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Evolution of dispersal and the analysis of a resource flourished population model with harvesting

Ishrat Zahan [a], Md. Kamrujjaman [b,*]

[a] Department of Mathematics, Bangladesh University of Engineering & Technology, Dhaka 1000, Bangladesh
[b] Department of Mathematics, University of Dhaka, Dhaka 1000, Bangladesh

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ABSTRACT

This study explores a spatially distributed harvesting model that signifies the outcome of the competition of two species in a heterogeneous environment. The model is controlled by reaction-diffusion equations with resource-based diffusion strategies. Two different situations are maintained by the harvesting effects: when the harvesting rates are independent in space and do not exceed the intrinsic growth rate; and when they are proportional to the time-independent intrinsic growth rate. In particular, the competition between both species differs only by their corresponding migration strategy and harvesting intensity. We have computed the main results for the global existence of solutions that represent either coexistence or competitive exclusion of two competing species depending on the harvesting levels and different imposed diffusion strategies. We also established some estimates on harvesting efforts for which coexistence is apparent. Also, some numerical results are exhibited in one and two spatial dimensions, which shed some light on the ecological implementation of the model.

1. Introduction

In population ecology, the most crucial point in species management with optimal resource distribution is the effects of harvesting, which is the common cause of species extinction. In the pursuit of sustainable resource management, it is crucial to prevent population extinction resulting from harvesting activities, a topic explored by Ainseba et al. in their study [1]. In an environmental approach, one of the most significant concerns in population dynamics is the effect of harvesting. Harvesting indicates reducing the population size due to hunting, fishing, or capturing, which shrinks the population density. The study of harvesting for one population was limited in [2–4], and in some situations, these are unable to explain the actual situation better. By taking into account resource-based diffusion for single species populations with the Gilpin-Ayala growth model and harvesting are explored by Zahan et al. in [5]. In this study, they identified conditions on the harvesting rate both for trivial and non-trivial equilibrium states and came to the conclusion that for small values of the Gilpin-Ayala parameter when enhanced effects of diffusion disallow the existence of non-trivial states even in some circumstances where the intrinsic growth rate exceeds harvesting at some locations in space where a logistic model permits a non-zero equilibrium density. However, X. Q. He and W. M. Ni in [6,7] used the classical Lotka-Volterra competition system to demonstrate how dispersal and spatial variations affect competition outcomes. In the first part of their study, they assumed the total resources were fixed. Their research revealed that a heterogeneous distribution of resources tends to be more efficacious compared

* Corresponding author.
E-mail addresses: ishratzahan@math.buet.ac.bd (I. Zahan), kamrujjaman@du.ac.bd (M. Kamrujjaman).

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to a homogeneous distribution, particularly in scenarios involving diffusion. They further expanded their investigation to encompass cases where both species exhibit heterogeneous carrying capacities, while the overall resource availability remains constant. In the second part of their study, they investigated a wider range of situations involving different strengths and distributions of resources and varying abilities of competition [8]. In this third part of their paper series, the researchers explore the combined impact of diffusion and spatial concentration on the overall dynamics of the classical Lotka-Volterra competition-diffusion system.

Additionally, the main objective of studying harvesting effects is to identify and establish the maximum sustainable policy that can be implemented in the long run while keeping the population size under control. Even more interesting situations arise when harvesting is applied to two or more interacting population dynamics [9–14] that represent either coexistence or competitive exclusion by others. In contrast, when both populations are over-exploited, it leads to overall extinction.

However, when both species compete for similar resources, and one or both species are harvested, it is essential to know what conditions or measurements on harvesting levels are required for surviving populations in competition. Harvesting efforts and diffusion strategies play an important role in sustaining the population in competition. Every interacting species follows an individual dispersal strategy that provides an evolutionary advantage or difficulty at a prescribed harvesting level. This can be obtained by exploring stability properties and harvesting levels on model solutions that give us estimates to promote coexistence or competitive exclusion of interacting species. However, while using the random diffusion technique, species attempt to relocate to the region with fewer resources available. In natural settings, that is not plausible. We have considered a resource-based diffusion method for both competing species to get around this problem. According to this diffusion strategy developed by E. Braverman and M. Kamrujjaman in 2016 [15,16], the organism's diffusive transport is considered proportional to the gradient of population density per unit of resources. In certain scenarios, notable disparities in resource distribution exist, rendering the standard diffusion model less accurate. In classical diffusion, organisms typically migrate from areas abundant in resources to those with limited resources. However, in random diffusion, the migration transport is proportional to the gradient of population density, resulting in migration patterns that are symmetric about the peak. This is not realistic when compared to field observations. Additionally, resources are often limited and environmental conditions are not always optimal. Factors such as food and water supply, climate change, space, mates, and habitat can limit population growth and make it resistant to environmental conditions. Along the lines of the above-mentioned observation, we have included an alternative type of diffusion strategy known as the resource-based diffusion strategy, where the diffusive transport of population is considered proportional to the gradient of population density per unit resource instead of just the population density. This means all organisms will diffuse according to the availability of maximum abundance, which is more realistic in nature. That is, we have tried to capture the reality that is observed in nature.

In the present study, we consider the model of a competitive system of coupled species that is isolated and spatially distributed in a heterogeneous environment. However, the species are being harvested with numerous harvesting exertions with similar resource-based diffusion strategies while competing for similar primary resources. It is also considered that both species' diffusion strategies are stipulated towards two positive distribution functions with different proportions of carrying capacities. Both are contemplating the same logistic type of growth function. Under these assumptions, the corresponding competition model with no-flux boundary conditions and non-trivial favorable initial conditions is defined.

$$
\begin{cases}
\frac{\partial u}{\partial t} = d_1 \Delta\left(\frac{u(t,x)}{M(x)}\right) + r(x)u(t,x)\left(1 - \frac{u(t,x)+v(t,x)}{K_u(x)}\right) - H_1(x)u(t,x),\ t>0,\ x\in\Omega,\\
\frac{\partial v}{\partial t} = d_2 \Delta\left(\frac{v(t,x)}{N(x)}\right) + r(x)v(t,x)\left(1 - \frac{v(t,x)+u(t,x)}{K_v(x)}\right) - H_2(x)v(t,x),\ t>0,\ x\in\Omega,\\
\mathbf{n}\cdot\nabla\left(\frac{u(t,x)}{M(x)}\right) = \mathbf{n}\cdot\nabla\left(\frac{v(t,x)}{N(x)}\right) = 0,\ x\in\partial\Omega,\\
u(0,x) = u_0(x),\ \ v(0,x) = v_0(x),\ x\in\Omega.
\end{cases}
\tag{1.1}
$$

Here, $u(t,x)$ and $v(t,x)$ denote the population densities and $H_1 > 0,\ H_2 > 0$ correspond to the harvesting effects for the first and second species, respectively. K_u, K_v are the species carrying capacity, and $r(x)$ is the intrinsic growth rate of competing species, respectively. Suppose that, $K_u(x) > 0,\ K_v(x) > 0,\ r(x) > 0$ and resource functions $M(x)$, $N(x)$ are in the class of $C^{1+\alpha}(\overline{\Omega})$, where Ω is an open non-empty isolated bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^{2+\alpha}$, where $0 < \alpha < 1$ for any $x \in \Omega$, which means that $r(x), K_u(x), K_v(x), M(x)$ and $N(x)$ are all positive in an open non-empty sub-domain of Ω. Here $d_1 > 0$, $d_2 > 0$ are the subsequent migration rate that depicts the species dispersal rates, and $\mathbf{n}$ is the outward normal to the boundary. The set $p_1 \times p_1$ relates to the range of the solutions to (1.1), which are determined by the subsequent upper and lower solutions technique, where $p_1 \times p_2$ is a bounded subset of $\mathbb{R}^2$. The following notations $\mathbb{A} = (0,\infty)\times\Omega$, $\overline{\mathbb{A}} = [0,\infty)\times\overline{\Omega}$, $\partial\mathbb{A} = (0,\infty)\times\partial\Omega$ are also convenient to introduce.

When it comes to distributing resources, it is expected that the percentage of population growth that is harvested will be directly proportional to the existing population and the intrinsic growth rate, which is distributed spatially and does not depend on time. This means that in areas with more resources, more effort can be put into harvesting, and there will be a greater expectation of a successful harvest.

In multi-species situations, the study of one harvested species [2,3,10,17] usually does not sufficiently define the population dynamics in some situations, as it ignores competition, mutualism, or predation of interacting species. However, for exploited populations, harvesting is a common reason for extinction. Further intriguing scenarios emerge when considering the application of harvesting to single or multiple interconnected populations. While two competing populations may coexist in the absence of harvesting, the introduction of harvesting measures can lead to the extinction of the harvested population. Also, if two competing species reveal similar resources and both are harvested, it is necessary to determine the relationship between the harvesting rates that preserve coexistence. For harvested populations, different diffusion strategies, model specifications, and harvesting rates may ultimately lead to species either competitive exclusion when one species survives in the competition or coexistence if both populations are over-exploited, then extinction of the species is possible. Different harvesting policies can be introduced for exploited species, for example, constant, proportional level, and threshold harvesting. In a study conducted in [4] by Korobenko et al., the spatial and temporal effects of a group of general growth functions with carrying capacity-driven diffusion strategy for single-species populations were analyzed. The research focused on the stability properties for the existence and extinction of species. Additionally, they established the existence of optimal harvesting efforts in the case of space-dependent carrying capacity.

In some species competition models, a general predator can serve as a source of harvesting to control invasion species management, as demonstrated by Madec et al. [18]. If the competition is varied by diffusion strategies or applied harvesting levels depending on intrinsic growth rate or not, then the extinction of one species by another is favored. In this situation, an estimate of harvesting levels that can lead the harvested species to survive in the competition is evaluated. Explicitly, by the interplay of steady-state population dispersion, diffusion strategy, intrinsic growth rate, and environmental carrying capacity, we will estimate some bounds on harvesting levels that are appropriate to get the required results. Examining solutions' stability properties will evaluate these bounds for the competition outcome. Also, we will study the models for ideal free pairs [15,19,20], which is the more interesting case for species coexistence.

In the present study, the major findings are as follows:

1. We prove the global existence of a couple of solutions of the model for the initial value problem when all parameters are time-independent.
2. We consider different diffusion strategies to examine the evolutionary advantage of the competition model and study the global existence of model solutions analytically by employing various conditions on diffusion strategy and harvesting efforts.
3. Both species are considered for two types of harvesting strategy. Case I: when harvesting effects are arbitrary and do not exceed time-independent intrinsic growth rates in the domain, and Case II: when harvesting efforts are proportional to time-independent intrinsic growth rates. We also establish some estimates for which coexistence necessarily happens.
4. Furthermore, we study numerically all the models both for one and two space dimensions, which has interesting ecological implications.

The development of the study is arranged as follows: Section 2 represents the persistence of positive and unique solutions for the coupled system. Section 3 provides an analytical study of global steady-state for two different conditions on harvesting effects and shows the main results of the study. In Section 4, we will compute these outcomes by numerical results for models both in one and two space dimensions to show the spatial effects for both cases on harvesting levels. Finally, the concluding words of the study are presented in Section 5.

As we know, the main interest of this study is based on the interaction of numerous migration strategies and the effect of intensity of harvesting at different levels for two interacting species. All through this segment, we estimate that $K_u(x), K_v(x), M(x), N(x), r(x), H_1$ and H_2 are positive in the domain. If we consider $g_l(x,u,v,K_u,H_1) = ur\overline{g}_l(x,u,v,K_u) - H_1 u$, and $g_L(x,v,u,K_v,H_2) = vr\overline{g}_L(x,v,u,K_v) - H_2 v$, where, $\overline{g}_l(x,u,v,K_u) = \left(1 - \frac{u+v}{K_u}\right)$ and $\overline{g}_L(x,v,u,K_v) = \left(1 - \frac{v+u}{K_v}\right)$, then the following assumption holds:

(l1) (g_l, g_L) is quasi-monotone non-increasing in $p_1 \times p_2$ as well as uniformly Hölder continuous on $(\overline{\Omega} \times \mathbb{R} \times \mathbb{R})$.

(l2) $g_l(x,u,v,K_u,H_1)$ is monotonically non-increasing in $\mathbb{R}^+$ while $r > H_1$ and $\overline{g}_l(x,u,v,K_u)$ is strictly monotonically decreasing in u and increasing in K_u as well as $g_L(x,u,v,K_v,H_2)$ is non-increasing in $\mathbb{R}^+$ when $r > H_2$, and $\overline{g}_L(x,u,v,K_v)$ is strictly monotonically decreasing in v and increasing in K_v.

For further analysis, it being also convenient to substitute $w = \frac{u(t,x)}{M(x)}$ and $z = \frac{v(t,x)}{N(x)}$, respectively then system (1.1) becomes reduced to

$$\begin{cases} \frac{\partial w}{\partial t} = \left(\frac{d_1}{M(x)}\right)\Delta w(t,x) + r(x)w\left(1 - \frac{M}{K_u}w - \frac{N}{K_u}z\right) - H_1 w, \; t>0, \; x \in \Omega, \\ \frac{\partial z}{\partial t} = \left(\frac{d_2}{N(x)}\right)\Delta z(t,x) + r(x)z\left(1 - \frac{M}{K_v}w - \frac{N}{K_v}z\right) - H_2 z, \; t>0, \; x \in \Omega, \\ \mathbf{n}\cdot\nabla w = \mathbf{n}\cdot\nabla z = 0, \; x \in \partial\Omega, \\ w(0,x) = w_0(x), \;\; z(0,x) = z_0(x), \; x \in \Omega. \end{cases} \tag{1.2}$$

Where, $\overline{g}_l^*(x,w,z,K_u,H_1) = \left(1 - \frac{M}{K_u}w - \frac{N}{K_u}z - H_1\right)$ and $\overline{g}_L^*(x,z,w,K_v,H_2) = \left(1 - \frac{M}{K_v}w - \frac{N}{K_v}z - H_2\right)$. Then it reduces to a system of regular diffusion with positive smooth space-dependent coefficients $\frac{d_1}{M(x)}$ and $\frac{d_2}{N(x)}$. Now, we will analyze the existence, uniqueness and positivity of the system (1.1). To do this, at first, we will revolve our observation to the model that reports the act of a system for mono and a couple of species.

2. Existence, uniqueness and positivity of solution

Consider the following resource-based diffusion model with a homogeneous Neumann boundary and positive initial conditions as

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta \left(\frac{u(t,x)}{M(x)} \right) + r(x)u \left(1 - \frac{u}{K_u} \right) - H_1(x)u, \ (t,x) \in \mathbb{A}, \\ \mathbf{n} \cdot \nabla \left(\frac{u(t,x)}{M(x)} \right) = 0, \ x \in \partial\Omega, \ \ u(0,x) = u_0(x), \ x \in \Omega. \end{cases} \tag{2.1}$$

We will now establish the existence and uniqueness results for the equation (2.1).

Theorem 1. *Let the non-negative and non-trivial initial function $u_0(x) \in C(\Omega)$ and $u_0(x) > 0$ in some open non-empty bounded sub-domain $\Omega_l \subset \Omega$. Then for any $t > 0$, there exists a unique solution $u(t,x)$ of the problem (2.1) and it is positive.*

Proof. For simplicity we considered, $w = \frac{u(t,x)}{M(x)}$. The function M is positive and bounded above for $x \in \overline{\Omega}$. So, $w(t,x)$ is well defined, which leads to model (2.1) in the following form

$$\begin{aligned} &\frac{\partial w}{\partial t} - \left(\frac{d_1}{M} \right) \Delta w = rw(t,x) \left(1 - \frac{Mw}{K_u} \right) - H_1 w, \ (t,x) \in \mathbb{A}, \ \ \mathbf{n} \cdot \nabla w = 0, \ x \in \partial\Omega, \\ &w(0,x) = w_0(x), \ x \in \Omega. \end{aligned} \tag{2.2}$$

Let us define the following function in some Ω,

$$G = \sup_{x\in\Omega} \left(\frac{M(x)}{K_u(x)} \right). \tag{2.3}$$

Where for some specific functions in the domain, either $K_u(x)$ is proportional to $M(x)$ or they are non-proportional. Then the system (2.2) becomes

$$\begin{aligned} &\frac{\partial w}{\partial t} - \left(\frac{d_1}{M} \right) \Delta w = rw(t,x)(1 - Gw) - H_1 w, \ (t,x) \in \mathbb{A}, \ \mathbf{n} \cdot \nabla w = 0, \ x \in \partial\Omega, \\ &w(0,x) = w_0(x), \ x \in \Omega. \end{aligned} \tag{2.4}$$

We will now apply the upper and lower solutions technique to model (2.4) to determine the existence and positiveness of the solution. It is apparent to investigate that $\underline{w} \equiv 0$ is a lower solution of the equation (2.4). Therefore, according to (see [21], Lemma 2.3.2) only an order pair of upper and lower solutions are needed to form for (2.4). Now, to construct the upper solution, let's consider a constant P^* so that

$$P^* \geq \max_{x\in\Omega_\infty} \left(\frac{u_0(x)}{M(x)} \right) \geq \max_{x\in\Omega} \left(\frac{u_0(x)}{M(x)} \right)$$

and $r(x)(1 - GP^*) < 0$ which can found for $u > K_u$. Since $u_0(x)$ is bounded in $\mathbb{A}$ and K_u is bounded from below, then $\max_{x\in\Omega} \left(\frac{u_0}{M} \right) < \infty$. Also, to construct the upper solution suppose $\mu \geq \sup_{x\in\Omega} |H_1(x)|$, and let $\overline{w} = P^* e^{\mu t}$. Then

$$\frac{\partial \overline{w}}{\partial t} - \frac{d_1}{M} \Delta \overline{w} = \mu \overline{w}$$

$$\overline{w} \left(r - G\overline{w} - H_1 \right) \leq \mu \overline{w}$$

and

$$\nabla \cdot \overline{w} = 0, \ \overline{w} \geq w(0,x).$$

Then consequently, $\overline{w}$ is an upper solution of (2.4) (see [21], Definition 2.3.1). Similarly, the right-hand side function

$$f_R(t,x,w,K_u,M) = rw \left(1 - \frac{M}{K_u} w \right) - H_1 w$$

is continuous and differentiable with respect to w and we define the maximal derivative of f_R in w for each t and x.

$$f_{R*}(t,x) = \sup\{-f_w, \ \underline{w} \leq \ w \leq \overline{w}\}, \ \ f_R^*(t,x) = \sup\{f_w, \ \underline{w} \leq \ w \leq \overline{w}\}.$$

Then the Lipschitz condition (see [21], equation 2.3.3) holds and by Lemma 2.3.2 (see [21]) there prevails a unique solution of the model (2.4) satisfying $w \in \langle \underline{w}, \overline{w} \rangle$. On taking the inverse substitution, $u(t,x) = Mw$, where $0 \leq u \leq M\overline{w}$. We get a unique solution of (2.4).

To show the positivity of u we substitute

$$S(t,x) := we^{\beta t} = \frac{u}{M}e^{\beta t}.$$

Where, $\beta := \overline{w}\sup_{x\in\overline{\Omega}}\left|r(x)G\right| + \sup_{x\in\overline{\Omega}}\left|H_1\right|,\ x\in\Omega$, and $\sup_{x\in\Omega} r > H_1$. So we have

$$\frac{\partial S}{\partial t} - \frac{d_1}{M}\Delta S = \left[rw\left(1-\frac{M}{K_u}w\right) - E_1 w + \beta w\right]e^{\beta t} = \left[r(1-Gw) - H_1 + \beta\right]S \geq 0,\ (t,x)\in\mathbb{A},$$
$$\mathbf{n}\cdot\nabla S = 0,\ (t,x)\in\partial\mathbb{A},\ S(0,x) = w(0,x)\geq 0,\ x\in\Omega.$$

As if $(t_0,x_0)\in\mathbb{A}$ then $w \geq \underline{w} \equiv 0$, so S is non-negative. Hence by Maximum principal $S>0$ in Ω_l and $S\in C(\mathbb{A})$. Therefore, we can conclude that the solution $w(t,x)$ of the model (2.1) is unique and positive which immediately follows the fact $u(t,x)$ is the solution of model (2.1) and it is unique and positive. □

Similarly, we can construct the existence and uniqueness result for Species II:

$$\begin{cases} \frac{\partial v}{\partial t} = d_2\Delta\left(\frac{v(t,x)}{N(x)}\right) + r(x)v\left(1-\frac{v}{K_v}\right) - H_2(x)v,\ (t,x)\in\mathbb{A}, \\ \mathbf{n}\cdot\nabla\left(\frac{v(t,x)}{N(x)}\right) = 0,\ x\in\partial\Omega,\ \ v(0,x) = v_0(x),\ x\in\Omega. \end{cases}$$

The following stated Theorem represents the existence and uniqueness of model (1.1) for the coupled system.

Theorem 2. *Let $u_0(x)$ and $v_0(x)$ be non-negative on Ω. Then for $u_0(x), v_0(x)\in C(\Omega)$, (u,v) is the unique solution of problem (1.1). Furthermore, if $(u(0,x),v(0,x))$ is non-trivial and non-negative, then for any $t>0$ both $u(t,x)>0$ and $v(t,x)>0$.*

Proof. To prove this we enacted ([21], Theorem 8.7.2) to the problem (1.2) which is obtained after substituting $w = \frac{u(t,x)}{M(x)}$, and $z = \frac{v(t,x)}{N(x)}$, respectively.

Next, we choose two constants ρ_w and ρ_z such that

$$\rho_w \geq \sup_{(t,x)\in\mathbb{A}_1}\frac{u_0(t,x)}{M(x)},\quad \text{and}\quad \rho_z \geq \sup_{x\in\Omega}\frac{v_0(x)}{N(x)}$$

so that $\overline{g}_l^*(t,x,\rho_w,0,K_u,H_1)<0$ and $\overline{g}_L^*(t,x,0,\rho_z,K_v,H_2)<0$, as $\overline{g}_l(x,Mw,Nz,K_u)$ and $\overline{g}_L(x,Nz,Mw,K_v)$ are monotonically non-increasing in $\mathbb{R}^+$. Since, $u(0,x)$ and $v(0,x)$ are bounded in $\overline{\Omega}$ and K_u, K_v are bounded from below, so we have $\sup_{x\in\Omega}\frac{u(0,x)}{M(x)}<\infty$, and $\sup_{x\in\Omega}\frac{v(0,x)}{N(x)}<\infty$.

It is simple to inspect that conditions (see [21], Theorem 8.7.2, equation 8.7.4) are assured for ρ_w and ρ_z specified above and

$$\left(\frac{u_0(x)}{M(x)},\frac{v_0(x)}{N(x)}\right)\in\mathbf{S}_\rho := \{(w,z)\in C\left([0,\infty)\times\overline{\Omega}\right) : (0,0)\leq(w,z)\leq(\rho_w,\rho_z)\}.$$

The functions, $g_l(x,w,z,K_u,H_1) = wr(x)\overline{g}_l(x,wM,zN,K_u) - H_1 w$, and $g_L(x,z,w,K_v,H_2) = zr(x)\overline{g}_L(x,zN,wM,K_v) - H_2 z$ are in C^1 and hence Lipschitz continuous in $\mathbf{S}_\rho$. Since, the function $g_l(\cdot,z)$ is non-increasing in $z\in\langle 0,\rho_z\rangle$ and the function $g_L(w,\cdot)$ is non-increasing in $w\in\langle 0,\rho_w\rangle$, therefore the vector function (g_l,g_L) is quasi-monotone non-increasing in $\mathbf{S}_\rho$.

Accordingly, from (see [21], Theorem 8.7.2) all the conditions are satisfied, so the unique solution (w,z) of (1.2) exists, and it is positive. Apparently, the unique positive solution of (1.1) is $(u,v)=(Mw,Nz)$. □

3. Steady states and global analysis

In this section of our study, we will examine two cases. Firstly, we will investigate the situation when both harvesting coefficients $H_1(x)$ and $H_2(x)$ are spatially dependent, arbitrary, and not proportional to the time-independent $r(x)$. We aim to investigate the global stability of equilibrium solutions under various conditions, including different diffusion strategies, carrying capacities, and the effects of harvesting. Secondly, we will analyze the global existence of steady-state when both harvesting coefficients $H_1(x)$ and $H_2(x)$ are proportional to the time-independent intrinsic growth rates, and we will consider different scenarios.

According to various studies [21–24], model (1.1) represents a sample of a monotone dynamical system. This states that, if the trivial equilibrium is unstable and repelling, and neither of the semi-trivial equilibria is stable, then there will be a globally stable coexistence equilibrium. However, the trivial equilibrium, coexistence equilibrium, and one of the associated semi-trivial equilibria will not be able to sustain stability, the remaining semi-trivial solution will be globally asymptotically stable.

Lemma 1. *For some $H_1<\sup r(x)$ and $H_2<\sup r(x)$, the zero equilibrium $(0,0)$ of the model (1.1) is unstable and repelling.*[1]

[1] Remark: If $H_1>\sup r(x)$ and $H_2>\sup r(x)$ in some $x\in\Omega_s\subset\Omega$, the $(0,0)$ equilibrium is an attractor.

The similar proof is available in [25,26] for $H_1 = H_2 \equiv 0$, so we have omitted it here.

Lemma 2. *[27] If $(0,0)$ of (1.1) is repeller, then one of the following three circumstances is held from a predetermined list:*

(a) a stable positive coexistence of (1.1) will exist, which yields both $(u^*,0)$ and $(0,v^*)$ are linearly unstable,
(b) $(u,v)\to(u^*,0)$, i.e. all positive solution converges to $(u^*,0)$ as $t\to\infty$,
(c) $(u,v)\to(0,v^*)$, i.e. all positive solution converges to $(0,v^*)$ as $t\to\infty$.

3.1. Case I: harvesting maps and growth rate are arbitrary

Consider the following equations with homogeneous Neumann boundary conditions

$$\begin{cases} \frac{\partial u}{\partial t} = d_1\Delta\left(\frac{u(t,x)}{M(x)}\right) + r(x)u\left(1-\frac{u+v}{K_u(x)}\right) - H_1(x)u,\ t>0,\ x\in\Omega, \\ \frac{\partial v}{\partial t} = d_2\Delta\left(\frac{v(t,x)}{N(x)}\right) + r(x)v\left(1-\frac{v+u}{K_v(x)}\right) - H_2(x)v,\ t>0,\ x\in\Omega, \\ \mathbf{n}\cdot\nabla\left(\frac{u}{M}\right) = \mathbf{n}\cdot\nabla\left(\frac{v}{N}\right) = 0,\ x\in\partial\Omega,\ u(0,x)=u_0(x),\ v(0,x)=v_0(x),\ x\in\Omega. \end{cases} \tag{3.1}$$

Now, we will study the global existence of model (3.1) under the different conditions on resource functions and harvesting coefficients. Let $u^*(x)$ and $v^*(x)$ be the stationary solutions of the first and second species corresponding to (3.1) when only one species survives so that the semi-trivial equilibrium $(u^*,0)$ and $(0,v^*)$ satisfies

$$d_1\Delta\left(\frac{u^*(x)}{M(x)}\right) + r\left(1-\frac{u^*(x)}{K_u(x)}\right)u^* - H_1(x)u^* = 0,\ x\in\Omega,\ \mathbf{n}\cdot\nabla\left(\frac{u^*}{M}\right)=0,\ x\in\partial\Omega.$$
$$d_2\Delta\left(\frac{v^*(x)}{N(x)}\right) + r\left(1-\frac{v^*(x)}{K_v(x)}\right)v^* - H_2(x)v^* = 0,\ x\in\Omega,\ \mathbf{n}\cdot\nabla\left(\frac{v^*}{N}\right)=0,\ x\in\partial\Omega.$$

The following Lemma 3 and Lemma 4 presented below represent the existence and extinction conditions for a positive equilibrium state when the carrying capacities and resource functions are spatially distributed.

Lemma 3. *[4,28] Let $f(t,x,u,K_u) = ru(1-\frac{u}{K_u})$ satisfy the properties (l1)-(l2) and $\sup_{x\in\Omega}[r(x)-H_1(x)]>0$ for all $x\in\Omega$, then for any $u_0(x)>0$ and $u_0\not\equiv 0$ all the solutions of the system (2.1) converge to a unique positive steady state $u^*(x)$ as $t\to\infty$.*

Lemma 4. *[4,28] Let $f(t,x,u,K_u) = ru(1-\frac{u}{K_u})$ satisfy the properties (l1)-(l2) and $\sup_{x\in\Omega}[r(x)-H_1(x)]<0$ for all $x\in\Omega$, then for any $u_0(x)>0$ and $u_0\not\equiv 0$ all the solutions of the system (2.1) converge to 0 as $t\to\infty$.*

Lemma 5. *Let $\frac{M}{K_u}\not\equiv Constant$ and $\frac{N}{K_v}\not\equiv Constant$ where K_u, K_v, M and N are non-constant and $K_u\equiv K_v\equiv K$ with $d_1\equiv d_2\equiv d$. Then for some $H_1, H_2\in\Omega_s\subset\Omega$ if $\sup_{x\in\Omega}[r(x)-H_1(x)]>0$ and $\sup_{x\in\Omega}[r(x)-H_2(x)]>0$ as well as $H_1(x)<H_2(x)$ in open, bounded and non-empty domain $x\in\Omega_s\subset\Omega$, the quasi-trivial equilibrium $(0,v^*)$ of system (3.1) is unstable.*

Proof. For any $x\in\Omega$, we have $H_1(x)<H_2(x)$ and so

$$r(x)-H_1(x) > r(x)-H_2(x).$$

For which,

$$\sup_{x\in\Omega}\left(r(x)-H_1(x)\right) > \sup_{x\in\Omega}\left(r(x)-H_2(x)\right).$$

Now at low densities, since $\sup_{x\in\Omega}(r(x)-H_2(x))>0$ by the property (l2) and Lemma 3 we get,

$$\sup_{x\in\Omega}\left[r(x)(1-\frac{v^*}{K})-H_1(x)\right] > \sup_{x\in\Omega}\left[r(x)(1-\frac{v^*}{K})-H_2(x)\right].$$

Which implies, $\sup_{x\in\Omega}[r(x)(1-\frac{v^*}{K})-H_2(x)]>0$ so that $\sup_{x\in\Omega}[r(x)(1-\frac{v^*}{K})-H_1(x)]>0$.

Furthermore, suppose for any $x\in\Omega$ and $t>0$

$$\left[r(x)\left(1-\frac{v^*}{K}\right)-H_1(x)\right] = G_h(x).$$

Consider the linearization of (3.1) over $(0,v^*)$ for the case $H_1<H_2$, we find

$$\frac{\partial u}{\partial t} = d\Delta\left(\frac{u}{M}\right) + G_h(x)u,\ t>0,\ x\in\Omega,$$
$$\frac{\partial v}{\partial t} = d\Delta\left(\frac{v}{N}\right) + v\overline{g}_L(x,v^*,0,K)r + v^*\overline{g}_{Lu}(x,v^*,0,K)ru + v^*\overline{g}_{Lv}(x,v^*,0,K)rv - H_2 v,\ t>0,\ x\in\Omega,$$
$$\mathbf{n}\cdot\nabla\left(\frac{u}{M}\right) = \mathbf{n}\cdot\nabla\left(\frac{v}{N}\right) = 0,\ x\in\partial\Omega.$$

Now to establish the unstability of $(0, v^*)$, it is enough to show that the principal eigenvalue of (3.1) is positive. Consider the linearized eigenvalue problem for the equation u we have,

$$d\Delta\left(\frac{\phi}{M}\right) + G_h(x)\phi = \sigma\phi,\ x\in\Omega,\ \ \mathbf{n}\cdot\nabla\left(\frac{\phi}{M}\right) = 0,\ x\in\partial\Omega. \tag{3.2}$$

By variational characterization of the eigenvalues ([22], Theorem 2.1[2]) the principal eigenvalue of (3.2) is defined by

$$\sigma_1 = \sup_{\phi\neq 0,\ \phi\in W^{1,2}} \frac{\int_\Omega -d\left|\nabla\left(\frac{\phi}{M}\right)\right|^2 dx + \int_\Omega \left(\frac{\phi^2}{M}\right) G_h(x)dx}{\int_\Omega \left(\frac{\phi^2}{M}\right) dx}.$$

Substituting $\phi = M$, and for $H_1 < H_2$ and $\sup_{x\in\Omega}\left[r\left(1-\frac{v^*}{K}\right) - H_1\right] > 0$ we have

$$\sigma_1 \geq \frac{\int_\Omega MG_h(x)dx}{\int_\Omega M dx} > 0.$$

Thus $\sigma_1 > 0$, which concludes the proof. □

Lemma 6. *Let $\frac{M}{K_u} \not\equiv Constant$ and $\frac{N}{K_v} \not\equiv Constant$ where K_u, K_v, M and N are non-constant and $K_u \equiv K_v \equiv K$ with $d_1 \equiv d_2 \equiv d$. For some $H_1, H_2 \in \Omega_s \subset \Omega$ if $\sup_{x\in\Omega}[r(x) - H_1(x)] > 0$ and $\sup_{x\in\Omega}[r(x) - H_2(x)] > 0$ as well as $H_1(x) < H_2(x)$ in open, bounded and non-empty domain $x \in \Omega_s \subset \Omega$, then system (3.1) has no stable coexistence state (u_c, v_c).*

Proof. Let us first suppose there exists coexistence state (u_c, v_c) of (3.1) then

$$\begin{cases} d\Delta\left(\frac{u_c(x)}{M(x)}\right) + \left(r(x)\left(1-\frac{u_c+v_c}{K}\right) - H_1(x)\right)u_c = 0,\ x\in\Omega,\ \mathbf{n}\cdot\nabla\left(\frac{u_c}{M}\right) = 0,\ x\in\partial\Omega, \\ d\Delta\left(\frac{v_c(x)}{N(x)}\right) + \left(r(x)\left(1-\frac{u_c+v_c}{K}\right) - H_2(x)\right)v_c = 0,\ x\in\Omega,\ \mathbf{n}\cdot\nabla\left(\frac{v_c}{N}\right) = 0,\ x\in\partial\Omega. \end{cases} \tag{3.3}$$

Adding both equations in (3.3), integrating over Ω, and applying the Neumann boundary conditions we get,

$$\int_\Omega \left(r(x)\left(1-\frac{u_c+v_c}{K}\right) - H_1(x)\right)u_c dx + \int_\Omega \left(r(x)\left(1-\frac{u_c+v_c}{K}\right) - H_2(x)\right)v_c dx = 0.$$

Since, $H_1(x) < H_2(x)$ and by our assumption $\sup_{x\in\Omega}[r(x) - H_1(x)] > \sup_{x\in\Omega}[r(x) - H_2(x)] > 0$.
It follows, $\sup_{x\in\Omega}\left[r(x)\left(1-\frac{u_c+v_c}{K}\right) - H_1(x)\right] > \sup_{x\in\Omega}\left[r(x)\left(1-\frac{u_c+v_c}{K}\right) - H_2(x)\right]$.
So,

$$\sup_{x\in\Omega}\left[r(x)\left(1-\frac{u_c+v_c}{K}\right) - H_1(x)\right] > 0. \tag{3.4}$$

Moreover, let

$$\left[r(x)\left(1-\frac{u_c+v_c}{K}\right) - H_1(x)\right] = G_c(x).$$

This is possible only when $u_c + v_c \not\equiv K$. Now we will impose only the case when $u_c + v_c \not\equiv K$ in some non-empty open domain Ω. Now consider the eigenvalue problem corresponding to positive eigenfunction ϕ we get,

$$d\Delta\left(\frac{\phi}{M}\right) + G_c(x)\phi = \sigma\phi,\ x\in\Omega,\ \ \mathbf{n}\cdot\nabla\left(\frac{\phi}{M}\right) = 0,\ x\in\partial\Omega.$$

[2] Remark: In this theorem, for investigating principal eigenvalue for the instability of equilibrium states the authors have considered the inf of σ_1 for which if the eigenvalue is negative then it would be unstable. But in our study, we have considered sup of σ_1, and it will be unstable if the principal eigenvalue is positive.

Then its principal eigenvalues according to ([22], Theorem 2.1) is

$$\sigma_1 = \sup_{\phi\neq 0,\, \phi\in W^{1,2}} \frac{\int\limits_\Omega -d\left|\nabla\left(\frac{\phi}{M}\right)\right|^2 dx + \int\limits_\Omega \left(\frac{\phi^2}{M}\right) G_c(x)dx}{\int\limits_\Omega \left(\frac{\phi^2}{M}\right) dx}.$$

Letting $\phi = M$ and for any $H_1 < H_2$ and by equation (3.4) at low population densities,

$$\sup_{x\in\Omega}\left[r(x)\left(1-\frac{u_c+v_c}{K}\right) - H_1(x)\right] > 0.$$

Therefore we get,

$$\sigma_1 \geq \frac{\int\limits_\Omega MG_c(x)dx}{\int\limits_\Omega M dx} > 0.$$

However, (w_c, z_c) is a steady state solution of (3.1), w_c satisfies,

$$d\Delta w_c + \overline{g}_l^*(x, w, z, K, H_1) w_c M(x) = 0,\ x\in\Omega,\ \mathbf{n}\cdot\nabla w_c = 0,\ x\in\partial\Omega,$$

and therefore, we get a positive principal eigenfunction of (3.1) along with principal eigenvalue 0. This contradicts $\sigma_1 > 0$, which concludes the proof. □

According to strong monotone dynamical system [21–24,27] and by Lemma 2 for $H_1 < H_2$, the following outcome is sketched by Lemma 1, Lemma 5 and Lemma 6.

Theorem 3. *Let $\frac{M}{K_u} \not\equiv Constant$ and $\frac{N}{K_v} \not\equiv Constant$ where K_u, K_v, M and N are non-constant and $K_u \equiv K_v \equiv K$ with $d_1 \equiv d_2 \equiv d$. For some $H_1, H_2 \in \Omega_s \subset \Omega$ if $\sup_{x\in\Omega}[r(x) - H_1(x)] > 0$ and $\sup_{x\in\Omega}[r(x) - H_2(x)] > 0$ as well as $H_1(x) < H_2(x)$ in open, bounded and non-empty domain $x \in \Omega_s \subset \Omega$, then $(u^*, 0)$ of system (3.1) is globally asymptotically stable.*

The following remark is followed by Theorem 3.

Remark 1. Let $\frac{M}{K_u} \not\equiv Constant$ and $\frac{N}{K_v} \not\equiv Constant$ where K_u, K_v, M and N are non-constant and $K_u \equiv K_v \equiv K$ with $d_1 \equiv d_2 \equiv d$. For some $H_1, H_2 \in \Omega_s \subset \Omega$ if $\sup_{x\in\Omega}[r(x) - H_1(x)] > 0$ and $\sup_{x\in\Omega}[r(x) - H_2(x)] > 0$, then for any $H_1(x) > H_2(x)$ in open, bounded and non-empty domain $x \in \Omega_s \subset \Omega$, $(0, v^*)$ of system (3.1) is globally asymptotically stable.

Lemma 7. *Let $\frac{M}{K_u} \not\equiv Constant$ and $\frac{N}{K_v} \not\equiv Constant$ where K_u, K_v, M and N are non-constant and $K_u \equiv K_v \equiv K$ with $d_1 \equiv d_2 \equiv d$. If for some $H_1, H_2 \in \Omega_s \subset \Omega$ if $\sup_{x\in\Omega}[r(x) - H_1(x)] > 0$ and $\sup_{x\in\Omega}[r(x) - H_2(x)] > 0$, then for any $H_1(x) \equiv H_2(x) \equiv H(x)$ and $K = \alpha M + \beta N$, where $\alpha > 0, \beta > 0$ in open, bounded and non-empty domain $x \in \Omega_s \subset \Omega$, $(u^*, 0)$ of (3.1) is not stable.*

Proof. Examine the linearized eigenvalue problem corresponding to the second equation in (3.1) over $(u^*, 0)$ with Neumann boundary conditions we get

$$d\Delta\left(\frac{\phi}{N}\right) + \left(r(x)\left(1-\frac{u^*}{K}\right) - H(x)\right)\phi = \sigma\phi,\ x\in\Omega,\ \ \mathbf{n}\cdot\nabla\left(\frac{\phi}{N}\right) = 0,\ x\in\partial\Omega. \tag{3.5}$$

The principal eigenvalue of (3.5) is defined as according to ([22], Theorem 2.1) we obtain

$$\sigma_1 = \sup_{\phi\neq 0,\, \phi\in W^{1,2}} \frac{\int\limits_\Omega -d\left|\nabla\left(\frac{\phi}{N}\right)\right|^2 dx + \int\limits_\Omega \left(\frac{\phi^2}{N}\right)\left(r(x)\left(1-\frac{u^*}{K}\right) - H(x)\right) dx}{\int\limits_\Omega \left(\frac{\phi^2}{N}\right) dx}.$$

Choosing $\phi = \sqrt{\beta} N(x)$ we find

$$\sigma_1 \geq \frac{\int\limits_\Omega \beta N\left(r\left(1-\frac{u^*}{K}\right) - H\right) dx}{\beta\int\limits_\Omega N dx}.$$

Since u^* is a steady state solution of (3.1) and $H_1(x) \equiv H_2(x) \equiv H(x)$, so we get

$$d\Delta\left(\frac{u^*(x)}{M(x)}\right)+r(x)u^*\left(1-\frac{u^*}{K}\right)-H(x)u^*=0,\ x\in\Omega,\ \mathbf{n}\cdot\nabla\left(\frac{u^*}{M(x)}\right)=0,\ x\in\partial\Omega. \tag{3.6}$$

As u^* and $M(x)$ both are non-negative, dividing the equation (3.6) by $\frac{u^*}{M(x)}$, integrating over Ω and using the corresponding Neumann boundary conditions we have,

$$\int_\Omega M(x)\left(r(x)\left(\frac{u^*}{K}-1\right)+H(x)\right)dx=\int_\Omega d\frac{\left|\nabla\left(\frac{u^*}{M(x)}\right)\right|^2}{\left(\frac{u^*}{M(x)}\right)^2}dx>0,\ \text{unless}\ \frac{M(x)}{K}\not\equiv Constant.$$

Hence,

$$\int_\Omega M(x)\left(r(x)\left(\frac{u^*}{K}-1\right)+H(x)\right)dx>0. \tag{3.7}$$

Now, for $K(x)=\alpha M(x)+\beta N(x)$ and using (3.7) for $H_1\equiv H_2\equiv H(x)$ we get

$$\begin{aligned}\sigma_1&\geq\frac{\int_\Omega \beta N(x)\left(r(x)\left(1-\frac{u^*}{K}\right)-H(x)\right)dx}{\beta\int_\Omega N(x)dx}\\&=\frac{1}{P_1}\int_\Omega \beta N(x)\left(r(x)\left(1-\frac{u^*}{K}\right)-H(x)\right)dx,\ \text{where we define}\ P_1=\beta\int_\Omega N(x)dx\\&=\frac{1}{P_1}\int_\Omega K\left(r(x)\left(1-\frac{u^*}{K}\right)-H(x)\right)dx+\frac{\alpha}{P_1}\int_\Omega\left(r(x)\left(\frac{u^*}{K}-1\right)+H(x)\right)M(x)dx>0,\ \text{unless}\ \frac{M(x)}{K}\not\equiv Constant.\end{aligned}$$

As in the right-hand side expression, the second term is positive according to (3.7) while the foremost term is positive by our assumption and following the similar process as in Lemma 5, thus $\sigma_1>0$, which implies that $(u^*,0)$ is not stable. □

Lemma 8. *Let $\frac{M}{K_u}\not\equiv Constant$ and $\frac{N}{K_v}\not\equiv Constant$ where K_u, K_v, M and N are non-constant and $K_u\equiv K_v\equiv K$ with $d_1\equiv d_2\equiv d$. If for some $H_1,H_2\in\Omega_s\subset\Omega$ if $\sup_{x\in\Omega}[r(x)-H_1(x)]>0$ and $\sup_{x\in\Omega}[r(x)-H_2(x)]>0$, then for any $H_1(x)\equiv H_2(x)\equiv H(x)$ and $K=\alpha M+\beta N$, where $\alpha>0,\beta>0$ in open, bounded and non-empty domain $x\in\Omega_s\subset\Omega$, $(0,v^*)$ of (3.1) is not stable.*

Proof. Taking the linearized eigenvalue problem associated with the leading equation in (3.1) around $(0,v^*)$ with Neumann boundary conditions yields

$$d\Delta\left(\frac{\phi}{M(x)}\right)+\left(r(x)\left(1-\frac{v^*}{K}\right)-H(x)\right)\phi=\sigma\phi,\ x\in\Omega,\ \mathbf{n}\cdot\nabla\left(\frac{\phi}{M(x)}\right)=0,\ x\in\partial\Omega. \tag{3.8}$$

The principal eigenvalue of (3.8) is defined as according to ([22], Theorem 2.1) we obtain

$$\sigma_1=\sup_{\phi\neq0,\,\phi\in W^{1,2}}\frac{\int_\Omega -d\left|\nabla\left(\frac{\phi}{M(x)}\right)\right|^2dx+\int_\Omega\left(\frac{\phi^2}{M(x)}\right)\left(r(x)\left(1-\frac{v^*}{K}\right)-H(x)\right)dx}{\int_\Omega\left(\frac{\phi^2}{M(x)}\right)dx}.$$

Choosing $\phi=\sqrt{\alpha}M(x)$ we find,

$$\sigma_1\geq\frac{\int_\Omega \alpha M(x)\left(r(x)\left(1-\frac{v^*}{K}\right)-H(x)\right)dx}{\alpha\int_\Omega M(x)dx}.$$

Since v^* is a stationary solution of (3.1) and both v^*, $N(x)$ are non-negative, then for $H_1(x)\equiv H_2(x)\equiv H(x)$ we get similarly as Lemma 7 is

$$\int_\Omega N(x)\left(r(x)\left(\frac{v^*}{K}-1\right)+H(x)\right)dx>0. \tag{3.9}$$

Now, for $K(x)=\alpha M(x)+\beta N(x)$ and using (3.9) for $H_1(x)\equiv H_2(x)\equiv H(x)$ we get

$$\sigma_1\geq\frac{\int_\Omega \alpha M\left(r\left(1-\frac{v^*}{K}\right)-H(x)\right)dx}{\alpha\int_\Omega N(x)dx}$$

$$= \frac{1}{Q_1}\int_{\Omega} \alpha M(x)\left(r(x)\left(1-\frac{u^*}{K}\right)-H(x)\right)dx, \text{ where we define } Q_1=\beta\int_{\Omega} M(x)dx$$

$$= \frac{1}{Q_1}\int_{\Omega} K\left(r(x)(\left(1-\frac{v^*}{K}\right)-H(x)\right)dx+\frac{\alpha}{Q_1}\int_{\Omega} N(x)\left(r(x)\left(\frac{v^*}{K}-1\right)+H(x)\right)dx>0, \text{ unless } \frac{N(x)}{K(x)}\not\equiv Constant.$$

According to (3.9), the second term in the expression is positive, while the first term is non-negative based on our assumption. By following the same process as in Lemma 5, we can conclude that σ_1 is positive. This implies that $(0,v^*)$ is not stable. □

By Lemma 7 and Lemma 8 both $(u^*,0)$ and $(0,v^*)$ are not stable, and the Lemma 1 is still valid. So, according to strong monotone dynamical system [21–24,27] and by Lemma 2 for $K(x)=\alpha M(x)+\beta N(x)$, where $\alpha>0$, $\beta>0$ in some non-empty open domain, (u^*,v^*) is the only coexistence equilibrium of (3.1) which concludes the following Theorem 4, where both M,N are linearly independent.

Theorem 4. *Let $\frac{M}{K_u}\not\equiv Constant$ and $\frac{N}{K_v}\not\equiv Constant$ where K_u, K_v, M and N are non-constant and $K_u\equiv K_v\equiv K$ with $d_1\equiv d_2\equiv d$. If for some $H_1,H_2\in\Omega_s\subset\Omega$ if $\sup_{x\in\Omega}[r(x)-H_1(x)]>0$ and $\sup_{x\in\Omega}[r(x)-H_2(x)]>0$, then for any $H_1(x)\equiv H_2(x)\equiv H(x)$ and $K=\alpha M+\beta N$, where $\alpha>0,\beta>0$ in open, bounded and non-empty domain $x\in\Omega_s\subset\Omega$, the coexistence state (u_c,v_c) is globally stable for the system (3.1).*

3.2. Case II: $H_1(x)$ and $H_2(x)$ are proportional to space dependent $r(x)$

Now consider the competition model with homogeneous Neumann boundary conditions as

$$\begin{cases} \frac{\partial u}{\partial t}=d_1\Delta\left(\frac{u(t,x)}{M(x)}\right)+r(x)u\left(1-\frac{u(t,x)+v(t,x)}{K_u(x)}\right)-\gamma_1 r(x)u(t,x),\ t>0,\ x\in\Omega,\\ \frac{\partial v}{\partial t}=d_2\Delta\left(\frac{v(t,x)}{N(x)}\right)+r(x)v\left(1-\frac{v(t,x)+u(t,x)}{K_v(x)}\right)-\gamma_2 r(x)v(t,x),\ t>0,\ x\in\Omega,\\ \mathbf{n}\cdot\nabla\left(\frac{u}{M}\right)=\mathbf{n}\cdot\nabla\left(\frac{v}{N}\right)=0,\ x\in\partial\Omega,\ u(0,x)=u_0(x),\ v(0,x)=v_0(x),\ x\in\Omega. \end{cases} \tag{3.10}$$

So that $H_1(x)=\gamma_1 r(x)$ and $H_2(x)=\gamma_2 r(x)$, where γ_1 and γ_2 is defined as harvesting rates corresponding to the first and second species in (3.10), respectively.

Now considering a system with different proportions of carrying capacity and without the effect of harvesting ($\gamma_1=\gamma_2=0$), then the modification of (3.10) is

$$\begin{cases} \frac{\partial u}{\partial t}=d_1\Delta\left(\frac{u(t,x)}{M(x)}\right)+r_1 r(x)u\left(1-\frac{u(t,x)+v(t,x)}{K_1(x)}\right),\ t>0,\ x\in\Omega,\\ \frac{\partial v}{\partial t}=d_2\Delta\left(\frac{v(t,x)}{N(x)}\right)+r_2 r(x)v\left(1-\frac{v(t,x)+u(t,x)}{K_2(x)}\right),\ t>0,\ x\in\Omega,\\ \mathbf{n}\cdot\nabla\left(\frac{u}{M}\right)=\mathbf{n}\cdot\nabla\left(\frac{v}{N}\right)=0,\ x\in\partial\Omega,\ u(0,x)=u_0(x),\ v(0,x)=v_0(x),\ x\in\Omega. \end{cases} \tag{3.11}$$

Suppose u^* and v^* be the stationary solutions of the first and second species corresponding to (3.11), so that $(u^*,0)$ and $(0,v^*)$ satisfy

$$d_1\Delta\left(\frac{u^*(x)}{M(x)}\right)+r_1 r(x)u^*\left(1-\frac{u^*(x)}{K_1(x)}\right)=0,\ x\in\Omega,\ \mathbf{n}\cdot\nabla\left(\frac{u^*}{M}\right)=0,\ x\in\partial\Omega, \tag{3.12}$$

$$d_2\Delta\left(\frac{v^*(x)}{N(x)}\right)+r_2 r(x)v^*\left(1-\frac{v^*(x)}{K_2(x)}\right)=0,\ x\in\Omega,\ \mathbf{n}\cdot\nabla\left(\frac{v^*}{N}\right)=0,\ x\in\partial\Omega, \tag{3.13}$$

respectively. For further analysis of system (3.10), we have extended the following three auxiliary results that will apply to complete the following discussion, which are already established in [19,15,25,26] for $K_u=K_v=K$.

Lemma 9. *[25,26] If $\frac{M(x)}{K_1(x)}\equiv Constant$, then $u^*(x)\equiv K_1(x)$ is the only solution of (3.12) and if $\frac{N(x)}{K_2(x)}\not\equiv Constant$ then (3.13) has only solution $v^*(x)$, which is unique and positive in $\overline{\Omega}$.*

Lemma 10. *[15,25,26] Suppose $M(x)\not\equiv Constant$ and $K_1(x)\not\equiv Constant$ while both are linearly independent in Ω. If $u^*(x)$ is a solution of (3.12) and it is positive then*

$$\int_{\Omega} r(x)M(x)\left(\frac{u^*}{K_1}-1\right)dx>0,\quad \textit{for any } x\in\Omega.$$

Then similar result valid for $v^*(x)$ in (3.13) i.e.,

$$\int_{\Omega} r(x)N(x)\left(\frac{v^*}{K_2}-1\right)dx>0,\quad \textit{for any } x\in\Omega,\quad \textit{unless } v^*(x)\equiv K_2(x).$$

Lemma 11. *[15,25] If $u^*(x)$ is a positive solution of (3.12) while $M(x) \not\equiv Constant$ and $\frac{M(x)}{K_1(x)} \not\equiv Constant$ on Ω then,*

$$\int_\Omega r(x)K_1(x)dx > \int_\Omega r(x)u^* dx.$$

As a special case of (3.11) consider the competitive model as

$$\begin{cases} \frac{\partial u}{\partial t} = d_1\Delta\left(\frac{u(t,x)}{M(x)}\right) + r_1 r(x)u(t,x)\left(1 - \frac{u(t,x)+v(t,x)}{aK(x)}\right),\ t>0,\ x\in\Omega, \\ \frac{\partial v}{\partial t} = d_2\Delta\left(\frac{u(t,x)}{N(x)}\right) + r_2 r(x)v(t,x)\left(1 - \frac{v(t,x)+u(t,x)}{K(x)}\right),\ t>0,\ x\in\Omega, \\ \mathbf{n}\cdot\nabla\left(\frac{u}{M}\right) = \mathbf{n}\cdot\nabla\left(\frac{v}{N}\right) = 0,\ x\in\partial\Omega,\ u(0,x)=u_0(x),\ v(0,x)=v_0(x),\ x\in\Omega. \end{cases} \tag{3.14}$$

The equation (3.14) belongs to the monotone dynamical system [19,22,24,27]. For equation (3.10), if one of the species becomes extinct, then either one semi-trivial solution becomes the equilibrium or both species coexist. If both species are harvested, they may become extinct when the harvesting effort exceeds the growth rate.
Let, $\gamma_1, \gamma_2 \in [0,1)$ then (3.10) becomes equivalent to (3.14) when $K_u(x) \equiv K_v(x) \equiv K(x)$ with

$$K \to (1-\gamma_2)K_u,\ K \to (1-\gamma_2)K_v,\ a = \frac{1-\gamma_1}{1-\gamma_2},\ r_1 = 1-\gamma_1,\ r_2 = 1-\gamma_2. \tag{3.15}$$

Theorem 5. *Suppose that $K_u(x) \equiv K_v(x) \equiv K(x)$ and $\frac{M(x)}{K_u(x)} \equiv Constant$. If $\gamma_1, \gamma_2 \in [0,1)$ and $\frac{N(x)}{K_v(x)} \not\equiv Constant$ with $d_1 \equiv d_2 \equiv d$ on x in Ω, then the succeeding three scenarios can hold for (3.10).*

(i) For any $\gamma_2 \in [0,1)$, there exists $\gamma \in (\gamma_2, 1)$ so that

$$\gamma \geq {\gamma_1}^* = 1 - \frac{\int_\Omega r v^*_{\gamma_2} dx}{\int_\Omega r K_v dx}. \tag{3.16}$$

Where $v^*_{\gamma_2}$ is a solution of (3.13) which satisfies $K_2(x) = (1-\gamma_2)K_v$ and $r_2 = \left(1-\gamma_2\right)$ so that for any $\gamma_1 \in (\gamma_2, \gamma)$, all solutions of (3.10) strongly persist, i.e. there exists a coexistence of (3.10).
(ii) For any $\gamma_2 \in [0,1)$, and $\gamma_1 \leq \gamma_2$ all solutions of (3.10) converge to $\left((1-\gamma_1)K_u, 0\right)$.
(iii) For any $\gamma_2 < 1$, there exists $\overline{\gamma} \in (\gamma_2, 1)$, such that for any $\gamma_1 \in (\overline{\gamma}, 1)$, all solutions of (3.10) converge to $(0, v^*_{\gamma_2})$. Where $K_2(x) = (1-\gamma_2)K_v$, $r_2 = \left(1-\gamma_2\right)$ and $(0, v^*_{\gamma_2})$ satisfies (3.13).

To prove the Theorem 5 at first, we will construct some auxiliary results for the system (3.14) following [29].

Lemma 12. *[29] Assume that $M(x)$, $N(x)$, $K(x)$ are non-constant and $\frac{M(x)}{K(x)} \equiv Constant$ where $\frac{N(x)}{K(x)} \not\equiv Constant$. Then for $a \in [1,\infty)$ the semi-trivial equilibrium $(aK, 0)$ of the system (3.14) is globally asymptotically stable on Ω.*

Lemma 13. *[29] Let $a \in (0,1)$, then $(aK, 0)$ of (3.14) is not stable.*

Let's discuss the case where a is in the range $(0,1)$ and determine whether both species can exist or only the second species can survive while the first species goes extinct, as described in [29].
Consequently, we will prove the existence of some a^*, so that it is inevitable to coexist for any $a \in (a^*, 1)$, where the numerical results on [29] in Example. 03, illustrates that for closely enough to one, *i.e.* for $a < 1$, there exists coexistence whereas for small enough a^* provides the solution $(0, v^*)$ is globally asymptotically stable.

Lemma 14. *Suppose that $\frac{M(x)}{K(x)} \equiv Constant$ but $\frac{N(x)}{K(x)} \not\equiv Constant$ with $d_1 \equiv d_2 \equiv d$ on Ω. Then there exists $a_1 \in (0,1)$ such that $(0, v^*)$ of (3.14) is unstable for $a \in (a_1, 1)$, where v^* satisfies (3.12) with $K_2(x) = K(x)$.*

Proof. Since $\frac{M(x)}{K(x)} \equiv Constant$, to manifest the instability of $(0, v^*)$ let us contemplate the linearization of the foremost equation of (3.14) around $(0, v^*)$ we get,

$$\frac{\partial u}{\partial t} = d\Delta\left(\frac{u}{M}\right) + r_1 r(x)\left(1 - \frac{v^*}{aK}\right)u,\ t>0,\ x\in\Omega.$$

For the linearization equation, the associated eigenvalue problem with standard boundary condition is given by

$$d\Delta\left(\frac{\phi}{M}\right) + r_1 r\left(1 - \frac{v^*}{aK}\right)\phi = \sigma\phi,\ x\in\Omega,\ \nabla\cdot\left(\frac{\phi}{M}\right) = 0,\ x\in\partial\Omega. \tag{3.17}$$

According to ([22], Theorem 2.1) the principal eigenvalue of (3.17) is defined by

$$\sigma_1 \int_\Omega \frac{\phi^2}{M} dx = \sup_{\phi\neq 0, \phi\in W^{1,2}} \left[-d \int_\Omega \left|\nabla(\frac{\phi}{M})\right|^2 dx + \int_\Omega r_1 r \frac{\phi^2}{M}\left(1-\frac{v^*}{aK}\right) dx \right].$$

Now, consider ψ_1 be the eigenfunction and since v^* does not depend on a, let $a=1$. Then the corresponding principal eigenvalues σ_1 is stated as

$$\sigma_1 = \sup_{\psi_1\neq 0, \psi_1\in W^{1,2}} \frac{-\int_\Omega d\left|\nabla\left(\frac{\psi_1}{M}\right)\right|^2 dx + \int_\Omega r_1 r\left(\frac{\psi_1^2}{M}\right) dx - \int_\Omega r_1 r\left(\frac{\psi_1^2}{M}\right)\frac{v^*}{K} dx}{\int_\Omega \frac{\psi_1^2}{M} dx}.$$

The steady-state $(0, v^*)$ will be unstable when we obtain such ψ_1 so that the assertion on the right-hand side is positive. Hence

$$a > a_1 := \int_\Omega r_1 r(x)\left(\frac{\psi_1^2}{M}\right)\frac{v^*}{K} dx \left[-\int_\Omega d\left|\nabla\left(\frac{\psi_1}{M}\right)\right|^2 dx + \int_\Omega r_1 r(x)\left(\frac{\psi_1^2}{M}\right) dx \right]^{-1}.$$

Hence for $a > a_1$, $\sigma_1 > 0$. So, the principal eigenvalue is positive.
Now, we will construct an estimate of a, which guarantees coexistence.
Since $\frac{M}{K} \equiv Constant$, taking $\phi = K(x)$, and letting $M = K$. Then the principal eigenvalue satisfies

$$\sigma_1 \geq \frac{\int_\Omega r_1 r(x) K\left(1-\frac{v^*}{aK}\right) dx}{\int_\Omega K dx}.$$

Assuming, $Q := \int_\Omega K dx$ and by Lemma 11

$$a^* = \frac{\int_\Omega r(x) v^* dx}{\int_\Omega r(x) K dx} < 1. \tag{3.18}$$

Then,

$$\sigma_1 \geq \left(\frac{r_1}{Q}\right)\left(\int_\Omega rK dx - \frac{1}{a}\int_\Omega r v^* dx\right).$$

So for $a \in (a^*, 1)$ and according to Lemma 11, we can conclude that σ_1 is positive. Hence, $(0, v^*)$ of (3.14) is unstable. □

Since, $(0,0)$ is unstable, according to Lemma 2 and, above all discussion on semi-trivial equilibrium solutions of (3.14) we can construct the following results.

Proposition 1. *If $\frac{M(x)}{K(x)} \equiv Constant$ and $\frac{N(x)}{K(x)} \not\equiv Constant$ with $d_1 \equiv d_2 \equiv d$ on Ω. Then for any $a_1 \in (0,1)$ there exists coexistence equilibrium of (3.14) if $a \in (a_1, 1)$, where $a_1 < a^*$ which defined in (3.18).*

Lemma 15. *Let for any $\frac{M(x)}{K(x)} \equiv Constant$ while $\frac{N(x)}{K(x)} \not\equiv Constant$ with $d_1 \equiv d_2 \equiv d$ on Ω. Then for any $a \in (0, a_2)$ the system (3.14) has no coexistence equilibrium whenever a_2 is defined in $a_2 \in (0,1)$.*

Proof. To prove Lemma 15, we will apply the method of upper and lower solutions technique and [22] to find the positive upper and lower bound.
Let us first assume that (u_c, v_c) is a coexistence solution of (3.14). Then suppose $K(x)$ is positive in $\overline{\Omega}$ and $K \in C^{1+\alpha}(\overline{\Omega})$.
Let the positive lower and upper bound of K is

$$0 < e := \inf_{x\in\overline{\Omega}} K(x) \quad \text{and} \quad E := \sup_{x\in\overline{\Omega}} K(x). \tag{3.19}$$

According to [4], if v^* is the positive stationary solution of (3.13), then there exists similar upper and lower positive bound such that

$$0 < l := \inf_{x\in\overline{\Omega}} v^*(x) \quad \text{and} \quad L := \sup_{x\in\overline{\Omega}} v^*(x). \tag{3.20}$$

So, we have $l \le E$, but we assume $v^* > K(x)$, which is contrary to the assumption. Thus we obtain, $\frac{l}{2E} \le 1$ for any x in Ω. Now, consider $\tilde{u} = \frac{lK}{2E} > 0$ be the upper solution of (3.14) and also recalling $M(x)$ is constant to $K(x)$ and assuming $\frac{\tilde{u}}{M}$ or $\frac{\tilde{u}}{K}$ is constant, $r_1 > 0$, and let $a \in (0, \frac{l}{2E})$. Then we obtain from (3.14)

$$0 = 0 + r_1 r\tilde{u}\left(1 - \frac{\tilde{u} + v_c}{aK}\right).$$

Therefore,

$$r_1 r\tilde{u}\left(1 - \frac{\tilde{u} + v_c}{aK}\right) \le r_1 r(x)\tilde{u}\left(1 - \frac{\tilde{u}}{aK}\right) = r_1 r(x)\tilde{u}\left(1 - \frac{lK}{2EaK}\right) = r_1 r\tilde{u}\left(1 - \frac{l}{2aE}\right) < 0,$$

for $a < \frac{l}{2E}$ in $x \in \Omega$, since we choose $a \in (0, \frac{l}{2E})$. So, $\tilde{u} = \frac{lK}{2E}$ is an upper solution of (3.14) by [21]. It is obvious to see zero is a lower solution of (3.14). From (3.19) and (3.20) we get,

$$0 \le u_c(x) \le \frac{lK(x)}{2E} \le \frac{l}{2} \le \frac{v^*(x)}{2} \le v_c(x)$$

Now for the second equation of (3.14) choosing $\tilde{v} = \frac{v^*}{2}$, which immediately shows that it is a lower solution, as v^* is a stable solution of (3.13), and N and K are linearly independent. So,

$$\begin{aligned} d\left[\Delta\left(\frac{\tilde{v}}{N}\right)\right] + r_2 r(x)\tilde{v}\left(1 - \frac{u_c + \tilde{v}}{K}\right) &\ge \frac{d}{2}\left[\Delta\left(\frac{v^*}{N}\right)\right] + r_2 r(x)\tilde{v}\left[1 - \frac{0.5v^* + 0.5v^*}{K}\right] \\ &= \frac{d}{2}\left[\Delta\left(\frac{v^*}{N}\right)\right] + r_2 r(x)\tilde{v}\left[1 - \frac{v^*}{K}\right] = 0 \end{aligned}$$

Thus, for $v_c \ge \frac{v^*}{2}$ and $a < \frac{l}{2E}$ from the first equation of (3.14) we get

$$0 = d\left[\Delta\left(\frac{u_c}{M}\right)\right] + r_1 r(x)u_c\left(1 - \frac{u_c + v_c}{aK}\right). \tag{3.21}$$

Then,

$$\begin{aligned} d\left[\Delta\left(\frac{u_c}{M}\right)\right] + r_1 r(x)u_c\left(1 - \frac{u_c + v_c}{K}\right) &\le d\left(\Delta\left(\frac{u_c}{M}\right)\right) + r_1 r(x)u_c\left(1 - \frac{v_c}{aK}\right), \\ &\le d\left(\Delta\left(\frac{u_c}{M}\right)\right) + r_1 r(x)u_c\left(1 - \frac{v^*}{2aK}\right), \\ &\le d\left(\Delta\left(\frac{u_c}{M}\right)\right) + r_1 r(x)u_c\left(1 - \frac{l}{2aE}\right), \\ &\le d\left(\Delta\left(\frac{u_c}{M}\right)\right) + 0, \quad \text{except} \quad u_c \equiv M. \end{aligned}$$

Now, integrating the inequality and using the boundary conditions and $u_c \not\equiv 0$, we get,

$$0 = \int_\Omega d\left(\Delta\left(\frac{u_c}{M}\right)\right)dx + \int_\Omega r_1 r(x)u_c\left(1 - \frac{u_c + v_c}{aK}\right)dx < \int_\Omega d\left(\Delta\left(\frac{u_c}{M}\right)\right)dx = 0,$$

which is a contradiction that concludes that there exists no coexistence of (3.14). □

After analyzing the results and discussion on the instability of the equilibria at $(0,0)$ and (u_c, v_c), the global stability of $(0, v*)$ is simply proven by Lemma 2, which leads to the following conclusion.

Proposition 2. *Assume $\frac{M(x)}{K(x)} \equiv Constant$ while $\frac{N(x)}{K(x)} \not\equiv Constant$ with $d_1 \equiv d_2 \equiv d$ on Ω. Then for any $a \in (0, a_2)$, where $a_2 \in (0,1)$ such that all solutions of (3.14) converge to $(0, v^*)$.*

The main result of Theorem 5 is now ready to proceed.

Proof. (Proof of Theorem 5)
Since, system (3.10) is become of the form of system (3.14) with (3.15) for $\gamma_2 \in [0,1)$. Then

(i) For fix $\gamma_2 \in [0,1)$, by Proposition 1 there exists $a_1 = \dfrac{1-\gamma}{1-\gamma_2} \in (0,1)$ such that for any $a_1 \le a^*$ there is $a \in (a_1, 1)$, whenever a^* is defined in (3.18), coexistence equilibrium of (3.10) is strongly exist, where $K(x)$ is defined as $K(x) = (1-\gamma_2)K_v$ with $r_2 = (1-\gamma_2)$. Since, $\gamma = 1 - a_1(1-\gamma_2)$ and $v^* = v^*_{\gamma_2}$ is also depend on γ_2 and the maximum bound $\gamma^* = 1 - a^*(1-\gamma_2)$ has of the form (3.16). So for any $\gamma_1 \in (\gamma_2, \gamma)$ which correlate with $a \in (a_1, 1)$ all solutions of the system (3.10) strongly persist by Proposition 1.

(ii) If $\gamma_2 \in [0,1)$ and $\gamma_1 \le \gamma_2$ then straight away we get $a \ge 1$. Then, $(u^*, 0)$ is stable and converges to $\big((1-\gamma_1)K_u, 0\big)$ according to Lemma 12.

(iii) For any $\gamma_2 \in [0,1)$, *i.e. for* $\gamma_2 < 1$, *by Proposition 2 there exists* $a_2 = \frac{1-\overline{\gamma}}{1-\gamma_2} \in (0,1)$ *such that for any* $a \in (0, a_2)$ *all solutions of* (3.10) *converge to* $(0, v^*) = (0, v^*_{\gamma_2})$. *Where* $\overline{\gamma} = 1 - a_2(1-\gamma_2) \in (0,1)$ *whenever* $\gamma_1 \in (\overline{\gamma}, 1)$. *Here* $v^*_{\gamma_2}$ *satisfies* (3.13) *so that* $K_2 = (1-\gamma_2)K_v$ *with* $r_2 = \big(1-\gamma_2\big)$.

Which concludes the proof. □

Next, following the situation of ideal free pair, i.e. when M and N form a linear combination of K. Then for

$$\frac{M}{K} \not\equiv Constant, \quad and \quad \frac{N}{K} \not\equiv Constant \tag{3.22}$$

neither $(K,0)$ nor $(0,K)$ with $\gamma_1 = \gamma_2 = 0$ is a solution of (3.10). So, for linearly independent M and N on Ω, and for $\alpha > 0$, $\beta > 0$, there exists

$$K(x) = \alpha M(x) + \beta N(x), \quad x \in \Omega. \tag{3.23}$$

Where K is the convex hull of M and N and equation (3.23) provides the system (3.10) has unique coexistence equilibrium $(\alpha M(x), \beta N(x))$.

Theorem 6. *Assume that* $K_u(x) \equiv K_v(x) \equiv K(x)$ *and* M *and* N *are not linearly dependent on* Ω *and* M, N *and* K *satisfy* (3.22) *and* (3.23) *with* $\alpha > 0$, $\beta > 0$. *Then for any* $\gamma_2 \in [0,1)$ *and* $d_1 \equiv d_2 \equiv d$ *in* Ω *the following three scenarios can hold for* (3.10).

(i) For any fix $\gamma_2 \in [0,1)$, *all solutions of* (3.10) *will strongly persist if* $\gamma_1 \in (\gamma, \overline{\gamma})$, *where* $\gamma \in [0, \gamma_2)$ *and* $\overline{\gamma} \in (\gamma_2, 1)$.

(ii) For any fix $\gamma_2 \in [0,1)$, *there exists* $\tilde{\gamma} \in (\gamma_2, 1)$ *such that while* $\gamma_1 \in (\tilde{\gamma}, 1)$, *then* $(0, v^*)$ *of* (3.10) *converges to* $(0, v^*_{\gamma_2})$, *whenever* $v^*_{\gamma_2}$ *satisfies* (3.13) *with* $K_2 = (1-\gamma_2)K_v(x)$, $r_2 = (1-\gamma_2)$.

(iii) For any fix $\gamma_2 \in (\gamma_2^*, 1)$, *where* $\gamma_2^* \in (0,1)$ *there exists* $\gamma_0(\gamma_2) \in [0, \gamma_2)$ *so that for any* $\gamma_1 \in [0, \gamma_0)$, $(u^*, 0)$ *of* (3.10) *converges to* $(u^*_{\gamma_1}, 0)$ *where* $u_{\gamma_1^*}$ *satisfies* (3.12) *with* $K_1(x) = (1-\gamma_1)K_u(x)$ *and* $r_1 = (1-\gamma_1)$.

In a similar way, to prove the main result of Theorem (6), at first, we will produce some auxiliary results followed by [29]. As in Theorem (5), for $\gamma_2 \in [0,1)$ we can present an estimate such that

$$\overline{\gamma} \ge \gamma_1{}^* = 1 - \frac{\int_\Omega M r v^*_{\gamma_2}/K_v dx}{\int_\Omega r M dx}, \tag{3.24}$$

all solutions of (3.10) strongly persist.

Lemma 16. *Let for any* $\alpha, \beta > 0$, $K(x) = \alpha M + \beta N$ *while* $\frac{M}{K} \not\equiv Constant$ *and* $\frac{N}{K} \not\equiv Constant$ *with* $d_1 \equiv d_2 \equiv d$. *Then for any* $a \in (0,1]$, $(u^*, 0)$ *of* (3.14) *is not stable.*

Proof. Assume the linearized eigenvalue problem associated with the second equation of (3.14) throughout $(u^*, 0)$ we get,

$$d\Delta\left(\frac{\phi}{N(x)}\right) + r_2 r(x)\left(1 - \frac{u^*}{K}\right)\phi = \sigma\phi, \; x \in \Omega, \; \nabla\cdot\left(\frac{\phi}{N}\right) = 0, \; x \in \partial\Omega. \tag{3.25}$$

The principal eigenvalue of (3.25) according to ([22], Theorem 2.1) is defined as

$$\sigma_1 = \sup_{\phi \ne 0, \phi \in W^{1,2}} \frac{-d\int_\Omega \left|\nabla(\frac{\phi}{N})\right|^2 + \int_\Omega r_2 r \frac{\phi^2}{N}\left(1 - \frac{u^*}{K}\right)dx}{\int_\Omega \frac{\phi^2}{N} dx}.$$

Using the boundary condition and choosing $\phi = N$, we observe that the principal eigenvalue is not less than σ_1, *i.e.*

$$\sigma_1 \ge \frac{r_2 \int_\Omega r N(1 - \frac{u^*}{K})dx}{\int_\Omega N dx}. \tag{3.26}$$

Now integrating (3.12) with $K_1 = aK$ and using the boundary condition for u^*, we get

$$0=\int_{\Omega} r_1 r u^* \left(1-\frac{u^*}{aK}\right)dx,$$
$$=r_1\int_{\Omega} r\left(u^*-aK+aK\right)\left(1-\frac{u^*}{aK}\right)dx,$$
$$=-r_1\int_{\Omega}\frac{r}{aK}\left(u^*-aK\right)^2 dx+ar_1\int_{\Omega} r(\alpha M+\beta N)\left(u^*-aK\right)^2 dx,\ \text{where}\quad K=\alpha M+\beta N$$
$$=-r_1\int_{\Omega}\frac{r}{aK}\left(u^*-aK\right)^2 dx-a\alpha r_1\int_{\Omega} rM\left(\frac{u^*}{aK}-1\right)dx+a\beta r_1\int_{\Omega} rN\left(1-\frac{u^*}{aK}\right)dx.$$

Thus,

$$-r_1\int_{\Omega}\frac{r}{aK}\left(u^*-aK\right)^2 dx-a\alpha r_1\int_{\Omega} rM(\frac{u^*}{aK}-1)dx+a\beta r_1\int_{\Omega} rN(1-\frac{u^*}{aK})dx=0. \tag{3.27}$$

If $\frac{M(x)}{K(x)}\not\equiv Constant$ and $\frac{N(x)}{K(x)}\not\equiv Constant$, then the substitution of u^* in (3.12) with $K_1=aK$ leads to conclusion that $u^*-aK\not\equiv 0$. Since from (3.27) we see that the first term in the expression is negative and according to Lemma 10 the next term in (3.27) is also negative, as the total sum is equal to zero, that immediately confirm that, $\int_{\Omega} rN(1-\frac{u^*}{aK})dx>0$.

Since, $a\in(0,1]$ then

$$\int_{\Omega} rN\left(1-\frac{u^*}{aK}\right)dx=\int_{\Omega} rN\left(1-\frac{u^*}{K}-\frac{u^*}{aK}+\frac{u^*}{K}\right)dx=\int_{\Omega} rN\left(1-\frac{u^*}{K}\right)dx+\frac{a-1}{a}\int_{\Omega} rN\frac{u^*}{K}dx>0.$$

While $\frac{a-1}{a}\le 0$, then the foremost term in the right side expression is positive, that is

$$\int_{\Omega} rN\left(1-\frac{u^*}{K}\right)dx>0. \tag{3.28}$$

Hence from (3.26) according to ([22], Theorem 2.1) the principal eigenvalues are given by using (3.28),

$$\sigma_1\ge\frac{r_2\int_{\Omega} rN(1-\frac{u^*}{K})dx}{\int_{\Omega} N dx}>0.$$

Since $\sigma_1>0$, so the equilibrium $(u^*,0)$ is unstable. □

Lemma 17. *Assume M and N are linearly independent and for some $\alpha>0$, $\beta>0$ equations (3.22) and (3.23) holds with $d_1\equiv d_2\equiv d$. Then for any $a\in(a_1,1]$, where $a_1\in(0,1)$, $(0,v^*)$ of (3.14) is unstable.*

The proof is similar to Lemma 16 and is omitted.

Lemma 18. *Let $M(x)$ and $N(x)$ be non-constant and for some $\alpha>0$, $\beta>0$, equations (3.22) and (3.23) holds with $d_1\equiv d_2\equiv d$. Then for $a\in(a_1,\infty)$, there exists $a_1\in(0,1)$ such that $(0,v^*)$ of (3.14) is unstable.*

Proof. Following the statement of Lemma 17, we can establish the stability for $a\in[1,\infty)$. So from Lemma 16, for particular a, $(u^*,0)$ is not stable, while for certain set of a by Lemma 16, $(0,v^*)$ is unstable. Hence, according to monotone dynamical system [19,22,24,27] when a satisfies both conditions, *i.e.* when both semi-trivial equilibria are not stable then by Lemma 2 coexistence solution will strongly persist. □

Proposition 3. *Let $\frac{M}{K}\not\equiv Constant$ and $\frac{N}{K}\not\equiv Constant$ and M,N,K are non-constant and for some $\alpha,\beta>0$, $K=\alpha M+\beta N$. Then for any $a_1\in(0,1)$ and $a_2\in(1,\infty)$ there exists $a\in(a_1,a_2)$ such that all solutions of (3.14) is strongly persist.*

Lemma 19. *Let for any $\alpha>0$, $\beta>0$ the equations (3.22) and (3.23) holds with $d_1\equiv d_2\equiv d$. Then for $a\in(0,a_2)$, where $a_2\in(0,1)$ there exists no coexistence equilibrium of the system (3.14).*

Proof. Analogous to the proof of Lemma 15, we assume that there retains a coexistence solution (u_c,v_c) of (3.14). Then K and M are positive in $x\in\Omega$ and $K,M\in C^{1+\alpha}(\overline{\Omega})$.

Let the positive lower and upper bound of K and M exist and also for $\frac{K}{M}$ we define,

$$0 < e_K := \inf_{x\in\Omega} \frac{K(x)}{M(x)} \quad \text{and} \quad E_K := \sup_{x\in\Omega} \frac{K(x)}{M(x)}. \tag{3.29}$$

Also, $P := \max_{x\in\Omega} K(x)$. Then, according to Lemma 15, if v^* is the positive stationary solution of (3.13), then its upper and lower bounds are

$$0 < l_K := \inf_{x\in\overline{\Omega}} v^*(x) \quad \text{and} \quad L_K := \sup_{x\in\overline{\Omega}} v^*(x) \quad \text{with} \quad K_2 = K. \tag{3.30}$$

Hence, $l_K \le P$. Thus we have $\frac{l_K}{2P} < 1$, and $e_K \le \frac{K(x)}{M(x)} \le E_K$.
Now considering $\tilde{u} = \frac{l_K e_K M}{2P}$ be the upper solution of (3.14) and assuming $v_c \ge 0$ and since $\frac{\tilde{u}}{M}$ is constant and $r_1 > 0$, so we get from the first equation of model (3.14)

$$\begin{aligned}
&0 = 0 + r_1 r\tilde{u}\left(1 - \frac{\tilde{u} + v_c}{aK}\right)\\
&r_1 r(x)\tilde{u}\left(1 - \frac{\tilde{u} + v_c}{aK}\right) \le r_1 r\tilde{u}\left(1 - \frac{\tilde{u}}{aK}\right)\\
&\qquad = r_1 r\tilde{u}\left(1 - \frac{l_K e_K M}{2PaK}\right)\\
&\qquad = r_1 r\tilde{u}\left(1 - \frac{l_K e_K}{2PE_K a}\right) < 0
\end{aligned}$$

for $a \in (0, a^{**})$ in Ω, where $a^{**} = \frac{l_L e_K}{2E_K P} < \frac{1}{2}$.
Thus, $\tilde{u} = \frac{l_K e_K M}{2P}$ is an upper solution of (3.14) by [21], and it is easy to check that zero is a lower solution of (3.14). So, from (3.29) and (3.30) we get

$$0 \le u_c(x) \le \frac{l_K e_K M}{2P} \le \frac{l_K}{2} \le \frac{v^*(x)}{2} \le v_c(x)$$

Similar to Lemma 15, we can choose $\tilde{v} = \frac{v^*}{2}$ is the lower solution [21] for the second equation of (3.14) with $K_2 = K$ as v^* is a stationary solution of (3.13). So,

$$\begin{aligned}
d\left(\Delta\left(\frac{\tilde{v}}{N}\right)\right) + r_2 r(x)\tilde{v}\left(1 - \frac{u_c + \overline{v}}{K}\right) &\ge \frac{d}{2}\left(\Delta\left(\frac{v^*}{N}\right)\right) + r_2 r(x)\tilde{v}\left(1 - \frac{0.5v^* + 0.5v^*}{K}\right),\\
&= \frac{d}{2}\left(\Delta\left(\frac{v^*}{N}\right)\right) + r_2 r(x)\tilde{v}\left(1 - \frac{v^*}{K}\right) = 0.
\end{aligned}$$

Thus, $v_c \ge \frac{v^*}{2}$ and $a \in (0, \frac{l_K e_K}{2E_K P})$, then from the first equation of (3.14) we get

$$0 = d\left(\Delta\left(\frac{u_c}{M}\right)\right) + r_1 r(x)u_c\left(1 - \frac{u_c + v_c}{aK}\right).$$

So,

$$\begin{aligned}
d\left(\Delta\left(\frac{u_c}{M}\right)\right) + r_1 r(x)u_c\left(1 - \frac{u_c + v_c}{aK}\right) &\le d\left(\Delta\left(\frac{u_c}{M}\right)\right) + r_1 r(x)u_c\left(1 - \frac{v_c}{aK}\right),\\
&\le d\left(\Delta\left(\frac{u_c}{M}\right)\right) + 0, \quad \text{except} \quad u_c \equiv M.
\end{aligned}$$

Now, integrating the inequality and applying the boundary conditions and $u_c \not\equiv 0$, we get,

$$0 = \int_\Omega d\left(\Delta\left(\frac{u_c}{M}\right)\right)dx + \int_\Omega r_1 r u_c\left(1 - \frac{u_c + v_c}{aK}\right)dx < \int_\Omega d\left(\Delta\left(\frac{u_c}{M}\right)\right)dx = 0.$$

That is a contradiction, which concludes that there exists no coexistence of (3.14). □

Lemma 20. *Let M and N be linearly independent and for any $\alpha, \beta > 0$, $K = \alpha M + \beta N$, where $\frac{M}{K} \not\equiv Constant$ and $\frac{N}{K} \not\equiv Constant$. Then for $a \in (a_3, \infty)$ whenever $a \in (1, \infty)$ there exists no coexistence equilibrium for system (3.14).*

Proposition 4. *Let $\frac{M}{K} \not\equiv Constant$ and $\frac{N}{K} \not\equiv Constant$ and M, N, K are non-constant and for some $\alpha > 0$, $\beta > 0$, $K = \alpha M + \beta N$. Then for $a_1 \in (0,1)$ there exists $a \in (0, a_1)$ such that all solutions of (3.14) converges to $(0, v^*)$, while for $a_2 \in (1, \infty)$, there exists $a \in (a_2, \infty)$ such that $(u^*, 0)$ is globally stable.*

We are now prepared to carry on the main result of Theorem 6.

Proof. (Proof of Theorem 6) Since system (3.10) is become of the form of system (3.14) with (3.15) for $\gamma_2 \in [0,1)$. Then

(i) For fix $\gamma_2 \in [0,1)$, by Proposition 3, there exists $a_1 = \frac{1-\overline{\gamma}}{1-\gamma_2} \in (1,\infty)$ and $a_2 = \frac{1-\gamma}{1-\gamma_2} \in (0,1)$ such that for $a \in (a_1, a_2)$, all solutions of (3.10) will strongly persist. That is for $\gamma = 1 - \max\{1 - (1-\gamma_2)a_2, 0\}$ and $\overline{\gamma} = 1 - (1-\gamma_2)a_1$, then $0 \le \gamma \le \gamma_2 < \overline{\gamma} < 1$. And by Proposition 3 for $\gamma \in (\gamma, \overline{\gamma})$ there exists coexistence equilibrium.

(ii) For any $\gamma_2 \in [0,1)$, by Proposition 4, there exists $a_1 \in (0,1)$ such that for $a \in (0, a_1)$ all solutions of (3.10) converge to $(0, v^*_{\gamma_2})$. Now, letting $a_1 = \frac{1-\tilde{\gamma}}{1-\gamma_2}$, so that $\tilde{\gamma} = 1 - a_1(1-\gamma_2) \in (\gamma_2, 1))$. Then for $\gamma \in (\tilde{\gamma}, 1)$, we get $a \in (0, a_1)$ and hence all solutions of (3.10) converge to $(0, v^*_{\gamma_2})$ with $K_2 = (1-\gamma_2)K_v$, $r_2 = (1-\gamma_2)$.

(iii) For any fix $\gamma_2 \in [0,1)$, by Proposition 4, there exists $a_1 \in [1,\infty)$ such that for any $a \in (a_1, \infty)$ all solutions of (3.10) converge to $(u^*_{\gamma_1}, 0)$. Thus for fix $\gamma_2 \in (\gamma_2^*, 1)$ let, $a_2 = \frac{1-\gamma_0}{1-\gamma_2}$ so that $\gamma_0 = 1 - a_2(1-\gamma_2)$. Then $a_2 > 1$ and $a \in (a_2, \infty)$ related to $0 < \gamma_0 < \gamma_2 < 1$ and $\gamma_1 \in [0, \gamma_2)$. Thus for $\gamma_1 \in [0, \gamma_0)$ all solutions of system (3.10) converge to $(u^*_{\gamma_1}, 0)$, where $K_1 = (1-\gamma_1)K_u$, and $r_1 = (1-\gamma_1)$ so that $u^*_{\gamma_1}$ satisfies (3.12).

Which concludes the proof. □

Theorem 7. *Suppose that $K_u(x) \equiv K_v(x) \equiv K(x)$, and $\frac{M(x)}{K_u(x)} \equiv Constant$ and $\frac{N(x)}{K_u(x)} \not\equiv Constant$. If $\gamma_1 \in [0,1)$ and $\gamma_2 \ge 1$ then for any $u_0(x) \not\equiv 0$ and $u_0(x) > 0$, all solutions of (3.10) converge to $((1-\gamma_1)K_u, 0)$, and similarly for any $v_0(x) \not\equiv 0$ and $v_0(x) > 0$ if $\gamma_2 \in [0,1)$ and $\gamma_1 \ge 1$ then all solutions of (3.10)converges to $(0, v^*_{\gamma_2})$. If $\gamma_1 \ge 1$ and $\gamma_2 \ge 1$, then the zero solution is global attractive.*

Proof. For any $\gamma_1 \in [0,1)$ and if we assume $\gamma_2 \ge 1$, then from the second equation of (3.10) we will get $\lim_{t\to\infty} v(t,x) = 0$, when $u(t,x) \ge 0$. So according to [4] by applying different inequalities the first equation of (3.10) immediately provide, $\lim_{t\to\infty} u(t,x) = (1-\gamma_1)K_u$ with $r_1 = (1-\gamma_1)$ for $\gamma_1 \in (0,1)$. Hence we can conclude that $(u^*_{\gamma_1}, 0) = \big((1-\gamma_1)K_u, 0\big)$ is globally asymptotically stable.
Similarly, for $\gamma_1 \ge 1$ and $\gamma_2 \in [0,1)$, the first equation of (3.10) immediately shows, $\lim_{t\to+\infty} u(t,x) = 0$, with independent of $v(t,x) \ge 0$, and all solutions of (3.10) converge to $(0, v^*_{\gamma_2})$ for $\gamma_2 \in [0,1)$. Thus the semi-trivial equilibrium $(0, v^*_{\gamma_2})$ will globally asymptotically stable.
In a similar manner, it is obvious that for $\gamma_1 \ge 1$, and $\gamma_2 \ge 1$ the zero solution is globally attractive. □

3.3. Maximum sustainable yield (MSY)

Theorem 8. *If the first species of system (3.10) is harvested and the other species is subject to culling, then for $\frac{M}{K} \equiv Constant$ and for any $N(x)$ the maximum sustainable yield for system (3.10) is attained at $\gamma_1 = 0.5$, $\gamma_2 = 0.5$ or $\gamma_2 \ge 0.5$ when $\frac{N}{K} \not\equiv Constant$. If both species are harvested then for $K = M + N$, i.e. when K is the convex hull of M and N then for $\gamma_1 = 0.5$, $\gamma_2 = 0.5$ leading to optimal sustainable yield for system (3.10), which is for both cases*

$$MSY := \frac{1}{4}\int_\Omega rK\,dx. \tag{3.31}$$

Proof. Let the maximum of the function, $g(t,x,u,v) = ru\left(1 - \frac{u+v}{K}\right)$ be attained at $u_{\max}$ and $v_{\max}$ where the function g satisfies $\frac{\partial(u_{\max}/M)}{\partial n} = 0$, for $(t,x) \in \partial\Omega$. We recall that [4] for a particular K the maximum of the function $g(y) = \left(1 - \frac{y}{K}\right)$ is attained at $y_{\max} = K/2$. Then for $K_u \equiv K_v \equiv K$, $d_1 \equiv d_2 \equiv d$, $H_1 = \gamma_1 r(x)$ and $H_2 = \gamma_2 r(x)$ from the first equation of (3.10) we get,

$$d\Delta\left(\frac{u_{\max}}{M}\right) + ru_{\max}\left(1 - \frac{u_{\max} + v_{max}}{K}\right) = H_1(x)u_{\max}(x),\ t > 0,\ x \in \Omega;\ \text{where } H_1(x) = \gamma_1 r(x). \tag{3.32}$$

Integrating (3.32) over Ω and using the Neumann boundary conditions, the maximum sustainable yield is

$$\begin{aligned} MYS := \int_\Omega \gamma_1 r(x)u_{\max}(x)dx &= \int_\Omega d\left[\Delta\left(\frac{u_{\max}}{M}\right)\right]dx + \int_\Omega r(x)u_{\max}\left(1 - \frac{u_{max} + v_{max}}{K}\right)dx \\ &\le \int_\Omega d\left[\Delta\left(\frac{u_{\max}}{M}\right)\right]dx + \int_\Omega r(x)u_{\max}\left(1 - \frac{u_{max}}{K}\right)dx \le 0 + \int_\Omega r(x)\frac{K}{2}\frac{1}{2}dx \le \frac{1}{4}\int_\Omega rK\,dx. \end{aligned}$$

If $u_{\max}$ is proportional to $K(x)$ that is, for $\frac{M(x)}{K(x)} \equiv Constant$, then the boundary condition is satisfied.

$$\int_{\Omega} \gamma_1 r(x) u_{\max}(x) dx \le \frac{1}{4} \int_{\Omega} rK dx.$$

So, for either for $\frac{M}{K} \equiv Constant$, and for any $N(x)$ the MYS is obtained for $\gamma_1 = 0.5$, $\gamma_2 > 0.5$ or for any $\gamma_2 \ge 0.5$ and $\frac{N}{K} \not\equiv Constant$. That is when the first species of system (3.10) is harvested, and other species are considered subject to culling then, the equilibrium solution $(0.5K, 0)$ leads to (3.31). Similarly, for the case of ideal free pair, so that if $K = M + N$, then MYS is,

$$\begin{aligned} MYS := &\int_{\Omega} \gamma_1 r(x) u_{\max}(x) dx + \int_{\Omega} \gamma_2 r(x) v_{\max}(x) dx \\ &= 0 + \int_{\Omega} r(x) u_{\max} \left(1 - \frac{u_{max} + v_{max}}{K}\right) dx + \int_{\Omega} r(x) v_{\max} \left(1 - \frac{u_{max} + v_{max}}{K}\right) dx \\ &= \int_{\Omega} r(x) \left(u_{\max} + v_{\max}\right) \left(1 - \frac{u_{max} + v_{\max}}{K}\right) dx \le \frac{1}{4} \int_{\Omega} rK dx \end{aligned}$$

So, when considering the ideal free pair MSY is achieved for a suitable choice of M and N so that $K = M(x) + N(x)$ and $\gamma_1 = \gamma_2 = 0.5$. That is when both species are harvesting. Which concludes the proof. □

4. Numerical methods and applications

4.1. Numerical methods

We used an implicit-explicit finite difference method, which resulted in a set of algebraic equations. The space and time were divided into a uniform grid size to discretize them. To solve the algebraic equations at every time step, we used the Crank-Nicolson method for 1-D in space and the Alternating-direction implicit (ADI) method for 2-D in space. We consider a uniform grid of equal spacing $\nabla x \equiv \nabla y \equiv h_x \equiv h_y$ on $\Omega = \{(x, y) | x_0 \le x \le x_n, y_0 \le y \le y_n\}$ with $h_x \equiv \frac{x_n - x_0}{N_x - 1}$, and $h_y \equiv \frac{y_n - y_0}{N_y - 1}$ where N_x and N_y are the number of grid points along x and y direction, respectively. Also, divide the time T into a line by a distance $\nabla t = h_t = \frac{T}{N_t}$ parallel to the x and y axis. The explanation procedure is sustained until the steady-state solution is reached. The convergence criterion for the solution procedure is defined as $\left|u^{n+1} - u^n\right| \le 10^{-7}$, where n is the number of iterations. We considered the spatial domain to be $[0, 1]$ in one space dimension and $[0, 1] \times [0, 1]$ in two space dimensions. In one-dimensional space, we can choose any length for a domain in the x-direction. However, we can consider a more general domain in two-dimensional space, but for simplicity's sake, we often restrict it to a square shape.

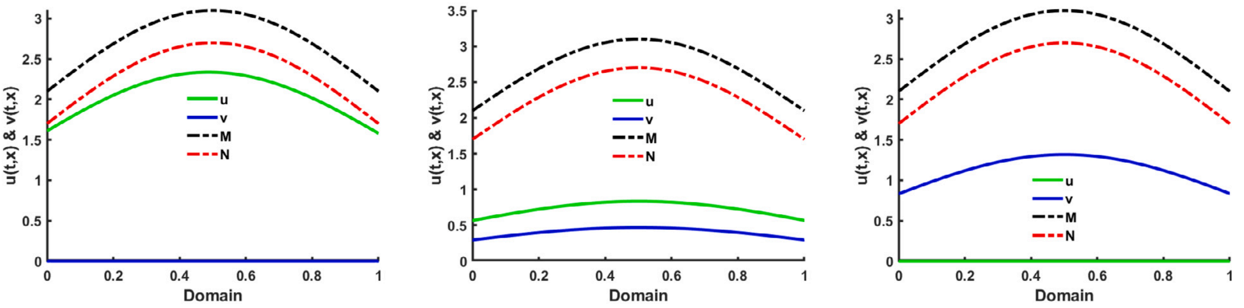
4.2. Applications for spatially distributed parametric values

In this segment, we will illustrate population density profiles for different values of model parameters in one and two space dimensions. We will also discuss their ecological implications. We have considered two cases for harvesting effects. For Case I, as per Theorem 3 and Theorem 4, any positive initial condition must result in $z(t, x) \to z(x)$ or $w(t, x) \to w(x)$ when H_1 and H_2 are unequal. For Case II, we have completed the theoretical analysis by numerically computing the existence of species when both H_1 and H_2 are proportional to space-dependent r. When capacity and resource functions are time-independent in two space dimensions, we will display the steady-state $u(x, y)$ and $v(x, y)$ through contour plots, which approach in the limit $t \to \infty$. We have used a range of different parametric values to assess the model's effectiveness through numerical computation.

4.3. Case of 1-space dimension

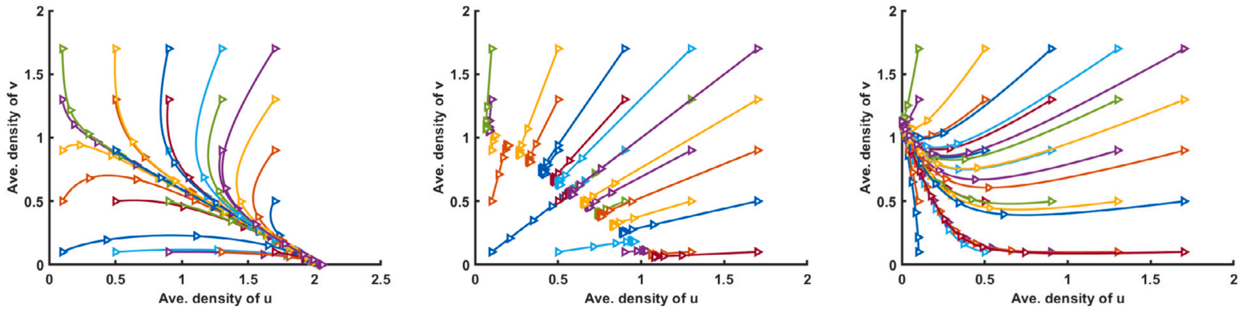
In this part of the section, we will analyze the numerical simulations for one dimension in space for both Case I and Case II.

Example 1. Consider the case of (3.1) for $K(x) = \cos(\pi x) + 2.5$ on $\Omega = (0, 1) \subset \mathbb{R}$ where both species follows the resource based diffusion strategy with $M(x) = \sin(\pi x) + 2.1$, $N(x) = \sin(\pi x) + 1.7$. Additionally let, $r = 1.0$, $u_0 = v_0 = 1.6$, $d_1 = d_2 = 1.0$ at $t = T = 1000$. Here Fig. 1 (a,b,c) presents the population density profiles of u and v over domain x for different space-dependent harvesting coefficients. In Fig. 1 (a), when $H_1 < H_2$, species v becomes extinct as predicted by Theorem 3 and Remark 1. Additionally, a non-trivial solution is found for species u. Conversely, in Fig. 1 (c), when $H_1 > H_2$, the opposite is observed. It is noted from Fig. 1 (b) that coexistence equilibrium is also possible when considering equal harvesting for both species. It is noteworthy that density profiles consistently align with their respective resource functions across all scenarios, regardless of initial values being non-negative and non-trivial. Implementing restrictions on the amount of species harvested from specific populations proves to be an effective strategy for preserving population stability during hunting endeavors. Our research shows that hunting can have localized impacts on populations, particularly on those that are resident and territorial. This study is a critical first step in comprehending the potential



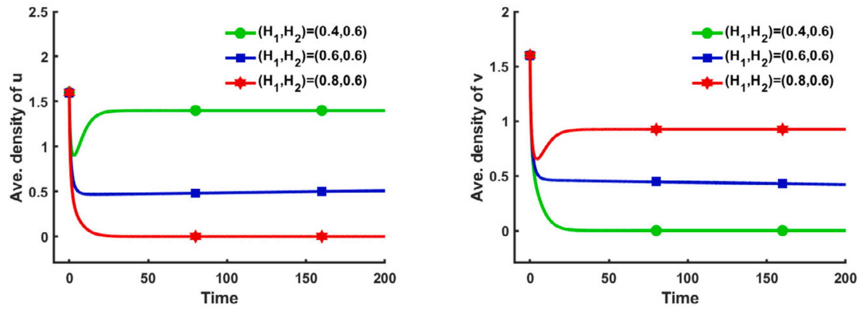
(a) (b) (c)

Fig. 1. Solution of (3.1) at $t = T = 1000$ where $K(x) = \cos(\pi x) + 2.5$, $M(x) = \sin(\pi x) + 2.1$, $N(x) = \sin(\pi x) + 1.7$, $r = 1.0$, $u_0 = v_0 = 1.6$, $d_1 = d_2 = 1.0$, $H_2 = 0.5 + 0.2\cos(\pi x)$ for (a) $H_1 = 0.1 + 0.3\cos(\pi x)$, (b) $H_1 = 0.5 + 0.2\cos(\pi x)$, and (c) $H_1 = 0.9 + 0.1\cos(\pi x)$ on $\Omega = (0, 1)$.



(a) (b) (c)

Fig. 2. Solution trajectories of space average density of u Vs v for different initial values (u_0, v_0) with $K(x) = \cos(\pi x) + 2.5$, $M(x) = \sin(\pi x) + 2.1$, $N(x) = \sin(\pi x) + 1.7$, $r = 1.0$, $d_1 = d_2 = 1.0$, $H_2 = 0.5 + 0.2\cos(\pi x)$ for (a) $H_1 = 0.1 + 0.3\cos(\pi x)$, (b) $H_1 = 0.5 + 0.2\cos(\pi x)$, and (c) $H_1 = 0.9 + 0.1\cos(\pi x)$ on $\Omega = (0, 1)$.



(a) (b)

Fig. 3. Space average density of (a) u, and (b) v for (3.1) for different values of harvesting coefficients on $\Omega = (0, 1)$ with $K(x) = \cos(\pi x) + 2.5$, $M(x) = \sin(\pi x) + 2.1$, $N(x) = \sin(\pi x) + 1.7$, $r = 1.0$, $u_0 = v_0 = 1.6$, $d_1 = d_2 = 1.0$.

regional impacts of hunting on various species management, where harvesting depends on the spatial location. However, Fig. 2 illustrates the solution trajectories for space average density of u Vs v for different initial values (u_0, v_0). We found that for unequal values of harvesting coefficient, there is a competitive exclusion for both species as in Fig. 2 (a) and Fig. 2 (c). Also, the coexistence solution is obvious for considering $H_1 = H_2$ in Fig. 2 (b).

Example 2. Assume the functions, $K(x) = \cos(\pi x) + 2.5$, $M(x) = \sin(\pi x) + 2.1$, and $N(x) = \sin(\pi x) + 1.7$ at $t = T = 1000$ for (3.1) where $r = 1.0$, $u_0 = v_0 = 1.6$, $d_1 = d_2 = 1.0$. We note that when the resource functions M and N are not proportional to K, a similar result as in Example 1 is found for considering different constant harvesting coefficients (equal/unequal) in Fig. 3 (a-b).

Example 3. Consider (3.1), concretely for $M(x) = 0.6\cos(\pi x) + 1.3$, $N(x) = 0.4\cos(\pi x) + 1.2$, and $K(x) = M(x) + N(x) = \cos(\pi x) + 2.5$, where M and N produce an ideal free pair on $\Omega \in (0, 1)$. Also set, $r = 1.0$, $u_0 = v_0 = 1.6$, $d_1 = d_2 = 1.0$ in Fig. 4 (a) $H_1 = H_2 = 0.0$, and in Fig. 4 (b) $H_1 = H_2 = 0.6$. We see that without harvesting effects, the coexistence solution will be obvious as (M, N) according to [15]. According to Theorem 4, even if we assume $H_1 = H_2 = 0.6$, a coexistence solution can still be achieved. It is important to

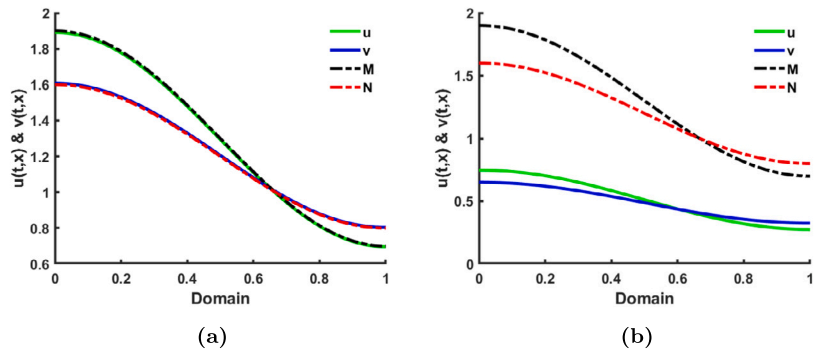


Fig. 4. Solution of (3.1) at $t = T = 1000$ where $M(x) = 0.6\cos(\pi x) + 1.3$, $N(x) = 0.4\cos(\pi x) + 1.2$, $K(x) = M(x) + N(x)$, $r = 1.0$, $u_0 = v_0 = 1.6$, $d_1 = d_2 = 1.0$ for (a) $H_1 = H_2 = 0.0$, and (b) $H_1 = H_2 = 0.6$ on $\Omega = (0, 1)$.

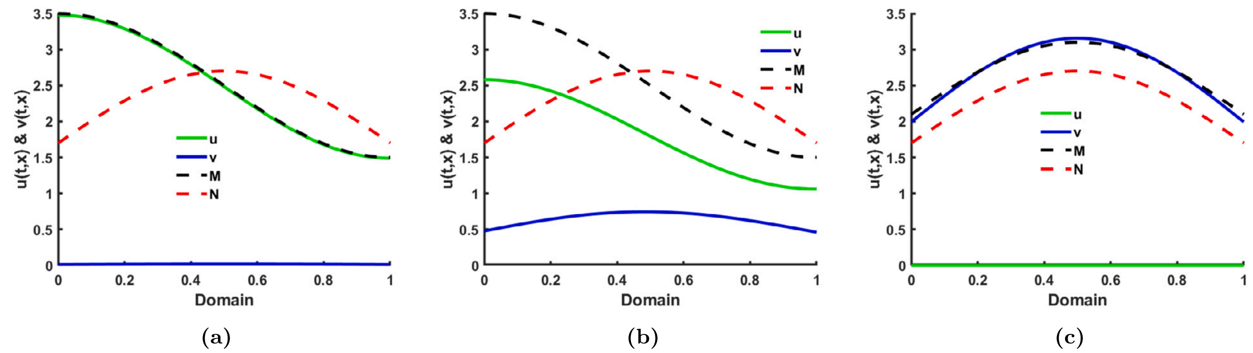


Fig. 5. Solution of (3.10) at $t = T = 2000$ where $K(x) = M(x) = \cos(\pi x) + 2.5$, $N(x) = \sin(\pi x) + 1.7$, $r = 1.0$, $u_0 = v_0 = 1.6$, $d_1 = d_2 = 1.0$, $\gamma_2 = 0.0$ for (a) $\gamma_1 = 0.0$, (b) $\gamma_1 = 0.02$, and (c) $\gamma_1 = 0.2$ on $\Omega = (0, 1)$.

note that the resource functions M and N are not linearly dependent, and the non-trivial steady state is more correlated with K, which forms an ideal free pair.

Example 4. Consider the model given in equation (3.10). The functions $K(x) = M(x) = \cos(\pi x) + 2.5$ and $N(x) = \sin(\pi x) + 1.7$ represent the carrying capacity and resource distribution of two species, respectively. The species u follows the carrying capacity-driven diffusion strategy, while v diffuses according to their resource distribution. The values of $r = 1.0$, $u_0 = v_0 = 1.6$, respectively. Also, $d_1 = d_2 = 1.0$, and $x \in \Omega \in (0, 1)$ for $t > 0$. We consider $t = T = 2000$, at which the solution is close to a steady state. Fig. 5 shows the spatial distribution of the solution for different values of $\gamma_1 \in [0, 1)$, assuming that γ_2 is fixed. We found that in the absence of harvesting efforts, a non-trivial solution exists for u. This shows that the species that disperse according to K-driven diffusion will survive in the competition, as shown in Fig. 5(a). However, when a small level of harvesting effort is incorporated on u, coexistence exists, as in Fig. 5 (b). Moreover, for $\gamma_1 = 0.2$, the trivial solution appears for u, as shown in Fig. 5 (c). This corresponds to the second and third parts of Theorem 5. To computer Fig. 6 numerically, we have used the functions $K_v(x) = K(x)$, $r(x)$ and $v_{\gamma_2}(x)$ in (3.16) where $v_{\gamma_2}(x)$ is the steady state solution of (3.13) that represents the values of γ_1^* for fixed values of γ_2. We represent the average scaled solutions of u and v as a function of harvesting rates $\gamma_1 \in [0, 1)$, where $\gamma_2 \in [0, 1)$ is considered fixed. Here, γ_1^* is assumed to be the lower bound for γ_1 for which coexistence is appeared and γ_1^{**} reveals the upper bound of γ for which coexistence is still occurs are presented in Fig. 6 (a-e) at $t = T = 2000$ that illustrate the first part of Theorem 5. Biologically, from this figure, we can estimate the sustainable level of harvesting that allows for coexistence in the environment. However, any excessive amount of harvested population can cause extinction, as demonstrated by this numerical explanation.

Example 5. Assume (3.10) at $t = T = 2000$ where M and N form an ideal free pair with $M(x) = 0.6\cos(\pi x) + 1.3$, $N(x) = 0.4\cos(\pi x) + 1.2$, $K(x) = M(x) + N(x)$. Also let, the initial values as $u_0 = v_0 = 1.6$ where $r = 1.0$ and $d_1 = d_2 = 1.0$.

Fig. 7 (a-e) displays the scaled average solutions of u and v as a function of γ_1 when considering $\gamma_2 \in [0, 1)$ is fixed and to compute the values of γ_1^* we have used equation (3.24) that provide the lower bound of $\overline{\gamma}$ for each fixed γ_2 that shows coexistence of species. Where to get γ_1^* in equation (3.24) we have used the functions $K_v(x) = K(x)$, $M(x)$, $r(x)$ and steady state solution v_{γ_2} corresponding to the second equation of (3.11), that correlate with Theorem 6, where M and N are linearly independent. Also, Fig. 8 (a-b) signifies the habit of scaled average population density of both species on harvesting rates $\gamma_1 \in [0, 1)$ and $\gamma_2 \in [0, 1)$.

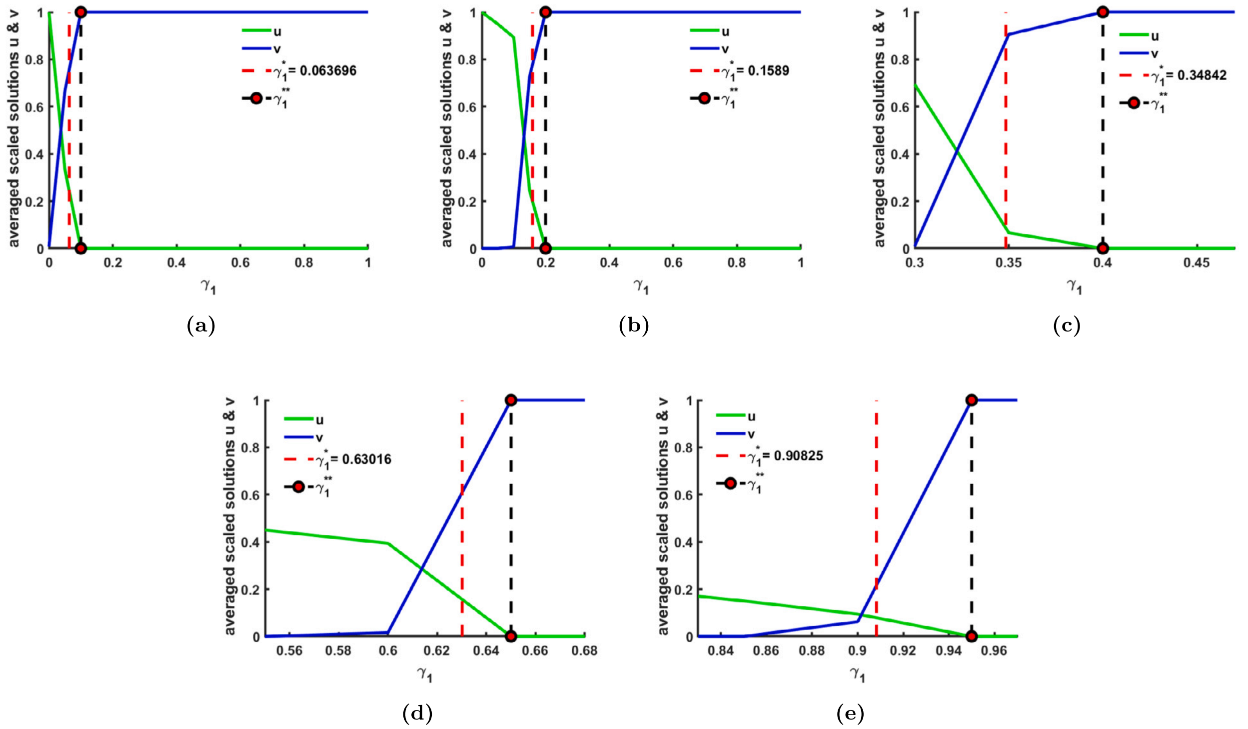


Fig. 6. Average scaled solutions of (3.10) at $t = T = 2000$ on $\Omega = (0,1)$ where $K(x) = M(x) = \cos(\pi x) + 2.5$, $N(x) = \sin(\pi x) + 1.7$, $r = 1.0$, $u_0 = v_0 = 1.6$, $d_1 = d_2 = 1.0$ for (a) $\gamma_2 = 0.0$, (b) $\gamma_2 = 0.1$, (c) $\gamma_2 = 0.3$, (d) $\gamma_2 = 0.6$, and (e) $\gamma_2 = 0.9$ with corresponding $\gamma_1^* = 0.063696$, 0.1589, 0.34842, 0.63016, 0.90825, respectively.

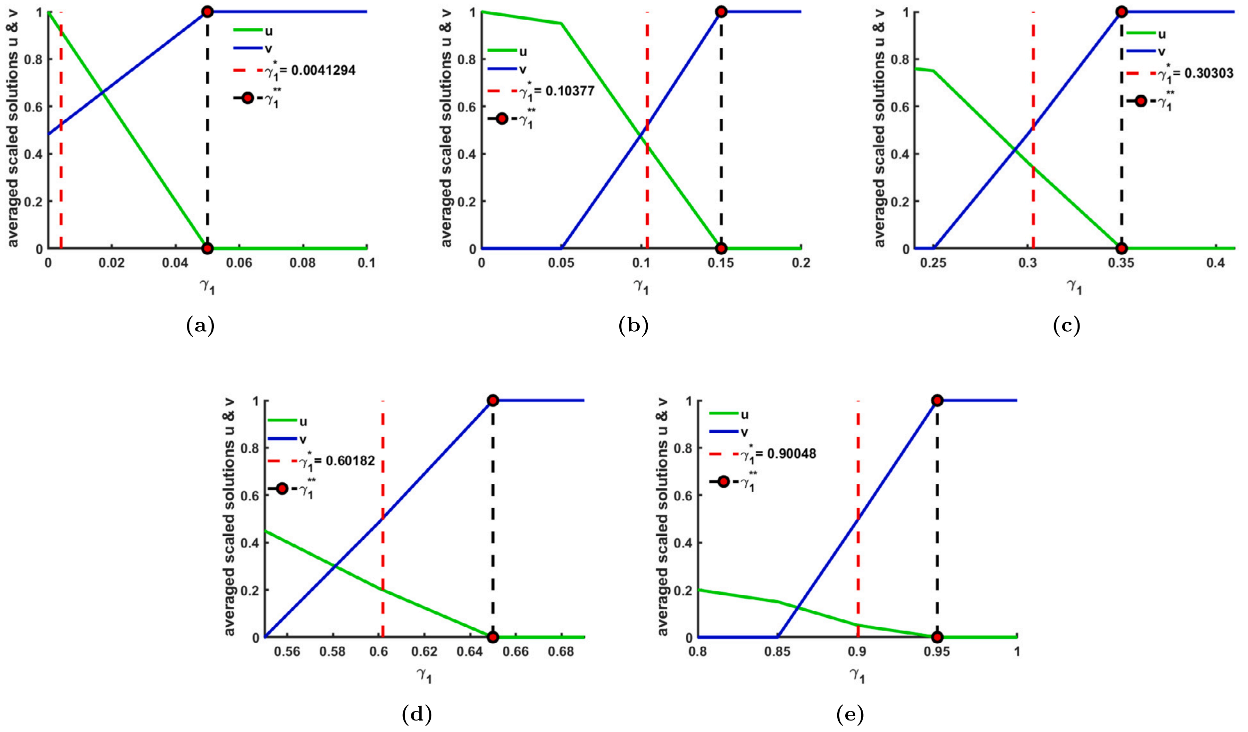


Fig. 7. Average scaled solutions of (3.10) at $t = T = 2000$ on $\Omega = (0,1)$ where $M(x) = 0.6\cos(\pi x) + 1.3$, $N(x) = 0.4\cos(\pi x) + 1.2$, $K(x) = M(x) + N(x)$, $r = 1.0$, $u_0 = v_0 = 1.6$, $d_1 = d_2 = 1.0$ for (a) $\gamma_2 = 0.0$, (b) $\gamma_2 = 0.1$, (c) $\gamma_2 = 0.3$, (d) $\gamma_2 = 0.6$, and (e) $\gamma_2 = 0.9$ with corresponding $\gamma_1^* = 0.0041294$, 0.10377, 0.30303, 0.60182, 0.90048, respectively.

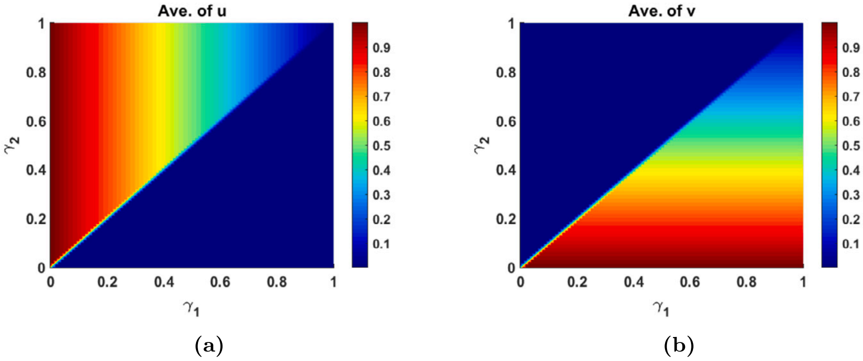


Fig. 8. Average scaled density of (a) u, and (b) v for (3.10) as a function of harvesting coefficients $\gamma_1 \in [0,1]$ and $\gamma_2 \in [0,1]$ on $\Omega \in (0,1)$ where $M(x) = 0.6\cos(\pi x) + 1.3$, $N(x) = 0.4\cos(\pi x) + 1.2$, $K(x) = M(x) + N(x)$, $r = 1.0$, $u_0 = v_0 = 1.6$, $d_1 = d_2 = 1.0$ at $t = T = 2000$.

4.4. Case of 2-space dimensions

In this part of the section, we will analyze the numerical simulations for two dimensions in space for both Case I and Case II.

Example 6. Consider the spatial functions, $K(x,y) = 2.5 + \cos(\pi x)\cos(\pi y)$, $M(x,y) = 2.1 + \sin(\pi x)\sin(\pi y)$, $N(x,y) = 1.7 + \sin(\pi x)\sin(\pi y)$ for model (3.1) with $r = 1.0$, $u_0 = v_0 = 1.6$, $d_1 = d_2 = 1.0$ on $\Omega = (0,1) \times (0,1)$ for different space dependent harvesting coefficient H_1, while $H_2 = 0.5 + 0.2\cos(\pi x)\cos(\pi y)$ is taken as fixed. Fig. 9 represents the contour profiles of u and v while their diffusive movement is directed towards positive resource functions M and N, respectively. We have computed the solution at $t = T = 400$, which is sufficient to get a steady state for this case. We see that the contour pattern followed shows a correlation with M and N rather than K, and the maximum population densities are located at the center of the domain. Due to the effects of harvesting at different levels in space, the dependence on population density is stronger for large diffusion. Indeed, the equilibrium profile is primarily influenced by the diffusion term, leading to a correlation with resource functions for both species. As we know, for two interacting species, the outcome is either a competitive exclusion or the coexistence of two species. Here we observed that for unequal values of harvesting coefficients, competitive exclusion is obvious (see, Fig. 9 (a,d) and Fig. 9 (c,f)), which justified the Theorem 3 and Remark 1 while coexistence is possible when considering the equal level of harvesting effects (see Fig. 9 (b,e)).

Example 7. Assuming (3.1), with $K(x,y) = 2.5 + \cos(\pi x)\cos(\pi y)$, $M(x,y) = 2.1 + \sin(\pi x)\sin(\pi y)$, $N(x,y) = 1.7 + \sin(\pi x)\sin(\pi y)$, $r = 1.0$, $u_0 = v_0 = 1.6$, and $d_1 = d_2 = 1.0$, we vary the constant value of the harvesting coefficient H_1 while keeping $H_2 = 0.6$ fixed on $\Omega \in (0,1) \times (0,1)$. The contours in Fig. 10 follow analogous patterns to the previous Example 6 by considering constant levels of harvesting.

Example 8. We consider the scenario of (3.10) with $K(x,y) = M(x,y) = 2.5 + \cos(\pi x)\cos(\pi y)$, $N(x,y) = 1.7 + \sin(\pi x)\sin(\pi y)$ at $t = T = 400$. Here, the diffusive migration of u follows the carrying capacity K, and the movement of other species is directed towards their resource distribution N. In Fig. 11, we have contour plots of u and v for different harvesting efforts $\gamma_1 \in [0,1)$ of u, while $\gamma_2 = 0$ remains fixed. We observe that without the harvesting effects, coexistence arises (see Fig. 11 (a,d)). Additionally, introducing a small quantity of harvesting effect ($\gamma_1 = 0.04$, see Fig. 11 (b,e)) on u leads to both species surviving, which demonstrates the evolutionary benefit of consuming these types of diffusion approaches, for which both species should exist in the competition.

On the other hand, in Fig. 11 (c,f)), we discover the existence of a semi-trivial equilibrium for v when the level of harvesting is considered at $\gamma_1 = 0.2$, even though others are not supposed to be harvested. We also notice that the contour plots of u correlate with K where the patterns of v correlate with N.

However, Fig. 12 (a-e) reveals the diagram of scaled average stationary solutions as a function of harvesting efforts $\gamma_1 \in [0,1)$ for numerous fixed levels of γ_2 on $\Omega \in (0,1) \times (0,1)$. Likewise, here we have estimated a lower level of harvesting γ_1^* of γ by equation (3.16) for two-dimensional cases in x and y for which coexistence of both species necessarily should exist. We have also computed numerically the upper bound γ_1^{**} in this case which is the upper estimate of γ_1 for which coexistence is yet possible.

Also, in Fig. 13 (a-b), we can see the average scaled population density of u and v for different levels of harvesting effects $\gamma_1 \in [0,1)$ and $\gamma_2 \in [0,1)$ at time $t = T = 1000$. The bisection area shows the coexistence of both u and v species, and the darker portion of this area represents a higher density of these species.

Example 9. Let $M(x,y) = 2.5\pi^2 e^{-(x-0.5)^2-(y-0.5)^2} + 1.2$, $N(x,y) = 1.5\pi^2 e^{-(x-1)^2-(y-1)^2} + 1.0$, $K(x,y) = M(x,y) + N(x,y)$, $r = 1.0$, $u_0 = v_0 = 1.6$, $d_1 = d_2 = 1.0$ on $\Omega = (0,1) \times (0,1)$ for the case of (3.10). As in a previous example, we found similar observations while considering resource functions and carrying capacity as a combination of two Gaussian functions and a positive constant for the case of an ideal free pair. Additionally, Fig. 14 (a-f) shows that the contour patterns for u follow M, which is maximum at the center, and the contour patterns of v follow N, which provides the highest population density at the top right corner for several values of γ_1 when $\gamma_2 = 0$.

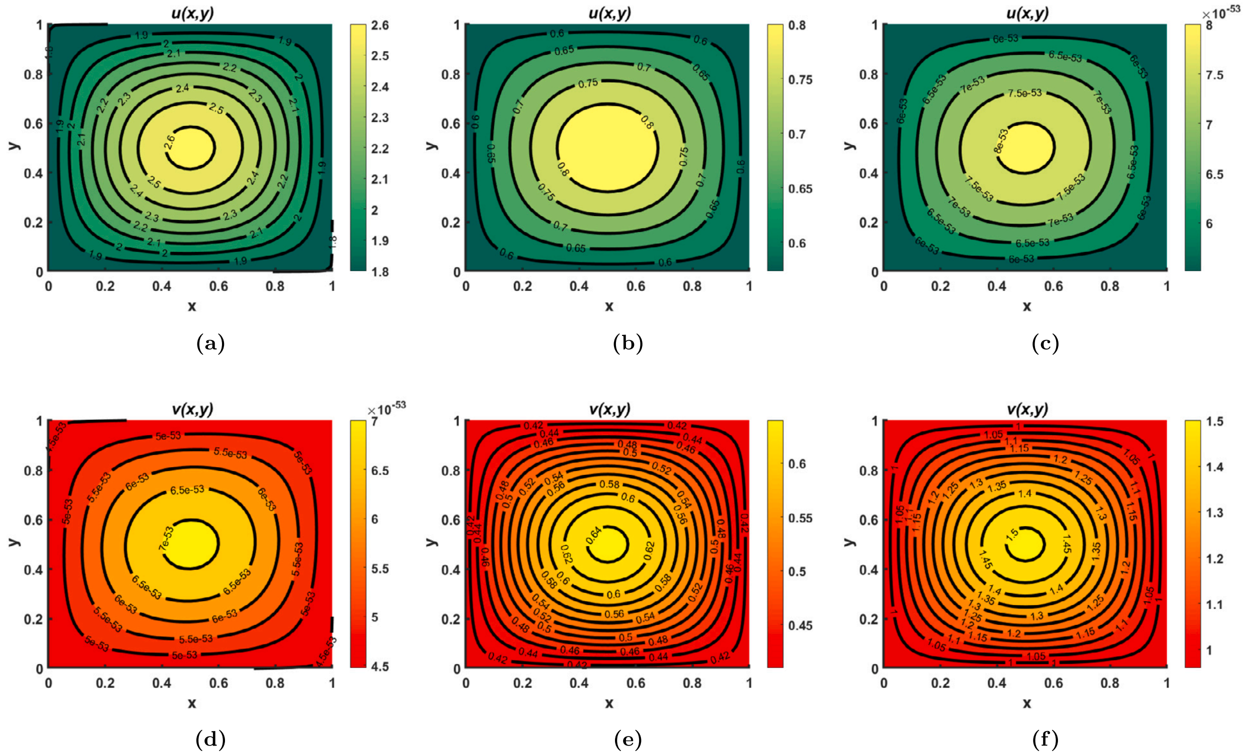


Fig. 9. Contour plots of u and v of (3.1) at $t = T = 400$ where $K(x,y) = 2.5 + \cos(\pi x)\cos(\pi y)$, $M(x,y) = 2.1 + \sin(\pi x)\sin(\pi y)$, $N(x,y) = 1.7 + \sin(\pi x)\sin(\pi y)$, $r = 1.0$, $u_0 = v_0 = 1.6$, $d_1 = d_2 = 1.0$, $H_2 = 0.5 + 0.2\cos(\pi x)\cos(\pi y)$ for (a,d) $H_1 = 0.1 + 0.3\cos(\pi x)\cos(\pi y)$, (b,e) $H_1 = 0.5 + 0.2\cos(\pi x)\cos(\pi y)$, and (c,f) $H_1 = 0.9 + 0.1\cos(\pi x)\cos(\pi y)$ on $\Omega = (0,1)\times(0,1)$.

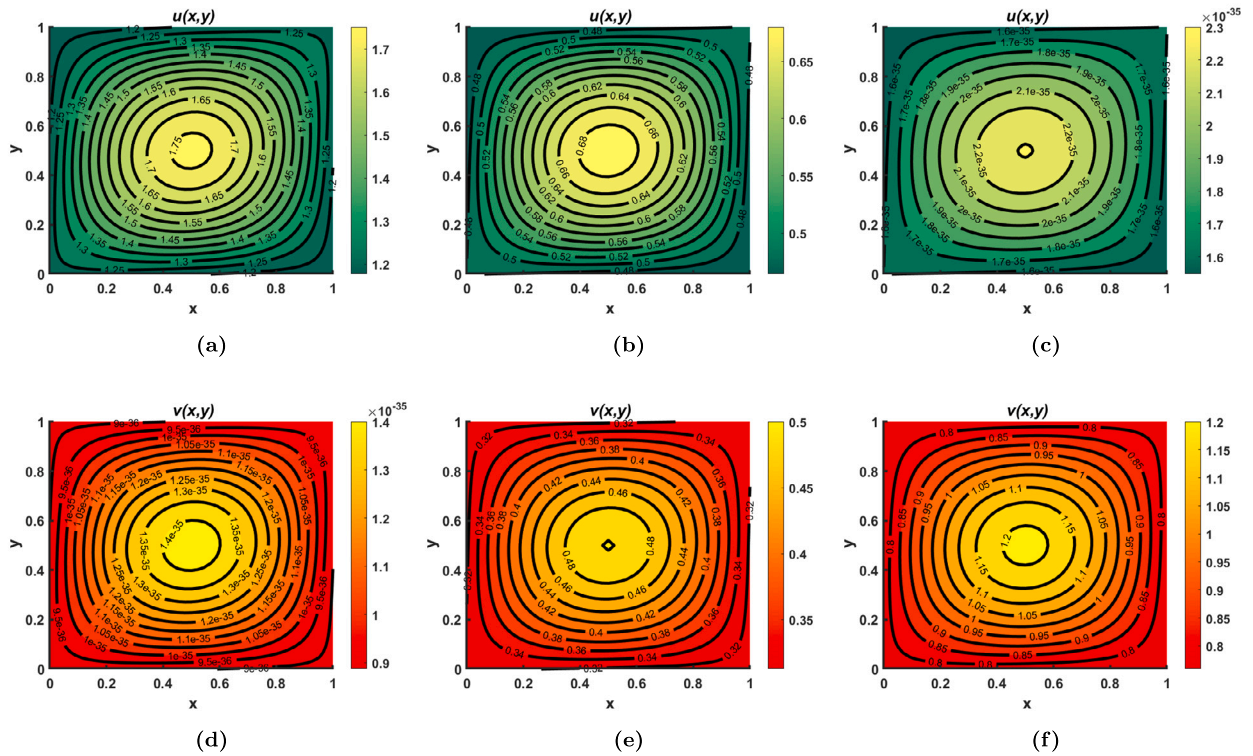


Fig. 10. Contour plots of u and v of (3.1) at $t = T = 400$ where $K(x,y) = 2.5 + \cos(\pi x)\cos(\pi y)$, $M(x,y) = 2.1 + \sin(\pi x)\sin(\pi y)$, $N(x,y) = 1.7 + \sin(\pi x)\sin(\pi y)$, $r = 1.0$, $u_0 = v_0 = 1.6$, $d_1 = d_2 = 1.0$, $H_2 = 0.6$ for (a,d) $H_1 = 0.4$, (b,e) $H_1 = 0.6$, and (c,f) $H_1 = 0.8$ on $\Omega = (0,1)\times(0,1)$.

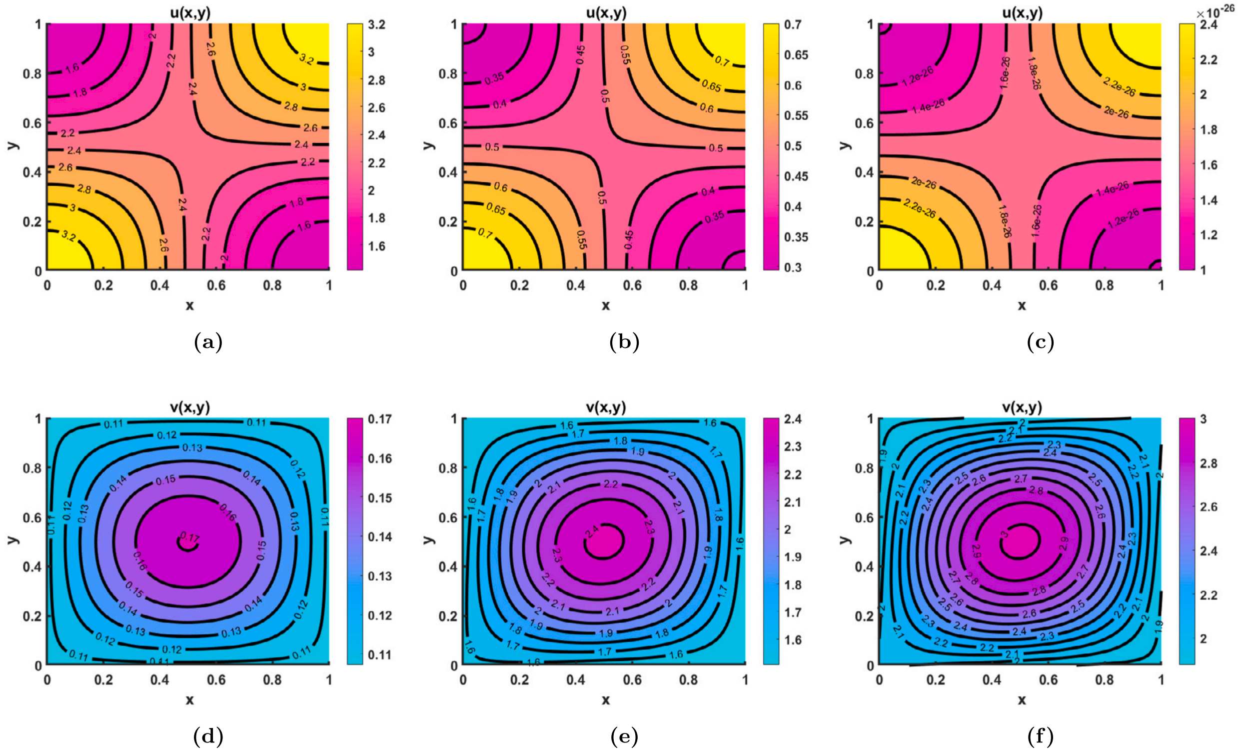


Fig. 11. Contour plots of u and v of (3.10) at $t = T = 400$ where $K(x,y) = M(x,y) = 2.5 + \cos(\pi x)\cos(\pi y)$, $N(x,y) = 1.7 + \sin(\pi x)\sin(\pi y)$, $r = 1.0$, $u_0 = v_0 = 1.6$, $d_1 = d_2 = 1.0$, $\gamma_2 = 0$ for (a,d) $\gamma_1 = 0$, (b,e) $\gamma_1 = 0.04$, and (c,f) $\gamma_1 = 0.2$ on $\Omega = (0,1) \times (0,1)$.

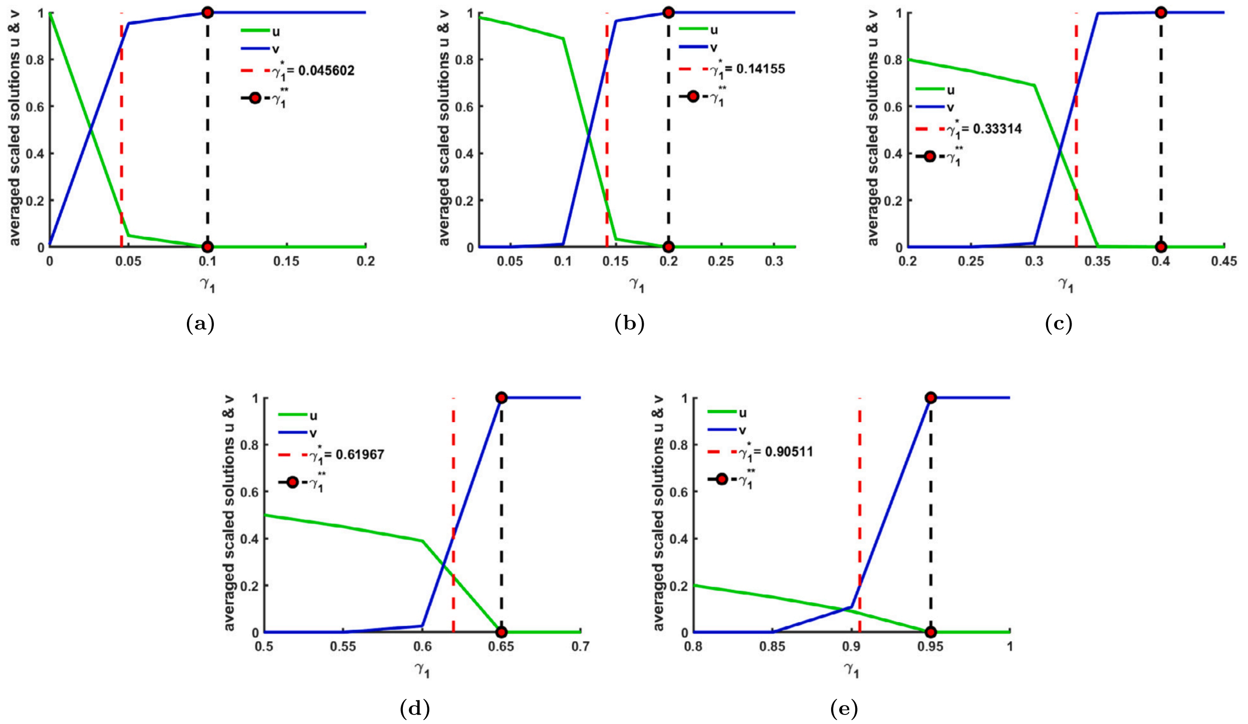


Fig. 12. Average scaled solutions of (3.10) at $t = T = 2000$ on $\Omega = (0,1) \times (0,1)$ where $K(x,y) = M(x,y) = 2.5 + \cos(\pi x)\cos(\pi y)$, $N(x,y) = 1.7 + \sin(\pi x)\sin(\pi y)$, $r = 1.0$, $u_0 = v_0 = 1.6$, $d_1 = d_2 = 1.0$ for (a) $\gamma_2 = 0.0$, (b) $\gamma_2 = 0.1$, (c) $\gamma_2 = 0.3$, (d) $\gamma_2 = 0.6$, and (e) $\gamma_2 = 0.9$ with corresponding $\gamma_1^* = 0.045602, 0.14155, 0.33314, 0.61967, 0.90511$, respectively.

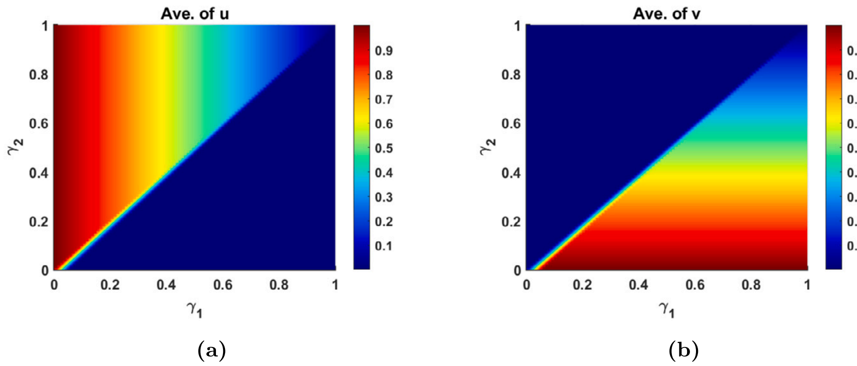


Fig. 13. Average scaled density of (a) u, and (b) v for (3.10) as a function of harvesting coefficients $\gamma_1 \in [0,1]$ and $\gamma_2 \in [0,1]$ on $\Omega \in (0,1) \times (0,1)$ where $K(x,y) = M(x,y) = 2.5 + \cos(\pi x)\cos(\pi y)$, $N(x,y) = 1.7 + \sin(\pi x)\sin(\pi y)$, $r = 1.0$, $u_0 = v_0 = 1.6$, $d_1 = d_2 = 1.0$ at $t = T = 1000$.

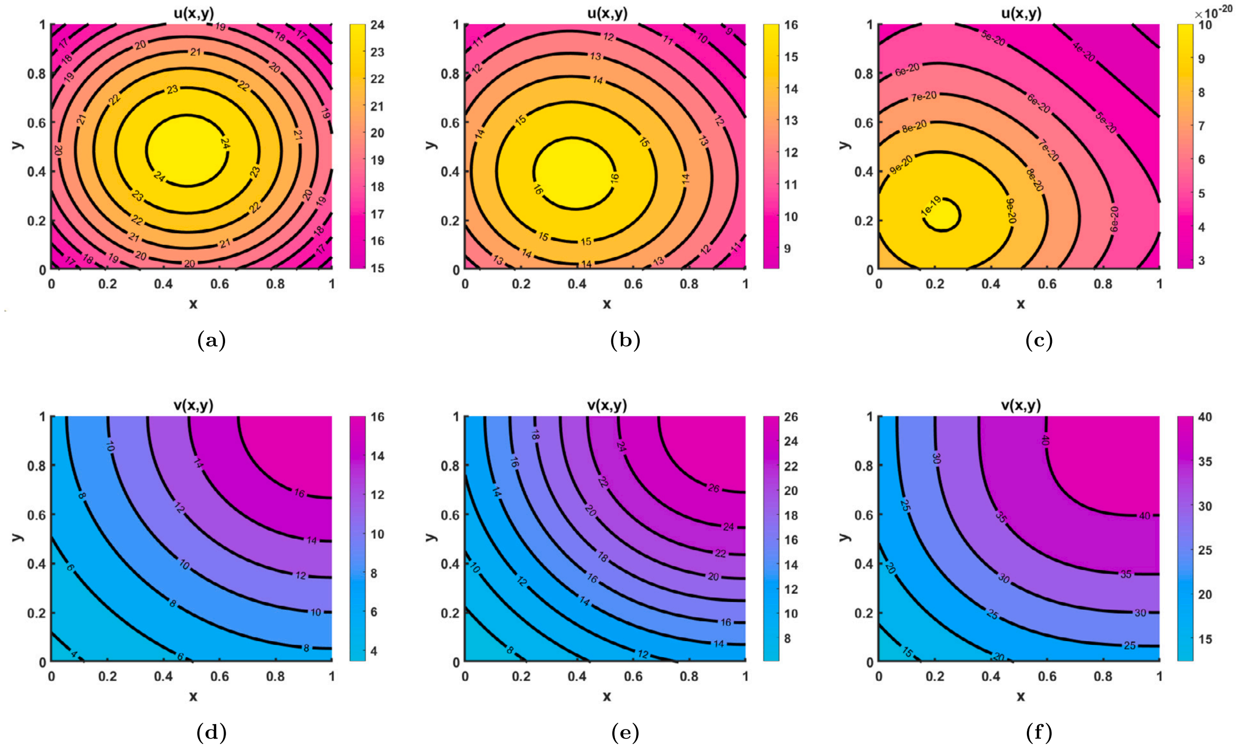


Fig. 14. Contour plots of u, and v of (3.10) at $t = T = 400$ where $M(x,y) = 2.5\pi^2 e^{-(x-0.5)^2-(y-0.5)^2} + 1.2$, $N(x,y) = 1.5\pi^2 e^{-(x-1)^2-(y-1)^2} + 1.0$, $K(x,y) = M(x,y) + N(x,y)$, $r = 1.0$, $u_0 = v_0 = 1.6$, $d_1 = d_2 = 1.0$, $\gamma_2 = 0.0$ for (a,d) $\gamma_1 = 0.0$, (b,e) $\gamma_1 = 0.015$, and (c,f) $\gamma_1 = 0.2$ on $\Omega = (0,1) \times (0,1)$.

Fig. 15 (a,b) illustrates how the average density of u and v depends on γ_1 and γ_2, where the darker part in the first quadrant of bisection region supports coexistence for $\gamma_2 = 0$.

5. Conclusion

Our research aimed to explore different harvesting strategies to gain insights into the management of species growth. We studied various diffusive strategies using reaction-diffusion equations for two competing species. We considered two cases for harvesting strategy: one where the harvesting coefficients are arbitrary, independent in space, and do not exceed the species' intrinsic growth rate, and another where the harvesting coefficients are proportional to the time-independent intrinsic growth rate.

We found that only one semi-trivial solution will be sustained for unequal harvesting levels that may be constant or space-dependent. However, for equal harvesting, coexistence is guaranteed in Case I. Furthermore, harvesting the invasive species could potentially safeguard the survival of the native population. Moreover, we conducted estimations on species extinction and provided certain bounds where coexistence is apparent.

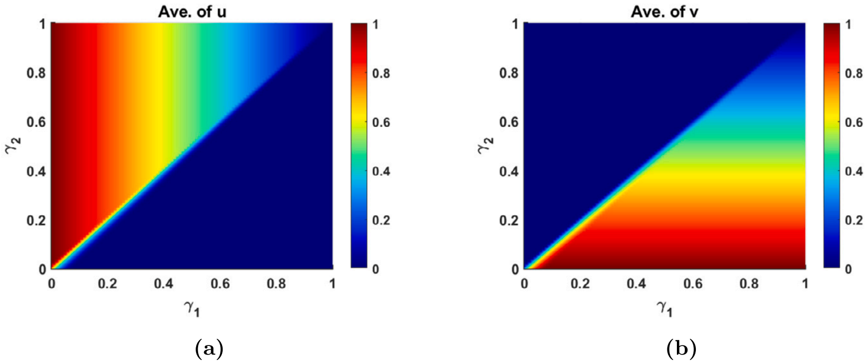


Fig. 15. Average scaled density of (a) u, and (b) v for (3.10) as a function of harvesting coefficients $\gamma_1 \in [0,1]$ and $\gamma_2 \in [0,1]$ on $\Omega \in (0,1)\times(0,1)$ where $M(x,y) = 2.5\pi^2 e^{-(x-0.5)^2-(y-0.5)^2} + 1.2$, $N(x,y) = 1.5\pi^2 e^{-(x-1)^2-(y-1)^2} + 1.0$, $K(x,y) = M(x,y) + N(x,y)$, $r = 1.0$, $u_0 = v_0 = 1.6$, $d_1 = d_2 = 1.0$ at $t = T = 1000$.

Our study is focused on the importance of harvesting levels that enable a species to maintain a sustainable population. In this context, diffusion strategy is a key factor for the survival of species in competitive environments. We performed numerical computations in one and two spatial dimensions, considering space-dependent parametric values. Our numerical analysis shows that both populations can coexist with restricted harvesting levels of the species. Additionally, we studied the ideal free pair models in both cases. Our theoretical and numerical studies aim to help readers better understand how species interact in complex environments.

In our study, we analyzed the theoretical results based on two harvesting levels for different diffusion strategies. However, we did not evaluate time periodic results, and we did not illustrate the situation when harvesting exceeds some local locations but not others theoretically. We have tried to demonstrate some of these situations through numerical illustration, which has not been analyzed theoretically. It is not feasible to provide a detailed answer to each point based solely on numerical calculations due to the vast number of parameters and functions involved. Our main focus is on the impact of different harvesting efforts and the significance of diffusion strategies. We strive to establish theoretical explanations to guide numerical analysis whenever possible and vice versa. In certain populations that are subject to harvesting, the harvesting process is not uniformly applied to all individuals but is restricted to certain individuals. This could have an impact on the birth rate. To minimize this effect of harvesting on the birth rate, sometimes the harvesting is biased towards certain species. Models of populations that account for such situations are more realistic than other models of populations subject to harvesting.

Ethical approval

No consent is required to publish this manuscript.

CRediT authorship contribution statement

Ishrat Zahan: Writing – original draft, Software, Resources, Methodology, Formal analysis, Data curation, Conceptualization. **Md. Kamrujjaman:** Writing – review & editing, Validation, Supervision, Software, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data is included in the manuscript/supplementary information.

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