

Received:

7 July 2018

Revised:

9 November 2018

Accepted:

14 January 2019

Cite as: P. Mdlongwa, B.O. Oluyede, A.K.A. Amey, A.F. Fagbamigbe, B. Makubate. Kumaraswamy log-logistic Weibull distribution: model, theory and application to lifetime and survival data.

Heliyon 5 (2019) e01144.

doi: [10.1016/j.heliyon.2019.e01144](https://doi.org/10.1016/j.heliyon.2019.e01144)



Kumaraswamy log-logistic Weibull distribution: model, theory and application to lifetime and survival data

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Abstract

We develop the new Kumaraswamy Log-Logistic Weibull (KLLoGW) distribution by combining the Kumaraswamy and Log-logistic Weibull distributions. This new model is flexible for modelling lifetime data. Some statistical properties including quantile function, hazard rate function, moments and conditional moments are presented. Model parameters are estimated via the method of maximum likelihood and a Monte Carlo simulation study conducted to assess the accuracy of the estimates. Finally, the model is applied to a real dataset.

Keywords: Applied mathematics, Computational mathematics

1. Introduction

New families of probability distributions can be generated by introducing one or more additional shape parameter(s) to existing distributions. It is often the case that

the role of the added shape parameter(s) is meant to introduce skewness, tail variation and improve the goodness-of-fit of the distribution.

The beta-G, Kumaraswamy-G, Zografos-Balakrishnan-G and Ristić-and-Balakrishnan G are some of the popular families of distributions (Cordeiro and de Castro [1], and Jones et al. [2]). For an arbitrary baseline cumulative distribution function (cdf) $G(x)$, the cdf of Kumaraswamy generalized distribution (Kum-G) is given by equation (1)

$$F(x) = 1 - (1 - (G(x))^a)^b, \quad (1)$$

and the corresponding probability density function (pdf) by equation (2)

$$f(x) = abg(x)(G(x))^{a-1}(1 - (G(x))^a)^{b-1}, \quad (2)$$

where $g(x)$ is the baseline pdf and $a > 0$, and $b > 0$ are shape parameters. Several authors have used this generalization to introduce new distributions. Examples include the Kumaraswamy Marshall–Olkin Log-Logistic (Kw-MOLLoG) distribution [3], Kw-Weibull distribution [4], Kw-generalized gamma distribution [5], Kw-log-logistic distribution [6], Kw-modified Weibull distribution [7], Kw-generalized Pareto distribution [8], Kumaraswamy linear exponential distribution [9], Kw-Lomax distribution [10], Kw-half-Cauchy distribution [11], Kw-generalized Rayleigh distribution [12] and Kw-Gompertz distribution [13] among others. The proposed distribution has a hazard function that covers a wide variety of shapes and infact we expect this generalization to have wider application in modelling data from various fields of research. The KLLoGW distribution and its sub-models are given in section 2. The quantile function, hazard and reverse hazard functions are presented is section 3. Moments, conditional moments, mean deviations, inequality measures, Rényi entropy, density of the order statistics and L-moments are also given in section 3. Maximum likelihood estimates and Monte Carlo simulation study are presented in section 4. Section 5 contain an application followed by a short concluding remark in section 6.

2. Model

2.1. Kumaraswamy log-logistic Weibull distribution

Consider the Log-logistic Weibull (LLoGW) distribution (see [14] for details) with cdf and pdf given by equations (3) and (4)

$$G_{\text{LLoGW}}(x) = 1 - (1 + x^c)^{-1} \exp(-\alpha x^\beta) \quad (3)$$

and

$$g_{\text{LLoGW}}(x) = \frac{e^{-\alpha x^\beta}}{1 + x^c} \left\{ \alpha \beta x^{\beta-1} + \frac{cx^{c-1}}{(1 + x^c)} \right\}, \quad (4)$$

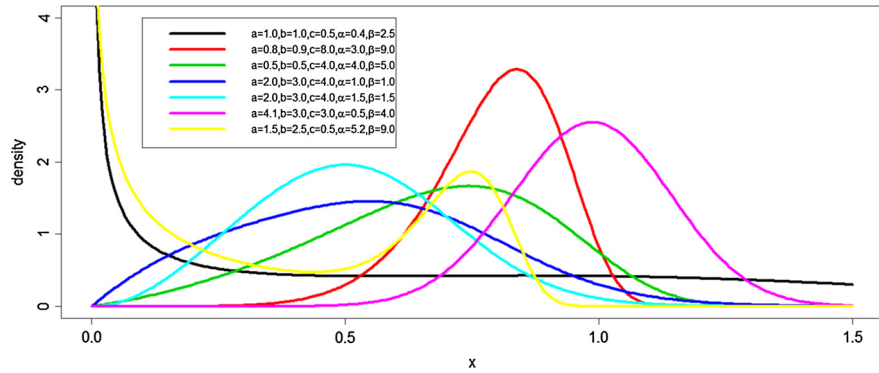


Figure 1. Plots of KLLoGW pdf for some parameter values.

respectively, where $c, \alpha, \beta > 0$, and $x \geq 0$. Substituting (3) into equation (1), we obtain the new KLLoGW distribution with cdf and pdf given by equations (5) and (6) as

$$F_{\text{KLLoGW}}(x; a, b, c, \alpha, \beta) = 1 - (1 - (1 - (1 + x^c)^{-1} e^{-\alpha x^\beta})^a)^b, \tag{5}$$

and

$$f_{\text{KLLoGW}}(x; a, b, c, \alpha, \beta) = ab \left(1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}\right)^{a-1} \times \left(1 - \left(1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}\right)^a\right)^{b-1} \times (1 + x^c)^{-2} e^{-\alpha x^\beta} \left(cx^{c-1} + \alpha\beta x^{\beta-1}(1 + x^c)\right), \tag{6}$$

respectively, for $a, b, c, \alpha, \beta > 0$. Plotted pdf are shown in Figure 1. The graphs show that the KLLoGW pdf can be decreasing, left or right skewed.

2.2. Series expansions

The generalized binomial theorem is applied to obtain a series expansion of the cdf. The binomial expansion is given by equation (7)

$$(1 - y)^r = \sum_{i=0}^{\infty} (-1)^i \binom{k}{i} v^i, \tag{7}$$

for $|y| < 1$ and any real number $k > 0$. Using equation (7) in equation (5), the expansion of the cdf is given in equation (8) as follows:

$$F_{\text{KLLoGW}}(x) = 1 - \sum_{i=0}^{\infty} \binom{b}{i} (-1)^i (G(x))^{ai}, \tag{8}$$

where $G(x)$ is the LLoGW cdf. Using equation (7) and applying the series expansion $e^{-y} = \sum_{k=0}^{\infty} \frac{(-1)^k y^k}{k!}$, the KLLoGW survival function, cdf and pdf are given by equations (9), (10), and (11), respectively as

$$\bar{F}_{\text{KLLoGW}}(x) = \sum_{i,j=0}^{\infty} \binom{b}{i} \binom{ai}{j} (-1)^{i+j} \bar{F}_{\text{ELLoGW}}(x), \tag{9}$$

$$F_{\text{KLLoGW}}(x) = 1 - \bar{F}_{\text{KLLoGW}}(x) = 1 - \sum_{i,j=0}^{\infty} \binom{b}{i} \binom{ai}{j} (-1)^{i+j} \bar{F}_{\text{ELLoGW}}(x), \tag{10}$$

and

$$f_{\text{KLLoGW}}(x) = \sum_{i,j,m=0}^{\infty} \binom{b}{i} \binom{ai}{j} \frac{(-1)^{i+j+m+1} (ai)^m \alpha^m x^{\beta m-1}}{m!} \times (1+x^c)^{-ai} \left(\beta m + (ai)(1+x^c)^{-1} c x^c \right), \tag{11}$$

where $\bar{F}_{\text{ELLoGW}}(x)$ is the survival function of the exponentiated LLoGW distribution.

Consequently, the cdf as well as the pdf of the KLLoGW distribution can be written as a infinite mixture of ELLoGW cdf's and pdf's, respectively. Thus, several mathematical and structural properties of the KLLoGW distribution follow immediately from those of the ELLoGW distribution.

2.3. Some sub-models of the KLLoGW distribution

The KLLoGW distribution contains several sub-models, some of which are listed below.

- The Kumaraswamy Log-logistic Exponential (KLLoGE) distribution is obtained when $\beta = 1$.
- The Kumaraswamy Log-logistic Rayleigh (KLLoGR) distribution is obtained when $\beta = 2$.
- If $\alpha \rightarrow 0^+$, the Kumaraswamy Log-logistic (KLLoG) distribution is the resulting distribution in the limit.
- If $a = 1$, a new lifetime distribution belonging to the frailty parameter family is obtained.
- If $a = 1, b = 1$, the parent Log-logistic Weibull (LLoGW) distribution is obtained.
- If $a = 1, b = 1$, and $\beta = 1$, the resulting distribution is the Log-logistic Exponential (LLoGE) distribution.
- When $a = 1, b = 1$, and $\beta = 2$, we have the Log-logistic Rayleigh (LLoGR) distribution.
- When $a = 1, b = 1$, and $\alpha \rightarrow 0^+$, the Log-logistic (LLoG) distribution is obtained in the limit.
- If $b = 1$, we have the Exponentiated Log-logistic Weibull (ELLoGW) distribution.

- If $b = 1, \beta = 1$, we obtain the Exponentiated Log-logistic Exponential (ELLoGE) distribution.
- If $b = 1, \beta = 2$, the Exponentiated Log-logistic Rayleigh (ELLoGR) distribution is obtained.
- If $b = 1$, and $\alpha \rightarrow 0^+$, the resulting distribution is the Exponentiated Log-Logistic (ELLoG) distribution in the limit.
- If $\beta = 1, c = 1$, the KLLoGW distribution becomes a 3 parameter distribution given by equation (12)

$$F(x; a, b, \alpha) = 1 - (1 - (1 - (1 + x)^{-1} e^{-\alpha x})^a)^b, \quad (12)$$

for $a, b, \alpha > 0$.

- If $\beta = 2$, and $c = 1$, then the KLLoGW distribution reduces to a 3 parameter distribution given by equation (13)

$$F(x; a, b, \alpha) = 1 - (1 - (1 - (1 + x)^{-1} e^{-\alpha x^2})^a)^b, \quad (13)$$

for $a, b, \alpha > 0$.

3. Methods

This section contains some statistical properties of the proposed KLLoGW distribution.

3.1. Quantile function

The KLLoGW quantile function can be obtained by inverting $F_{\text{KLLoGW}}(x) = u, 0 \leq u \leq 1$, where $F_{\text{KLLoGW}}(x)$ is given in (5). The quantiles of the KLLoGW distribution is obtained by numerically solving equation (14)

$$\log \left(1 - \left(1 - (1 - u)^{\frac{1}{b}} \right)^{\frac{1}{a}} \right) + \alpha x^\beta + \log(1 + x^c) = 0. \quad (14)$$

Therefore, random numbers can be generated from equation (14). Some KLLoGW quantiles are listed in Table 1.

3.2. Hazard and reverse Hazard functions

The KLLoGW hazard and reverse hazard functions are given in equations (15) and (16), respectively, that is,

$$h_{F_{\text{KLLoGW}}}(x) = \frac{f_{\text{KLLoGW}}(x)}{F_{\text{KLLoGW}}(x)}$$

Table 1. KLLoGW Quantiles for Selected Parameter Values.

<i>u</i>	<i>(a, b, c, α, β)</i>				
	(2.0 2.0 3.0 0.5 2.0)	(4.0 3.0 5.0 4.0 0.8)	(1.0 0.5 0.5 2.0 2.0)	(0.5 3.0 1.0 3.0 3.0)	(0.5 2.0 2.0 1.0 1.0)
0.1	0.5119100	0.08638097	0.05191901	0.001192354	0.002630526
0.2	0.6115194	0.11899910	0.19785200	0.005163070	0.011094335
0.3	0.6875387	0.14728917	0.35186384	0.012710173	0.026328019
0.4	0.7554795	0.17489755	0.49453238	0.025107134	0.049677758
0.5	0.8217849	0.20375676	0.63286849	0.044180702	0.082844155
0.6	0.8910735	0.23574345	0.77494461	0.073163756	0.128698996
0.7	0.9689506	0.27362093	0.93073957	0.117262122	0.192447899
0.8	1.0655445	0.32297172	1.11748872	0.185402083	0.286029727
0.9	1.2108779	0.40076263	1.38340087	0.296873034	0.447627980

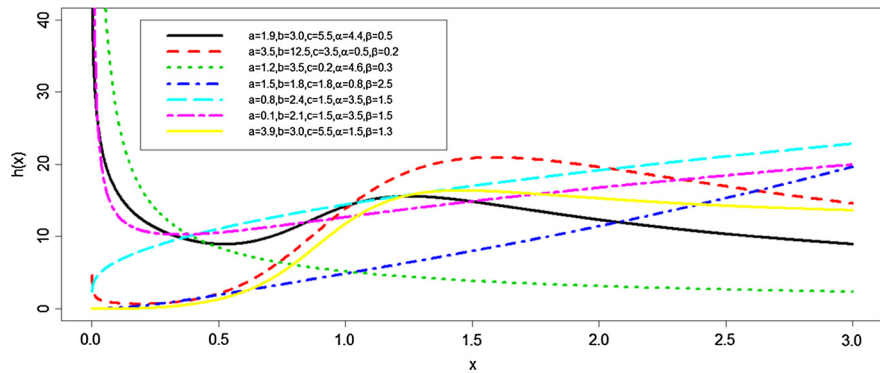


Figure 2. Graphs of KLLoGW hazard function for some parameter values.

$$\begin{aligned}
 &= ab \left(1 - (1 + x^c)^{-1} e^{-\alpha x^\beta} \right)^{a-1} \\
 &\times (1 + x^c)^{-2} e^{-\alpha x^\beta} \left(c x^{c-1} + \alpha \beta x^{\beta-1} (1 + x^c) \right) \\
 &\times \left(1 - \left(1 - (1 + x^c)^{-1} e^{-\alpha x^\beta} \right)^a \right)^{-1}, \tag{15}
 \end{aligned}$$

and

$$\begin{aligned}
 \tau_{F_{KLLoGW}}(x) &= ab \left(1 - (1 + x^c)^{-1} e^{-\alpha x^\beta} \right)^{a-1} \\
 &\times (1 + x^c)^{-2} e^{-\alpha x^\beta} \left(c x^{c-1} + \alpha \beta x^{\beta-1} (1 + x^c) \right) \\
 &\times \frac{\left(1 - \left(1 - (1 + x^c)^{-1} e^{-\alpha x^\beta} \right)^a \right)^{b-1}}{1 - \left(1 - (1 + x^c)^{-1} e^{-\alpha x^\beta} \right)^a}^b. \tag{16}
 \end{aligned}$$

Graphs of the hazard function for some selected values of the parameters are shown in Figure 2.

The graphs of the hazard function exhibit shapes that include increasing, decreasing, bathtub and bathtub followed by upside down bathtub. The KLLoGW hazard function is flexibility for modelling non-monotonic empirical hazard behaviors.

3.3. Moments and conditional moments

Moments and conditional moments are crucial in the study of some of the most important properties of a distribution. Using the binomial expansion (7) and the series expansion $e^{-y} = \sum_{k=0}^{\infty} \frac{(-1)^k y^k}{k!}$, the r^{th} moment of the KLLoGW distribution is given in equation (17) as follows:

$$\begin{aligned}
 E(X^r) &= \int_0^{\infty} x^r f_{\text{KLLoGW}}(x) dx \\
 &= \int_0^{\infty} x^r ab \left(1 - (1+x^c)^{-1} e^{-\alpha x^\beta} \right)^{a-1} \\
 &\quad \times (1+x^c)^{-2} e^{-\alpha x^\beta} \left(cx^{c-1} + \alpha \beta x^{\beta-1} (1+x^c) \right) \\
 &\quad \times \left(1 - \left(1 - (1+x^c)^{-1} e^{-\alpha x^\beta} \right)^a \right)^{b-1} dx \\
 &= \sum_{j,k,l,m=0}^{\infty} \binom{a-1}{j} \binom{b-1}{k} \binom{ak}{l} \frac{(-1)^{j+k+l+m} abc(j\alpha + \alpha + l\alpha)^m}{m!} \\
 &\quad \times \int_0^{\infty} x^{r+\beta m+c-1} (1+x^c)^{-j-l-2} dx \\
 &\quad + \sum_{j,k,l,m=0}^{\infty} \binom{a-1}{j} \binom{b-1}{k} \binom{ak}{l} \frac{(-1)^{j+k+l+m} ab\beta\alpha^{m+1}(j+l+1)^m}{m!} \\
 &\quad \times \int_0^{\infty} x^{r+\beta m+\beta-1} (1+x^c)^{-j-l-1} dx.
 \end{aligned}$$

By setting $w = (1+x^c)^{-1}$, we obtain

$$\begin{aligned}
 E(X^r) &= \sum_{j,k,l,m=0}^{\infty} \binom{a-1}{j} \binom{b-1}{k} \binom{ak}{l} \frac{(-1)^{j+k+l+m} ab(j\alpha + \alpha + l\alpha)^m}{m!} \\
 &\quad \times B\left(j+l+5 - \left(\frac{r+\beta m}{c}\right), \frac{r+\beta m+c}{c}\right) \\
 &\quad + \sum_{j,k,l,m=0}^{\infty} \binom{a-1}{j} \binom{b-1}{k} \binom{ak}{l} \frac{(-1)^{j+k+l+m} ab\beta\alpha^{m+1}(j+l+1)^m}{m!c} \\
 &\quad \times B\left(-\frac{1}{c}(r+\beta m+\beta-1) + j+l+4, \frac{1}{c}(r+\beta m+\beta)\right), \tag{17}
 \end{aligned}$$

where $B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$ is the complete beta function. Tables 2 and 3 lists the first six moments as well as the standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) of the KLLoGW distribution for selected values of the parameters, by fixing $(\alpha = 1.5$ and $\beta = 0.5)$

Table 2. Moments of the KLLoGW distribution values when $\alpha = 1.5$ and $\beta = 0.5$.

μ'_s	$a = 2.0, b = 1.0, c = 3.5$	$a = 0.5, b = 0.5, c = 0.5$	$a = 1.5, b = 0.5, c = 0.5$	$a = 1.5, b = 1.5, c = 0.5$
μ'_1	0.1312023	0.2207560	0.2207560	0.05421813
μ'_2	0.1754878	0.2723792	0.2723792	0.09262975
μ'_3	0.2008259	0.2915226	0.2915226	0.11789461
μ'_4	0.2162316	0.3013457	0.3013457	0.13478358
μ'_5	0.2263072	0.3073002	0.3073002	0.14671096
μ'_6	0.2333092	0.3112900	0.3112900	0.15554052
SD	0.3978363	0.4729122	0.4729122	0.29948313
CV	3.0322348	2.1422397	2.1422397	5.52367166
CS	2.1641446	1.2542050	1.2542050	3.84006373
CK	5.1125569	2.3280431	2.3280431	13.77659050

Table 3. Moments of the KLLoGW distribution values when $a = b = 1.5$ and $c = 3.5$.

μ'_s	$\alpha = 0.5, \beta = 0.5$	$\alpha = 1.5, \beta = 0.5$	$\alpha = 0.5, \beta = 1.5$	$\alpha = 1.5, \beta = 1.5$
μ'_1	0.2123674	0.3260763	0.07071062	0.0811697
μ'_2	0.2401710	0.3915203	0.13737234	0.2061925
μ'_3	0.2607722	0.4271667	0.17996517	0.2911043
μ'_4	0.2744934	0.4489942	0.20814338	0.3477296
μ'_5	0.2840812	0.4634773	0.22789742	0.3870302
μ'_6	0.2911100	0.4736783	0.24243541	0.4154215
SD	0.4416685	0.5340361	0.36383011	0.4467706
CV	2.0797380	1.6377642	5.14533939	5.5041550
CS	1.4730616	0.7452903	3.14634406	2.7132961
CK	2.9396956	1.3240783	9.20464029	6.5568132

and ($a = b = 1.5$ and $c = 3.5$), respectively. The R software package was used to obtain these values. The SD, CV, CS and CK are given by equations (18)–(21), respectively,

$$SD = \sqrt{\mu'_2 - \mu^2}, \quad CV = \frac{SD}{\mu} = \frac{\sqrt{\mu'_2 - \mu^2}}{\mu} = \sqrt{\frac{\mu'_2}{\mu^2} - 1}, \tag{18}$$

$$CV = \frac{SD}{\mu} = \frac{\sqrt{\mu'_2 - \mu^2}}{\mu} = \sqrt{\frac{\mu'_2}{\mu^2} - 1}, \tag{19}$$

$$CS = \frac{E[(X - \mu)^3]}{[E(X - \mu)^2]^{3/2}} = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{(\mu'_2 - \mu^2)^{3/2}}, \tag{20}$$

and

$$CK = \frac{E[(X - \mu)^4]}{[E(X - \mu)^2]^2} = \frac{\mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4}{(\mu'_2 - \mu^2)^2}, \tag{21}$$

respectively.

The r^{th} conditional moment is given by equation (22) as follows:

$$E(X^r | X > t) = \frac{1}{F(t)} \int_t^\infty x^r f_{KLLoGW}(x) dx$$

$$\begin{aligned}
 &= \frac{1}{F(t)} \left\{ \sum_{j,k,l,m=0}^{\infty} \binom{a-1}{j} \binom{b-1}{k} \binom{ak}{l} \right. \\
 &\quad \times \frac{(-1)^{j+k+l+m} abc(j\alpha + \alpha + l\alpha)^m}{m!} \\
 &\quad \left. \times \int_t^{\infty} x^{r+\beta m} (1+x^c)^{-j-l-2} (cx^{c-1} + \alpha\beta x^{\beta-1}(1+x^c)) dx \right\}.
 \end{aligned}$$

Now, let $w = (1 + x^c)^{-1}$, then

$$\begin{aligned}
 E(X^r | X > t) &= \frac{1}{F(t)} \left\{ \sum_{j,k,l,m=0}^{\infty} \binom{a-1}{j} \binom{b-1}{k} \binom{ak}{l} \right. \\
 &\quad \times \frac{(-1)^{j+k+l+m} ab(j\alpha + \alpha + l\alpha)^m}{m!} \\
 &\quad \times B_{(1+t^c)^{-1}} \left(j+l+5 - \left(\frac{r+\beta m}{c} \right), \frac{r+\beta m+c}{c} \right) \\
 &\quad + \sum_{j,k,l,m=0}^{\infty} \binom{a-1}{j} \binom{b-1}{k} \binom{ak}{l} \\
 &\quad \times \frac{(-1)^{j+k+l+m} ab\beta\alpha^{m+1} (j+1+l)^m}{m!c} \\
 &\quad \left. \times B_{(1+t^c)^{-1}} \left(-\frac{1}{c}(r+\beta m+\beta-1) + j+l+4, \frac{1}{c}(r+\beta m+\beta) \right) \right\}. \tag{22}
 \end{aligned}$$

3.4. Mean deviations

The mean deviation about the mean and the mean deviation about the median are defined by equation (23)

$$\delta_1(x) = \int_0^{\infty} |x - \mu| f_{\text{KLLoGW}}(x) dx \quad \text{and} \quad \delta_2(x) = \int_0^{\infty} |x - M| f_{\text{KLLoGW}}(x) dx, \tag{23}$$

respectively, where $\mu = E(X)$ and $M = \text{Median}(X)$ denotes the median. Note that $\delta_1(x)$ and $\delta_2(x)$ can be rewritten as in equation (24), that is,

$$\delta_1(x) = 2\mu F_{\text{KLLoGW}}(\mu) - 2\mu + 2T(\mu) \quad \text{and} \quad \delta_2(x) = -\mu + 2T(M), \tag{24}$$

where $T(\mu)$ is given by equation (25)

$$\begin{aligned}
 T(\mu) &= \int_{\mu}^{\infty} x f_{\text{KLLoGW}}(x) dx \\
 &= \sum_{j,k,l,m=0}^{\infty} \binom{a-1}{j} \binom{b-1}{k} \binom{ak}{l} \frac{(-1)^{j+k+l+m} ab(j\alpha + \alpha + l\alpha)^m}{m!}
 \end{aligned}$$

$$\begin{aligned} & \times B_{(1+\mu^c)^{-1}}\left(j+l+5-\left(\frac{\beta m+1}{c}\right), \frac{\beta m+c+1}{c}\right) \\ & + \sum_{j,k,l,m=0}^{\infty} \binom{a-1}{j} \binom{b-1}{k} \binom{ak}{l} \frac{(-1)^{j+k+l+m} ab\beta\alpha^{m+1}(j+1+l)^m}{m!c} \\ & \times B_{(1+\mu^c)^{-1}}\left(-\frac{1}{c}(1+\beta m+\beta-1)+j+l+4, \frac{1}{c}(1+\beta m+\beta)\right). \end{aligned} \tag{25}$$

3.5. Lorenz and Bonferroni curves

The well-known income inequality measures, referred to as Bonferroni and Lorenz curves for the KLLoGW distribution are given by equation (26)

$$\begin{aligned} B(p) &= \frac{1}{p\mu} \int_0^q x f_{\text{KLLoGW}}(x) dx = \frac{1}{p\mu} [\mu - T(q)] \quad \text{and} \\ L(p) &= \frac{1}{\mu} \int_0^q x f_{\text{KLLoGW}}(x) dx = \frac{1}{\mu} [\mu - T(q)], \end{aligned} \tag{26}$$

where $T(q)$ is given by equation (27)

$$\begin{aligned} T(q) &= \int_q^{\infty} x f_{\text{KLLoGW}}(x) dx \\ &= \sum_{j,k,l,m=0}^{\infty} \binom{a-1}{j} \binom{b-1}{k} \binom{ak}{l} \frac{(-1)^{j+k+l+m} ab(j\alpha + \alpha + l\alpha)^m}{m!} \\ & \times B_{(1+q^c)^{-1}}\left(j+l+5-\left(\frac{\beta m+1}{c}\right), \frac{\beta m+c+1}{c}\right) \\ & + \sum_{j,k,l,m=0}^{\infty} \binom{a-1}{j} \binom{b-1}{k} \binom{ak}{l} \frac{(-1)^{j+k+l+m} ab\beta\alpha^{m+1}(j+1+l)^m}{m!c} \\ & \times B_{(1+q^c)^{-1}}\left(-\frac{1}{c}(\beta m+\beta)+j+l+4, \frac{1}{c}(1+\beta m+\beta)\right), \end{aligned} \tag{27}$$

is the first incomplete moment and $q = F^{-1}(p)$, $0 \leq p \leq 1$.

3.6. Distribution of order statistics

We present the pdf of the i^{th} order statistics in this subsection. The distribution of the i^{th} order statistic from the KLLoGW distribution is presented in equation (28) as follows:

$$f_{i:n}(x) = \frac{n! f_{\text{KLLoGW}}(y)}{(i-1)!(n-i)!} [F_{\text{KLLoGW}}(x)]^{i-1} [1 - F_{\text{KLLoGW}}(x)]^{n-i}$$

$$= \frac{n!f_{\text{KLLoGW}}(x)}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} [F_{\text{KLLoGW}}(x)]^{j+i-1}. \tag{28}$$

Note that by using the binomial expansion $[1 - F_{\text{KLLoGW}}(x)]^{n-i} = \sum_{m=0}^{n-i} \binom{n-i}{m} (-1)^m [F_{\text{KLLoGW}}(x)]^m$, in (28), we obtain equation (29)

$$\begin{aligned} f_{i:n}(x) &= \frac{1}{B(i, n-i+1)} \sum_{m=0}^{n-i} \binom{n-i}{m} \frac{(-1)^m}{m+i} (m+i) [F_{\text{KLLoGW}}(x)]^{m+i-1} f_{\text{KLLoGW}}(x) \\ &= \sum_{m=0}^{n-i} w_{i,m} f_{m+i}(x), \end{aligned} \tag{29}$$

where $f_{m+i}(x)$ is the exponentiated KLLoGW pdf with parameters a, b, c, α, β and $m+i$.

The t^{th} moment of the i^{th} order statistics from the KLLoGW distribution may be computed from the result of Barakat and Abdelkader [15], that is,

$$E(X_{i:n}^t) = t \sum_{p=n-i+1}^n (-1)^{p-n+i-1} \binom{p-1}{n-i} \binom{n}{p} \int_0^\infty x^{t-1} [1 - F_{\text{KLLoGW}}(x)]^p dx. \tag{30}$$

Note:

$$\begin{aligned} [1 - F_{\text{KLLoGW}}(x)]^p &= \left[1 - \left[1 - \left(1 - \left[1 - (1+x^c)^{-1} e^{\alpha x^\beta} \right]^a \right)^b \right] \right]^p \\ &= \sum_{j,k,m=0}^\infty (-1)^{j+k+m} \binom{p}{j} \binom{j}{k} \binom{bk}{m} \left[1 - (1+x^c)^{-1} e^{-\alpha x^\beta} \right]^m \\ &= \sum_{j,k,m,l=0}^\infty (-1)^{j+k+m+l} \binom{p}{j} \binom{j}{k} \binom{bk}{m} \binom{m}{l} (1+x^c)^{-l} e^{-\alpha l x^\beta}. \end{aligned} \tag{31}$$

Now,

$$\begin{aligned} \int_0^\infty x^{t-1} \left(\overline{F}_{\text{KLLoGW}}(x) \right)^p dx &= \sum_{j,k,m,l,r=0}^\infty \binom{p}{j} \binom{j}{k} \binom{bk}{m} \binom{m}{l} \frac{(-1)^{j+k+m+l+r} (\alpha l)^r}{r!} \\ &\quad \times \int_0^\infty x^{t-1+\beta r} (1+x^c)^{-l} dx. \end{aligned} \tag{32}$$

Let $y = (1+x^c)^{-1}$, then

$$\begin{aligned} \int_0^\infty x^{t-1+\beta r} (1+x^c)^{-l} dx &= \frac{1}{c} \int_0^1 y^{l-\frac{1}{c}(t+\beta r)-1} (1-y)^{\frac{1}{c}(t+\beta r)-1} dy \\ &= \frac{1}{c} B\left(a - \frac{1}{c}(t+\beta r), \frac{1}{c}(t+\beta r)\right). \end{aligned} \tag{33}$$

Consequently, the t^{th} moment of the i^{th} order statistic, equation (34), follows from equations (30)–(33) as

$$E(X_{i:n}^t) = t \sum_{p=n-i+1}^n \sum_{j,k,m,l,r=0}^{\infty} \frac{(-1)^{p-n+i+j+k+m+l+r-1} (\alpha l)^r}{r! c} \times \binom{p}{j} \binom{j}{k} \binom{bk}{m} \binom{m}{l} B\left(l - \frac{1}{c}(t + \beta r), \frac{1}{c}(t + \beta r)\right). \tag{34}$$

3.7. L-moments

Some statistical description of probability distributions are based on linear combination of order statistics, particularly, L -moments (see Hoskings [16] for additional details) and are given by

$$\lambda_{k+1} = \frac{1}{k+1} \sum_{j=0}^k (-1)^j \binom{k}{j} E(X_{k+1-j:k+1}), \quad k = 0, 1, 2, \dots \tag{35}$$

The L -moments of the KLLoGW distribution can be obtained from equation (35). The first four L -moments are given by $\lambda_1 = E(X_{1:1})$, $\lambda_2 = \frac{1}{2}E(X_{2:2} - X_{1:2})$, $\lambda_3 = \frac{1}{3}E(X_{3:3} - 2X_{2:3} + X_{1:3})$ and $\lambda_4 = \frac{1}{4}E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4})$, respectively.

3.8. Rényi entropy

Rényi entropy is a generalization of the well known Shannon entropy and is defined as

$$I_R(v) = (1 - v)^{-1} \log \left(\int_0^{\infty} [f(x; c, k, \alpha, \beta, \theta)]^v dx \right), \quad v \neq 1, v > 0. \tag{36}$$

Note that

$$f_{\text{KLLoGW}}^v(x) = \sum_{j,k,r,m=0}^{\infty} \sum_{s=0}^v \binom{av-v}{j} \binom{bv-v}{k} \binom{ak}{r} \binom{v}{s} \times \frac{(-1)^{j+k+ak+m} (ab)^v (\alpha\beta)^{v-s} c^s (-\alpha(avj-vj) - ak\alpha - \alpha v)^m}{m!} \times \int_0^{\infty} x^{sc+\beta v-\beta s-v} (1+x^c)^{-avj-vj-ak-2v+v-s} dx. \tag{37}$$

Now let $x = (1 + y^c)^{-1}$, then

$$\int_0^{\infty} x^{sc+\beta v-\beta s-v} (1+x^c)^{-avj-vj-ak-2v+v-s} dx = \frac{1}{c} \int_0^1 y^{avj+vj+ak+2v-v+s-\frac{1}{c}(sc+\beta v-\beta s-v-1)+1}$$

$$\begin{aligned} & \times (1 - y)^{\frac{1}{c}(sc+\beta v-\beta s-v+1)-1} dy \\ & = \frac{1}{c} B\left(avj + vj + ak + 2v - v + s - \frac{1}{c}(z) + 2, \frac{1}{c}(z) \right), \end{aligned} \tag{38}$$

where $z = sc + \beta v - \beta s - v + 1$. Consequently, from equations (36)–(38) Rényi entropy reduces to equation (39)

$$\begin{aligned} I_R(v) &= (1 - v)^{-1} \log \left[\sum_{j,k,r,m=0}^{\infty} \sum_{s=0}^v \binom{av - v}{j} \binom{bv - v}{k} \binom{ak}{r} \binom{v}{s} \right. \\ & \quad \times \frac{(-1)^{j+k+ak+m} (ab)^v (\alpha\beta)^{v-s} c^{s-1} (-\alpha(avj - vj) - ak\alpha - \alpha v)^m}{m!} \\ & \quad \left. \times B\left(avj + vj + ak + 2v - v + s - \frac{1}{c}(z) + 2, \frac{1}{c}(z) \right) \right] \end{aligned} \tag{39}$$

for $v \neq 1$, and $v > 0$.

4. Calculation

4.1. Maximum likelihood estimation

Let $\Delta = (a, b, c, \alpha, \beta)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\Delta)$ for a single observation of x of X is

$$\begin{aligned} \ell(\Delta) &= \ln(ab) + (a - 1) \ln \left(1 - (1 + x^c)^{-1} e^{-\alpha x^\beta} \right) - 2 \ln(1 + x^c) - \alpha x^\beta \\ & \quad + \ln \left(cx^{c-1} + \alpha\beta x^\beta (1 + x^c) \right) + (b - 1) \ln \left(1 - \left(1 - (1 + x^c)^{-1} e^{-\alpha x^\beta} \right)^a \right). \end{aligned} \tag{40}$$

The first partial derivatives of equation (40) with respect to $\Delta = (a, b, c, \alpha, \beta)^T$ are given by equations (41)–(45)

$$\begin{aligned} \frac{\partial \ell}{\partial a} &= \frac{1}{a} + \ln \left(1 - (1 + x^c)^{-1} e^{-\alpha x^\beta} \right) \\ & \quad - \frac{(b - 1) \left(1 - (1 + x^c)^{-1} e^{-\alpha x^\beta} \right)^a \ln \left(1 - (1 + x^c)^{-1} e^{-\alpha x^\beta} \right)}{1 - \left(1 - (1 + x^c)^{-1} e^{-\alpha x^\beta} \right)^a}, \end{aligned} \tag{41}$$

$$\frac{\partial \ell}{\partial b} = \frac{a}{ab} + \ln \left(1 - \left(1 - (1 + x^c)^{-1} e^{-\alpha x^\beta} \right)^a \right), \tag{42}$$

$$\begin{aligned} \frac{\partial \ell}{\partial c} &= \frac{(b - 1)x^c(1 + x^c)^{-2} e^{-\alpha x^\beta} \ln(x)}{1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}} + \frac{x^{c-1} + cx^{c-1} \ln(x) + \alpha\beta x^{\beta+c} \ln(x)}{cx^{c-1} + \alpha\beta x^\beta (1 + x^c)} \\ & \quad - \frac{2x^c \ln(x)}{1 + x^c} + \frac{a \left(1 - (1 + x^c)^{-1} e^{-\alpha x^\beta} \right)^{a-1} e^{-\alpha x^\beta} x^c \ln(x)}{1 - \left(1 - (1 + x^c)^{-1} e^{-\alpha x^\beta} \right)^a}, \end{aligned} \tag{43}$$

$$\begin{aligned} \frac{\partial \ell}{\partial \alpha} &= \frac{(a-1)\alpha x^\beta (1+x^c)^{-1} e^{-\alpha x^\beta}}{1 - (1+x^c)^{-1} e^{-\alpha x^\beta}} - x^\beta + \frac{\beta x^\beta (1+x^c)}{c x^{c-1} + \alpha \beta x^\beta (1+x^c)} \\ &+ \frac{\alpha(b-1) \left(1 - (1+x^c)^{-1} e^{-\alpha x^\beta}\right)^{a-1} x^\beta (1+x^c)^{-1} e^{-\alpha x^\beta}}{1 - \left(1 - (1+x^c)^{-1} e^{-\alpha x^\beta}\right)^a}, \end{aligned} \tag{44}$$

and

$$\begin{aligned} \frac{\partial \ell}{\partial \beta} &= \frac{\alpha(a-1)(1+x^c)^{-1} e^{-\alpha x^\beta} x^\beta \ln(x)}{1 - (1+x^c)^{-1} e^{-\alpha x^\beta}} + \frac{\alpha x^\beta (1+x^c) + \alpha \beta (1+x^c) x^\beta \ln(x)}{c x^{c-1} + \alpha \beta x^\beta (1+x^c)} \\ &- \alpha x^\beta \ln(x) + \frac{\alpha(b-1) \left(1 - (1+x^c)^{-1} e^{-\alpha x^\beta}\right)^{a-1} \alpha (1+x^c)^{-1} e^{-\alpha x^\beta} x^\beta \ln(x)}{1 - \left(1 - (1+x^c)^{-1} e^{-\alpha x^\beta}\right)^a}. \end{aligned} \tag{45}$$

The total log-likelihood function for a random sample of n observations: x_1, x_2, \dots, x_n drawn from the KLLoGW distribution is $\ell_n^* = \sum_{i=1}^n \ell_i(\Delta)$, where $\ell_i(\Delta)$, $i = 1, 2, \dots, n$ is given by equation (40). The maximum likelihood estimates of the parameters, denoted by $\hat{\Delta}$ is obtained by solving the nonlinear equation $\left(\frac{\partial \ell_n^*}{\partial a}, \frac{\partial \ell_n^*}{\partial b}, \frac{\partial \ell_n^*}{\partial c}, \frac{\partial \ell_n^*}{\partial \alpha}, \frac{\partial \ell_n^*}{\partial \beta}\right)^T = \mathbf{0}$, numerically using Newton–Raphson procedure.

4.2. Asymptotic confidence intervals

Let $\hat{\Delta} = (\hat{a}, \hat{b}, \hat{c}, \hat{\alpha}, \hat{\beta})$ be the maximum likelihood estimate of $\Delta = (a, b, c, \alpha, \beta)$. Under the usual regularity conditions (see [17]), $\sqrt{n}(\hat{\Delta} - \Delta) \xrightarrow{d} N_5(\mathbf{0}, I^{-1}(\Delta))$, where $I(\Delta)$ is the expected Fisher information matrix. The asymptotic results remain valid if $I(\Delta)$ is replaced by $J(\hat{\Delta})$, the observed information matrix evaluated at $\hat{\Delta}$. Consequently, the multivariate normal distribution $N_5(\mathbf{0}, J(\hat{\Delta})^{-1})$, where the mean vector $\mathbf{0} = (0, 0, 0, 0, 0)^T$, can be readily applied to construct approximate interval estimates for the individual parameters of the KLLoGW distribution.

4.3. Monte Carlo simulation study

The performance of the maximum likelihood estimates of the KLLoGW model parameters is assessed by conducting various simulations. We use equation (14) to generate random data from the KLLoGW distribution. The simulation study is repeated $N = 1,000$ times, each for samples of size $n = 35, 65, 95, 200, 400, 800$, and 1000. The selected vector parameter values are $I : (a = 0.8, b = 1, c = 4, \alpha = 0.4, \beta = 1.5)$ and $II : (a = 1.2, b = 1.2, c = 2.5, \alpha = 0.45, \beta = 1.1)$. Three quantities were computed in this simulation study, namely: the mean estimate,

Table 4. Monte Carlo Simulation Results.

Parameter	Sample Size	(0.8,1.0,4.0,0.4,1.5)			(1.2,1.2,2.5,0.45,1.1)		
		Mean	Bias	RMSE	Mean	Bias	RMSE
a	35	2.454937	1.334761	5.802126	3.226858	2.173275	9.257023
	65	2.241960	1.290336	5.680329	2.779228	1.672261	7.272027
	95	2.009842	1.027036	4.696777	2.595130	1.428049	7.290296
	200	1.688558	0.895372	3.010665	2.272676	0.978310	3.986035
	400	1.587665	0.796579	2.540174	2.024520	0.6651463	2.449244
	800	1.404696	0.5138077	1.697704	1.699791	0.6102496	2.182203
	1000	1.307541	0.424941	1.297141	1.656817	0.504970	2.015271
b	35	1.685210	0.8534831	3.088545	2.687211	1.386376	4.739939
	65	1.533564	0.606782	2.801664	2.212098	0.882724	3.275099
	95	1.447680	0.348033	1.472674	1.957700	0.558066	2.445108
	200	1.240854	0.241497	1.410947	1.526717	0.415181	2.183907
	400	1.189828	0.188696	0.842366	1.311741	0.188230	1.082836
	800	1.155526	0.121918	0.552137	1.285358	0.126265	0.876679
	1000	1.130740	0.117301	0.479060	1.273334	0.086942	0.736860
c	35	4.303821	0.389029	2.698139	2.804231	0.351035	1.966502
	65	4.151255	0.174111	1.784346	2.788904	0.270322	1.385068
	95	4.094082	0.139632	1.429894	2.679923	0.348234	1.275419
	200	4.156561	0.1100711	1.087140	2.754832	0.227211	0.883935
	400	4.024893	0.038950	0.873098	2.807567	0.217896	0.722707
	800	3.958511	0.012178	0.722316	2.744339	0.213442	0.655903
	1000	3.998927	-0.007509	0.683523	2.731031	0.185827	0.584460
α	35	0.605396	0.303405	1.153181	0.780438	0.376760	1.415956
	65	0.752877	0.289684	0.954995	0.766231	0.345303	1.176068
	95	0.705679	0.289401	0.864894	0.750330	0.305884	1.086413
	200	0.673274	0.288780	0.731268	0.718121	0.277071	0.911422
	400	0.643451	0.231968	0.579367	0.702698	0.210257	0.753339
	800	0.567870	0.147111	0.392132	0.620924	0.184757	0.650396
	1000	0.537333	0.120583	0.354184	0.605390	0.140297	0.575529
β	35	3.122728	1.299114	4.219852	2.631582	1.582363	3.925881
	65	2.224872	0.687505	3.040250	2.253443	0.999029	2.929135
	95	1.979420	0.588904	2.580203	2.050823	0.996898	2.918666
	200	1.792706	0.180765	1.528497	1.77265	0.684818	2.126652
	400	1.545018	0.057145	0.942048	1.622692	0.458916	1.612678
	800	1.492630	0.040096	0.768656	1.467334	0.292346	1.162711
	1000	1.527526	0.037205	0.758161	1.374958	0.266078	0.989198

average bias and Root Mean Squared Error (RMSE). The average bias and RMSEs are given in equation (46):

$$ABias(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}}, \quad (46)$$

where $\theta = a, b, c, \alpha, \beta$. In Table 4, the Mean, Average Bias and RMSE values for the different sample sizes are presented. The results show that in general, the mean estimates converges to the true parameter value with increasing sample size since the RMSEs decay toward zero. Also, for all the parametric values, the biases tend to decrease with increasing sample size.

5. Example

5.1. Application

An application to real life data is given in this subsection. We present the parameter estimates (standard error in parentheses), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Cramér–von Mises (W^*), Anderson–Darling (A^*), and sum of squares (SS) from the probability plots for the KLLoGW distribution, its nested models and several non-nested models. All computations were done using R software. The KLLoGW distribution is compared with the non-nested distributions, namely: log-logistic Weibull Poisson (LLoGWP) [18], gamma log-logistic Weibull (GLLoGW) [19], beta modified Weibull (BetaMW) [20], Burr Weibull Logarithmic (BWL) [21], beta Weibull Poisson (BetaWP) [22], gamma-Dagum (GD) [23] and exponentiated Kumaraswamy Dagum (EKD) [17] distributions. The pdfs of LLoGWP, GLLoGW, BWL, BetaMW, EKD, BetaWP (BWP), and GD are given by equations (47)–(53), respectively,

$$f_{\text{LLoGWP}}(x) = \frac{\theta e^{\theta(1-(1+(x/s)^c)^{-1}e^{-\alpha x^b})}}{e^\theta - 1} \left(1 + \left(\frac{x}{s}\right)^c\right)^{-1} e^{-\alpha x^b} \times \left[\left(1 + \left(\frac{x}{s}\right)^c\right)^{-1} \frac{c}{s} \left(\frac{x}{s}\right)^{c-1} + \alpha x^{\beta-1} \right], \quad (47)$$

for $s, c, \alpha, \beta, \theta > 0$, and $x > 0$,

$$f_{\text{GLLoGW}}(x) = \frac{1}{\Gamma(\delta)\theta^\delta} (1+x^c)^{-1} e^{-\alpha x^\beta} [(1+x^c)^{-1} c x^{c-1} + \alpha \beta x^{\beta-1}] \times (-\log[1 - (1+x^c)^{-1} e^{-\alpha x^\beta}])^{\delta-1} [1 - (1+x^c)^{-1} e^{-\alpha x^\beta}]^{(1/\theta)-1}, \quad (48)$$

for $c, \alpha, \beta, \delta, \theta > 0$, and $x > 0$,

$$f_{\text{BWL}}(x; c, k, \alpha, \beta, \theta) = \frac{\theta e^{-\alpha x^\beta} (1+x^c)^{-k-1} (k c x^{c-1} + \alpha \beta x^{\beta-1} (1+x^c))}{-(1-\theta(1+x^c)^{-k} e^{-\alpha x^\beta}) \log(1-\theta)}, \quad (49)$$

for $x, c, k, \alpha, \beta > 0$ and $0 < \theta < 1$,

$$f_{\text{BetaMW}}(x) = \frac{\alpha x^{\gamma-1} (\gamma + \lambda x) \exp(\lambda x)}{B(a, b)} e^{-b \alpha x^\gamma \exp(\lambda x)} (1 - e^{-\alpha x^\gamma \exp(\lambda x)})^{a-1}, \quad (50)$$

for $\alpha, \gamma, \lambda, a, b > 0$ and $x > 0$,

$$f_{\text{EKD}}(x) = \alpha \lambda \delta \phi \theta x^{-\delta-1} (1 + \lambda x^{-\delta})^{-\alpha-1} [1 - (1 + \lambda x^{-\delta})^{-\alpha}] \phi^{-1} \times \{1 - [1 - (1 + \lambda x^{-\delta})^{-\alpha}] \phi\}^{\theta-1}, \quad (51)$$

for $\alpha, \lambda, \delta, \phi, \theta > 0$, and $x > 0$,

$$f_{\text{BetaWP}}(x) = \frac{\alpha \beta \lambda x^{\alpha-1} e^{\lambda e^{-\beta x^\alpha}} - \lambda - \beta x^\alpha (e^\lambda - 1)^{2-a-b} (e^\lambda - e^{\lambda e^{-\beta x^\alpha}})^{a-1} (e^{\lambda e^{-\beta x^\alpha}} - 1)^{b-1}}{B(a, b)(1 - e^{-\lambda})}, \quad (52)$$

Table 5. Data on failure times of Kevlar 49/epoxy strands with pressure at 90%.

0.01	0.01	0.02	0.02	0.02	0.03	0.03	0.04	0.05	0.06	0.07	0.07	0.08
0.09	0.09	0.10	0.10	0.11	0.11	0.12	0.13	0.18	0.19	0.20	0.23	0.24
0.24	0.29	0.34	0.35	0.36	0.38	0.40	0.42	0.43	0.52	0.54	0.56	0.60
0.60	0.63	0.65	0.67	0.68	0.72	0.72	0.72	0.73	0.79	0.79	0.80	0.80
0.83	0.85	0.90	0.92	0.95	0.99	1.00	1.01	1.02	1.03	1.05	1.10	1.10
1.11	1.15	1.18	1.20	1.29	1.31	1.33	1.34	1.40	1.43	1.45	1.50	1.51
1.52	1.53	1.54	1.54	1.55	1.58	1.60	1.63	1.64	1.80	1.80	1.81	2.02
2.05	2.14	2.17	2.33	3.03	3.03	3.34	4.20	4.69	7.89			

for $a, b, \alpha, \beta, \lambda > 0$, and $x > 0$, and

$$f_{GD}(x) = \frac{\lambda\beta\delta x^{-\delta-1}}{\Gamma(\alpha)\theta^\alpha} (1 + \lambda x^{-\delta})^{-\beta-1} (-\log[1 - (1 + \lambda x^{-\delta})^{-\beta}])^{\alpha-1} \times [1 - (1 + \lambda x^{-\delta})^{-\beta}]^{(1/\theta)-1}, \tag{53}$$

for $\lambda, \beta, \delta, \alpha, \theta > 0$, and $x > 0$, respectively.

5.2. Time to failure of kevlar 49/epoxy strands tested at various stress level

These measurements consist of 101 data points that represent the stress-rupture life of kevlar 49/epoxy strands which are subjected to constant sustained pressure at the 90% stress level until all have failed (see Cooray and Ananda [24]). The data (failure times in hours), were originally presented by Andrews and Herzberg [25] and Barlow et al. [26]. The data are presented in Table 5.

The estimates of the parameter, as well as the goodness-of-fit statistics and results for this data are given in Table 6. Plots of the fitted densities and histogram are given in Figures 3(a), 3(b), 3(c) and 3(d). The data is right skewed as shown on the plots. The observed probability versus predicted probability plots of the KLLoGW distribution, it's sub-models and non-nested models are presented in Figures 4(a), 4(c) and 4(d). Figures 3(a) and 4(a) showed that KLLoGW distribution does provide the best fit.

The estimated variance-covariance matrix for the KLLoGW distribution is

$$\begin{bmatrix} 47027618 & 222.270000 & -1084.050000 & 26221 & -278.220000 \\ 222.270000 & 0.030520 & -0.187500 & 0.020970 & -0.003810 \\ -1084.050000 & -0.187500 & 1.504400 & 0.026380 & 0.021520 \\ 26221 & 0.02097 & 0.026380 & 15.028400 & -0.145100 \\ -278.220000 & -0.003810 & 0.021520 & -0.145100 & 0.001930 \end{bmatrix}$$

and the 95% asymptotic confidence intervals are respectively, given by $a \in (1733.34 \pm 1.96 \times 2168.5850)$, $b \in (0.4940 \pm 1.96 \times 0.1747)$, $c \in (4.2477 \pm 1.96 \times 1.2265)$, $\alpha \in (8.4738 \pm 1.96 \times 3.8766)$ and $\beta \in (0.07046 \pm 1.96 \times 0.0439)$.

Table 6. Estimates of Models for time to failure of kevlar 49/epoxy strands data.

Model	Estimates					Statistics							
	<i>a</i>	<i>b</i>	<i>c</i>	α	β	$-2 \log L$	AIC	AICC	BIC	W*	A*	SS	
KLLoGW	1733.34 (6857.67)	0.4940 (0.1747)	4.2477 (1.2266)	8.4738 (0.01714)	8.4738 (3.8766)	191.1	201.1	201.7	214.1	0.0212	0.1534	0.0207	
KLLoGE	2.7004 (1.1108)	2.3948 (2.8321)	0.3485 (0.1123)	0.3976 (0.4075)	1 –	204.4	212.4	212.8	222.9	0.1292	0.7910	0.1253	
KLLoGR	0.2734 (0.06916)	0.5547 (0.1061)	2.7501 (0.5447)	0.06908 (0.07194)	2 –	205.9	213.9	214.3	224.3	0.1635	0.9753	0.1674	
ELLoGW	1.7650 (0.6653)	1 –	0.4598 (0.2118)	0.7387 (0.3461)	1.0910 (0.2533)	204.9	212.9	213.3	223.3	0.1319	0.8073	0.1259	
ELLoGE	1.9878 (0.4241)	1 –	0.4051 (0.1378)	0.8586 (0.1636)	1 –	205.0	211.0	211.2	218.8	0.1447	0.8635	0.1393	
ELLoGR	0.6750 (0.1493)	1 –	1.3884 (0.3509)	0.06580 (0.051701)	2 –	210.1	216.1	216.4	224.0	0.2610	1.4415	0.2938	
LLoGW	1 –	1 –	2.1998 (0.3761)	0.4113 (0.09196)	0.5413 (0.1021)	207.5	213.5	213.7	221.3	0.1070	0.7446	0.2699	
LLoGE	1 –	1 –	1.0389 (0.4245)	0.4245 (0.09814)	1 –	210.2	214.2	314.4	219.5	0.3099	1.6712	0.4327	
LLoGR	1 –	1 –	0.9114 (0.1423)	0.1423 (0.03759)	2 –	213.3	217.3	217.5	22.6	0.1776	1.1049	0.2475	
LLoGWP	2.8622 (0.4671)	1.2383 (0.1695)	4.1372 (1.2496)	0.1071 (0.04320)	50.5347 (63.5950)	193.9	203.9	204.5	217.0	0.0267	0.2231	0.0269	
GLLoGW	<i>c</i> 0.2365 (0.2965)	α 0.2591 (0.3727)	β 0.9648 (0.3741)	δ 4.3962 (10.719)	θ 0.1396 (0.3332)	204.1	214.01	214.6	227.1	0.1322	0.7996	0.2569	
BetaMW	<i>a</i> 108.86 (0.0002)	<i>b</i> 25.631 (0.0009)	α 1.6632 (0.0279)	γ 0.0534 (0.0075)	λ 0.0343 (0.0089)	207.3	217.3	217.9	230.38	0.1955	1.1190	0.1916	
BWL	<i>c</i> 5.8331 (2.0868)	<i>k</i> 0.2335 (0.1268)	α 0.4260 (0.3959)	β 0.7695 (0.1738)	θ 0.7290 (0.5967)	196.1	206.1	206.8	219.2	0.0429	0.3107	0.0419	
BetaWP	<i>a</i> 0.0745 (0.0177)	<i>b</i> 8.3950 (0.3903)	α 0.8247 (0.2761)	β 1.0698 (0.2005)	θ 374.51 (0.0015)	202.4	212.04	212.7	225.12	0.1141	0.7016	0.1093	
EKD	α 0.0179 (0.0279)	λ 6.7074 (6.5531)	δ 3.7493 (0.9489)	ϕ 0.8577 (0.2534)	θ 8.4926 (13.328)	199.8	209.8	210.5	222.9	0.053	0.4193	0.0581	
GD	λ 49.090 (111.98)	β 0.3216 (0.2286)	δ 9.9094 (5.0965)	α 0.2055 (0.1636)	θ 6.8443 (6.1399)	196.6	206.6	207.2	219.7	0.0367	0.2895	0.0352	

Note. Standard errors are in parentheses.

The likelihood ratio (LR) test statistic for testing of the hypothesis H_0 : KLLoGE against H_a : KLLoGW and H_0 : KLLoGR against H_a : KLLoGW are 12.5 (p-value = 0.000407) and 13.3 (p-value = 0.000265), respectively. We conclude that there are significant differences between the fit of the KLLoGE distribution and KLLoGW distribution as well as between the fit of the KLLoGR distribution and KLLoGW distribution. The LR test statistic for testing the hypothesis H_0 : ELLoGW against H_a : KLLoGW and H_0 : ELLoGE against H_a : KLLoGW are 14.8 (p-value = 0.00012) and 13.9 (p-value = 0.000959), respectively. Therefore, we conclude that

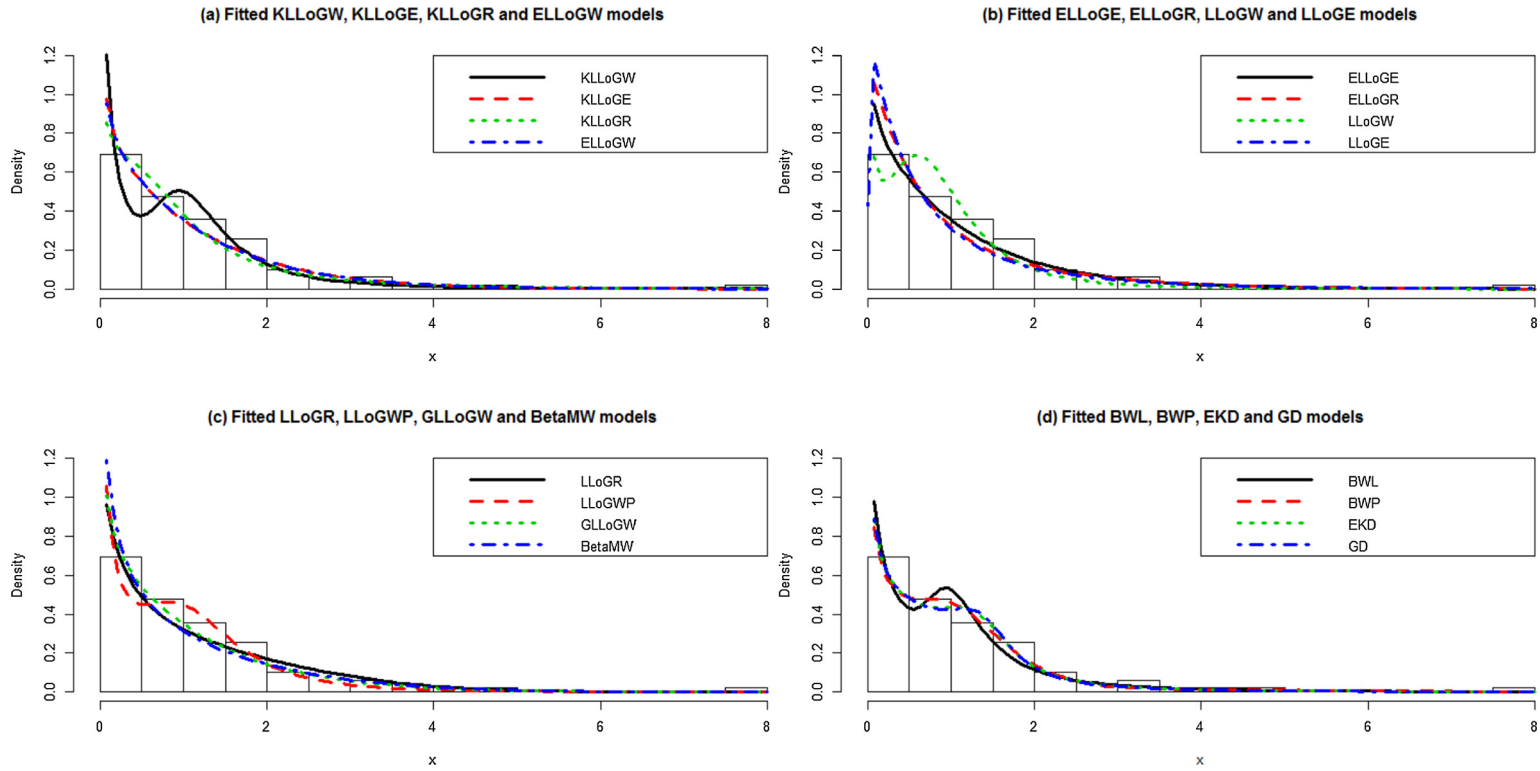


Figure 3. Plots of fitted densities for time to failure of Kevlar 49/epoxy strands data. (a) Fitted KLLoGW, KLLoGE, KLLoGR & ELLoGW models. (b) Fitted ELLoGE, ELLoGR, LLoGW & LLoGE models. (c) Fitted LLoGR, LLoGWP, GLLoGW & BetaMW models. (d) Fitted BWL, BWP, EKD & GD models.

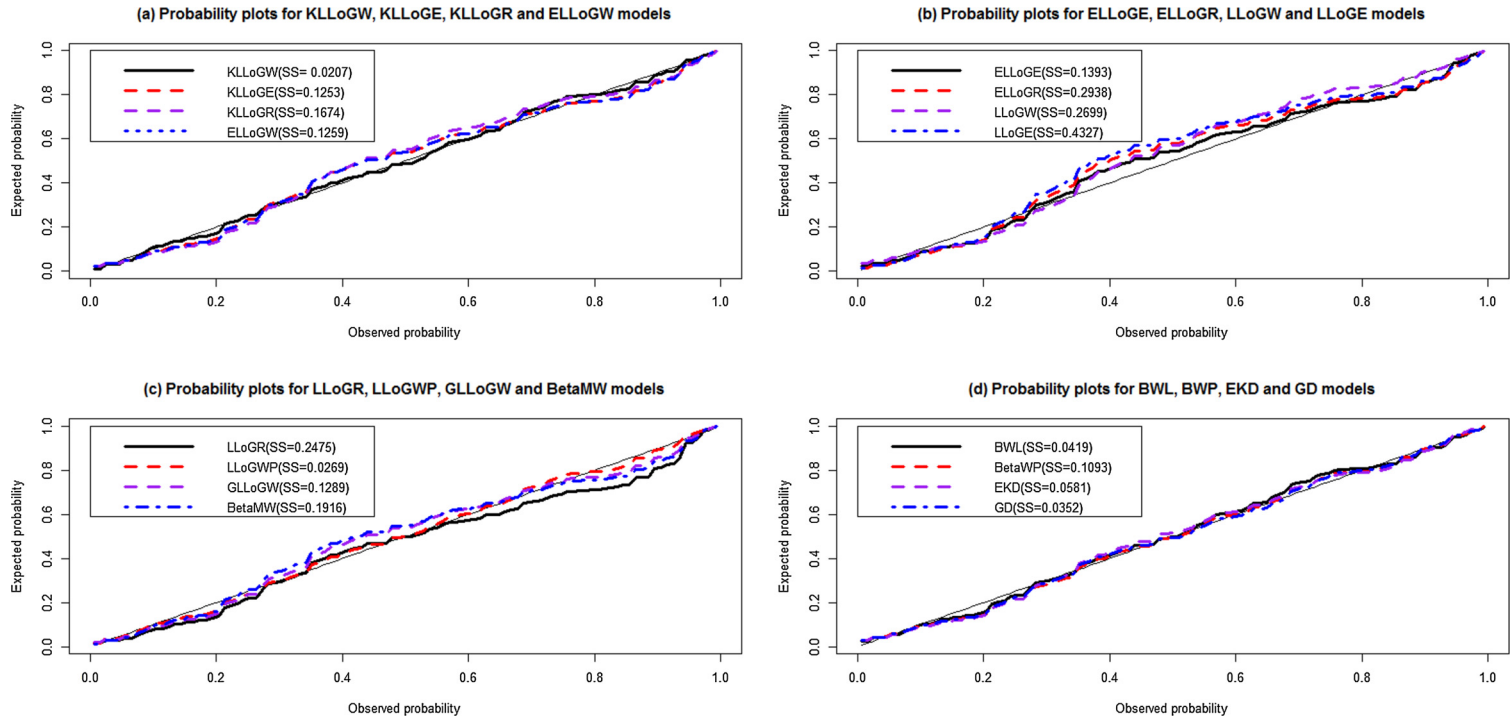


Figure 4. Probability Plots for time to failure of Kevlar 49/epoxy strands data. (a) Probability Plots for KLLoGW, KLLoGE, KLLoGR & ELLoGW models. (b) Probability Plots for ELLoGe, ELLoGR, LLoGW & LLoGE models. (c) Probability Plots for LLoGR, LLoGWP, GLLoGW & BetaMW models. (d) Probability Plots for BWL, BWP, EKD & GD models.

the fits of the ELLoGW and KLLoGW distributions as well as those of the ELLoGE and KLLoGE distributions are significantly different. The LR test statistic of the hypothesis H_0 : ELLoGR against H_a : KLLoGW and H_0 : LLoGW against H_a : KLLoGW are 19 (p-value = 0.000075) and 16.4 (p-value = 0.000275), respectively. There are significant differences between the fits of the ELLoGR distribution and KLLoGW distribution as well as between LLoGW distribution and KLLoGW distribution based on the LR test. The values of the goodness of fit-statistics W^* and A^* as well as the values of AIC and BIC clearly indicate that the KLLoGW distribution is better than the sub-models, as well as the non-nested LLoGWP, GLLoGW, BetaMW, BWL, BetaWP, EKD and GD distributions. The value of the sum of squares from Table 6 and the probability plot is smallest ($SS = 0.0207$) for the KLLoGW distribution.

6. Conclusion

We have developed a new lifetime distribution called the Kumaraswamy Log-logistic Weibull (KLLoGW) distribution and studied its statistical properties in detail. The hazard function of the new model is quite flexible. Estimates of the parameters are presented and simulations indicate that the MLEs are consistent. The new distribution provided a better fit when compared with its nested and several non-nested distributions in fitting a real data set. The new distribution is applicable in a variety of areas dealing with lifetime data including engineering, environmental sciences, and reliability among others.

Declarations

Author contribution statement

Broderick Oluyede: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Wrote the paper.

Precious Mdlongwa, Alphonse Amey, Adeniyi Fagbamigbe, Boikanyo Makubate: Performed the experiments; Contributed analysis tools or data; Wrote the paper.

Funding statement

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Competing interest statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

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