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The generalized harmonic potential theorem in the presence of a time-varying magnetic field

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We investigate the evolution of the many-body wave function of a quantum system with time-varying effective mass, confined by a harmonic potential with time-varying frequency in the presence of a uniform time-varying magnetic field, and perturbed by a time-dependent uniform electric field. It is found that the wave function is comprised of a phase factor times the solution to the unperturbed time-dependent Schrödinger equation with the latter being translated by a time-dependent value that satisfies the classical driven equation of motion. In other words, we generalize the harmonic potential theorem to the case when the effective mass, harmonic potential, and the external uniform magnetic field with arbitrary orientation are all time-varying. The results reduce to various special cases obtained in the literature, particularly to that of the harmonic potential theorem wave function when the effective mass and frequency are both static and the external magnetic field is absent.

The harmonic potential theorem (HPT)¹ concerning the many-body system trapped in an external harmonic potential describes the evolution of the wave function (WF) under the influence of an arbitrary external uniform electric field. It plays a significant role in time-dependent (TD) phenomena of quantum many-body systems. In particular, in the TD theories such as the TD density functional theory^{2–5}, TD quantal density functional theory^{6,7} and also in Bose-Einstein condensation (BEC)⁸. Soon after being discovered, the theorem was investigated in the presence of a static magnetic field but in the absence of harmonic potential by Vignale⁹, whereby he proved the theorem by the observation that an applied uniform TD electric field can be eliminated by transforming the static harmonically trapped system to an accelerated reference frame according to the classical equation of motion of the mass center. Furthermore, he stated that HPT is valid even in the case when the uniform magnetic field is time-varying (TV) and gave the explicit form of the TD potential in the accelerated frame. Recently, there has been considerable interest in the study of the theorem itself¹⁰. It has been generalized to the harmonic potential whose frequencies are TV¹¹, the explicit form of the HPT WF in the presence of a perpendicular static magnetic field is also obtained¹².

In the meantime, a TV effective mass could simulate the input or removal of energy from the system¹³. If quantum many-body systems interact with TV environment such as temperature, pressure, stress and energy, the effective masses will be modified. Thus, if the environment changes as time goes by, one would have a system with TV effective masses¹⁴. Thus, the study of harmonically trapped quantum systems with TV masses, particularly the model of TD harmonic oscillators (TDHO)^{15,16} has been extensively investigated (see, e.g.^{17–29}). Moreover, there has been considerable interest in the quantum systems in a TV electromagnetic field^{18,30–38}. The quantum problems that combine a TV effective mass and a TV external magnetic field have also been investigated. For instance, the Feynman's propagator for a charged particle with TV mass in a TV magnetic or electromagnetic field were obtained^{39,40}. Nevertheless, these studies all focused on the single-particle problems.

Besides its tremendous theoretical interest, the model of TDHO can be mapped to (i) a system of electric and magnetic fields in the interior of a Fabry-pérot cavity^{41–43}, or (ii) in a cavity filled with a medium with TV dielectric constant⁴⁴, or (iii) Paul traps systems^{45–47} where atomic particles are trapped in TV electromagnetic fields, or (iv) the effective Hamiltonian for dissipative systems^{24,48} confined in quantum dots⁴⁹, if one applies TV electromagnetic fields to some of above mentioned systems, say the dissipative systems in quantum dots⁵⁰, then one will obtain the model Hamiltonian (see eq. (1) below), or (v) a Bose-Einstein condensate subject to a rotating harmonic potential⁵¹ (see the Discussion below).

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In light of above facts, in this work we try to investigate the HPT in the most general form, i.e. the case when the effective mass, confining frequencies, and the external uniform magnetic field are all TV. After stating the generalized HPT, we then give the proof via two different approaches, i.e. the operator method and the accelerated frame approach⁹. We show that the WF is still comprised of a phase factor times the solution to the unperturbed TD Schrödinger equation with the latter being translated by a TD value that satisfies the classical driven equation of motion.

Results

Hamiltonian and the generalized HPT. Consider a system of N identical particles with a TV effective mass $m = m(t)$ under an external TV magnetic field with arbitrary orientation $\mathbf{B}(t) = (B_1(t), B_2(t), B_3(t))$, confined in an external TD harmonic potential $v(\mathbf{r}, t) = \frac{1}{2} \tilde{\mathbf{r}} \cdot \mathbf{K}(t) \cdot \mathbf{r}$, with $\mathbf{K}(t)$ an symmetric positive 3×3 matrix. The TD harmonic potential can be used to describe many experimental situations. For instance in BEC experiments, it can describe rotating the quadratic trap⁵¹, modulated trapping⁵², or reflecting fact that the trap is perturbed to obtain the response spectrum of the condensate⁵³⁻⁵⁵. The two-body interaction between the particles $u(\mathbf{r}_i - \mathbf{r}_j)$ can be of arbitrary form. A uniform TD driving electric field $\mathbf{E}(t) = \mathbf{f}(t)/q$ is turned on at time $t=0$ with $q = -e$ the charge of an electron. Thus, in the coordinate representation the Hamiltonian reads

$$\hat{H}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N; t) = \hat{H}_{E=0}(t) + e\mathbf{E}(t) \cdot \sum_j \mathbf{r}_j, \tag{1}$$

where the unperturbed component is

$$\hat{H}_{E=0}(t) = \sum_{i=1}^N \left(\frac{1}{2m(t)} \hat{\pi}_i^2 + \frac{m(t)}{2} \tilde{\mathbf{r}}_i \cdot \mathbf{K}(t) \cdot \mathbf{r}_i \right) + \sum_{(i,j)} u(\mathbf{r}_i - \mathbf{r}_j), \tag{2}$$

with $\tilde{\mathbf{r}}_i$ the transpose of the position vector \mathbf{r}_i , $\hat{\pi}_i$ is the physical momentum operator

$$\hat{\pi}_i = -i\hbar \nabla_i - \frac{q}{c} \mathbf{A}(\mathbf{r}_i, t). \tag{3}$$

Choosing the symmetry gauge such that the vector potential $\mathbf{A}(\mathbf{r}_i, t) = (\mathbf{B}(t) \times \mathbf{r}_i)/2$, and substituting eq. (3) into eq. (2) yields

$$\hat{H}_{E=0}(t) = \sum_{i=1}^N \left(\frac{1}{2m(t)} \hat{\mathbf{p}}_i^2 + \frac{m(t)}{2} \tilde{\mathbf{r}}_i \cdot \mathcal{K}(t) \cdot \mathbf{r}_i + \frac{e}{2m(t)c} \mathbf{B}(t) \cdot \hat{\mathbf{L}}_i \right) + \sum_{(i,j)} u(\mathbf{r}_i - \mathbf{r}_j), \tag{4}$$

where $\hat{\mathbf{L}}_i$ is the angular momentum operator of the i -th particle, and

$$\mathcal{K}(t) = \mathbf{K}(t) + \frac{q^2 \mathbf{D}(t)}{4m^2(t)c^2}, \tag{5}$$

is still a 3×3 real positive symmetric matrix with

$$\mathbf{D}(t) \equiv \begin{pmatrix} \mathbf{B}^2 - B_1^2 & -B_1 B_2 & -B_1 B_3 \\ -B_1 B_2 & \mathbf{B}^2 - B_2^2 & -B_2 B_3 \\ -B_1 B_3 & -B_2 B_3 & \mathbf{B}^2 - B_3^2 \end{pmatrix}. \tag{6}$$

The model above considered is obvious an open system. Generally there exist two main approach for such systems. The first is so called system-plus-bath approach⁵⁶⁻⁵⁸, and the second one is the effective Hamiltonian approach^{24,59,60}. The effective Hamiltonian usually has a TV mass that arises due to the interaction between the system and the bath⁴⁸, and governed by the TD Schrödinger equation within the adiabatic approximation. Hence, the Hamiltonian of eq. (1) can be regarded as an effective Hamiltonian for some open system and obey the following TD Schrödinger equation,

$$i\hbar \frac{\partial \Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N; t)}{\partial t} = \hat{H}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N; t) \Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N; t). \tag{7}$$

The core of the generalized HPT is the solution to the TD Schrödinger equation eq. (7). We refer to this solution as HPT WF. The generalized HPT states that the following WF

$$\begin{aligned} \Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N; t) = & \exp \left\{ \frac{1}{i\hbar} \int_0^t \left[\frac{\mathbf{P}_\xi^2(\tau)}{2M(\tau)} - \frac{M(\tau)}{2} \tilde{\xi}(\tau) \cdot \mathcal{K}(\tau) \cdot \xi(\tau) \right] d\tau + \frac{i}{\hbar} \mathbf{P}_\xi(t) \cdot \mathbf{R} \right\} \\ & \times \Psi_0(\mathbf{r}_1 - \xi(t), \mathbf{r}_2 - \xi(t), \dots, \mathbf{r}_N - \xi(t); t), \end{aligned} \tag{8}$$

satisfies eq. (7), where $\Psi_0(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N; t)$ is the solution of the following unperturbed Schrödinger equation:

$$i\hbar \frac{\partial \Psi_0(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N; t)}{\partial t} = \hat{H}_{E=0}(t) \Psi_0(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N; t). \tag{9}$$

Note that the phase factor in eq. (8) has the form similar to classical action. In eq. (8), $M(t) = Nm(t)$, $\xi(t)$ is the translation vector, $\mathbf{P}_\xi(t)$ the corresponding momentum vector (see eq. (26) below), and $\mathbf{R} = \sum_i \mathbf{r}_i/N$ the center of mass coordinate. The translation vector $\xi(t)$ satisfies the classical equation of motion

$$\ddot{\xi}(t) + \gamma(t) \left(\dot{\xi}(t) + \frac{1}{2} \xi(t) \times \omega_L(t) \right) + \dot{\xi}(t) \times \omega_L(t) + \mathbf{K}(t) \cdot \xi(t) = \frac{\mathbf{F}(t)}{M(t)} + \frac{1}{2} \dot{\omega}_L(t) \times \xi(t), \tag{10}$$

with $\gamma(t) = \frac{d[\ln M(t)]}{dt}$, $\omega_L(t) = Ne\mathbf{B}(t)/(M(t)c) = e\mathbf{B}(t)/(m(t)c)$ the cyclotron frequency, and $\mathbf{F}(t) = N\mathbf{f}(t)$. Eq. (10) is just the classical equation of motion for a harmonically trapped particle with a TV mass in the presence of TV external magnetic field $\mathbf{B}(t)$, perturbed by an external force $\mathbf{F}(t)$.

Proof of the theorem via derivation. Next we prove the generalized HPT by derivation. Using the center of mass (CM) and relative coordinates and momentums^{61–63}

$$\mathbf{R}^{(1)} \equiv \mathbf{R} = \frac{1}{N} \sum_i \mathbf{r}_i, \quad \hat{\mathbf{P}}^{(1)} \equiv \hat{\mathbf{P}} = \sum_i \hat{\mathbf{p}}_i, \tag{11}$$

and

$$\begin{aligned} X^{(2)} &= x_1 - x_2, \\ X^{(3)} &= x_1 + x_2 - 2x_3, \dots, \\ X^{(N)} &= x_1 + x_2 + \dots + x_{N-1} - (N-1)x_N, \end{aligned} \tag{12}$$

and similarly for $Y^{(2)}, \dots, Y^{(N)}, Z^{(2)}, \dots, Z^{(N)}$, and $P^{(2)}, \dots, P^{(N)}$, the Hamiltonian of eq. (1) can be decomposed into the CM and relative motion parts,

$$\hat{H}(t) = \hat{H}_{cm}(t) + \hat{H}_{rel}(t), \tag{13}$$

where

$$\hat{H}_{cm}(t) = \hat{H}_0(t) + \hat{H}_1(t), \tag{14}$$

with

$$\hat{H}_0(t) = \frac{1}{2M(t)} \hat{\mathbf{P}}^2 + \frac{M(t)}{2} \tilde{\mathbf{R}} \cdot \mathcal{K}(t) \cdot \mathbf{R} + \frac{\omega_L(t)}{2} \cdot \hat{\mathbf{L}}_{\mathbf{R}}, \tag{15}$$

and

$$\begin{aligned} \hat{H}_1(t) &= -\mathbf{F}(t) \cdot \mathbf{R} \\ &= -Nq(E_x(t)X + E_y(t)Y + E_z(t)Z), \end{aligned} \tag{16}$$

is the perturbation term due to the external electric field, and $\hat{\mathbf{L}}_{\mathbf{R}}$ the angular momentum operator for the CM coordinate \mathbf{R} . The relative motion part $\hat{H}_{rel}(t)$ contains only the relative coordinates, hence $[\hat{H}_{cm}(t), \hat{H}_{rel}(t)] = 0$. Consequently, the CM motion and the relative motion are separable. Therefore, the total WF of the Hamiltonian is the product of the WFs of CM motion and relative motions:

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N; t) = \Phi(\mathbf{R}, t) \varphi_{rel}(\mathbf{R}^{(2)}, \dots, \mathbf{R}^{(N)}; t). \tag{17}$$

The relative motion WF $\varphi_{rel}(\mathbf{R}^{(2)}, \dots, \mathbf{R}^{(N)}; t)$ and the CM motion WF $\Phi(\mathbf{R}, t)$ satisfy their own Schrödinger equations, respectively, with certain initial conditions. In the following, we shall focus on the CM motion Hamiltonian of eq. (14), and try to find its WF $\Phi(\mathbf{R}, t)$. Similar to the structure of the HPT WF, let us assume

$$\Phi(\mathbf{R}, t) = e^{iS(\mathbf{R}, t)} \Phi_0(\mathbf{R} - \xi(t), t), \tag{18}$$

where $\Phi_0(\mathbf{R}, t)$ is the WF for the unperturbed CM motion Hamiltonian, i.e. which satisfies the following Schrödinger equation,

$$i\hbar \frac{\partial \Phi_0(\mathbf{R}, t)}{\partial t} = \hat{H}_0(t) \Phi_0(\mathbf{R}, t). \tag{19}$$

Next, we shall seek the analytical expressions for $S(\mathbf{R}, t)$ and determine the translated vector $\xi(t)$ which leads $\Phi(\mathbf{R}, t)$ to satisfy its own Schrödinger equation,

$$i\hbar \frac{\partial \Phi(\mathbf{R}, t)}{\partial t} = \hat{H}_{cm}(t) \Phi(\mathbf{R}, t). \quad (20)$$

Inserting eq. (18) into eq. (20) and using eq. (14), we have

$$\begin{aligned} i\hbar \frac{\partial \Phi}{\partial t} &= i\hbar e^{iS} \left(i \frac{\partial S}{\partial t} \Phi_0(\mathbf{R} - \xi(t), t) + (-\dot{\xi} \cdot \nabla_{\mathbf{R}}) \Phi_0(\mathbf{R} - \xi(t), t) \right) \\ &\quad + e^{iS} \hat{H}_0(\mathbf{R} - \xi(t), t) \Phi_0(\mathbf{R} - \xi(t), t) \\ &= (\hat{H}_0 + \hat{H}_1) e^{iS} \Phi_0(\mathbf{R} - \xi(t), t) \\ &= \{e^{iS} (\hat{H}_0 + \hat{H}_1) + [\hat{H}_0 + \hat{H}_1, e^{iS}]\} \Phi_0(\mathbf{R} - \xi(t), t), \end{aligned} \quad (21)$$

Note here we have used that fact that $\frac{\partial R}{\partial t} = 0$ since R now is the eigenvalue of the coordinate operator \hat{R} , whose hat has been dropped since we work in the coordinate representation. With the ansatz that the phase factor can be cast into the following form,

$$iS(\mathbf{R}, t) = A_0(t) + \mathbf{A}_1(t) \cdot \mathbf{R}, \quad (22)$$

then we have the commutator

$$\begin{aligned} [\hat{H}_0 + \hat{H}_1, e^{iS}] &= \left[\frac{1}{2M} \hat{\mathbf{P}}^2 + \frac{\omega_L(t)}{2} \cdot \hat{\mathbf{L}}_{\mathbf{R}}, e^{iS} \right] \\ &= -\frac{\hbar^2}{2M(t)} e^{iS} (2\mathbf{A}_1(t) \cdot \nabla_{\mathbf{R}} + \mathbf{A}_1^2(t)) - \frac{i\hbar}{2} \omega_L(t) \cdot (\mathbf{R} \times \mathbf{A}_1(t)) e^{iS}. \end{aligned} \quad (23)$$

Making use of the following expression

$$\begin{aligned} \hat{H}_0(\mathbf{R} - \xi(t), t) &= \hat{H}_0(\mathbf{R}, t) + \frac{M(t)}{2} (\tilde{\mathbf{R}} \cdot \mathcal{K}(t) \cdot (-\xi(t)) \\ &\quad + \tilde{\xi}(t) \cdot \mathcal{K}(t) \cdot \xi(t) - \tilde{\xi}(t) \cdot \mathcal{K}(t) \cdot \mathbf{R}) \\ &\quad + \frac{1}{2} \omega_L(t) \cdot (-\xi(t) \times \hat{\mathbf{P}}), \end{aligned} \quad (24)$$

and inserting eqs (23) and (24) into eq. (21) yields

$$\begin{aligned} &-\hbar \frac{\partial S}{\partial t} - i\hbar \dot{\xi} \cdot \nabla_{\mathbf{R}} + \frac{M(t)}{2} (-\tilde{\xi}(t) \cdot \mathcal{K}(t) \cdot \mathbf{R} + \tilde{\xi}(t) \cdot \mathcal{K}(t) \cdot \xi(t) \\ &\quad - \tilde{\xi}(t) \cdot \mathcal{K}(t) \cdot \mathbf{R}) - \frac{i\hbar}{2} \xi(t) \times \omega_L(t) \cdot \nabla_{\mathbf{R}} \\ &= -\mathbf{F}(t) \cdot \mathbf{R} - \frac{\hbar^2}{2M(t)} (2\mathbf{A}_1(t) \cdot \nabla_{\mathbf{R}} + \mathbf{A}_1^2(t)) - \frac{i\hbar}{2} \mathbf{A}_1(t) \times \omega_L(t) \cdot \mathbf{R}. \end{aligned} \quad (25)$$

By comparing the coefficients of $\nabla_{\mathbf{R}}$ on both sides of eq. (25), we have

$$\mathbf{A}_1(t) = \frac{iM}{\hbar} \left(\dot{\xi}(t) + \frac{1}{2} \xi(t) \times \omega_L(t) \right) \equiv \frac{i}{\hbar} \mathbf{P}_{\xi}(t), \quad (26)$$

and similarly for the coefficients of \mathbf{R} , we obtain

$$i\hbar \dot{\mathbf{A}}_1(t) - M \tilde{\xi}(t) \cdot \mathcal{K}(t) = -\mathbf{F}(t) - \frac{i\hbar}{2} \mathbf{A}_1(t) \times \omega_L(t), \quad (27)$$

and

$$i\hbar \dot{\mathbf{A}}_0(t) + \frac{M(t)}{2} \tilde{\xi}(t) \cdot \mathcal{K}(t) \cdot \xi(t) = -\frac{\hbar^2}{2M} \mathbf{A}_1^2(t). \quad (28)$$

Inserting eq. (26) into eqs (27) and (28), we immediately find that the translation vector satisfies eq. (10). Note that $\frac{1}{2} \dot{\omega}_L(t) \times \xi(t)$ is the induced electric force by the TV magnetic field, and

$$i\hbar \dot{A}_0(t) = \int_0^t \left[\frac{\mathbf{P}_{\xi}^2(\tau)}{2M(t)} - \frac{M(t)}{2} \tilde{\xi}(\tau) \cdot \mathcal{K}(\tau) \cdot \xi(\tau) \right] d\tau. \quad (29)$$

is just the classical action without the electric field term. Hence, from eqs (17), (18), (22), (10) and (29), we obtain the final WF of eq. (8). In other words, we have proved via derivation that the HPT WF eq. (8) is the solution of the TD Schrödinger equation eq. (7).

The HPT WF is the key result of this paper. Note that if one requires that the initial state is the eigenstate of the unperturbed Hamiltonian, i.e. $\Psi(t=0) = \Psi_0(t=0)$, then usually one has the initial conditions: $\xi(0) = 0, \dot{\xi}(0) = 0$. We stress that the HPT WF can reduce to various special cases existed in literature^{1,9-12}. Thus, we have extended the HPT to the case when the quantum systems have a TV effective mass and TV confining frequencies, in the presence of a uniform TV magnetic field with arbitrary orientation.

The Hamiltonian and wave function in the accelerated frame. Inspired by the method of Vignale⁹, we next show that our results can also be obtained by transforming the system to an accelerated reference frame. Making the acceleration transformations

$$\mathbf{r}_i \rightarrow \mathbf{r}'_i = \mathbf{r}_i + \xi(t); \quad t = t' \quad (30)$$

on the system with $\xi(t)$ governed by eq. (10), hence the connection between the original WF $\Psi(t)$ and the accelerating WF $\Psi'(t)$ is

$$\Psi'(t) = \hat{U}_{\xi(t)} \Psi(t). \quad (31)$$

The WFs $\Psi(t)$ and $\Psi'(t)$ in the above equation satisfy the following Schrödinger equations respectively:

$$i\hbar \frac{\partial \Psi(t)}{\partial t} = \hat{H}(t) \Psi(t), \quad i\hbar \frac{\partial \Psi'(t)}{\partial t} = \hat{H}'(t) \Psi'(t). \quad (32)$$

And the explicit form of the unitary operator $\hat{U}_{\xi(t)}$ is⁶⁴

$$\begin{aligned} \hat{u}_i(t) &= \exp\left[\frac{i}{\hbar} \hat{\mathbf{p}}_i \cdot \xi(t)\right] \exp\left[-\frac{i}{\hbar} \mathbf{r}_i \cdot \frac{\mathbf{P}_\xi(t)}{N}\right]; \\ \hat{U}_{\xi(t)} &= \prod_i \hat{u}_i(t) = \exp\left[\frac{i}{\hbar} \hat{\mathbf{P}} \cdot \xi(t)\right] \exp\left[-\frac{i}{\hbar} \mathbf{R} \cdot \mathbf{P}_\xi(t)\right]. \end{aligned} \quad (33)$$

From eqs (8), (31) and (33), the connection of the accelerating WF and the WF in the absence of the external electric field can be written in a simpler form as

$$\Psi'(t) = \exp\left[\frac{1}{i\hbar} \int_0^t \left[\frac{\mathbf{P}_\xi^2(\tau)}{2M(\tau)} - \frac{M(\tau)}{2} \tilde{\xi}(\tau) \cdot \mathcal{K}(\tau) \cdot \xi(\tau) \right] d\tau\right] \Psi_0(t). \quad (34)$$

Applying $\hat{U}_{\xi(t)}$ to the Schrödinger equation eq. (32), yields the explicit expression of the accelerating Hamiltonian,

$$\begin{aligned} \hat{H}'(t) &= \hat{U}_{\xi(t)} \hat{H} \hat{U}_{\xi(t)}^{-1} - i\hbar \hat{U}_{\xi(t)} \frac{\partial}{\partial t} \hat{U}_{\xi(t)}^{-1} \\ &= \hat{H}_{\mathbf{E}=0}(t) + \frac{N}{2} m(t) \dot{\xi}^2(t) - N \frac{e}{2c} \dot{\xi}(t) \times \mathbf{B}(t) \cdot \xi(t) \\ &\quad - N \frac{m(t)}{2} \tilde{\xi}(t) \cdot \mathbf{K}(t) \cdot \xi(t), \end{aligned} \quad (35)$$

after a long calculation, above Hamiltonian can be recast into the following form

$$\hat{H}'(t) = \hat{H}_{\mathbf{E}=0}(t) + i\hbar \dot{A}_0(t), \quad (36)$$

thus we immediately recognized that the WFs for $\hat{H}'(t)$ and $\hat{H}_{\mathbf{E}=0}(t)$ only differ by a phase which is exactly eq. (34). On the other hand, from eqs (34) and (36), one can readily see that the uniform time dependent electric field is eliminated by performing the acceleration transformations. This proves again the generalized harmonic potential theorem.

Discussion

In summary, we have presented the detailed analytical form of the evolution of the WF for a quantum system with TV effective mass trapped in a harmonic potential with TV frequency, in the presence of a TV uniform magnetic field with arbitrary orientation, and driven by a TD **uniform** electric field. It is found that the WF is comprised of a phase factor times the solution to the unperturbed TD Schrödinger equation with the latter being translated by a TD value that satisfies the classical equation of motion for a driven harmonic oscillator with TV mass in the presence of an external TV magnetic field. The analytical form of the phase is also given. The results can reduce to various special cases existed in the literature. We also show that our results can be obtained by transforming the system to an accelerated frame. Moreover, we stress that our results are applicable to both the fermionic and bosonic systems with general effective masses and external magnetic fields that can be described by some smooth functions of time, since the derivations do not rely on the statistical properties of the WF or any specifically choice of the TD terms and parameters. However, the external TD electric field must be uniform.

Finally, we briefly discuss some real physical systems that our results might shed lights on. Notice that if one identifies the angular velocity $\Omega = \frac{eB(t)}{2m(t)c}$ and the gravity with the external driving force, then the model Hamiltonian of eq. (1) can be used to describe atoms trapped in a harmonic potential rotating instantaneous around the z axis⁵¹, the related experiment has been done at ENS⁶⁵. In above case of a vertical axis of rotation, the only effect of gravity is a displacement of the equilibrium position⁶⁶ thus can be ignored. When the axis of rotation was tilted away from the trap axis such those experiments done in refs 67,68, the effect of gravity must be taken into account. For instance, in a uniformly rotating trap, it can causes resonances hence the escape of the center of mass for a collection of interacting particles from the trap⁶⁶. Our results implies that even the rotation is tilted and TV, the effect of gravity is solely to transport rigidly the center of mass, or density distribution of the system. This is expected to be confirmed experimentally.

References

- Dobson, J. F. Harmonic-potential theorem: Implications for approximate many-body theories. *Physical Review Letters* **73**, 2244 (1994).
- Runge, E. & Gross, E. K. Density-functional theory for time-dependent systems. *Physical Review Letters* **52**, 997 (1984).
- Ullrich, C. A. *Time-dependent density-functional theory: concepts and applications* (OUP Oxford, 2011).
- Marques, M. *Time-dependent density functional theory* vol. 706 (Springer Science & Business Media, 2006).
- Gross, E., Dobson, J. & Petersilka, M. Density functional theory of time-dependent phenomena. In *Density functional theory II* 81–172 (Springer, 1996).
- Qian, Z. & Sahni, V. Physics of transformation from Schrödinger theory to Kohn-Sham density-functional theory: Application to an exactly solvable model. *Physical Review A* **57**, 2527 (1998).
- Sahni, V. *Quantal density functional theory* (Springer, 2004).
- Dalfovo, F., Giorgini, S., Pitaevskii, L. P. & Stringari, S. Theory of Bose-Einstein condensation in trapped gases. *Reviews of Modern Physics* **71**, 463 (1999).
- Vignale, G. Center of mass and relative motion in time dependent density functional theory. *Physical Review Letters* **74**, 3233 (1995).
- Jin-Wang, C., Tao, Y. & Xiao-Yin, P. A New Proof for the Harmonic-Potential Theorem. *Chinese Physics Letters* **30**, 020303 (2013).
- Li, Y.-Q., Pan, X.-Y. & Sahni, V. Wave function for time-dependent harmonically confined electrons in a time-dependent electric field. *The Journal of Chemical Physics* **139**, 114301 (2013).
- Zhu, H.-M., Chen, J.-W., Pan, X.-Y. & Sahni, V. Wave function for harmonically confined electrons in time-dependent electric and magnetostatic fields. *The Journal of Chemical Physics* **140**, 024318 (2014).
- Remaud, B. & Hernandez, E. Damping of wave packet motion in a general time-dependent quadratic field. *Journal of Physics A: mathematical and general* **13**, 2013 (1980).
- Choi, J.-R. Exact quantum theory of noninteracting electrons with time-dependent effective mass in a time-dependent magnetic field. *Journal of Physics: Condensed Matter* **15**, 823 (2003).
- Dekker, H. Classical and quantum mechanics of the damped harmonic oscillator. *Physics Reports* **80**, 1–110 (1981).
- Um, C.-I., Yeon, K.-H. & George, T. F. The quantum damped harmonic oscillator. *Physics Reports* **362**, 63–192 (2002).
- Kanai, E. On the quantization of the dissipative systems. *Progress of Theoretical Physics* **3**, 440–442 (1948).
- Lewis Jr, H. R. Classical and quantum systems with time-dependent harmonic-oscillator-type Hamiltonians. *Physical Review Letters* **18**, 510 (1967).
- Hasse, R. W. On the quantum mechanical treatment of dissipative systems. *Journal of Mathematical Physics* **16**, 2005–2011 (1975).
- Dhara, A. & Lawande, S. Time-dependent invariants and the Feynman propagator. *Physical Review A* **30**, 560 (1984).
- Abdalla, M. S. & Colegrave, R. Harmonic oscillator with strongly pulsating mass under the action of a driving force. *Physical Review A* **32**, 1958 (1985).
- Dantas, C. M., Pedrosa, I. & Baseia, B. Harmonic oscillator with time-dependent mass and frequency and a perturbative potential. *Physical Review A* **45**, 1320 (1992).
- Pedrosa, I. Exact wave functions of a harmonic oscillator with time-dependent mass and frequency. *Physical Review A* **55**, 3219 (1997).
- Yu, L. H. & Sun, C.-P. Evolution of the wave function in a dissipative system. *Physical Review A* **49**, 592 (1994).
- Ji, J.-Y., Kim, J. K. & Kim, S. P. Heisenberg-picture approach to the exact quantum motion of a time-dependent harmonic oscillator. *Physical Review A* **51**, 4268 (1995).
- Pedrosa, I. A., Serra, G. & Guedes, I. Wave functions of a time-dependent harmonic oscillator with and without a singular perturbation. *Physical Review A* **56**, 4300 (1997).
- Kim, H.-C., Lee, M.-H., Ji, J.-Y. & Kim, J. K. Heisenberg-picture approach to the exact quantum motion of a time-dependent forced harmonic oscillator. *Physical Review A* **53**, 3767 (1996).
- Song, D.-Y. Periodic Hamiltonian and Berry's phase in harmonic oscillators. *Physical Review A* **61**, 024102 (2000).
- Harari, G., Ben-Aryeh, Y. & Mann, A. Propagator for the general time-dependent harmonic oscillator with application to an ion trap. *Physical Review A* **84**, 062104 (2011).
- Lewis Jr, H. R. & Riesenfeld, W. An Exact Quantum Theory of the Time-Dependent Harmonic Oscillator and of a Charged Particle in a Time-Dependent Electromagnetic Field. *Journal of Mathematical Physics* **10**, 1458–1473 (1969).
- Horing, N. J. M., Cui, H. & Fiorenza, G. Nonrelativistic Schrödinger Green's function for crossed time-dependent electric and magnetic fields. *Physical Review A* **34**, 612 (1986).
- Sökmen, I. Exact path-integral solution of the ring-shaped potential. *Physics Letters A* **115**, 249–252 (1986).
- Wang, S.-J. Nonadiabatic Berry's phase for a spin particle in a rotating magnetic field. *Physical Review A* **42**, 5107 (1990).
- Wagh, A. G. & Rakhecha, V. C. Exact fixed-angle spinor evolutions via the rotating-frame formalism. *Physical Review A* **48**, R1729 (1993).
- Calvo, M. Classical and quantum theory of a two-dimensional electron gas in time-dependent magnetic fields. *Physical Review B* **54**, 5665 (1996).
- Lin, Q.-g. Charged particles in a rotating magnetic field. *Physical Review A* **63**, 012108 (2000).
- Laroze, D. & Rivera, R. An exact solution for electrons in a time-dependent magnetic field. *Physics Letters A* **355**, 348–351 (2006).
- Sadhukhan, M., Panigrahi, P. & Deb, B. Dynamics of a hydrogen atom under a strong, time-dependent magnetic field. *EPL (Europhysics Letters)* **91**, 23001 (2010).
- Nassar, A. B. & Berg, R. T. Propagator for the time-dependent three-dimensional charged harmonic oscillator in a time-varying magnetic field. *Physical Review A* **34**, 2462 (1986).
- de Souza Dutra, A. & Cheng, B. K. Feynman's propagator for a charged particle with time-dependent mass in a crossed time-varying electromagnetic field. *Physical Review A* **39**, 5897 (1989).
- Colegrave, R. & Abdalla, M. S. A canonical description of the Fabry-Perot cavity. *Journal of Modern Optics* **28**, 495–501 (1981).
- Colegrave, R. & Abdalla, M. S. Harmonic oscillator with exponentially decaying mass. *Journal of Physics A: Mathematical and General* **14**, 2269 (1981).

43. Cheng, B. K. Exact propagator of the harmonic oscillator with a time-dependent mass. *Physics Letters A* **113**, 293–296 (1985).
44. Agarwal, G. & Kumar, S. A. Exact quantum-statistical dynamics of an oscillator with time-dependent frequency and generation of nonclassical states. *Physical Review Letters* **67**, 3665 (1991).
45. Paul, W. Electromagnetic traps for charged and neutral particles. *Reviews of Modern Physics* **62**, 531 (1990).
46. Brown, L. S. Quantum motion in a Paul trap. *Physical Review Letters* **66**, 527 (1991).
47. Glauber, R. J. Quantum optics. In *Quantum Optics* (1969).
48. Lai, M.-Y., Pan, X.-Y. & Li, Y.-Q. Wave function for dissipative harmonically confined electrons in a time-dependent electric field. *Physica A: Statistical Mechanics and its Applications* **453**, 305–316 (2016).
49. Mshvelidse, G. *et al.* Optical properties of AgI quantum dots embedded in a glass matrix. *Physica Status Solidi B Basic Research* **207**, 369–376 (1998).
50. Baskoutas, S., Politis, C., Rieth, M. & Schommers, W. Dissipative dynamics of a parabolic confined particle in the presence of magnetic field. *Physica A: Statistical Mechanics and its Applications* **292**, 238–254 (2001).
51. Sinha, S. & Castin, Y. Dynamic instability of a rotating Bose-Einstein condensate. *Physical Review Letters* **87**, 190402 (2001).
52. Quinn, E. & Haque, M. Modulated trapping of interacting bosons in one dimension. *Physical Review A* **90**, 053609 (2014).
53. Garca-Ripoll, J. J., Pérez-García, V. M. & Vekslerchik, V. Construction of exact solutions by spatial translations in inhomogeneous nonlinear Schrödinger equations. *Physical Review E* **64**, 056602 (2001).
54. Garca-Ripoll, J. J. & Pérez-García, V. M. Barrier resonances in Bose-Einstein condensation. *Physical Review A* **59**, 2220 (1999).
55. Garca-Ripoll, J. J., Pérez-García, V. M. & Torres, P. Extended parametric resonances in nonlinear Schrödinger systems. *Physical Review Letters* **83**, 1715 (1999).
56. Weiss, U. *Quantum dissipative systems* vol. 10 (World Scientific, 1999).
57. Caldeira, A. O. & Leggett, A. J. Influence of dissipation on quantum tunneling in macroscopic systems. *Physical Review Letters* **46**, 211 (1981).
58. Caldeira, A. O. & Leggett, A. J. Quantum tunnelling in a dissipative system. *Annals of Physics* **149**, 374–456 (1983).
59. Yu, L. H. Exponential decay of wavelength in a dissipative system. *Physics Letters A* **202**, 167–175 (1995).
60. Shao, Z.-Q., Li, Y.-Q. & Pan, X.-Y. Dissipation-induced transition of a simple harmonic oscillator. *The Journal of Chemical Physics* **141**, 224110 (2014).
61. Yip, S. K. Magneto-optical absorption by electrons in the presence of parabolic confinement potentials. *Physical Review B* **43**, 1707 (1991).
62. Yu, A. F. & Gurevitch, V. L. *Zh. Eksp. Teor. Fiz* **41**, 512 (1961).
63. Maksym, P. & Chakraborty, T. Quantum dots in a magnetic field: Role of electron-electron interactions. *Physical Review Letters* **65**, 108 (1990).
64. Klink, W. Quantum Mechanics in Noninertial Reference Frames. *Annals of Physics* **260**, 27–49 (1997).
65. Madison, K., Chevy, F., Bretin, V. & Dalibard, J. Stationary states of a rotating Bose-Einstein condensate: Routes to vortex nucleation. *Physical Review Letters* **86**, 4443 (2001).
66. Bialynicki-Birula, I. & Sowiński, T. Gravity-induced resonances in a rotating trap. *Physical Review A* **71**, 043610 (2005).
67. Marago, O. *et al.* Observation of the scissors mode and evidence for superfluidity of a trapped Bose-Einstein condensed gas. *Physical Review Letters* **84**, 2056 (2000).
68. Smith, N., Heathcote, W., Krueger, J. & Foot, C. Experimental observation of the tilting mode of an array of vortices in a dilute Bose-Einstein condensate. *Physical Review Letters* **93**, 080406 (2004).

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Author Contributions

X.-Y.P. conceived the original ideas, M.-Y.L. performed the calculations. All authors wrote the manuscript and clarified the ideas.

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