interview for

Constructing complex quantification schemas to establish formal schemas.

42. R: If $0 < |x - 2| < \delta$, for every real number x, then $|x^2 - 4| < 1$, consider the following. For x sufficiently closer to 2 , we have $|x^2 - 4| = |(x - 2)(x + 2)| \leq 5|x - 2| < 1$, this implies $|x - 2| < \frac{1}{5}$, then $\delta = \frac{1}{5}$, do you think that the choice of δ is correct?

43. Taye: Yes I think so!

44. R: Really ? What if $|x^2 - 4| < 10$?

45. Taye: $\delta = \frac{10}{5} = 2$

46. R: Really?, Ok, suppose $\delta = 2$, then $|x - 2| < \delta = 2$, and let x = 3.9, then $|x^2 - 4| = 11.21 < 10$, why this happen? use GeoGebra to explore the relationship between $|x - 2|$ and $|x^2 - 4|$ using a slider? Consider the value of $|x^2 - 4|$ for the case $|x - 2| < 1$, and $|x - 2| < 2$?

47. Taye: Um,..., Ok, I noticed that $|x^2 - 4| \leq 5|x - 2|$ whenever $|x - 2| < 1$, and fails whenever $|x - 2| < 2$. It means, for a $\delta > 1$ the estimation does not work. So, $\delta = \frac{1}{5}$ is smaller than 1 is a choice for $|x^2 - 4| < 1$, and $\delta = 1 < 2$ is a choice for $|x^2 - 4| < 10$.

48. R: What if $0 < |x - 2| < \delta$, for every real number x, then $|x^2 - 4| < \varepsilon$, for every positive real number ε?

49. Taye: Ok, you know, $\delta = \frac{\varepsilon}{5}$, using the assumption $|x - 2| < 1$, so, δ is the smaller of the two values.

50. R: Do you mean $\delta = \min\left(\frac{\varepsilon}{5}, 1\right)$, minimum of $\frac{\varepsilon}{5}$ and 1?

51. Taye: Yeap!

52. R: Ok, how do you describe $"\lim_{x \to a}f(x) = L"$ using the previous approach?

53. Taye: For any given $\varepsilon > 0$, by finding a $\delta > 0$ if exists which could be dependent on ε, that satisfy the property, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

From the excerpts, Taye successfully established quantification schema inductively based on the patterns on each particular cases (43, 45, 47, 49, 51 & 53). The thematization of this create a mental object; and again using reflective abstraction [35] a formal schema was exhibited for the learning of limit. Similarly, Alex, Biru, Hana and Sara also successfully exhibited manifestation of similar cognitive structures which were germane to formal conception of limit based on step 3 and step 4 in the genetic decomposition (Figure 3). They were capable of constructing informal, coordinated and formal schemas using the affordances of GeoGebra which might trigger logico mathematical thinking for learning the concept of limit. They elicited a coherent and viable reasons in the learning process. Moreover, whilst learning the concept of limit using slider in GeoGebra, students boost their mental constructions of the concept, and somehow minimized cognitive difficulties like diductical obstacles (due to the nature of teaching) and epistemological obstacles (due to the nature of the concepts). Overall, the analysis articulated students' understanding of the concept of limit.

Note that the names: Alex, Biru, Hana, Sara and Taye in this study are pseudonym for the sake of protecting students' anonymity.

3.8.2. Quantitative data analysis

The quantitative data was organized and analyzed using the SPSS statistical software package. Descriptive statistics and independent samples t-test were employed to analyze the data collected using pretest and posttest on conceptual understanding of the notion of limit.

3.8.2.1. Pretest result. Table 2 shows that the participants' average performances in both groups were almost the same on the pretest. The independent samples t-test also showed that there was no statistically significant difference between the students' average results in the control group (M = 4.23, SD = 1.14) and experimental group (M = 4.25, SD = 1.15, t = .059, df = 48, p > .05). This indicated that the participants' conceptual understanding of the notion of limit was similar before the intervention.

3.8.2.2. Posttest result. The diagnostic test items were categorized into three groups: informal conception of limit, concepts demanding coordinated schema and formal conception of limit as indicated in Table 3.

Table 3 indicates that students in the experimental group correctly responded to each Item and its corresponding reason associated with informal conception of limit (Item 4 (100), Item 5 (100)). The participants in the experimental group (Item 1 (FT 45.83, ST 50); Item 2 (FT (79.17), ST (79.17))) showed better percentage of correct responses to concepts requiring coordinated schema as compared to the participants in the control group (Item 1 (FT 30.77, ST 38.46); Item 2 (FT (46.15), ST (34.62))). Moreover, the students' percentage of correct responses to formal conception of limit in Item 3 was better in the experimental group (FT (87.5), ST (83.33)) in comparison to the students in the control group (FT (46.15), ST (53.85)). Besides, the students' answers to the second tier questions were completely at odds with answers to the first tier in the

Table 2. Independent samples t-test result of students' Pretest on conceptual understanding.

Group	Mean(M)	N	SD	t	df	p
Control	4.23	26	1.14			
Experimental	4.25	24	1.15	0.059	48	.953

Table 3. Number and percentage of students who correctly responded on each tier of each item in the posttest.

Category	Item	Tier	Experimental Group		Control Group	
			N0 of student	Percent	N0 of student	Percent
Informal conception of limit	Item 4	First tier (FT)	24	100	22	84.62
		second tier (ST)	24	100	23	88.46
	Item 5	First tier (FT)	24	100	20	76.92
		second tier (ST)	24	100	15	57.69
Concepts demanding coordinated schema	Item 1	First tier (FT)	11	45.83	8	30.77
		second tier (ST)	12	50	10	38.46
	Item 2	First tier (FT)	19	79.17	12	46.15
		second tier (ST)	19	79.17	9	34.62
Formal conception of limit	Item 3	First tier (FT)	21	87.5	12	46.15
		second tier (ST)	20	83.33	14	53.85
	Item 6	First tier (FT)	3	12.5	2	7.69
		second tier (ST)	3	12.5	6	23.1

case of the control group (Item 4 (FT (84.62), ST (88.46)); Item 5 (FT (76.92), ST (57.69)); Item 1 (FT (30.77), ST (38.46)); Item 2 (FT (46.15), ST (34.62)); Item 3 (FT (46.15), ST (53.85)); Item 6 (FT (7.69), ST (23.1))) relative to the experimental group (Item 4 (FT (100), ST (100)); Item 5 (FT (100), ST (100)); Item 1 (FT (45.83), ST (50)); Item 2 (FT (79.17), ST (79.17)); Item 3 (FT (87.5), ST (83.33)); Item 6 (FT (12.5), ST (12.5))). Furthermore, percentage of the students who selected correct answers for both tiers in each Item was larger for the experimental group compared with the control group (Table 4). However, most of the students in both groups responded incorrectly to Item 6 (Table 3 & Table 4). This indicates that Item 6 could be a challenging problem for majority of the students. Regardless of this, from Table 3 and Table 4, it is possible to claim that students' conceptual understanding of limit was better in the experimental group as compared to the control group.

Each item in the diagnostic test had a weight of two points (1 point for each tier) and totally the test was corrected (evaluated) out of twelve points.

Table 5 indicates that the participants in the experimental group (M = 8.33, SD = 1.81) performed better than the participants in the control group (M = 6.04, SD = 1.08) on the posttest. An independent samples t-test for equal variance assumption using Levene's test (F = 2.613, p > 0.05, t = 5.500) also showed that there was a statistically significant difference between the participants' average results in the control and experimental groups (t = 5.500, df = 48, p < .05) (Table 5). This indicates that the students in the experimental group have better conceptual understanding of limit as compared to students in the control group. Moreover, an effect size of 1.55 using standardized Cohen's d indicates a very large effect based on Cohen's criteria [40]. This effect size indicates that the mean of the experimental group is 1.55 standard deviations higher than the mean of the participants in the control group. This shows that GeoGebra integrated with multi-teaching approaches guided by the APOS theory has a strong positive impact on the learning of limit.

4. Discussion

The purpose of this study was to enhance students' conceptual understanding of limit using GeoGebra integrated with multi-teaching approaches within an APOS paradigm. To this end, students' views regarding benefits of GeoGebra integrated with multi-teaching approaches for learning the concept of limit were scrutinized; students' interview in line with their reasoning while doing activities based on a genetic decomposition model grounded in the APOS theory was investigated; and students' quantitative data on the concept of limit were examined.

With regard to the students' view, the results disclosed that the students acquired better visualization (S2 & S3) and improved their conceptual understanding of limit (S5 & S23) using GeoGebra in a multi-teaching environment. In this regard, results of different research endeavors pointed out that the GeoGebra software increases the students' visualization and understanding in mathematics learning [28, 41]. For example, Saha et al. [41] documented that the GeoGebra software increases students' visualization for learning coordinate geometry and also enhances their performance. In addition, Takači et al. [28] pointed out that the GeoGebra software adds substantial visualization for the learning of functions and their graphs. In their study, posttest results evidenced that first year physics and chemistry University students scored significantly higher results through GeoGebra as a visual tool in a collaborative learning environment as compared to a collaborative learning environment without GeoGebra. The finding of Karadag and McDougall [15] also revealed that GeoGebra software provides a great opportunity to link the dynamic visual and analytic representation for better understanding of calculus contents. Similarly, the study of Nobre et al. [31] also indicated that the implementation of GeoGebra software increases students' dynamic visualization and understanding of limit and other calculus concepts. Apart from this, results of students' views of the present study also portrayed that the learning environment encourages

Table 4. Percentage of students who selected correct answers for both tiers in each item.

Group	N	Item 1	Item 2	Item 3	Item 4	Item 5	Item 6
Control	26	30.77	34.62	38.46	84.62	57.69	3.85
Experimental	24	41.67	79.17	79.2	100	100	12.5

Table 5. Independent samples t-test result of students' posttest on conceptual understanding.

Group	Mean(M)	N	SD	F	Sig.	t	df	p	d
Control	6.04	26	1.08						
Experimental	8.33	24	1.81	2.613	.113	5.500	48	.000	1.55

their participation (S7 & S8) and enjoyment (S17 & S13) in both as an individual and teamwork (S16 & S9). Moreover, results of students' views revealed that GeoGebra integrated with multi-teaching approaches increased their interest and motivation (S24 & S18), and boosted their imagination in mathematics learning (S15 & S11). Besides, only few students reported that they were struggling to manage the GeoGebra software (S6 & S4). The finding is in consonance with different research studies [31, 42], which incorporated GeoGebra during mathematics learning. For instance, Nobre et al. [31] confirmed that the GeoGebra software has a positive contribution to students' motivation while learning calculus contents. Tatar and Zengin [42] also disclosed that students have positive opinions about the use of GeoGebra for learning definite integrals. Furthermore, employing different teaching approaches, which involve effective technology that stimulate students' interest in mathematics, inevitably improve their understanding and achievement [9, 10].

With regard to the students' interviews, the results disclosed that the students elucidated the different steps of conceptual understanding of limit based on the stated genetic decomposition model, which is grounded in the APOS theory. The students exhibited the action, process, and their coordination to form a mental object during the conception of limit. The versatility and interactivity of GeoGebra, and the flexible learning environment activated the students' inquisitive mind and prompted them to delve into deep learning. For example, the finding from Taye's interview for determining limit candidate indicated that he successfully interiorized the actions and evoked the process conception of limit. It is also apparent from the students' interviews that the visualization empowered by GeoGebra integrated with multi-teaching approaches enabled them to coordinate the $x-$ process along the $x-$ axis and the $y-$ process along the $y-$ axis simultaneously in a dynamic manner with the use of slider. Consequently, the variations along the $y-$ axis ($|f(x)-L|$) and the variations along the $x-$ axis ($|x-a|$) were coordinated to establish quantification schema in a dynamic manner with ease. For example, Biru articulated process conception of limit and coordinated schemas so as to establish a single quantification schema in alignment with the genetic decomposition. In addition, the interview results of Hana uncovered that thematization of quantification schemas establishes validation of limit candidate as a cognitive object. Apart from this, the interview results confirmed that Taye successfully managed the construction of complex quantification schemas to establish formal schema of limit. We feel that, this was the toughest, but a key step for constructing quantification schema and validating the limit. Cottrill et al. [7] also concluded that the difficulty arises from formal conception of limit was due to weak dynamic conception of limit. We also hold the same position in this regard; nevertheless, the integrated approach helped us to develop a strong coordinated schema so as to overcome the cognitive obstacles while moving from the intuitive approach to the formal conception of limit. The interview results also confirmed that once the students realized the concept, finding δ for a given ε to validate limit candidate of any function was found to be technical, and was not any longer a big deal. We believe that this is a better alternative approach for learning the concept of limit. Our result is in agreement with Swinyard [21] and Swinyard and Larsen [22]; however, they used limit at infinity to substantiate students' learning of limit at a point, and, in fact, it was tested using only two students and four students, respectively. In their study, $ay-$ first principle of limit at infinity guides characterization of limit at a point to reinvent the formal definition. They argued that the dynamic (intuitive) approach ($x-$ first principle) at a point is an obstacle

to the formal conception of limit. In contrast to this, our result revealed that the dynamic conception of limit found to be fundamental for students' formal conception of limit. The physical operations, students' computer activities using GeoGebra, and paper and pencil work on purposefully organized activities (genetic decomposition model) grounded in the APOS theory in a multi-learning environment enabled the students to synthesize a viable and coherent reasoning of formal conception of limit.

With regard to the quantitative data, the pretest result indicated that there is no statistically significant difference between students' mean scores of the experimental and control groups on conceptual understanding of limit. This uncovered that students had similar background information on the concept of limit before the treatment. However, posttest results revealed that there is a statistically significant difference between students' mean scores of the experimental and control groups on conceptual understanding of limit. Students in the experimental group scored significantly higher than students in the control group on the posttest. Students in the experimental group also showed better consistency in responding to two tier diagnostic test than students in the control group (Table 3). Moreover, percentage of students who correctly answered both tiers of each Item in the posttest was higher in the experimental group compared to the students in the control group (Table 4). For example, percentage of the students who correctly responded to diagnostic posttest Items requiring informal conception of limit was higher in the experimental group as compared to the students in the control group. Therefore, the findings of the present study revealed that GeoGebra integrated with the multi-teaching approaches improved the students' conceptual understanding of limit better than the traditional method which dominantly used chalk and talk approach. Hence, it could be possible to argue that employing purposefully organized activities guided by the APOS theory integrated with the GeoGebra software in a multi-learning environment played a significant role for students' learning. Apart from this, based on our findings, it is clear that the students' interest and motivation might play a significant role in improving their learning and achievement [9]. The findings of the present study are also in agreement with different empirical studies [28, 32], which got positive impact on students' learning in mathematics using GeoGebra.

In sum, the students' positive views, manifestations of coherent reasons, their capability of mental constructions (object and schemas), and providing correct and consistent responses to two tier Items, similar background information and higher scores on the posttest all together could elucidate the students' conceptual understanding of limit in the experimental group. Therefore, employing GeoGebra in multi-teaching environment based on an APOS paradigm enhances students' conceptual understanding of limit. Over and above, the findings also indicate the effectiveness of the APOS paradigm when it is integrated with GeoGebra for learning difficult mathematical concepts.

5. Limitations

The present study found invaluable findings for teaching and learning of mathematics. However, it has some limitations which may have little impact on the results. Firstly, the teachers who taught the experimental and control groups had different years of teaching experience even if both had master's degrees in teaching mathematics. Secondly, the researchers were always guiding the teacher teaching the experimental group so as to implement each and every activity based on the design. Therefore, these may have different impacts on the outcome of the study.

6. Conclusions

In the present study, the impact of GeoGebra integrated with multi-teaching approaches on students' conceptual understanding of limit was explored using students' views, interviews and two-tier diagnostic tests. Moreover, a genetic decomposition model based on the APOS theory was employed to gauge students' cognition of the concept of limit. The findings were of paramount importance to the teaching and learning of mathematics.

One of the significant findings emerging out of this study was the effectiveness of GeoGebra integrated with multi-teaching approaches on cultivating students' positive views in the learning process. The students were capable of constructing informal, coordinated and formal schemas using the affordances of GeoGebra which might trigger logico-mathematical thinking for learning the concept of limit. Besides, students elicited coherent and viable reasons while making mental constructions and their coordination in the learning process. Additionally, GeoGebra integrated with the multi-teaching approaches guided by APOS theory improved the students' conceptual understanding of limit better than the traditional method which dominantly used chalk and talk approach. The students did formal proofs of limit successfully using the affordances of the learning environment. Furthermore, the findings indicated that paper and pencil work becomes more effective while it is delivered simultaneously with technology affordances in the same classroom. Overall, the findings of the present study could significantly contribute to solving the prevailing problem associated with the concept of limit for mathematics students; and developing technology integrated curriculums in Ethiopian Universities.

7. Recommendations

7.1. Recommendations for future research

Future large scale studies on the same and other related concepts in mathematics, could further probe the efficiency and drawbacks of GeoGebra integrated with multi-teaching approaches based on the APOS theory.

7.2. Recommendations for different stakeholders in education

The findings of the present study have very import implications for practice of teaching mathematics. Therefore, curriculum designers and policy makers should give much emphasis to GeoGebra integrated curriculums and APOS theory to enhance students' learning. Moreover, the government should make resources available for teaching and learning of mathematics through technology. Apart from this, appropriate training should be given for teachers at any level of the education system with regard to GeoGebra and multi-teaching approaches.

Declarations

Author contribution statement

Mulat Gebeyehu Baye: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Mulugeta Atnafu Ayele: Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Tadele Ejigu Wondimuneh: Conceived and designed the experiments; Performed the experiments; Wrote the paper.

Data availability statement

Data will be made available on request.

Declaration of interests statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

References

[1] M.L. Bittinger, D.J. Ellenbogen, S.J. Surgent, Calculus and its Applications, global edition, Pearson, 2015.

[2] C. Nagle, Transitioning from introductory calculus to formal limit conceptions, Learn. Math. 33 (2013) 2–10.

[3] B. Cornu, Limits, in: D. Tall (Ed.), Adv. Math. Think., Kluwer Academic Publishers, 2002, pp. 153–166.

[4] D.G. Denbel, Students' misconceptions of the limit concept in a first calculus course, J. Educ. Pract. 5 (2014) 24–41.

[5] D. Bressoud, I. Ghedamsi, V. Martinez-Luaces, G. Törner, Teaching and Learning of Calculus, Springer, 2016.

[6] B. Güçler, Examining the discourse on the limit concept in a beginning-level calculus classroom, Educ. Stud. Math. 82 (2013) 439–453.

[7] J. Cottrill, E. Dubinsky, D. Nichols, K. Schwingendorf, K. Thomas, D. Vidakovic, Understanding the limit concept: beginning with a coordinated process scheme, J. Math. Behav. 15 (1996) 167–192.

[8] M. Hohenwarter, K. Jones, Ways of linking geometry and algebra: the case of GeoGebra, in: D. Küchemann (Ed.), Proc. Br. Soc. Res. Into Learn. Math., British Society for Research into Learning Mathematics, 2007, pp. 126–131.

[9] Y. Wassie, G. Zergaw, Some of the potential affordances, challenges and limitations of using GeoGebra in mathematics education, Eurasia J. Math. Sci. Technol. Educ. 15 (2019).

[10] N. Arbain, N.A. Shukor, The effects of GeoGebra on students achievement, Proc. Soc. Behav. Sci. 172 (2015) 208–214.

[11] I. Arnon, J. Cottrill, E. Dubinsky, A. Oktac, S. Fuentes, M. Trigueros, K. Weller, APOS Theory: A Framework for Research and Curriculum Development in Mathematics Education, Springer, 2014.

[12] E. Dubinsky, Actions, processes, objects, schemas (APOS) in mathematics education, in: S. Lerman (Ed.), Encycl. Math. Educ., Springer, 2014, pp. 8–11.

[13] J. Piaget, Biology and Knowledge (trans: Walsh P.), University of Chicago Press, 1971.

[14] E. Dubinsky, The cognitive effect of computer experiences on learning abstract mathematical concepts, Korkeakoulujen Atk-Uutiset. 2 (1984) 41–47.

[15] Z. Karadag, D. McDougall, GeoGebra as a cognitive tool: where cognitive theories and technology meet, in: L. Bu R. Schoen (Ed.), Model. Learn. Pathways to Math. Underst. Using GeoGebra, Sense Publishers, 2011, pp. 169–181.

[16] Q. Li, M. Xin, A meta-analysis of the effects of computer technology on school students' mathematics learning, Educ. Psychol. Rev. 22 (2010) 215–243.

[17] S.R. Williams, Models of limit held by college calculus students, J. Res. Math. Educ. 22 (1991) 219–236.

[18] G.T. Bagni, The historical roots of the limit notion: cognitive development and the development of representation registers, Can. J. Sci. Math. Technol. Educ. 5 (2005) 453–468.

[19] K. Juter, Students' concept development of limits, in: D. Pantazi- Pitta, G. Philippou (Eds.), Proc. Fifth Congr. Eur. Soc. Res. Math. Educ., Department of Education, University of Cyprus, Larnaca, Cyprus, 2007, pp. 2320–2329.

[20] J. Bezuidenhout, Limits and continuity: some conceptions of first-year students, Int. J. Math. Educ. Sci. Technol. 32 (2001) 487–500.

[21] C. Swinyard, Reinventing the formal definition of limit: the case of Amy and Mike, J. Math. Behav. 30 (2011) 93–114.

[22] C. Swinyard, S. Larsen, Coming to understand the formal definition of limit: insights gained from engaging students in reinvention, J. Res. Math. Educ. 43 (2012) 465–493.

[23] B. Nurwahyu, Concept image and concept definition of student's concept understanding, in: M. Mustangin, A.H. Fathani (Eds.), Proc. Int. Semin. Math. Educ.

Graph Theory, Department of Mathematics Education Faculty of Teacher Training and Education Islamic University of Malang, Malang, 2014, pp. 17–26.

[24] J. Westbrook, N. Durrani, R. Brown, D. Orr, J. Pryor, J. Boddy, F. Salvi, Pedagogy, Curriculum, Teaching Practices and Teacher Education in Developing Countries, Department for International Development, UK, 2013.

[25] C.E. Rizi, M. Najafipour, F. Haghani, S. Dehghan, The effect of the using the brainstorming method on the academic achievement of students in grade five in Tehran elementary schools, Proc. Soc. Behav. Sci. 83 (2013) 230–233.

[26] M. Gordon, The misuses and effective uses of constructivist teaching, Teach. Teach. Theory Into Pract. 15 (2009) 737–746.

[27] M.A. Gülpinar, B.Ç. Yeğen, Interactive lecturing for meaningful learning in large groups, Med. Teach. 27 (2005) 590–594.

[28] D. Takači, G. Stankov, I. Milanovic, Efficiency of learning environment using GeoGebra when calculus contents are learned in collaborative groups, Comput. Educ. 82 (2015) 421–431.

[29] H.I. Yung, F. Paas, Effects of computer-based visual representation on mathematics learning and cognitive load, J. Educ. Technol. Soc. 18 (2015) 70–77.

[30] A. Arcavi, The role of visual representations in the learning of mathematics, Educ. Stud. Math. 52 (2003) 215–241.

[31] C.N. Nobre, M.R. Meireles, N.V. Junior, M.N. Resende, L.E. Costa, R.C. Rocha, The use of Geogebra software as a calculus teaching and learning tool, Inf. Educ. 15 (2016) 253–267.

[32] Y. Zengin, H. Furkan, T. Kutluca, The effect of dynamic mathematics software GeoGebra on student achievement in teaching of trigonometry, Proc. Soc. Behav. Sci. 31 (2012) 183–187.

[33] E. Dubinsky, K. Weller, M.A. McDonald, A. Brown, Some historical issues and paradoxes regarding the concept of infinity: an APOS-based analysis, Educ. Stud. Math. 58 (2005) 335–359.

[34] A. Maharaj, An APOS analysis of students' understanding of the concept of a limit of a function, Pythagoras 71 (2010) 41–52.

[35] E. Dubinsky, Reflective abstraction in advanced mathematical thinking, in: D. Tall (Ed.), Adv. Math. Think., Kluwer Academic Publishers, 2002, pp. 95–123.

[36] J.w. Creswell, J.D. Creswell, Research Design: Qualitative, Quantitative, and Mixed Methods Research Designs, Sage Publication, 2018.

[37] A. Hilton, G. Hilton, S. Dole, M. Goos, Development and application of a two-tier diagnostic instrument to assess middle-years students' proportional reasoning, Math. Educ. Res. J. 25 (2013) 523–545.

[38] D. George, P. Mallery, IBM SPSS Statistics 25 Step by Step, fifteenth ed., Taylor & Francis Group, 2019.

[39] G. Harel, J. Kaput, The role of conceptual entities and their symbols in building advanced mathematical concepts, in: D. Tall (Ed.), Adv. Math. Think., Kluwer Academic Publishers, 2002, pp. 82–94.

[40] G. Cumming, Understanding the New Statistics: Effect Sizes, Confidence Intervals, and Meta-Analysis, Taylor & Francis Group, 2012.

[41] R.A. Saha, A.F.M. Ayub, R.A. Tarmizi, The effects of GeoGebra on mathematics achievement: enlightening coordinate geometry learning, Proc. Soc. Behav. Sci. 8 (2010) 686–693.

[42] E. Tatar, Y. Zengin, Conceptual understanding of definite integral with GeoGebra, Comput. Sch. 33 (2016) 120–132.