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REVIEW ARTICLE

Study of a Nonlinear System of Fractional Differential Equations with Deviated Arguments Via Adomian Decomposition Method

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Abstract
This paper studies a system of nonlinear fractional differential equations (FDEs) with deviated arguments. Many linear and nonlinear problems are faced in the real-life. Generally, linear problems are solved quickly, but some difficulties appear while solving nonlinear problems. Our purpose is to approximate those solutions numerically via the Adomian decomposition method (ADM). Here, our main goal is to apply the ADM to solve higher-order nonlinear system of FDEs with deviated arguments. We prove the existence and uniqueness of the solution using Banach contraction principle. Moreover, we plot the figures of ADM solutions using MATLAB.

Keywords Fractional differential equations · Adomian decomposition method · Caputo fractional derivative · Deviated arguments · Nonlinear system · Existence uniqueness

Introduction

"Science is a differential equation," said Alan Turing, and Paul Ormerod "Baseball players or cricketers do not need to be able to solve explicitly the nonlinear differential equations which govern the flight of the ball. They just catch it." Nonlinear differential equations describe many real-world physical phenomena. To understand the nature of these phenomena, we must first solve differential equations. "In order to solve this differential equation, you look at it until a solution occurs to you," George Polya explained. In 1980, George Adomian proposed a new iterative scheme known as ADM. This method provides analytical solutions to both linear and nonlinear differential equations.

Classical differential equations cannot adequately describe more and more phenomena as science and technology advance. Various physical processes, for example, have memory and heritability properties that the classical local differential operators cannot adequately repre-

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sent. Many great mathematicians such as Euler, Liouville, Riemann, Caputo, and Letnikov developed a new excellent tool to describe these nonlocal processes (fractional differential equations described by nonlocal operators) [21, 23]. Fractional Calculus was born in 1695. G.F.A. de L Hospital thought about what happens if the order is 1/2. And in 1697, G.W. Leibniz used fractional derivatives of order 1/2 and made some remarks on it. However, in 1819, S.F. Lacroix mentioned the derivative of arbitrary order in his text on differential and integral calculus. N.H. Abel gave the first application of fractional calculus in 1823.

Fractional calculus has been included in recent studies on CoViD-19 [3, 17, 20]. Many studies have been conducted in the field of mathematics, and it has been shown that differential equations using fractional operators are effective in demonstrating epidemic models linked to many infectious illnesses [7, 12]. Two leading implementations of fractional calculus are in epidemiological and biomathematical models [6, 18, 19]. Ahmad et al. [2] performed simulations of a fractional model for CoViD-19 transmission, taking into account various values of the non-integer order derivative and came to the conclusion that the value of = 0.97 best matched the actual data. Furthermore, Zhang et al. [29] created a non-integer order model for the dynamics of CoViD-19. The authors investigated the stability of the system and reproduction number.

FDEs with deviated arguments have many applications in science and engineering, including fractals theory, chemistry, biology, physics, neural network, weather prediction model, etc. [19, 26]. Brauer et al. [5] presented logistic equations, which are particularly applicable to epidemic systems. Xu et al. [26] explored the effect of numerous time delays on fractional-order neural network bifurcation. There are various methods to solve fractional differential equations: the homotopy perturbation method [9, 13, 14, 22], the Adomian decomposition method [1, 8, 15, 16, 25, 27], the polynomial least square method [4], and so on [24].

In [10], Duan et al. provided a review of ADM and its application to FDEs. Evans and Raslan [11] applied the Adomian method to solve a particular ordinary delay differential equations in which the delay is located in the linear or nonlinear part, where the history function is not necessary. In [16], Li and Pang provided an application of ADM to a nonlinear system. In [25], Saeed and Rahman studied the ADM for solving the system of delay differential equation. In [27], Ziada studied the nonlinear system of fractional differential equations via ADM, and the fractional order rabies model was solved as an application. However, in [28] Ziada studied the analytical and numerical solutions of a multi-term nonlinear differential equation with deviated arguments.

Motivated by the works of Ziada [27, 28] as well as Saeed and Rahman [25], we construct the system (1)–(2). We extend the work of [25] for higher order nonlinear FDEs with deviated arguments.

Delay differential equations are far more complicated than traditional ordinary differential equations, they explain many processes found in several fields such as biology, medicine, chemistry, economics, engineering and physics. Systems of FDEs have many applications in engineering and science, including electrical networks, control theory, fractals theory, viscoelasticity, optical and neural network systems. This paper aims to discuss the approximate solution of a nonlinear system of FDEs with deviated arguments via ADM (an algorithm that uses a decomposition technique). Here, our main goal is to apply the ADM to solve higher-order nonlinear systems of FDEs with deviated arguments. This method has numerous advantages. It is very simple to use and can solve a wide range of nonlinear systems, such as ordinary and partial differential equations, fractional delay differential equations, and so on. It avoids the Picard method's time-consuming integrations. It decomposes the solution into a series with easily computed components. It has the advantage of converging to the exact solution.

The remaining paper is designed as "Formulation of the Problem" section introduces the formulation of the problem and ADM's iterative scheme. "Basic Definitions" section contains some useful basic definitions. In the next two sections, we prove the existence, uniqueness and convergence of the solution of the system (1)–(2). Numerical examples have been provided in "Numerical Examples" section. The conclusion is added in the last section.

Formulation of the Problem

Consider the following higher-order nonlinear system of FDEs with deviated arguments

$$ {}^{C}\mathcal{D}_t^{q_i} y_i(t) = \mathcal{F}_i\big(t, y_1(t), \ldots, y_n(t), y_1(\nu_1(t)), \ldots, y_n(\nu_n(t))\big), \quad t \in J_0 = [0, T], \tag{1} $$

with initial conditions

$$ y_i^{(j)}(0) = c_{ij}, \quad \nu_i(t) \le t, \quad y_i(t) = \Psi_i(t), \quad t \le 0, \tag{2} $$

where ${}^{C}\mathcal{D}_t^{q_i}$ denote the Caputo derivative of order $n-1 < q_i \le n, \ i = 1, 2, \ldots, n, \ j = 0, 1, 2, \ldots, n-1$. Here, we use Caputo fractional derivative amongst a variety of definitions for fractional order derivatives as it is suitable for describing various phenomena, since the initial values of the function and its integer order derivatives have to be specified. $\mathcal{F}_i$ are nonlinear operators that satisfy Lipschitz condition with Lipschitz constant L_i, such as

$$ \big|\big(\mathcal{F}_i y_i\big)(t) - \big(\mathcal{F}_i z_i\big)(t)\big| = L_i\left(\sum_{i=1}^{n} |y_i(t) - z_i(t)| + \sum_{i=1}^{n} |y_i(\nu_i(t)) - z_i(\nu_i(t))|\right). \tag{3} $$

$y_i(t) \in C(J_0)$ are unknown functions, $\Psi_i(t)$ are given continuous functions and c_{ij} are given constants. In order to solve the problem (1) with (2) by using the ADM, performing the fractional integral $\mathcal{I}_t^{q_i}$ to both sides of (2), we have

$$ y_i(t) = \sum_{j=0}^{n-1} c_{ij}\frac{t^j}{j!} + \mathcal{I}_t^{q_i}\mathcal{F}_i\big(t, y_1(t), \ldots, y_n(t), y_1(\nu_1(t)), \ldots, y_n(\nu_n(t))\big). \tag{4} $$

Adomian's method defines the solution by series

$$ y_i(t) = \sum_{m=0}^{\infty} y_{i,m}(t). \tag{5} $$

So that, the components $y_{i,m}$ will be determined recursively. Moreover, the method defines the nonlinear term $\mathcal{F}_i\big(t, y_1(t), \ldots, y_n(t), y_1(\nu_1(t)), \ldots, y_n(\nu_n(t))\big)$ by the Adomian polynomials

$$ \mathcal{F}_i\big(t, y_1(t), \ldots, y_n(t), y_1(\nu_1(t)), \ldots, y_n(\nu_n(t))\big) = \sum_{m=0}^{\infty} \mathcal{A}_{i,m}(y_{i,0}, y_{i,1}, \ldots, y_{i,m}), \tag{6} $$

where $\mathcal{A}_{i,m}$ are Adomian polynomials that can be generated for all forms of nonlinearity as

$$ \mathcal{A}_{i,m} = \left[\frac{1}{m!}\frac{d^m}{d\lambda^m}\mathcal{F}_i\left(t, \sum_{m=0}^{\infty}\lambda^m y_{1,m}(t), \ldots, \sum_{m=0}^{\infty}\lambda^m y_{n,m}(t), \right.\right. $$
$$ \left.\left.\sum_{m=0}^{\infty}\lambda^m y_{1,m}(\nu_1(t)), \ldots, \sum_{m=0}^{\infty}\lambda^m y_{n,m}(\nu_n(t))\right)\right]_{\lambda=0}, \tag{7} $$

where λ is a parameter.
In view of (5) and (6), (4) becomes

$$\sum_{m=0}^{\infty} y_{i,m}(t) = \sum_{j=0}^{n-1} c_{ij}\frac{t^j}{j!} + \mathcal{I}_t^{q_i}\sum_{m=0}^{\infty}\mathcal{A}_{i,m}. \tag{8}$$

To determine the components $y_{i,m}(t),\ m \geq 0$. First we identify the zero component $y_{i,0}$ by the terms $\sum_{j=0}^{n-1} c_{ij}\frac{t^j}{j!}$ and $\mathcal{I}_t^{q_i} f_i(t)$, where $f_i(t)$ represent the non-homogeneous parts of $\mathcal{F}_i\big(t, y_1(t), \ldots, y_n(t), y_1(\nu_1(t)), \ldots, y_n(\nu_n(t))\big)$. Thus, the recurrence relation is

$$y_{i,0} = \sum_{j=0}^{n-1} c_{ij}\frac{t^j}{j!} + \mathcal{I}_t^{q_i} f_i(t), \tag{9}$$

$$y_{i,m+1} = \mathcal{I}_t^{q_i}\mathcal{A}_{i,m}, \quad m = 0, 1, 2, \ldots. \tag{10}$$

We can approximate the solution y_i by the truncated series
$N_{i,k} = \sum_{m=0}^{k-1} y_{i,m}, \quad \lim_{k\to\infty} N_{i,k} = y_i(t).$

Basic Definitions

Definition 3.1 [21, 23] (a) Caputo fractional derivative

$$\big({}^C\mathcal{D}_{0+}^{q} y\big)(t) = \big(\mathcal{I}_{0+}^{n-q}\mathcal{D}^n y\big)(t),$$

where

$$\big(\mathcal{I}_{0+}^{q} y\big)(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} y(s)ds.$$

(b)

$$\mathcal{I}^q t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+q+1)} t^{\mu+q}, \quad q > 0, \mu > -1, t > 0.$$

(c)

$$\mathcal{I}^{q_1}\mathcal{I}^{q_2} y = \mathcal{I}^{q_1+q_2} y, \quad q_1, q_2 > 0.$$

Existence and Uniqueness of Solution

Define the operator $\mathcal{P} : \mathcal{Y} \to \mathcal{Y}$, where $\mathcal{Y}$ is the Banach space $\big(C(J_0), \|\cdot\|\big)$, the space of all continuous functions on J_0 equipped with the norm $\|y\| = \sup_{t\in J_0} |y(t)|$.

Theorem 4.1 *If F_i satisfy the Lipschitz condition (3) and $0 < \gamma < 1$, where $\gamma = \frac{2nLT^{q_i}}{\Gamma(q_i+1)}$, $L = \sup\{L_1, L_2, \ldots, L_n\}$, then the system (1)–(2) has a unique solution $y_i \in \mathcal{Y}$ on J_0.*

Proof Define the operator $\mathcal{P} : \mathcal{Y} \to \mathcal{Y}$ as

$$\big(\mathcal{P}y_i\big)(t) = \sum_{j=0}^{n-1} c_{ij}\frac{t^j}{j!} + \frac{1}{\Gamma(q_i)}\int_0^t (t-s)^{q_i-1}\mathcal{F}_i\big(t, y_1(t), \dots, y_n(t), y_1(\nu_1(t)), \dots, y_n(\nu_n(t))\big)ds. \tag{11}$$

Let $y_i, z_i \in \mathcal{Y}$, $i = 1, 2, \dots, n$, then

$$\begin{aligned}
&\big|\big(\mathcal{P}y_i\big)(t) - \big(\mathcal{P}z_i\big)(t)\big| \\
&= \bigg|\frac{1}{\Gamma(q_i)}\int_0^t (t-s)^{q_i-1}\Big[\mathcal{F}_i\big(t, y_1(t), \dots, y_n(t), y_1(\nu_1(t)), \dots, y_n(\nu_n(t))\big) \\
&\quad -\mathcal{F}_i\big(t, z_1(t), \dots, z_n(t), z_1(\nu_1(t)), \dots, z_n(\nu_n(t))\big)\Big]ds\bigg| \\
&\le \frac{1}{\Gamma(q_i)}\int_0^t (t-s)^{q_i-1}\Big|\mathcal{F}_i\big(t, y_1(t), \dots, y_n(t), y_1(\nu_1(t)), \dots, y_n(\nu_n(t))\big) \\
&\quad -\mathcal{F}_i\big(t, z_1(t), \dots, z_n(t), z_1(\nu_1(t)), \dots, z_n(\nu_n(t))\big)\Big|ds \\
&\le \frac{1}{\Gamma(q_i)}\int_0^t (t-s)^{q_i-1}L_i\left(\sum_{i=1}^n |y_i(t) - z_i(t)| + \sum_{i=1}^n |y_i(\nu_i(t)) - z_i(\nu_i(t))|\right)ds \\
&\Rightarrow \sup_{t\in J_0}\big|\big(\mathcal{P}y_i\big)(t) - \big(\mathcal{P}z_i\big)(t)\big| \\
&\le \frac{L}{\Gamma(q_i)}\sum_{i=1}^n \sup_{t\in J_0}\int_0^t (t-s)^{q_i-1}\big(|y_i(t) - z_i(t)| \\
&\quad +|y_i(\nu_i(t)) - z_i(\nu_i(t))|\big)ds
\end{aligned}$$

$$\begin{aligned}
\big\|\big(\mathcal{P}y_i\big)(t) - \big(\mathcal{P}z_i\big)(t)\big\| &\le \frac{2nL}{\Gamma(q_i)}\|y_i - z_i\|\int_0^t (t-s)^{q_i-1}ds \\
&\le \frac{2nLT^{q_i}}{q_i\Gamma(q_i)}\|y_i - z_i\| \\
&\le \frac{2nLT^{q_i}}{\Gamma(q_i+1)}\|y_i - z_i\| \\
&\le \gamma\|y_i - z_i\|.
\end{aligned}$$

Since $0 < \gamma < 1$, therefore the mapping $\mathcal{P}$ is contraction. By Banach contraction principle, there exists a unique solution $y_i \in \mathcal{Y}$. This completes the proof. □

Proof of Convergence

Theorem 5.1 *The series solution (5) of the system (1)–(2) using ADM converges if* $|y_{i,1}(t)| < \infty$ *and* $0 < \delta < 1$, *where* $\delta = \frac{LT^{q_i}}{\Gamma(q_i+1)}$, $\quad L = \sup\{L_1, L_2, \dots, L_n\}$.

Proof Define the sequence of partial sum $\{S_{i,p}\}$, as $S_{i,p} = \sum_{m=0}^{p} y_{i,m}(t)$.

Since,

$$\mathcal{F}_i\big(t, y_1(t), \ldots, y_n(t), y_1(\nu_1(t)), \ldots, y_n(\nu_n(t))\big) = \sum_{m=0}^{\infty} \mathcal{A}_{i,m}.$$

So, we have

$$\mathcal{F}_i\big(t, S_{1,p}(t), \ldots, S_{n,p}(t), S_{1,p}(\nu_1(t)), \ldots, S_{n,p}(\nu_n(t))\big) = \sum_{m=0}^{p} \mathcal{A}_{i,m}.$$

Further, we prove that $\{S_{i,p}\}$ is a Cauchy sequence in $\mathcal{Y}$. Let $\{S_{i,p}\}, \{S_{i,q}\}$ be two arbitrary partial sums such that $p \geq q$, then

$$\begin{aligned}
\|S_{i,p} - S_{i,q}\| &= \sup_{t\in J_0} |S_{i,p} - S_{i,q}| \\
&= \sup_{t\in J_0} \left| \sum_{m=q+1}^{p} y_{i,m}(t) \right| \\
&= \sup_{t\in J_0} \left| \sum_{m=q+1}^{p} \frac{1}{\Gamma(q_i)} \int_0^t (t-s)^{q_i-1} \mathcal{A}_{i,m-1} ds \right| \\
&= \sup_{t\in J_0} \left| \frac{1}{\Gamma(q_i)} \int_0^t (t-s)^{q_i-1} \sum_{m=q+1}^{p} \mathcal{A}_{i,m-1} ds \right| \\
&= \sup_{t\in J_0} \left| \frac{1}{\Gamma(q_i)} \int_0^t (t-s)^{q_i-1} \sum_{m=q}^{p-1} \mathcal{A}_{i,m} ds \right| \\
&= \sup_{t\in J_0} \left| \frac{1}{\Gamma(q_i)} \int_0^t (t-s)^{q_i-1} \Big[\mathcal{F}_i\big(S_{i,p-1}\big) - \mathcal{F}_i\big(S_{i,q-1}\big)\Big] ds \right| \\
&\leq \frac{1}{\Gamma(q_i)} \sup_{t\in J_0} \int_0^t (t-s)^{q_i-1} \Big|\mathcal{F}_i\big(S_{i,p-1}\big) - \mathcal{F}_i\big(S_{i,q-1}\big)\Big| ds \\
&\leq \frac{L_i}{\Gamma(q_i)} \big\|S_{i,p-1} - S_{i,q-1}\big\| \int_0^t (t-s)^{q_i-1} ds \\
&\leq \frac{LT^{q_i}}{\Gamma(q_i+1)} \big\|S_{i,p-1} - S_{i,q-1}\big\| \\
&\leq \delta \big\|S_{i,p-1} - S_{i,q-1}\big\|.
\end{aligned}$$

Let $p = q + 1$, then
$\big\|S_{i,q+1} - S_{i,q}\big\| \leq \delta \big\|S_{i,q} - S_{i,q-1}\big\| \leq \delta^2 \big\|S_{i,q-1} - S_{i,q-2}\big\| \leq \ldots \leq \delta^q \big\|S_{i,1} - S_{i,0}\big\|$.
Using triangle inequality, we have

$$\|S_{i,p} - S_{i,q}\| \leq \delta^q \left[\frac{1-\delta^{p-q}}{1-\delta}\right] \|y_{i,1}\|.$$

Since $0 < \delta < 1$ and $p \geq q$, then $1 - \delta^{p-q} \leq 1$. Hence,

$$\|S_{i,p} - S_{i,q}\| \leq \left[\frac{\delta^q}{1-\delta}\right] \sup_{t\in J_0} |y_{i,1}(t)|.$$

Since $|y_{i,1}(t)| < \infty$, therefore $\|S_{i,p} - S_{i,q}\| \to 0$ as $q \to \infty$. Hence, $\{S_{i,p}\}$ is a Cauchy sequence in $\mathcal{Y}$ and thus the series (5) converges. The proof is completed. □

Numerical Examples

Example 1 Consider the following nonlinear system

$$\begin{cases} \mathcal{D}^q y_1(t) = y_1\left(\frac{t}{3}\right) + 3y_2^2\left(\frac{t}{2}\right), \\ \mathcal{D}^q y_2(t) = y_2(t)y_3(t), \\ \mathcal{D}^q y_3(t) = y_1^2(t) + t y_3(t), \end{cases} \tag{12}$$

subject to the initial conditions

$$y_1(0) = 0, \;\; y_2(0) = 1, \;\; y_3(0) = 1,$$

where $q \in (0, 1]$, $t \in [0, 2]$.

On applying ADM to system (12), we obtain the following scheme

$$y_{1,0}(t) = 0, \qquad y_{1,m+1}(t) = \mathcal{I}_t^q\left(y_{1,m}\left(\frac{t}{3}\right)\right) + \mathcal{I}_t^q\left(3\mathcal{A}_{1,m}(t)\right), \tag{13}$$

$$y_{2,0}(t) = 1, \qquad y_{2,m+1}(t) = \mathcal{I}_t^q\left(\mathcal{A}_{2,m}(t)\right), \tag{14}$$

$$y_{3,0}(t) = 1, \qquad y_{3,m+1}(t) = \mathcal{I}_t^q\left(\mathcal{A}_{3,m}(t)\right) + \mathcal{I}_t^q\left(t y_{3,m}(t)\right), \tag{15}$$

where $\mathcal{A}_{1,m}(t)$, $\mathcal{A}_{2,m}(t)$ and $\mathcal{A}_{3,m}(t)$ represent the Adomian polynomials of nonlinear terms $y_2^2\left(\frac{t}{2}\right)$, $y_2(t)y_3(t)$ and $y_1^2(t)$, respectively.

Using the relations (13)–(15), the first four terms of the series solutions are

$$\begin{aligned} y_1(t) &= \frac{3}{\Gamma(q+1)}t^q + \frac{3^{2-q}2^{1-q}}{[\Gamma(2q+1)]^2}t^{4q} + \frac{3}{2^{2q}\Gamma(3q+1)}\left[\frac{\Gamma(2q+1)}{[\Gamma(q+1)]^2} + 2\right]t^{3q} \\ &\quad + \frac{3^{2-5q}2^{1-q}\Gamma(4q+1)}{[\Gamma(2q+1)]^2\Gamma(5q+1)}t^{5q} + \ldots, \end{aligned} \tag{16}$$

$$\begin{aligned} y_2(t) &= 1 + \frac{1}{\Gamma(q+1)}t^q + \frac{1}{\Gamma(2q+1)}t^{2q} + \frac{1}{\Gamma(2q+2)}t^{2q+1} \\ &\quad + \frac{1}{\Gamma(3q+2)}\left[\frac{\Gamma(2q+2)}{\Gamma(q+1)\Gamma(q+2)} + 1\right]t^{3q+1} + \frac{(2+q)}{\Gamma(3q+3)}t^{3q+2} \\ &\quad + \frac{1}{\Gamma(3q+1)}t^{3q} + \ldots, \end{aligned} \tag{17}$$

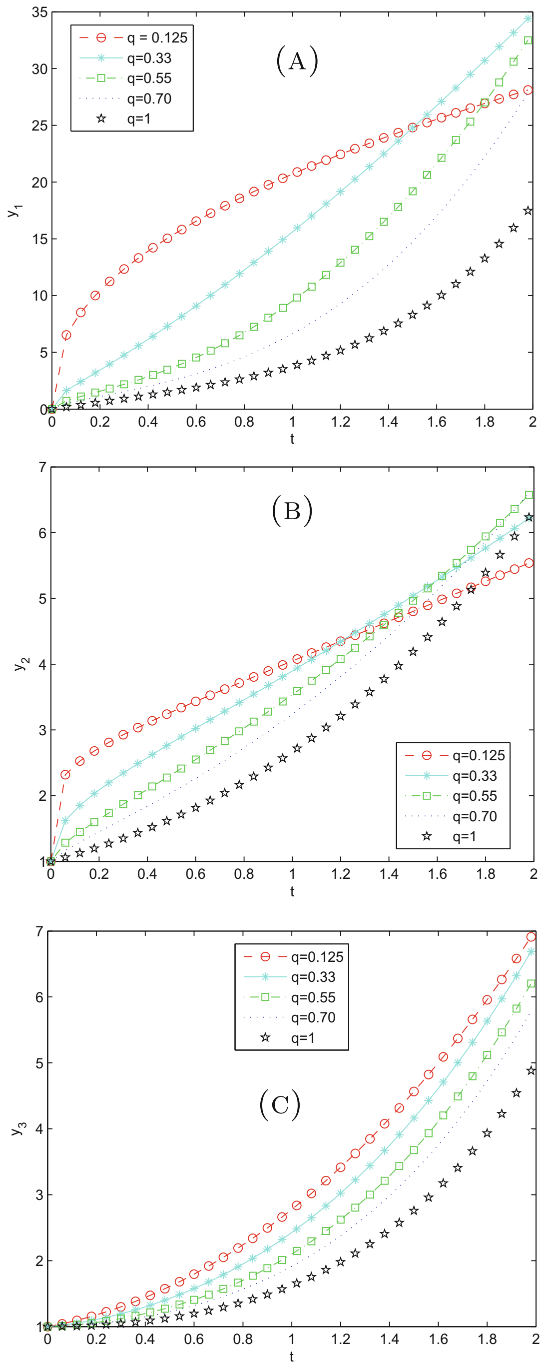
$$\begin{aligned} y_3(t) &= 1 + \frac{1}{\Gamma(q+2)}t^{q+1} + \frac{(2+q)}{\Gamma(2q+3)}t^{2q+2} + \frac{9\Gamma(2q+1)}{[\Gamma(q+1)]^2\Gamma(3q+1)}t^{3q} \\ &\quad + \frac{(2+q)\Gamma(2q+4)}{\Gamma(2q+3)\Gamma(3q+4)}t^{3q+3} + \ldots. \end{aligned} \tag{18}$$

Figure 1A–C show ADM solution of y_1, y_2 and y_3 at different values of q ($q = 0.125, 0.33, 0.55, 0.70, 1$), respectively.

Example 2 Consider the following system

$$\begin{cases} \mathcal{D}^{0.75} y_1(t) = y_1^2(t) + t^3, \\ \mathcal{D}^{1.25} y_2(t) = y_1^4(t) + y_2\left(\frac{t}{7}\right), \\ \mathcal{D}^{2.5} y_3(t) \;\, = y_2\left(\frac{t}{5}\right) + y_3^3(t) - t, \end{cases} \tag{19}$$

Fig. 1 **A**–**C**, ADM Sol. of y_1, y_2, y_3



subject to the initial conditions

$$y_1(0)=0,\ y_2(0)=0,\ y_2'(0)=0,$$
$$y_3(0)=0,\ y_3'(0)=0,\ y_3''(0)=0,$$

$t\in[0,2]$.

On applying ADM to system (19), we have the following recursive relations

$$y_{1,0}(t)=\frac{\Gamma(4)}{\Gamma(4.75)}t^{3.75},\qquad y_{1,m+1}(t)=\mathcal{I}_t^{0.75}\big(\mathcal{A}_{1,m}(t)\big), \tag{20}$$

$$y_{2,0}(t)=0,\qquad y_{2,m+1}(t)=\mathcal{I}_t^{1.25}\big(\mathcal{A}_{2,m}(t)\big)+\mathcal{I}_t^{1.25}\Big(y_{2,m}\Big(\frac{t}{7}\Big)\Big), \tag{21}$$

$$y_{3,0}(t)=-\frac{1}{\Gamma(4.5)}t^{3.5},\qquad y_{3,m+1}(t)=\mathcal{I}_t^{2.5}\Big(y_{2,m}\Big(\frac{t}{5}\Big)\Big)+\mathcal{I}_t^{2.5}\big(\mathcal{A}_{3,m}(t)\big), \tag{22}$$

where $\mathcal{A}_{1,m}(t)$, $\mathcal{A}_{2,m}(t)$ and $\mathcal{A}_{3,m}(t)$ represent the Adomian polynomials of nonlinear terms $y_1^2(t)$, $y_1^4(t)$ and $y_3^3(t)$, respectively.

Using the relations (20)–(22), the first few terms of the series solution are

$$\begin{aligned}
y_1(t)=&\frac{\Gamma(4)}{\Gamma(4.75)}t^{3.75}+\left[\frac{\Gamma(4)}{\Gamma(4.75)}\right]^2\cdot\frac{\Gamma(8.5)}{\Gamma(9.25)}t^{8.25}\\
&+2\left[\frac{\Gamma(4)}{\Gamma(4.75)}\right]^3\cdot\frac{\Gamma(8.5)}{\Gamma(9.25)}\cdot\frac{\Gamma(13)}{\Gamma(13.75)}t^{12.75}\\
&+\left[\frac{\Gamma(4)}{\Gamma(4.75)}\right]^4\cdot\frac{\Gamma(8.5)}{\Gamma(9.25)}\cdot\frac{\Gamma(17.5)}{\Gamma(18.25)}\left[\frac{\Gamma(8.5)}{\Gamma(9.25)}+\frac{4\Gamma(13)}{\Gamma(13.75)}\right]t^{17.25}\\
&+\ldots,
\end{aligned}\tag{23}$$

$$\begin{aligned}
y_2(t)=&\left[\frac{\Gamma(4)}{\Gamma(4.75)}\right]^4\cdot\frac{\Gamma(16)}{\Gamma(17.25)}t^{16.25}+4\left[\frac{\Gamma(4)}{\Gamma(4.75)}\right]^5\cdot\frac{\Gamma(8.5)}{\Gamma(9.25)}\cdot\frac{\Gamma(20.5)}{\Gamma(21.75)}t^{20.75}\\
&+\left[\frac{\Gamma(4)}{\Gamma(4.75)}\right]^4\cdot\frac{\Gamma(16)}{\Gamma(18.5)}\cdot\Big(\frac{1}{7}\Big)^{16.25}t^{17.5}\\
&+2\left[\frac{\Gamma(4)}{\Gamma(4.75)}\right]^6\cdot\frac{\Gamma(8.5)}{\Gamma(9.25)}\cdot\frac{\Gamma(25)}{\Gamma(26.25)}\left[\frac{3\Gamma(8.5)}{\Gamma(9.25)}+\frac{4\Gamma(13)}{\Gamma(13.75)}\right]t^{25.25}\\
&+4\left[\frac{\Gamma(4)}{\Gamma(4.75)}\right]^5\cdot\frac{\Gamma(8.5)}{\Gamma(9.25)}\cdot\frac{\Gamma(20.5)}{\Gamma(23)}\cdot\Big(\frac{1}{7}\Big)^{20.75}t^{22}\\
&+\left[\frac{\Gamma(4)}{\Gamma(4.75)}\right]^4\cdot\frac{\Gamma(16)}{\Gamma(19.75)}\cdot\Big(\frac{1}{7}\Big)^{33.75}t^{18.75}+\ldots,
\end{aligned}\tag{24}$$

$$\begin{aligned}
y_3(t)=&-\frac{1}{\Gamma(4.5)}t^{3.5}-\left[\frac{1}{\Gamma(4.5)}\right]^3\cdot\frac{\Gamma(11.5)}{\Gamma(14)}t^{13}\\
&+\left[\frac{\Gamma(4)}{\Gamma(4.75)}\right]^4\cdot\frac{\Gamma(16)}{\Gamma(17.25)}\cdot\frac{1}{\Gamma(4.5)}\cdot\Big(\frac{1}{5}\Big)^{16.25}t^{3.5}\\
&-3\left[\frac{1}{\Gamma(4.5)}\right]^5\cdot\frac{\Gamma(11.5)}{\Gamma(14)}\cdot\frac{\Gamma(21)}{\Gamma(23.5)}t^{22.5}+\ldots.
\end{aligned}\tag{25}$$

Figure 2 shows the ADM solution of y_1, y_2, y_3.

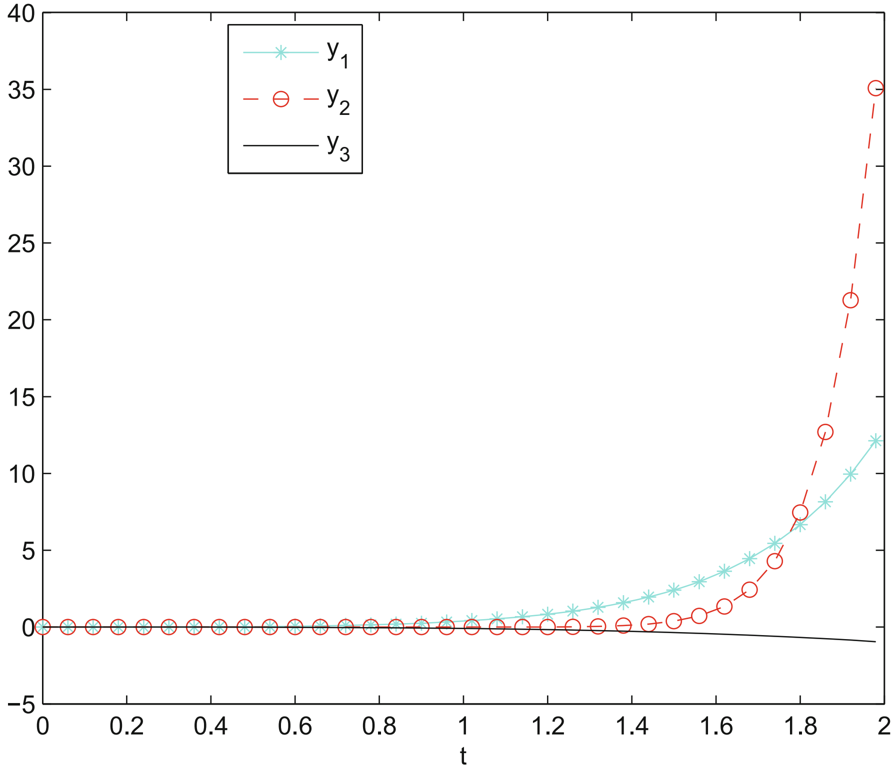


Fig. 2 ADM Sol. of y_1, y_2, y_3

Example 3 Consider the following system

$$\begin{cases} \mathcal{D}^p y_1(t) = y_1\left(\frac{t}{3}\right) + y_1^2(t) - \frac{\Gamma(3+p)}{\Gamma(3)} t^2, \\ \mathcal{D}^q y_2(t) = y_2(t)\sin(y_1(t)) + t y_1\left(\frac{t}{4}\right), \end{cases} \tag{26}$$

subject to the initial conditions

$$y_1(0) = 0, \;\; y_1'(0) = 0, \;\; y_2(0) = 0,$$

where $p \in (1, 2]$, $q \in (0, 1]$, $t \in [0, 2]$.

Applying ADM to system (26), we get the following scheme

$$y_{1,0}(t) = -t^{p+2}, \qquad y_{1,m+1}(t) = \mathcal{I}_t^p\left(y_{1,m}\left(\frac{t}{3}\right)\right) + \mathcal{I}_t^p\left(\mathcal{A}_{1,m}(t)\right), \tag{27}$$

$$y_{2,0}(t) = 0, \qquad y_{2,m+1}(t) = \mathcal{I}_t^q\left(\mathcal{A}_{2,m}(t)\right) + \mathcal{I}_t^q\left(t y_{1,m}\left(\frac{t}{4}\right)\right), \tag{28}$$

where $\mathcal{A}_{1,m}(t)$ and $\mathcal{A}_{2,m}(t)$ represent the Adomian polynomials of nonlinear terms $y_2^2(t)$ and $y_2(t)\sin(y_1(t))$, respectively.
Using the relations (27)–(28), the first few terms of the series solution are

$$y_1(t) = -t^{p+2} - \frac{1}{3^{p+2}} \cdot \frac{\Gamma(p+3)}{\Gamma(2p+3)} t^{2p+2} + \frac{\Gamma(2p+5)}{\Gamma(3p+5)} t^{3p+4}$$

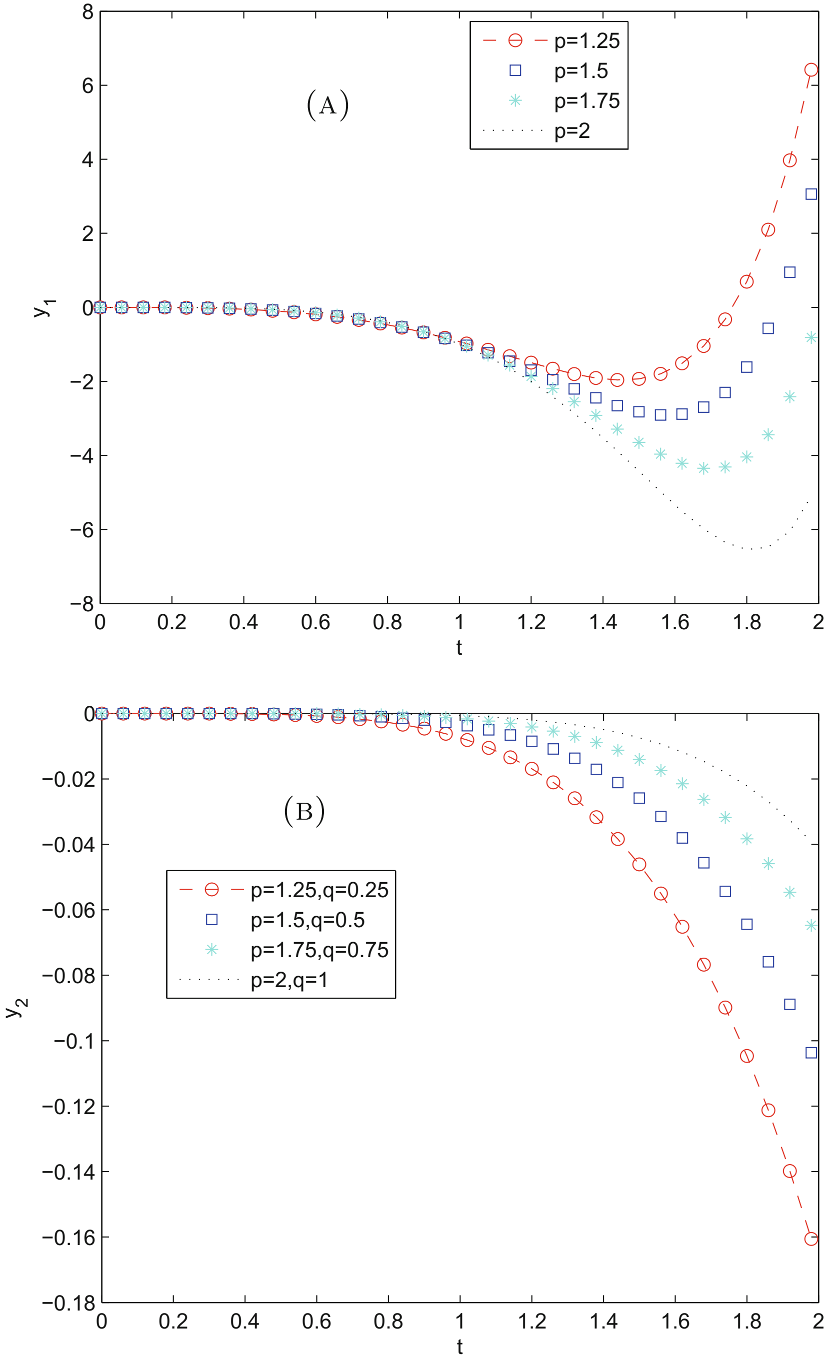


Fig. 3 **A**, **B**, ADM Sol. of y_1, y_2

$$
\begin{aligned}
&-\frac{1}{3^{3p+4}}\cdot\frac{\Gamma(p+3)}{\Gamma(3p+3)}t^{3p+2}+\frac{1}{3^{3p+4}}\cdot\frac{\Gamma(2p+5)}{\Gamma(4p+5)}t^{4p+4}\\
&+\frac{2}{3^{p+2}}\cdot\frac{\Gamma(p+3)}{\Gamma(2p+3)}\cdot\frac{\Gamma(3p+5)}{\Gamma(4p+5)}t^{4p+4}\\
&-2\cdot\frac{\Gamma(2p+5)}{\Gamma(3p+5)}\cdot\frac{\Gamma(4p+7)}{\Gamma(5p+7)}t^{5p+6}+\ldots, \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
y_2(t)=&-\frac{1}{4^{p+2}}\cdot\frac{\Gamma(p+4)}{\Gamma(p+q+4)}t^{p+q+3}+\frac{1}{4^{3p+4}}\cdot\frac{\Gamma(2p+5)}{\Gamma(3p+5)}\cdot\frac{\Gamma(3p+6)}{\Gamma(3p+q+6)}t^{3p+q+5}\\
&-\frac{1}{3^{p+2}}\cdot\frac{1}{4^{2p+2}}\cdot\frac{\Gamma(p+3)}{\Gamma(2p+3)}\cdot\frac{\Gamma(2p+4)}{\Gamma(2p+q+4)}t^{2p+q+3}+\ldots. \qquad (30)
\end{aligned}
$$

Figure 3A, B show the ADM solution of y_1, y_2.

Example 4 Consider the following nonlinear system

$$
\begin{cases}
\mathcal{D}^{5/2}y_1(t)=y_2\left(\frac{t}{3}\right)+y_1(t)y_2(t)+\Gamma(7/2)\cos(\Gamma(9/2)),\\
\mathcal{D}^{3/2}y_2(t)=ty_1\left(\frac{t}{5}\right)+\pi^2\tan(3)y_2^2(t),
\end{cases} \qquad (31)
$$

subject to the initial conditions

$$y_1(0)=1,\ y_1'(0)=-1,\ y_1''(0)=-1,$$

$$y_2(0)=0,\ y_2'(0)=1,$$

$t\in[0,4]$. Applying ADM to system (31) leads to the following scheme

$$
\begin{aligned}
&y_{1,0}(t)=1-t-\frac{t^2}{2}+\cos(\Gamma(9/2))t^{5/2},\\
&y_{1,m+1}(t)=\mathcal{I}_t^{5/2}\left(y_{2,m}\left(\frac{t}{3}\right)\right)+\mathcal{I}_t^{5/2}\left(\mathcal{A}_{1,m}(t)\right), \qquad (32)\\
y_{2,0}(t)=t,\quad &y_{2,m+1}(t)=\mathcal{I}_t^{3/2}\left(ty_{1,m}\left(\frac{t}{5}\right)\right)+\pi^2\tan(3)\mathcal{I}_t^{3/2}\left(\mathcal{A}_{2,m}(t)\right), \qquad (33)
\end{aligned}
$$

where $\mathcal{A}_{1,m}(t)$ and $\mathcal{A}_{2,m}(t)$ represent the Adomian polynomials of nonlinear terms $y_1(t)y_2(t)$ and $y_2^2(t)$, respectively.

Using the relations (32)–(33), the first few terms of the series solution are

$$
\begin{aligned}
y_1(t)=&\,1-t-\frac{t^2}{2}+\cos(\Gamma(9/2))t^{5/2}+\frac{4}{3}\cdot\frac{1}{\Gamma(9/2)}t^{7/2}-\frac{\Gamma(3)}{\Gamma(11/2)}t^{9/2}\\
&-\frac{1}{2}\cdot\frac{\Gamma(4)}{\Gamma(13/2)}t^{11/2}+\cos(\Gamma(9/2))\cdot\frac{\Gamma(9/2)}{\Gamma(7)}t^6+\ldots, \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
y_2(t)=&\,t+\frac{1}{\Gamma(7/2)}t^{5/2}+\frac{\Gamma(3)}{\Gamma(9/2)}\left[\pi^2\tan(3)-\frac{1}{5}\right]t^{7/2}-\frac{1}{50}\cdot\frac{\Gamma(4)}{\Gamma(11/2)}t^{9/2}\\
&+\frac{\Gamma(9/2)}{\Gamma(6)}\left[\cos(\Gamma(9/2))\cdot\left(\frac{1}{5}\right)^{5/2}+\frac{4}{3}\cdot\frac{1}{\Gamma(9/2)\Gamma(7/2)}+\frac{2\pi^2\tan(3)}{\Gamma(7/2)}\right]t^5\\
&+\frac{4\pi^2\tan(3)}{\Gamma(9/2)}\left[\pi^2\tan(3)-\frac{1}{5}\right]\frac{\Gamma(11/2)}{\Gamma(7)}t^6+\ldots. \qquad (35)
\end{aligned}
$$

Figure 4 shows the ADM solution of y_1, y_2.

In general, finding the exact solution of every differential equation is difficult, particularly the higher-order non-linear fractional differential equations with deviated arguments. The question is how one can check the accuracy of the method in the absence of the exact solution. Therefore, we give the following example with the known exact solution and show the accuracy of the proposed method.

Example 5 Consider the following nonlinear system

$$\begin{cases} \mathcal{D}^p y_1(t) = -2y_2\left(\frac{t}{2}\right)y_1\left(\frac{t}{2}\right), \\ \mathcal{D}^p y_2(t) = 1 - 2y_2^2\left(\frac{t}{2}\right), \end{cases} \tag{36}$$

subject to the initial conditions

$$y_1(0) = 1, \ \ y_2(0) = 0,$$

where $0 < p \leq 1, t \in [0, 1]$, which has the exact solution $y_1(t) = \cos t$ and $y_2(t) = \sin t$ at $p = 1$ (Fig. 5).

On applying ADM to system (36), we obtain the following scheme

$$y_{1,0}(t) = 1, \qquad y_{1,m+1}(t) = -2\mathcal{I}_t^p\left(\mathcal{A}_{1,m}(t)\right), \tag{37}$$

$$y_{2,0}(t) = 0, \qquad y_{2,m+1}(t) = -2\mathcal{I}_t^p\left(\mathcal{A}_{2,m}(t)\right), \tag{38}$$

Table 1 The exact and numerical values of the solution y_1 of example 5

t	Exact solution	ADM solution	Error
0	1.0000	1.0000	0
0.1	0.9950	0.9950	0.0000
0.2	0.9801	0.9801	0.0000
0.3	0.9553	0.9553	0.0000
0.4	0.9211	0.9211	0.0000
0.5	0.8776	0.8776	0.0000
0.6	0.8253	0.8254	0.0001
0.7	0.7648	0.7650	0.0002
0.8	0.6967	0.6971	0.0004
0.9	0.6216	0.6223	0.0007
1	0.5403	0.5417	0.0014

Table 2 The exact and numerical values of the solution y_2 of example 5

t	Exact solution	ADM solution	Error
0	0	0	0
0.1	0.0998	0.0998	0.0000
0.2	0.1987	0.1987	0.0000
0.3	0.2955	0.2955	0.0000
0.4	0.3894	0.3893	0.0001
0.5	0.4794	0.4792	0.0003
0.6	0.5646	0.5640	0.0006
0.7	0.6442	0.6428	0.0014
0.8	0.7174	0.7147	0.0027
0.9	0.7833	0.7785	0.0048
1	0.8415	0.8333	0.0081

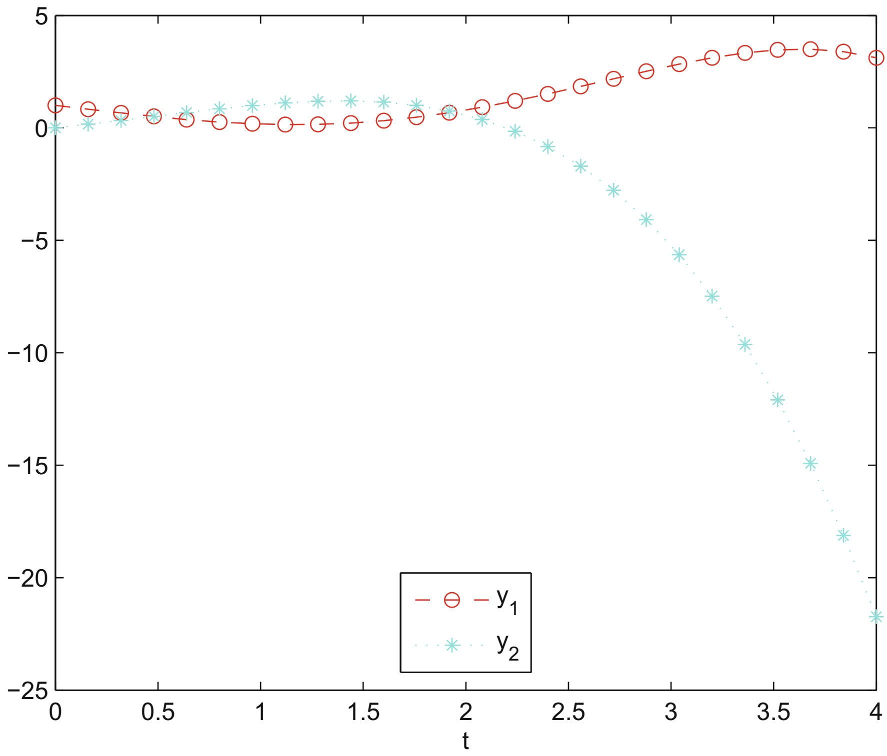


Fig. 4 ADM Sol. of y_1, y_2

where $\mathcal{A}_{1,m}(t)$ and $\mathcal{A}_{2,m}(t)$ represent the Adomian polynomial of nonlinear terms $y_2\left(\frac{t}{2}\right)y_1\left(\frac{t}{2}\right)$ and $y_2^2\left(\frac{t}{2}\right)$, respectively. Using the relations (37)–(38), the first few terms of the series solution are

$$\begin{aligned} y_1(t) &= 1 - \frac{1}{2^{p-1}\Gamma(2p+1)}t^{2p} + 2\left[\frac{1}{2^{4p-1}\Gamma(p+1)\Gamma(2p+1)}\right. \\ &\quad \left.+\frac{\Gamma(2p+1)}{2^{5p-1}[\Gamma(p+1)]^2\Gamma(3p+1)}\right]\frac{\Gamma(3p+1)}{\Gamma(4p+1)}t^{4p} - \frac{16}{[2^{4p}\Gamma(p+1)]^2} \\ &\quad \left[\frac{\Gamma(3p+1)}{2^{p}\Gamma(2p+1)\Gamma(4p+1)} + \frac{\Gamma(2p+1)}{2^{2p}\Gamma(p+1)\Gamma(4p+1)}\right. \\ &\quad \left.+\frac{\Gamma(2p+1)\Gamma(4p+1)}{2^{3p}\Gamma(p+1)\Gamma(3p+1)\Gamma(5p+1)}\right]\frac{\Gamma(5p+1)}{\Gamma(6p+1)}t^{6p} + \ldots, \end{aligned} \tag{39}$$

$$\begin{aligned} y_2(t) &= \frac{1}{\Gamma(p+1)}t^{p} - \frac{2}{[2^{p}\Gamma(p+1)]^2}\cdot\frac{\Gamma(2p+1)}{\Gamma(3p+1)}t^{3p} \\ &\quad +\frac{8}{2^{6p}[\Gamma(p+1)]^3}\cdot\frac{\Gamma(2p+1)}{\Gamma(3p+1)}\cdot\frac{\Gamma(4p+1)}{\Gamma(5p+1)}t^{5p} \\ &\quad -\frac{8}{2^{4p}[\Gamma(p+1)]^4}\cdot\frac{\Gamma(2p+1)}{\Gamma(3p+1)}\left[\frac{\Gamma(2p+1)}{\Gamma(3p+1)} + \frac{4\Gamma(4p+1)}{2^{8p}\Gamma(5p+1)}\right]\frac{\Gamma(6p+1)}{\Gamma(7p+1)}t^{7p} \\ &\quad +\ldots. \end{aligned} \tag{40}$$

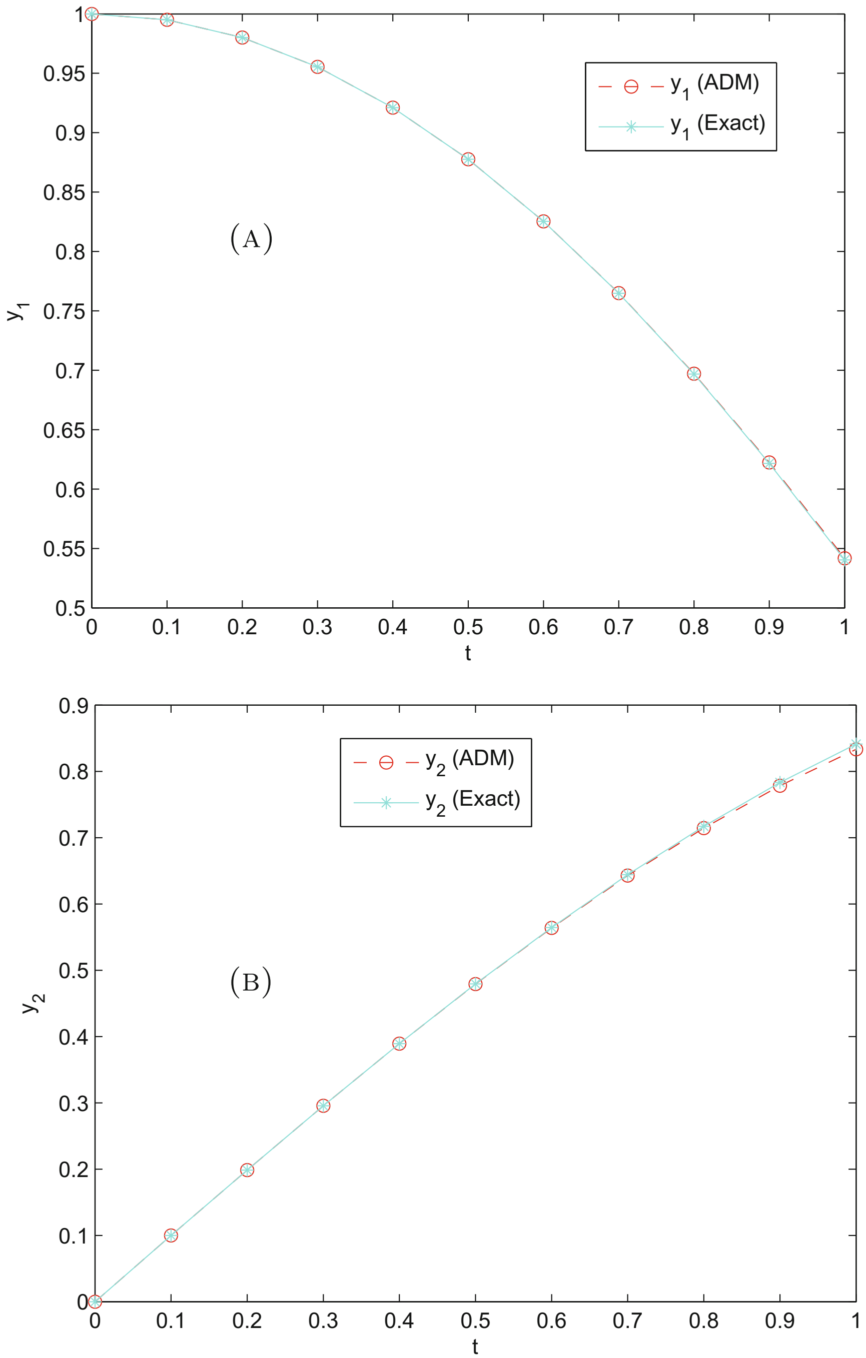


Fig. 5 **A**, **B**, ADM and exact sol. of y_1, y_2

Tables 1 and 2 show the exact and numerical values of y_1 and y_2. Figure 5 A, B show the solutions of y_1 and y_2, respectively.

Conclusion

The focus of this paper is to approximate the solution of a nonlinear system of FDEs with deviated arguments using a simple method. Using Banach contraction principle, we prove the existence and uniqueness of the solution. As we know that in real-life, many linear and nonlinear problems occur in the form of a differential equation. Some difficulties occur while solving nonlinear FDEs. Therefore, we apply the ADM method to solve the higher-order nonlinear system of FDEs with deviated arguments and provide some numerical examples to show the effectuality of the method. We plot the figures of ADM solutions using MATLAB.

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Data Availability Not applicable.

Code Availability Not applicable.

Declarations

Conflict of Interest The authors declare no conflict of interest.

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