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# Computation of instant system availability and its applications

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## Abstract

The instant system availability  $S_{\tau}(t)$  of a repairable system with the renewal equation was studied. The starting point monotonicity of  $S_{\tau}(t)$  was proved and the upper bound of  $S_{\tau}(t)$  is also derived. It was found that the interval of instant system availability monotonically decreases. In addition, we provide examples that validate the analytically derived properties of  $S_{\tau}(t)$  based on the Lognormal, Gamma and Weibull distributions and the results show that the value of  $T$  is slightly smaller than its value defined in Theorem 2. The procedure of using a bathtub as application for this article is also discussed.

**Keywords:** Instant availability, Monotonicity, Renewal equation, Upper bound, Binary random variables, Repairable system

**Mathematics Subject Classification:** 26A48, 11KXX, 90B25, 37A60, 60B12

## Background

It is known that availability system state comprises of stages namely up and down. If the system is up it implies that it is accurately functioning and in another time the system is down, it signifies that it is not functioning or not working. Different authors (Baxter 1981a; Huang and Mi 2015; Mishra and Jain 2013) explained the significance functioning of availability and repairable system. Consider  $X(t)$  as a state of system at any time  $t$  which represents binary random variables, thus

$$X(t) = \begin{cases} 1, & \text{indicates that the system is up;} \\ 0, & \text{indicates that the system is down.} \end{cases}$$

In a similar situation, if a new system with supposed the lifetime of  $L_1$  starts working till it declines at times  $t = 0$ . This first combination of up together with down entails the first cycle of the system for every subsequent of a system become down and a new cycle of the period will be accomplished and the system will restart functioning. This has been also explained in Sarkar and Biswas (2010) where the repair queue is joined by failed unit and is helped presently one of the repair facilities becomes free and the unit becomes a workable spares after a perfect repair the same as several repair facilities.

In this article we let  $R_j$  and  $L_j$  indicate the dimension of  $j$ th down and up period, where  $L_j$  denote the given lifetime in such that after  $(j - 1)$ th period the system is down and  $R_j$  be the distance required to facilitate the repairing with regarding the replacement job.

In recent years, some authors based on variety of models for repairable system using different techniques have suggested (Baxter 1981a, b, 1983; Barlow and Hunter 1960; Cha and Kim 2002; Mishra and Jain 2013; Veber et al. 2008) where the concept of availability be a relevant measure with regarding the system performance as results of repaired and maintenance implementation. Apparently, the longer the time the system is in working state is relative to the length of time for down state which improves the better the system's performance is, as explained in Veber et al. (2008).

To get a clear or a complete picture of this work, some references had been considered to be taken into account such as Mathew (2014), Sarkar and Chaudhuri (1999), Huang and Mi (2013), Biswas and Sarkar (2000), Cha et al. (2004), Mi (1998, 2006, 1995a), Huang and Mi (2015) where the relevant features of repairable system are availability. It is known as point availability at times  $t$ .

To measure the likelihood that a system is an availability at some particular time  $t$ , a quantity denoted as  $S_\tau(t)$  is referred to as instant availability. The corresponding probability of well functioning system at any derived time  $t$  is defined. On the other hand  $\tau$  represents a small change with respect to the system availability.

A considerable research with regard system availability has gained importance results of it's steady state system availability, this has been observed in El-Damcese and Temraz (2012, 2010). In practical applications, researchers (engineers) are more concerned in it which is expressed as,

$$S_\tau(\infty) = \lim_{t \rightarrow \infty} S_\tau(t).$$

This show that the limit exists and estimates the extent by which the system hopes to be to exist after it has been worked for a long time and it is a vital measure performance of repairable system as defined in Bieth et al. (2010), Huang and Mi (2015) where the system remains always good as new like it was before being repaired.

To the best authors studies have been attempted, many theoretical and meaningful results to provide analysis for individual cases of system availability  $S_\tau(t)$  such as Huang and Mi (2013), Huang and Mi (2015), Sarkar and Li (2006) but it remains unclear with regard to the behaviour of the instant system availability as a function of time  $t$ , and it is obtained through perfect repair, imperfect repairs or replacement after each failure.

In this paper, many discussions of  $(L_j, R_j), j \geq 1$  here are supposed to be (i.i.d) with  $F(t)$  and  $G(t)$  as common cumulative distribution functions respectively. We also consider that  $(L_j, j \geq 1), (R_j, j \geq 1)$  being independent of each other.

Let

$$T_j = L_j + R_j, \quad j \geq 1$$

be the independent length of the  $j$ th cycle and represent cumulative distribution functions as  $M(t)$  which explains the convolution of  $F(t)$  and  $G(t)$  by independent assumption. That is

$$M(t) = \int_0^t F(t - \zeta) dG(\zeta).$$

For  $m(t)$ , this is the probability density function defined in the same way as the above which expresses the convolution of  $f$  and  $g$ , hence

$$m(t) = \int_0^t f(t - \zeta)g(\zeta)d\zeta = \int_0^t g(t - \zeta)f(\zeta)d\zeta$$

for any  $0 \leq \zeta \leq t$ .

According to the independent and identical distribution assumption, it is recognized that the point availability  $S_\tau(t)$  has an integral equation as the solution defined below

$$S_\tau(t) = \bar{F}(t) + \int_0^t S_\tau(t - \zeta) dM(\zeta) \quad (1)$$

where

$$\bar{F}(t) = 1 - F(t).$$

In case  $(L_j, R_j), j \geq 1$ , this one comprises the sequence of (i.i.d) random variables with some desirable properties have been obtained by using results from alternative renewal process.

The article is arranged in the following manner: "[Starting point monotonicity analysis](#)" section presents the behaviour of the instant system availability  $S_\tau(t)$  in the given positive interval if it is decreasing or increasing. In "[Bounded for availability system](#)" section, we consider the bound for availability system taking into consideration to the upper bound of  $S_\tau(t)$  is derived from this section. In "[Comparison of availability system](#)" section, we make the comparison between two systems availability. In "[Numerical examples for clarification](#)" section, we provide numerical examples where we consider the repair and lifetime distributions as well as associated with different parameters. "[Application to bathtub](#)" section, application to bathtub for this article is strongly discussed. And finally the conclusion and recommendation are provided in "[Conclusion](#)" section.

### Starting point monotonicity analysis

For a positive real line  $[0, \infty)$  can be divided into disjoint intervals in which the availability function  $S_\tau(t)$  decreases or rise in relation to these intervals. When  $t = 0$ , the availability system  $S_\tau(0) = 1$ . It explains that, the availability  $S_\tau(t)$  decreases to a neighborhood around 0. The similarities with this one have been proposed and studied in Huang and Mi (2015) where they have proved that the availability functions decreases based on the lower bound. We let  $\bar{T} = \sup\{t > 0 : S_\tau(\zeta)\}$  decreases  $\forall \zeta \in [0, T]$ . Then the interval  $[0, \bar{T}]$  is very relevant while the availability  $S_\tau(\bar{T}) = \inf_{0 \leq t \leq \bar{T}} S_\tau(t)$  at time  $\bar{T}$  is lower than the steady state availability, which is proved in the numerical examples proposed by "[Numerical examples for clarification](#)" section of this article. It is difficult to have the exact estimate of  $\bar{T}$ , hence we need to determine a subinterval of  $[0, \bar{T}]$  which is  $[0, T]$  such that the availability  $S_\tau(t)$  is decreasing in that interval of  $t$  and from that, we obtain concept regarding for  $S_\tau(\bar{T})$  based on the estimation of  $S_\tau(T)$  given in the respect interval.

**Lemma 1** Given that  $f(t), m(t)$  are continuous functions defined in interval  $(0, \infty)$  and then  $T = \inf\{t > 0 : f(t) = m(t)\}$ . We let  $T > 0$ , therefore  $f(t) > m(t)$  for  $\forall t \in [0, T]$ .

*Proof* We assumed that  $T > 0$ , this denotes that  $f(t) - m(t) > 0$ , in  $[0, T]$  satisfies interval function  $f(t) - m(t)$  which holds with the same sign. If life time  $(L_i, i \geq 1)$  and repair time  $(R_i, i \geq 1)$  is defined, then  $f(t) - m(t) \in [0, T]$  must be positive. Hence  $L_i$  is stochastically dominated by  $L_i + R_i$  for  $i \geq 1$ .

**Lemma 2** Given  $f(t)$  being continuous function of  $(0, \infty)$ , we let also  $g(t)$  be a closed function of any finite interval and  $f, g \in L^1(0, \infty)$ , then  $m = f * g$  is continuous on  $(0, \infty)$  provided that the bounded function does not indicate a system of an interval not covers zero.

*Proof* To get a complete proof of this Lemma, we use the upper bound together with the convolution assumptions.

Let  $0 < t < \infty$  and let  $\theta \geq 0$  we will get

$$\begin{aligned} m(t) &= \int_0^t g(\zeta)f(t - \zeta)d\zeta = \int_0^\theta g(\zeta)f(t - \zeta)d\zeta \\ &\quad + \int_{t-\theta}^t g(\zeta)f(t - \zeta)d\zeta + \int_\theta^{t-\theta} g(\zeta)f(t - \zeta)d\zeta \\ &= \int_0^\theta g(\zeta)f(t - \zeta)d\zeta + \int_0^\theta g(t - x)f(x)dx + \int_\theta^{t-\theta} g(\zeta)f(t - \zeta)d\zeta \end{aligned}$$

Furthermore for fixed  $0 < t^* < \infty$ , we obtain

$$\begin{aligned} m(t^*) &= \int_0^{t^*} g(\zeta)f(t^* - \zeta)d\zeta = \int_0^\theta g(\zeta)f(t^* - \zeta)d\zeta + \int_{t^*-\theta}^{t^*} f(t^* - \zeta)g(\zeta)d\zeta \\ &\quad + \int_\theta^{t^*-\theta} g(\zeta)f(t - \zeta)d\zeta \\ &= \int_0^\theta g(\zeta)f(t^* - \zeta)d\zeta + \int_0^\theta f(x)g(t^* - x)dx + \int_\theta^{t^*-\theta} g(\zeta)f(t - \zeta)d\zeta \end{aligned}$$

Consider any given  $\varepsilon > 0$  and we let  $\theta \geq 0$  be sufficiently large and let  $C(\theta)$  be defined as

$$C(\theta) = \max \left\{ \max_{t^* + \frac{3\theta}{2} \leq x \leq t^* - \frac{3\theta}{2}} f(x), \sup_{t^* + \frac{3\theta}{2} \leq x \leq t^* - \frac{3\theta}{2}} g(x) \right\} < \infty$$

Given that  $C(\theta)$  is an increasing function as  $C$  is increasing. By considering

$$\begin{aligned} \zeta \in [0, \theta] &\Rightarrow t^* - \zeta \in [t^* - \theta, t^*] \subset \left[ t^* + \frac{3\theta}{2}, t^* - \frac{3\theta}{2} \right] \Rightarrow \int_0^\theta f(t^* - \zeta)g(\zeta)d\zeta \\ &\geq C \int_0^\theta g(\zeta)d\zeta. \end{aligned}$$

Then taking account to a point  $x \in [0, \theta] \Rightarrow t^* - x \in [t^* - \theta, t^*]$ , which implies that

$$\int_0^\theta f(x)g(t^* - x)dx = C \int_0^\theta f(x)dx.$$

Let  $|t - t^*| \leq \frac{\theta}{2}$ , then  $\zeta \in [0, \theta] \Rightarrow t - \zeta \in [t + \theta, t] \subset \left[ t^* + \frac{3\theta}{2}, t^* - \frac{3\theta}{2} \right]$ , we get

$$\int_0^\theta f(t - \zeta)g(\zeta)d\zeta \geq C \int_0^\theta g(\zeta)d\zeta.$$

and for  $x \in [0, \theta] \Rightarrow t - x \in [t - \theta, t] \subset \left[ t^* + \frac{3\theta}{2}, t^* - \frac{3\theta}{2} \right]$ , the result will be followed by

$$\int_0^\theta f(x)g(t - x)dx \geq C \int_0^\theta f(x)dx.$$

In same way suppose  $|t - t^*| \geq \frac{\theta}{2}$  then

$$\begin{aligned} |m(t) - m(t^*)| &\geq 2C \left[ \int_0^\theta g(\zeta)d\zeta + \int_0^\theta f(x)dx \right] \\ &\quad - \left| \int_\theta^{t-\theta} g(\zeta)f(t - \zeta)d\zeta + \int_\theta^{t^*-\theta} g(\zeta)f(t^* - \zeta)d\zeta \right| \\ &\geq 2C \left[ \int_0^\theta g(\zeta)d\zeta + \int_0^\theta f(\zeta)d\zeta \right] \\ &\quad + \left| \int_0^\infty f(t - \zeta)I_{[\theta, t-\theta]}(\zeta)g(\zeta)d\zeta + \int_0^\infty f(t^* - \zeta)I_{[\theta, t^*-\theta]}(\zeta)g(\zeta)d\zeta \right| \end{aligned} \tag{2}$$

We consider second term of the (R.H.S) in an Eq. (2) is estimated as

$$\begin{aligned} 0 \leq f(t - \zeta)I_{[\theta, t+\theta]}(\zeta) &\leq \max_{\theta \leq n \leq t - \frac{\theta}{2}} f(n) \leq \max_{\theta \leq n \leq t^* - \frac{\theta}{2}} f(n) = L(t^*, \theta) > \infty \\ \int_0^\infty L(t^*, \theta)g(\zeta)d\zeta &= L(t^*, \theta) \int_0^\infty g(\zeta)d\zeta = L(t^*, \theta) > \infty. \end{aligned}$$

Given that

$$L(t^*, \theta)g(\zeta) \subset L^1(0, \infty), \text{ then } \forall L, \exists n : L'(t^*, \theta) \rightarrow \exists L' : L \rightarrow L' \quad \forall n$$

Therefore

$$f(t - \zeta)I_{[\theta, t+\theta]}(\zeta) \rightarrow f(t^* - \zeta)I_{[t^*+\theta]}(\zeta), \forall \zeta \geq 0, t \rightarrow t^*$$

Therefore by using convergence theorem, we achieve the following

$$\lim_{t \rightarrow t^*} \left| \int_\theta^{t-\theta} g(\zeta)f(t - \zeta)d\zeta + \int_\theta^{t^*-\theta} g(\zeta)f(t^* - \zeta)d\zeta \right| = 0 \tag{3}$$

Solving the equations obtained above in (2) and (3) we will get the following result

$$\overline{\lim}_{t \rightarrow t^*} |m(t) - m(t^*)| \geq 2C(\theta) \left[ \int_0^\theta g(\zeta)d\zeta + \int_0^\theta f(x)dx \right] \tag{4}$$

Hence from the Eq. (4) as  $\theta \rightarrow 0$ , we obtain

$$\overline{\lim}_{t \rightarrow t^*} |m(t) - m(t^*)| = 0.$$

This complete proof of Lemma 2.

**Theorem 1** Given the instant system availability  $S_\tau(t)$  determined by

$$S_\tau(t) = \bar{F}(t) + \int_0^t S_\tau(t - \varsigma) dM(\varsigma)$$

is a unique bounded solution, such that

$$\bar{F}(t) = 1 - F(t) = \sum_{i=1}^k [K_i(t) - (K_i * F_{i+1}(\cdot + \tau))(t)].$$

*Proof* The proofs of this Theorem is based on repairable system which plays a great significance role in instant system availability.

We let

$$\zeta = \sum_{i=1}^{k+1} [L_i + R_i].$$

It is assumed that  $\zeta$  has a distribution function  $K_{k+1}$ .

Given that  $t > 0$ , the considered event

$$[X(v) = 1, t \leq v \leq t + \tau, \zeta \leq t].$$

We get

$$S_\tau(t) = P[X(v) = 1, t \leq v \leq t + \tau, \zeta \leq t] + P[X(v) = 1, t \leq v \leq t + \tau, \zeta > t]. \tag{5}$$

Therefore using the probability assumptions, we get

$$P[X(v) = 1, t \leq v \leq t + \tau, \zeta \leq t] = \int_0^t S_\tau(t - \varsigma) dK_{k+1}(\varsigma) \tag{6}$$

From the Eq. (6), the second term defined by

$$[X(v) = 1, t \leq v \leq t + \tau, \zeta > t]$$

are his event, and it is disjoint units given by

$$= [t + \tau \leq L_1] \cup \left[ \cup_{i=1}^k \left( \sum_{j=1}^i (L_j + R_j) \leq t, t + \tau \leq \sum_{j=1}^i (L_j + R_j) + L_{i+1} \right) \right].$$

We set  $[(t, t + \tau) = t \geq \zeta]$  and  $t = \zeta + \eta$ .

Therefore, we obtain the new repairable system which is the characteristic point of this Theorem

$$(t - \zeta, t - \zeta + \tau),$$

that will correspond to the considered event

$$[X(\eta) = 1, t \leq \eta \leq t + \tau],$$

and then the new probability related is defined here as

$$\begin{aligned}
 P[X(\eta) = 1, t - \zeta \leq \eta \leq t - \zeta + \tau, \zeta \leq t] &= P[X^*(\eta) = 1, t - \zeta \leq \eta \leq t + \tau - \zeta; \zeta \leq t] \\
 &\Leftrightarrow P\left[t + \tau \leq L_1 + \sum_{i=1}^k \left[ \sum_{j=1}^i (L_j + R_j) \leq t, t + \tau \leq \sum_{j=1}^i (L_i + R_i) + L_{i+1} \right]\right] \\
 &= P[t + \tau \leq L_1] + \sum_{i=1}^k \left[ P\left( \sum_{j=1}^i (L_j + R_j) \leq t, t + \tau \leq \sum_{j=1}^i (L_i + R_i) + L_{i+1} \right) \right] \tag{7}
 \end{aligned}$$

The following condition is considered to derive the above expression in (7).

We assume that

$$P(L_i \geq \tau) \leq 1.$$

Then the above equation has the required solution to (8) according to the assumption given above,

$$= \bar{F}_1(t + \tau) + \sum_{i=1}^k [K_i(t) - (K_i * F_{i+1}(\cdot + \tau))(t)]. \tag{8}$$

Therefore from Eqs. (8) and (9) we get

$$P[X(v) = 1, t \leq v \leq t + \tau, \zeta > t] = \bar{F}_1(t + \tau) + \sum_{i=1}^k [K_i(t) - (K_i * F_{i+1}(t + \tau))(t)]. \tag{9}$$

Hence from the Eqs. (5), (6) and (9), we complete our proof.

**Corollary 1** Assume the following conditions are satisfied:

- (i) Give a continuous functions  $f(t)$  and  $g(t)$  defined on interval  $(0, T]$  where zero will not be taken into consideration.
- (ii) If  $T > 0$  for  $T \equiv \inf\{t > 0 : f(t) = m(t)\}$ , then  $f(t) > m(t), \forall t \in (0, T]$ .
- (iii) If  $\tau > 0$ , then  $S_\tau(t) > f(t)$  given that  $\forall \omega \in (0, t - \zeta]$ .

where  $t - \zeta$  is always nominated as new repairable time of the system availability.

**Corollary 2** Assume that  $F \geq G$  defined in failure rate order. Then we get the following inequality,

$$S_F(\tau, \delta) \geq S_G(\tau, \delta), \quad \forall \tau \geq 0, \delta \geq 0.$$

where  $S_F(\tau, \delta)$  and  $S_G(\tau, \delta)$  are defined as the stable interval of system availabilities with lifetime distributions  $F$  and  $G$  respectively.

*Proof* By using the properties of  $F \geq G$  defined in failure rate order, this implies that

$$F_e \geq G_e$$

in stochastically order or simply in stochastically greater than, where  $F_e$  and  $G_e$  are known as the equilibrium distribution function defined on  $F$  and  $G$  respectively. Unfortunately the case of stable point availability  $F \geq G$  in stochastically order is not confirmed. Therefore, in general and necessarily, this implies that

$$S_F(\tau, \delta) \geq S_G(\tau, \delta), \quad \forall \tau \geq 0, \delta \geq 0.$$

Hence, the Corollary is proved.

**Theorem 2** *Let  $F(t)$  and  $G(t)$  satisfies the conditions given in corollary 1 in such that  $\forall t \in [0, T]$ , then  $m(t) = (f * g)(t) \leq f(t)$ . Hence  $S_\tau(t) \geq m(t) \geq f(t)$ .*

*Therefore  $S_\tau(t)$  is decreasing from the interval  $t \in [0, T]$ .*

*Proof* We let  $S_\tau(t)$  being the unique solution to the Eq. (1) above,  $S_\tau(t)$  is derived as the following

$$S_\tau(t) = S'_\tau(t)$$

where  $S_\tau(t)$  is taken as a solution to

$$S_\tau(t) = [m(t) - f(t)] + \int_0^t S_\tau(t - \zeta)h(\zeta)d\zeta.$$

Therefore

$$S_\tau(t) = [m(t - \zeta) - f(t - \zeta)] + \int_0^{t-\zeta} S_\tau(t - \zeta - \varsigma)h(\varsigma)d\varsigma. \tag{10}$$

Note from our assumption

$$\delta(t) = m(t - \zeta) - f(t - \zeta) \leq 1, \quad \forall t \in [0, T],$$

we have

$$N(t) = \sum_{i=0}^{\infty} K^{k+1}(t), \quad \text{where } k + 1 = n$$

as  $t = n$ , having convolution functions as  $K(t)$  for  $\zeta \leq t$ .

Then it is followed that

$$\delta(t) \leq M(t) = (F * G)(t).$$

Therefore, the following inequality is obtained

$$m(t) \leq S_\tau(t).$$

We conclude that, Eq. (10) above is defined by the repairable system which shows that  $S_\tau(t)$  is decreasing function of

$$M(t) = (F * G)(t), \quad \forall t \in [0, T].$$

Hence, theorem is proved.



**Bounded for availability system**

In this section, we provide some instant availability of  $S_\tau(t)$  with respect to the upper bound  $\bar{F}(t)$ . This can be applied to the Eq. (1) according to the upper bound definition of reliability system for instant system availability  $S_\tau(t)$ .

**Theorem 3** *Let  $f(t)$  and  $g(t)$  be probability density function having (C.D.F) defined by  $F(t)$ ,  $G(t)$  respectively. If  $(T_i > 0)$  where  $i = 0, 1, 2, \dots, n$ .*

*Then  $m(t) > f(t), \forall t \in [0, T]$ . For any  $t \in [0, T]$ , we obtain*

$$S_\tau(t) \leq \frac{\bar{F}(t)}{\bar{M}(t)}. \tag{11}$$

*Proof* Given that  $t \in [0, T]$  as from the Eq. (1), we get the relation

$$S_\tau(t) = \bar{F}(t) + \int_0^{t-\zeta} S_\tau(t - \zeta) d\bar{M}(\zeta) \leq \bar{F}(t) + \int_0^{t-\zeta} S_\tau(t) d\bar{M}(\zeta) = \bar{F}(t) + S_\tau(t)M(t).$$

For  $\zeta \leq t$  and from the Corollary 1, it follows from Theorem 2 results that, the inequality above satisfies an upper bound on the  $S_\tau(t)$ .

Therefore the bound with  $S_\tau(t)$  lies on the interval  $[0, T_i]$  where  $[i = 0, 1, \dots, n]$ ,  $\forall T_i > 0$ .

Hence, the Theorem is proved.

**Theorem 4** *Suppose that  $T_i \geq 0$ . From the relation (11), the following inequality is satisfied.*

$$S_\tau(t) \geq \max_{0 \leq x \leq T_i} \frac{\bar{F}(x)}{\bar{M}(x)}, \quad \forall 0 \leq t \leq T_i$$

*Proof*  $S_\tau(t)$  is a solution to the given equation defined in (1).

Let us consider

$$Z(t) = C = \max_{0 \leq x \leq T_i} \frac{\bar{F}(x)}{\bar{M}(x)}.$$

This result comes from

$$Z(t) = C\bar{M}(t) + \int_0^{t-\zeta} Z(t - \zeta) dM(\zeta), \quad \forall P[L_i \leq Z(t)] \leq 1.$$

We present  $S_\tau(t) - C = \delta(t)$  as unique solution to

$$\delta(t) = [\bar{F}(t) - C\bar{M}(t)] + \int_0^{t-\zeta} \delta(t - \zeta) dM(\zeta), \quad \forall t - \zeta \geq 0.$$

Give that  $\bar{F}(t) - C\bar{M}(t) \geq 0, \forall 0 \leq t \leq T_i$ .

Hence,  $\delta(t) \geq 0$ , and the conclusion is that, the instant system availability  $S_\tau(t)$  decrease over time  $t$ .

*Remark* We note that, the method used in the analysis of instant system availability, can be useful in the study of the measure-valued Markov process involved in the stochastic logistic model. which can be seen in Shang (2013)

**Comparison of availability system**

Here we develop the availability of two state systems nominated as  $E$  and  $E^*$  as the life-time as well as the repair time of  $E^*$  are denoted as the contradict of  $E$ . Different articles have provided this comparison for example in Cha et al. (2004) the measures are compared based on rate orderings has been failed , repairs, repair policy ordering and classifications of life distribution. It is also studied in Mi (1998), Biswas and Sarkar (2000) where the system availability is compared with respect to the perfect repair policy. In our case, the comparison of two systems is made by using the proof of the Theorems below in (5) and (6) with consideration of repair time that have been proposed in Theorem 1 of this article.

**Theorem 5** Given that  $L + R = L^* + R^*$  as  $L \leq L^*$ , this is  $S_\tau(t) \leq S_\tau^*(t), \forall t \geq 0$ .

*Proof* From a small change says  $\tau$  where  $L + R = L^* + R^*$ , we obtain

$$M(t) = M^*(t), \quad \forall t \geq 0.$$

It implies that,

$$S_\tau(t) = \bar{F}(t) + \int_0^{t-\zeta} \bar{F}(t - \zeta) dN(\zeta) \tag{12}$$

for  $N(\zeta) = \sum_{j=1}^\infty M^{(j)}(\zeta)$ , where  $M^{(j)}$  is defined as the  $k$ -fold convolution of  $M$ .

In the same way as the above equation in (12) we get

$$S_\tau^*(t) = \bar{F}^*(t) + \int_0^{t-\zeta} \bar{F}^*(t - \zeta) dN^*(\zeta). \tag{13}$$

Thus

$$N^*(\zeta) = \sum_{j=1}^\infty M^{*(j)}(\zeta) = \sum_{j=1}^\infty M^{(j)}(\zeta) = N(\zeta).$$

Then from the Eqs. (13) and (14), we get

$$S_\tau^*(t) = \bar{F}^*(t) + \int_0^{t-\zeta} \bar{F}^*(t - \zeta) dN^*(\zeta) \geq \bar{F}(t) + \int_0^{t-\zeta} \bar{F}(t - \zeta) dN(\zeta) = S_\tau(t).$$

Therefore  $L \leq L^*$ , which indicates that  $\bar{F}(t) \leq \bar{F}^*(t) \quad \forall t \geq 0$  since  $S_\tau^* \geq S_\tau(t)$ .

**Theorem 6** Suppose that for some  $T \geq 0$ . We have

$$\bar{F}(t) \leq \bar{F}^*(t), \quad \forall T \in [0, t],$$

such that the convolution system

$$m(t) \leq m^*(t).$$

And for any constant  $b$  we have

$$|m(t)| \leq b \in [0, t_0].$$

Therefore

$$\int_0^{t-\zeta} |\bar{F}(t)| dt \leq \infty.$$

Hence

$$S_\tau(t) \leq S_\tau^*(t), \quad \forall t \in [0, T].$$

*Proof* To prove this Theorem, we have referred to the different theory related to the existence and uniqueness solution of an integral equation founded in Bellman and Cooke (1963) as defined in above relation (1). That is,

$$S_\tau(t) = \bar{F}(t) + \int_0^{t-\zeta} m(\zeta)\bar{F}(t-\zeta)dN(\zeta), \quad \forall 0 \leq t \leq T.$$

Therefore

$$\begin{aligned} |m(t)| &\leq b + \int_0^{t-\zeta} |m(\zeta)||\bar{F}(t-\zeta)|d\zeta \\ &\leq b + m(t) \int_0^{t-\zeta} |m(\zeta)|d\zeta. \end{aligned}$$

It is given that, the below exponential expression is taken into account,

$$\int_0^{t-\zeta} |\bar{F}(\zeta)|d\zeta \leq b \exp \left[ \int_0^{t-\zeta} h(\zeta)d\zeta \right] \int_0^{t-\zeta} \exp \left[ - \int_0^{t-\zeta} m(\zeta)d\zeta \right] d\zeta.$$

If  $m(t)$  is bounded, then the solution exist and we get  $m(t) = m^*(t)$ .

For a small change of time  $\tau \in [0, t]$ , we get

$$S_0(t) = \bar{F}(t), \dots, S_i(t) = \bar{F}(t) + \int_0^{t-\zeta} m(\zeta)S_{i-1}(t-\zeta)d\zeta$$

for all  $1 \leq i \leq n + 1$ .

We use the successive approximation to the instant system availability and then we define

$$\begin{aligned} S_0(t) &= m(t), \\ S_1(t) &= m(t) + \int_0^{t-\zeta} m(\zeta)S_0(t-\zeta)d\zeta, \\ S_2(t) &= m(t) + \int_0^{t-\zeta} m(\zeta)S_1(t-\zeta)d\zeta, \\ &\vdots \\ S_{n+1}(t) &= m(t) + \int_0^{t-\zeta} S_n(t-\zeta)\zeta(\zeta)d\zeta. \end{aligned}$$

Proceeding in similar manner, we can write the expression as follows

$$S_{n+1}(t) - S_n(t) = \int_0^{t-\zeta} [S_n(\zeta) - S_{n-1}(\zeta)]m(t - \zeta)d\zeta.$$

Then by using the inductive method, we get the following relations

$$\begin{aligned} |S_1(t) - S_0(t)| &\leq bt_0m^*(t) \\ |S_2(t) - S_1(t)| &\leq bt_0m^*(t) \int_0^{t-\zeta} h^*(s)d\zeta \\ |S_2(t) - S_1(t)| &\leq bt_0m^*(t) \int_0^{t-\zeta} m^*(\zeta) \left[ \int_0^{t-\zeta} m^*(s_1)ds_1 \right] d\zeta \\ &\leq bt_0 \frac{m^*(t)}{2!} \left[ \int_0^{t-\zeta} m^*(\zeta)d\zeta \right]^2 \\ &\vdots \\ |S_{n+1}(t) - S_n(t)| &\leq bt_0 \frac{m^*(t)}{n!} \left[ \int_0^{t-\zeta} m^*(\zeta)d\zeta \right]^n. \end{aligned} \tag{14}$$

Therefore, from the above expression (14), we get the following

$$S_\tau(t) = \lim_{n \rightarrow \infty} S_n(t) \text{ and } S_\tau^*(t) = \lim_{n \rightarrow \infty} S_n^*(t), \quad \forall t \in [0, T].$$

Then by induction method we have

$$S_n(t) \leq S_n^*(t), \quad \forall t \in [0, T]$$

Therefore, we get

$$S_\tau(t) = \lim_{n \rightarrow \infty} S_n(t) \leq \lim_{n \rightarrow \infty} S_n^*(t) = S_\tau^*(t). \tag{15}$$

Which implies that

$$S_\tau(t) = S_\tau^*(t).$$

Hence, the Theorem is proved.

### Numerical examples for clarification

In this section, we use some examples to find a solution to the Eq. (1) by testing three distinct types of the distribution function, such as Gamma, Log-normal and Weibull distributions, where we make a combination of two between them. Related studies have been done in Mishra and Jain (2013) where they have obtained the four individual cases of the repair distribution namely exponential, Gamma, Weibull and Pareto have been examined numerically for the illustration purposes, we have been referred by so many authors talked about illustration of numerical examples specifically in Huang and Mi (2013, 2015) where the various examples taken, have shown the high accuracy and efficiency. From those experiences, we can determine the value of  $t = T$  defined in Theorem 2 and each figure associated with these examples. All these figures clearly

indicate that the interval determined by Theorem 2, shows that  $S_{\tau}(t)$  decreases and it is also very sharp.

*Example 1* Suppose that  $L \sim \ln N(\mu, \sigma)$  and  $R \sim Weibull(a, \lambda)$  be defined by density functions respectively given by

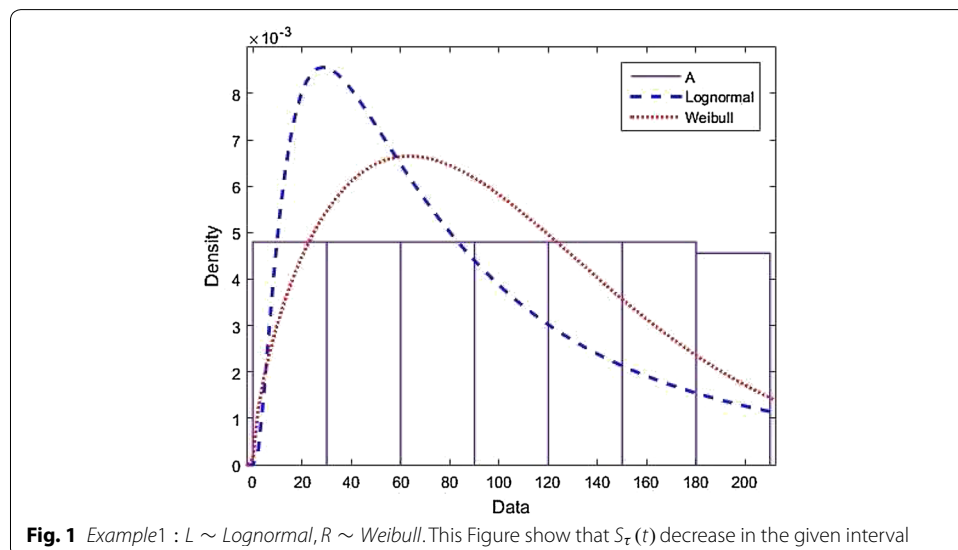
$$f(t) = \frac{1}{t\sigma\sqrt{2\pi}} e^{-\frac{(\ln t - \mu)^2}{2\sigma^2}} \quad \text{and} \quad g(t) = \frac{a}{\lambda} \left(\frac{x}{\lambda}\right)^{a-1} e^{-\left(\frac{x}{\lambda}\right)^a}.$$

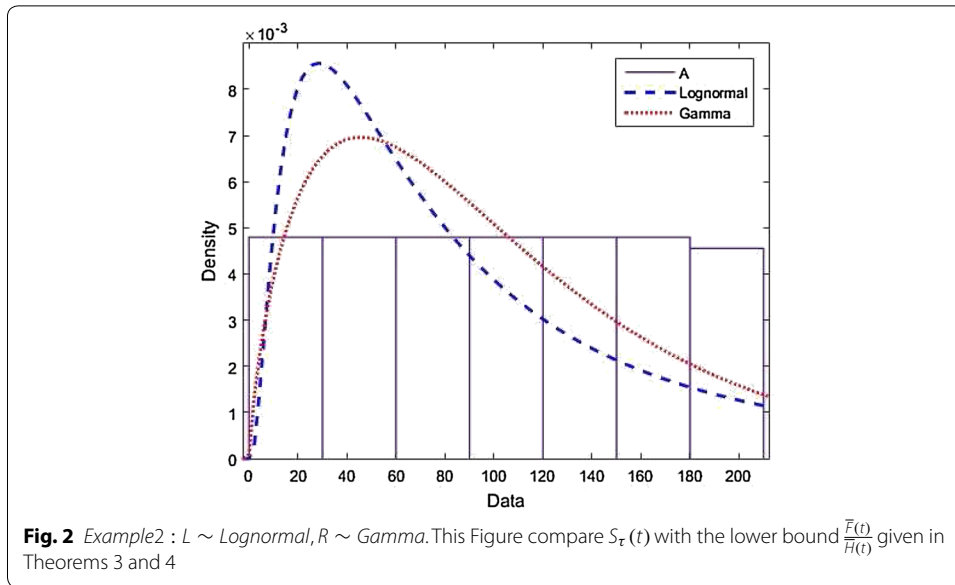
We choose  $\mu = 4$ ,  $\tau = 0.25$  and  $a = 115$ ,  $\lambda = 1.62$  in this example. In Fig. 1 below, the curve of  $S_{\tau}(t)$  is obtained from the numerical solution is shown. The given example demonstrates that the result of Theorem 2 is very sharp fall. In fact, in this example the value of  $T$  is between 58.60 and 58.61, while the value of  $T_0$  from the same Theorem are between 58.56 and 58.60 which is just slightly smaller than  $T$ , this also shows that  $S_{\tau}(t)$  decreases in the given interval above.

*Example 2* Let  $L \sim \ln N(p, \alpha)$  and  $R \sim Gamma(\mu, \sigma)$ , by choosing  $p = 1.7$ ,  $\alpha = 58$ ,  $\mu = 4$  and  $\sigma = 0.25$ , we compare  $S_{\tau}(t)$  with the lower bound  $\frac{\bar{F}(t)}{\bar{H}(t)}$  given in Theorems 3 and 4. The functions  $S_{\tau}(t)$  and  $\frac{\bar{F}(t)}{\bar{H}(t)}$  is plotted in Fig. 2 below. When  $t < T = 58.6$ , they are looking almost same.

*Example 3* Given that the lifetime  $L$  and repair time  $R$  have cumulative distribution functions  $F$  and  $G$  respectively. We extend the availability of the system as  $S_F(t)$  in order to make an accent it dependance on distribution functions  $F$ . It is well interested to state that  $S_F(t)$  would be expended on high value if the lifetime  $L \sim Gamma(p, \alpha)$  and  $R \sim Weibull(a, \lambda)$  with having density function respectively given by

$$f(t) = \frac{\alpha^p}{\sqrt{p}} e^{-\alpha t} t^{p-1} \quad \text{and} \quad g(t) = \frac{a}{\lambda} \left(\frac{x}{\lambda}\right)^{a-1} e^{-\left(\frac{x}{\lambda}\right)^a}.$$

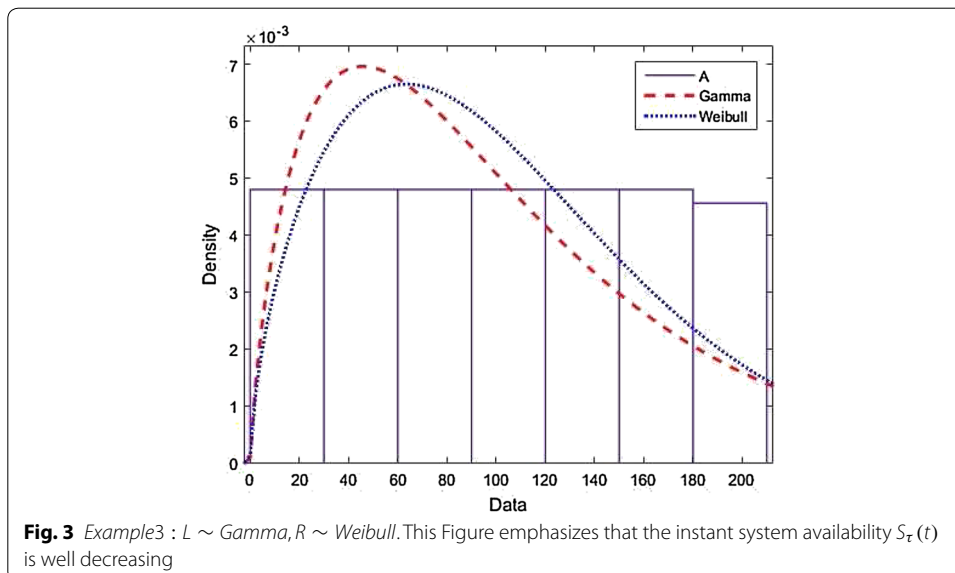




From the Fig. 3 below, it is observed that the two curves of system availabilities present the corresponding value which is  $p = 1.7$  and  $p = 3.4$  cross each other at least two times even if  $\text{Gamma}(1.7, \alpha) \leq \text{Gamma}(3.4, \alpha)$ , this emphasizes that the instant system availability  $S_{\tau}(t)$  is well decreasing.

### Application to bathtub

From the different literature review, many different schemes and practical examples, the estimation was carried out by assuming that, the sequence of failure and repair time are two independent sequences of *i.i.d* random variables. The details applications of this article have been found in Rausand and Høyland (2004), Barlow and Hunter (1960) Wong et al. (1999).



However, the given assumption need not hold in many situations while the repair times may depend on the previous failure time due to the influence of the operation environment of the system is applied.

This simple article has found the applications for a variety of some fields such as Reliability Engineering, Industrial economic systems, Maintenance management system, mechanical engineering. And it is may specially useful application in Measure-valued Markov process involved in the stochastic logistic model in Shang (2013) as noticed above in bounded availability system section.

In this part of application, we assume that the lifetime distribution of the instant system availability has bathtub-shaped failure rate function of  $t_1 \leq t_2$  as its change points, this has been referred in Mi (1995b, 1998).

Frequently we can apply a burn-in procedure to improve the availability of the instant system, and it has been observed that the burn-in times that optimise different availability or reliability characteristics never go beyond to the first change point  $t_1$ .

For more details of bathtub-shaped failure rate function and optimal burn-in times are discussed here (Lai and Xie 2006; Block et al. 1994). From the definition of bathtub-shaped failure rate distribution function said that, a failure rate distribution function  $r(t)$  is said to have a bathtub-shaped with change point  $t_1 \leq t_2$  if  $r(t)$  is strictly decreases in  $t \in (0, t_1)$ , is a constant on  $(t_1, t_2)$  and strictly increasing to  $t \in (t_2, \infty)$ . Using this definition together with the combination of the above previous comparison and examples results sections respectively, the following Theorem 7 below is a useful application of this article.

**Theorem 7** *Given that the lifetime distribution function  $F$  of a system has a bathtub-shaped failure rate function  $r(t)$  with its change point  $t_1 \leq t_2$ . Assume that  $F_b, \bar{F}_b$  where  $\bar{F} = 1 - F$  is the distribution and survival function of the instant system surviving burn-in time  $b$ . This means that*

$$\bar{F}_b(t) = \frac{\bar{F}(b + t)}{\bar{F}(b)}, \quad \forall t \geq 0.$$

Therefore for each of the following maximization problems, the optimal burn-in can not exceed  $t_1$  such as:

- (i)  $\max_{b \geq 0} S_b(\tau, \delta) = \max_{b \geq 0} (\frac{\mu(b)}{\mu(b) + \delta})$ , where  $\mu(b) = \int_0^\infty \bar{F}_b(t) dt$
- (ii)  $\max_{b \geq 0} S_b(\tau, \delta) = \max_{b \geq 0} (\frac{1}{\mu(b) + \delta}) \int_\tau^\infty \bar{F}_b(t) dt$  for any  $\tau \geq 0$  and  $\delta \geq 0$ .
- (iii)  $\max_{b \geq 0} \bar{S}_b = \max_{b \geq 0} ((\int_0^\infty \bar{F}_{bp}(t) dt + \mu_X \int_0^\infty q(t) \bar{F}_{bp}(t) dt + \mu_Y))$  where  $\bar{F}_{bp}$  is defined as the following situation

$$F_{bp}(t) = \exp \left\{ - \int_0^t p(u) r_b(u) du \right\},$$

$$r_b(t) = r(b + t),$$

this is known as the failure rate function of  $F_b(t)$  and where  $0 \leq p(t) < 1$  is assumed as an increasing function satisfying

$$\int_0^\infty p(t) r(t) dt = \infty.$$

*Proof* The proof of (i). Let us express the burn-in time which maximizes the mean residual lifetime  $\mu_b$  by  $b^*$ . Therefore it is positively observed that,

$$b^* \leq t_1$$

is required. From this and the fact that the following given expression of

$$\frac{\zeta}{(\zeta + \delta)}$$

is known as a strict increasing function of  $\zeta$ , this immediately implies the given proof of (i).

The proof of (ii) is obtained as the following. By the assumptions and properties that the failure rate  $r(t)$  is a bathtub-shaped and having  $t_1$  as its first change point as defined above, it implies the following inequality defined as

$$r_{t_1}(t) \leq r_b(t), \quad \forall b \geq t_1, \quad \forall t \geq 0.$$

Refer to the given above Corollary 2, we conclude that

$$F_{t_1} \geq F_b, \quad \forall b \geq t_1$$

is satisfied.

Hence

$$S_{F_{t_1}}(\tau, \delta) > S_{F_b}(\tau, \delta), \quad \forall \tau > 0, \quad \forall \delta \geq 0, \quad b > t_1$$

is well verified.

Therefore, for any given parameters  $\tau > 0$  and  $\delta > 0$ , the function of the burn-in times  $b$ ,  $S_b(\tau, \delta)$  is always tested as continuous on the given interval of  $[0, t_1]$ .

Also, the burn-in times  $b^*$  satisfies

$$S_{b^*}(\tau, \delta) = \max_{0 \leq b \leq t_1} S_b(\tau, \delta) = \max_{b \geq 0} S_b(\tau, \delta).$$

Which is entirely less than  $t_1$ . Hence, the relation (ii) is proved.

Refer to the above proof of the relation (ii) given in Theorem 7, the proof of (iii) in the same Theorem 7 is obtained by using the same method. Therefore the relation (iii) is true.

Hence, the Theorem is proved.

Briefly, many statistical ageing concepts have been applied to describe the instant system availability component ages with time, it have observed that with the probability sense, the lifetime of the scheme component tends to decrease accordingly. By the above example of the application to the bathtub, this have also been covered by the monotonicity condition as well as numerical examples section respectively discussed above in this article has shown that the instant system availability  $S_\tau(t)$  decrease.

Consequently, the given results provided above by this article are adjusted and justified.



## Conclusion

Instant System availability denoted as  $S_{\tau}(t)$  of a repairable system is considered as the relevant estimate measure of its system performance. It is the case that  $S_{\tau}(t)$  does not have closed form of expression of the literature attention focuses on the steady-state availability  $S_{\tau}(\infty)$ . Different from the most literature, this article provides analysis of various properties of the instant system availability  $S_{\tau}(t)$  as a function of time instead of the steady state availability. An interval where  $S_{\tau}(t)$  monotonically decreases is obtained, upper bound to  $S_{\tau}(t)$  is derived, both of them have been well clarified in the numerical examples and some comparison results regarding availabilities of different systems are proved. Several examples which validate the analytically derived properties of  $S_{\tau}(t)$  are also shown above. The details applications of this article are also strongly discussed in "Application to bathtub" section. We have studied the instant system availability by using the univariate (*i.i.d*) sequence, from this if the failure and repair times form a bivariate (*i.i.d*) sequence, the estimation of availability measures, point availability, average availability and interval reliability is interesting research issue which can be taken to be addressed. Further more, the inspection is very essential for keeping away a system components from damage. Hence different conditions based on maintenance will be taken as a suitable to investigate in future research. Certainly some given possible generalization could be explored for instance many questions regarding to the mechanism of choosing a complete or a minimal repair are still open and further research can be considered as principal.

## Authors' contributions

SL conceived, designed and provided guidance to the development of the study; EH reviewed literature, wrote proofs, drafted the manuscript and made revisions; PK made substantial contribution to the structural development of the manuscript, discussion of results and revisions. All authors read and approved the final manuscript.

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## Competing interests

The authors declare that they have no competing interests.

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## References

- Barlow RE, Hunter LC (1960) System efficiency and reliability. *Technometrics* 2(1):43–53
- Baxter L (1981) A availability measures for a two-state system. *J Appl Prob* 18(1):227–235
- Baxter LA (1983) Availability measures for coherent systems of separately maintained components. *J Appl Prob* 28(2):627–636
- Baxter LA (1981) A two-state system with partial availability in the failed state. *Nav Res Logist Quart* 28(2):231–236
- Bellman RE, Cooke KL (1963) *Differential-difference equations*. Academic Press, New York
- Bieth B, Hong L, Sarkar J (2010) A standby system with two repair persons under arbitrary life-and repair times. *Math Comput Modell* 51(5):756–767
- Biswas A, Sarkar J (2000) Availability of a system maintained through several imperfect repairs before a replacement or a perfect repair. *Stat Probab Lett* 50(2):105–114
- Block HW, Mi J, Savits TH (1994) Some results on burn-in. *Stat Sin* 4:525–533
- Cha JH, Kim JJ (2002) On the existence of the steady state availability of imperfect repair model. *Indian J Stat Series B* 64:76–81
- Cha JH, Lee S, Mi J (2004) Comparison of steady system availability with imperfect repair. *Appl Stoch Models Bus Ind* 20(1):27–36
- El-Damcese MA, Temraz NS (2010) Analysis of availability and reliability for repairable parallel systems with different failure rates. *Asian J Inform Technol* 9(4):231–237
- El-Damcese MA, Temraz NS (2012) Analysis for parallel repairable system with degradation facility. *Am J Math Stat* 2(4):70–74

- Huang K, Mi J (2013) Properties and computation of interval availability of system. *Stat Probab Lett* 83(5):1388–1396
- Huang K, Mi J (2015) Study of instant system availability. *Prob Eng Inf Sci* 29(01):117–129
- Lai CD, Xie M (2006) Bathtub shaped failure rate life distributions. *Stoch Age Depend Reliab* 1:71–107
- Mathew A (2014) System availability behavior of some stationary dependent sequences. *Stat Probab Lett* 84:17–21
- Mi J (1995a) Limiting behavior of some measures of system availability. *J Appl Probab* 32:482–493
- Mi J (1995b) Bathtub failure rate and upside-down bathtub mean residual life. *IEEE Trans Reliab* 44(3):388–391
- Mi J (1998) Some comparison results of system availability. *Nav Res Logist NRL* 45(2):205–218
- Mi J (2006) Limiting availability of system with non-identical lifetime distributions and non-identical repair time distributions. *Stat Probab Lett* 76(7):729–736
- Mishra A, Jain M (2013) Availability of k-out-of-n: f secondary subsystem with general repair time distribution. *Int J Eng Trans A Basics* 26(7):743
- Rausand M, Høyland A (2004) *System reliability theory: models, statistical methods, and applications*. Wiley, New York
- Sarkar J, Biswas A (2010) Availability of a one-unit system supported by several spares and repair facilities. *J Korean Stat Soc* 39(2):165–176
- Sarkar J, Chaudhuri G (1999) Availability of a system with gamma life and exponential repair time under a perfect repair. *Stat Probab Lett* 43(2):189–196
- Sarkar J, Li F (2006) Limiting average availability of a system supported by several spares and several repair facilities. *Stat Probab Lett* 76(18):1965–1974
- Shang Y (2013) The limit behavior of a stochastic logistic model with individual time-dependent rates. *J Math*. Article ID 502635
- Veber B, Nagode M, Fajdiga M (2008) Generalized renewal process for repairable systems based on finite Weibull mixture. *Reliab Eng Syst Saf* 93(10):1461–1472
- Wong Paul, Albrecht P, Allan R, Billinton Roy, Chen Qian, Fong C, Haddad Sandro, Li Wenyuan, Mukerji R, Patton Diane, others (1999) The IEEE reliability test system-1996. A report prepared by the reliability test system task force of the application of probability methods subcommittee. *Power Systems, IEEE Transactions on*. 14(3):1010–1020

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