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Multiplicative processes as a source of fat-tail distributions

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ABSTRACT

Many works in the literature are in favour of the universality of power-laws. However, more recently, a significant amount of research has exposed more subtle details about this subject. In this paper we present two generalisations that aim to solve these possible antagonistic conceptions in a mathematical framework. The first generalisation allows us to show that a vast range of power-laws can be produced through multiplicative process (for example through a single positive feedback mechanism). The second generalisation shows, by solving Lambert's transcendental equations, that the space of solutions where the Pareto distribution is equal to the distribution produced by the power multiplicative transformation (for a number of input distributions) is infinite.

1. Introduction

Power-laws have attracted the attention of many researchers who have attempted to show their universality in well-known compilation works, such as [1, 2, 3]. However, they have also attracted criticism, mainly for two reasons: a) In some cases a power-law is inferred showing that when plotting a curve in log-log scale a straight line emerges. This method lacks precision and does not usually resist any analysis of statistical significance. b) Sometimes the range where the power-law is fulfilled is small and distortions appear outside that range (delimited by cut-offs). There is controversy when considering whether these distortions are inevitable limitations of the real world and, therefore, they do not invalidate the models or, on the contrary, they are an indication that there is no a power-law present.

Benguigui & Marinov [4] produced a comprehensive survey, encompassing 89 papers, in which power-laws were identified in very diverse disciplines. Their analysis was done first in a visual way, determining the characteristics of the graphic representations in three formats: Zipf, PDF (Probability Density Function) and CDF (Cumulative Distribution Function). Cut-offs to reject some data in each graphic were established and then the works were classified according to whether they were strong power-laws, weak power-laws, or false power-laws. After a more detailed mathematical analysis, the authors showed that most of them were not strong power-laws. They also reported that many distributions (LogNormal, Weibull, LogLogistic and others) can be approximated to a power-law when a certain arbitrary cut-off on the data is accepted.

Many researchers on this subject have conjectured that because the same type of distribution appears in so many different fields (economy, geology, incomes, city populations, vocabulary, social networks, etc.), then there should be a universal underlying phenomenon behind it. This observed universality of power-law distributions makes the search and understanding of them very attractive but it is also possible to believe that they are present where they do not exist. Several explanations have been proposed for the universality of power-laws, such as self-organised criticality [5], where cascades of phenomena that follow power-laws occur when there is a phase transition between a completely ordered (solid) system and a completely disordered (liquid) system; preferential attachment [6], exemplified by the saying "the rich get richer", and as being a positive feedback we want to study and generalise in the present work; hidden variables [7]; exponential growth with exponential killing processes [8], where two exponential processes, one for growth and one for observation (or termination) are combined to give a power-law process; and, exponential growth with exponential diffusion [9], which is very similar to the previous one. It is interesting to note that the growth-diffusion processes had already been studied by Turing much earlier [10], showing that they generate complexity, even though the mathematical treatment of both works is very different. Tsallis' work in the field of statistical mechanics generalized the classical Boltzmann-Gibbs entropy [11]. Maximization of the generalized Tsallis entropy, subject to certain conditions, lead to the Tsallis probability distributions. Many practical problems have been found to be well modeled by the Tsallis distributions. Since the Tsallis exponential distribution can be seen as

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a special case of the Pareto distribution for $q > 1$, there will be cases where the Tsallis and Pareto distributions will be indistinguishable. In fact, some authors call it the Tsallis-Pareto distribution. Hanel shows in [12] a derivation of the Tsallis distribution from first principles of statistical mechanics.

In the work by Broido & Clauset [13], a vast majority of the phenomena that were thought to be power-laws are questioned, showing that it is not too important if a power-law is found but it is important if there is a phenomenon that exhibits fat tails. Thus, the universality of power-laws must be reformulated in favour of fat-tail distributions, which are captured very well (as we show in this work) by Pareto distributions, for example. There is a crucial observation, which we are going to fulfill here. Many phenomena "in the wild" do not have an observable data production mechanism; that is, only its output data can be observed but not its underlying mechanism. All power-laws are fat-tails, but not all fat-tails are power-laws. However, as we are going to show, it is easy to find a parametrisation of a Pareto distribution that approximates many fat-tail distributions. So, even if we do not know the phenomenon that generates fat-tail data, we can in many situations approximate it fairly well by a Pareto distribution.

Mitzenmacher in [14] described a basic multiplicative process, in the context of biology, that produces a Lognormal distribution. The author shows with the help of an example that small changes to the Lognormal generative process can lead to a generative process with power law distributions. Sato [15] showed in the context of financial time series, that a system modeled by a first-order stochastic differential equation perturbed with multiplicative noise as well as additive noise has a state solution with a Tsallis Gaussian probability distribution function. Awazu observed in [16] that Normal, Power Laws or some intermediate distributions appear depending on the strength of a feedback. For strong feedbacks, the result is a power law; when there is no feedback the result is normal; and in a certain intermediate zone, the result is a mixture of both distributions, which is characterized by a differential stochastic equation. In [17] the behaviour of several cryptocurrencies is analyzed and it is validated that in the long-run they follow a fat-tailed law that is caused by a herding phenomenon that can be viewed as positive feedback too. In this paper, we are going to show on a mathematical basis, in a general way, that the probability distribution function of many different phenomena that results from multiplicative transformations can be approached very well by a Pareto distribution. A secondary goal is to show that a Pareto distribution is a good approximation to data which exhibits a fat-tail distribution.

In Section 2, we will show how the exponential transformation produces exact Pareto distributions or similar distributions for several input distributions. In section 3, we study the distributions obtained by the power transformation and how a Pareto distribution can be indistinguishable under certain assumptions in many cases from the output distribution. In Section 4, the multiplicative Kolmogorov transformation is presented as the basis of a comparison to the power transformation. In Section 5, the main conclusions of this work are summarised.

2. Exponential transformation

The well-known 80-20 rule attributed to Pareto states (for example) that 80% of the world's wealth belongs to 20% of the population, but this is a rule formulated informally. Later, distributions were formulated that fulfill that characteristic in what is known today as Pareto distributions. One of its most relevant aspects is that the average value is very sensitive to the exact values of the sample; that is, it does not converge or, if it does, then the standard deviation does not. This is due to the phenomenon of the "fat tail", which means that as the values of the sampling points become smaller, their probability of occurrence decreases but slowly, so that there is no value from which the following values can be ignored. All values are important and they all contribute to the average value of the distribution.

There are several variants of the Pareto distribution. Pareto IV is one of the most flexible and best suited to many datasets because it provides more adjustment parameters to approximate, for instance, a LogNormal distribution, as we will show in Section 3. It should be noted, however, that in a strict mathematical sense, Pareto IV is not a pure power-law (see Appendix A).

The exponential killed process is a well-known example of an exponential transformation [8], which refers to a process with exponential distribution observed randomly through an exponential transformation giving as a result a Pareto probability distribution function. The term *killed* comes from the fact that every new observation is independent of any previous observation. In the next subsections, the derivation of Pareto I and Pareto IV probability distribution functions based on kill exponential processes is presented.

2.1. The kill process and the Pareto I distribution

Let X be a decaying exponential random experiment whose PDF (Probability Density Function) is given by

$$f_X(x) = \alpha e^{-\alpha x} \qquad \alpha \in \mathbb{R}_{>0},\ x > 0 \tag{1}$$

and let Y be the random variable obtained through the transformation

$$Y = ke^X \qquad k \in \mathbb{R}_{>0} \tag{2}$$

The process described by Eq. (2) is referred as a killed process because every time that an event is consumed, a new one starts. For instance, let us consider a sample of X with $\alpha = 0.7$ equal to {0.23, 0.38, 0.34}. The transformation given by Eq. (2) with $k = 1.2$ will produce the sample of Y equal to {1.51, 1.75, 1.69}. Our goal is to find the probability distribution of Y in terms of y. Using the derived distributions concept [18, pp. 40, ch. 3], we can express the CDF (Cumulative Density Function) of Y in terms of the CDF of X as

$$F_Y(y) = P(Y \le y) = P(ke^X \le y) = P\left[X \le \mathrm{Log}\left(\frac{y}{k}\right)\right] = F_X\left[\mathrm{Log}\left(\frac{y}{k}\right)\right] \tag{3}$$

Since the CDF of the exponential decaying experiment X is given by

$$F_X(x) = 1 - e^{-\alpha x} \qquad x \ge 0 \tag{4}$$

the PDF of the resulting derived distribution is the derivative of $F_Y(y)$ in Eq. (3); that is,

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy}\left[1 - e^{-\alpha \mathrm{Log}\left(\frac{y}{k}\right)}\right] = k^\alpha y^{-(\alpha+1)}\alpha \tag{5}$$

which corresponds to the PDF of the distribution commonly known as Pareto I with parameter k and shape factor α [19, pp. 50].

2.2. The kill process and the Pareto IV distribution

Now we consider a more elaborated transformation. Let Y be the derived exponential transformation given by

$$Y = k\left(e^X - 1\right)^\gamma + \mu \qquad k, \gamma \in \mathbb{R}_{>0},\ \mu \in \mathbb{R} \tag{6}$$

Following the same line of reasoning described by Eq. (3), we can express the CDF of Y in terms of the CDF of X as

$$F_Y(y) = F_X\left\{\mathrm{Log}\left[\left(\frac{y-\mu}{k}\right)^{1/\gamma} + 1\right]\right\} \tag{7}$$

Using Eq. (4) we finally get the PDF for the transformed variable Y as

$$\begin{aligned} f_Y(y) = \frac{dF_Y(y)}{dy} &= \frac{d}{dx}\left\{1 - \mathrm{Exp}\left(-\alpha \mathrm{Log}\left[\left(\frac{y-\mu}{k}\right)^{1/\gamma} + 1\right]\right)\right\} \\ &= \frac{\alpha (y-\mu)^{\frac{1}{\gamma}-1}}{\gamma k^{1/\gamma}\left[\left(\frac{y-\mu}{k}\right)^{1/\gamma} + 1\right]^{\alpha+1}} \end{aligned} \tag{8}$$

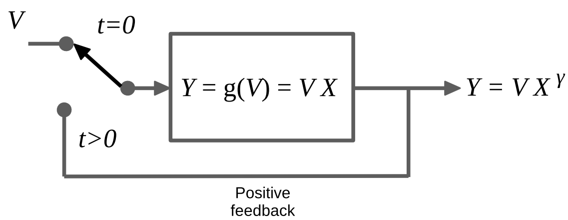

Fig. 1. Positive feedback process.

which corresponds to the PDF of the distribution that is commonly known as Pareto IV with parameter k, shape parameter γ and location parameter μ. As we have just shown, the kill process produces an accurate Pareto distribution when X has an exponential distribution.

If in Eq. (2) we take X with Normal distribution with CDF $\frac{1}{2}\text{erfc}\left(\frac{\mu-x}{\sqrt{2}\sigma}\right)$, and $k=1$, then we have

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dx}\left[\frac{1}{2}\text{erfc}\left(\frac{\mu - y^{1/\gamma}}{\sqrt{2}\sigma}\right)\right] = \frac{e^{-\frac{(\mu-\log(y))^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma y} \tag{9}$$

which is exactly the probability distribution function of a LogNormal distribution with parameters μ and σ and, as is shown in subsection 3.5, in this case there exists a space of solutions where a LogNormal distribution is indistinguishable from a Pareto IV distribution. Similarly, it can also be observed that when X in Eq. (2) has a LogNormal distribution with parameters μ and σ, a distribution of the form $\frac{e^{-\frac{\{\log[\log(x)]-\mu\}^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x \log(x)}$ is obtained. However, although in this case under certain parameters the resulting PDF can be very similar to Pareto IV, there is no space of solutions where both are indistinguishable. Thus, we see that the transformation of Eq. (2) is able to produce distributions very similar to Pareto for certain input distributions.

3. Power transformation

Consider a positive feedback as shown in Fig. 1. Feedback is defined in a process when at least one output is connected to at least one input. Loosely speaking, positive feedback is characterised as a process that auto-amplifies its output: the more there is of something, the more there will be in the future. In contrast, negative feedback tends to stabilise the output: it tends to eliminate disturbances. In all engineering areas, negative feedback is used extensively to be able to synthesise stable and predictable systems. It is also used to try to cancel any possible positive feedback that may appear, for the same reasons. However, in non-designed systems (physical, chemical, biological and even economic and social, as long as there is no centralised control), positive feedback plays a very important role (sometimes undesirable, sometimes creative).

The positive-feedback transformation process can be observed in many situations of real life. For instance, when a person gets some wealth, they invest that wealth and increase their savings. Imagine a group of people who take their money to their respective bank. Each bank has a different (albeit similar) interest rate, and each person chooses a bank at random, and stays there for T periods. If all of the people start with exactly the same money, then after T periods we will see that a power-law distribution appears. In general, when there is positive feedback, the successive output values of the process are multiplied by the previous ones, which is what we are studying in this article. As we will see, the result can be suitably modeled by Pareto IV distribution.

Other examples of positive feedback are the reproductive growth of populations of individuals without resource constraints: the more individuals there are, the more individuals there will be; the mechanism of preferential attachment [20] in social networks: the more friends you have, the more friends you will meet; and the more quotes a paper has, the more citations they will receive.

Going back to Fig. 1, the variable X is an intrinsic parameter of the system. For example, if the system were a bank, then V would be the initial capital, and X would be the interests of a period, while γ would be the number of periods.

Initially, we are going to assume a fixed and constant initial capital (then, we will relay that assumption). The variable X is not constant but is a random variable with a certain distribution. In the banking example, all banks offer a similar but not equal interest, which will follow a certain distribution (in banks, it is typically a normal distribution).

We define the power transformation as the transformation given by

$$Y = kX^\gamma \qquad k \in \mathbb{R}_{>0}, \gamma > 0 \tag{10}$$

where X is a random variable with probability distribution function $f_X(x)$.

Many situations in real life can be modeled as a positive feedback process, as shown in Fig. 1. We will not consider this yet but sometimes there is an initial input distribution $f_V(v)$ which is multiplied many times by $f_X(x)$. Initially, we will consider $f_V(v) = k$ (constant); that is, people who open their bank account all with the same amount of money. The interest offered by each bank follows a distribution X. All people save their money during the same number of periods γ, at the end of which, looking at the wealth of people we will find a distribution Y. Now, let us analyze what distribution Y emerges, depending on the initial distribution X. (In Appendix C we will discuss the case when $f_V(v)$ is not constant but a uniform distribution, that is, people with a random initial amount of money; but we can anticipate that the results are similar.)

3.1. Power transformation for X having an exponential distribution

Let us consider the power transformation of Eq. (10) where the random variable X has the decaying exponential PDF given by equation (1). It is worth noting that the exponential distribution is a special case of the Weibull distribution with parameters $\{1, \frac{1}{\lambda}\}$. Similar to what we did in equation (3), we express the CDF of Y in terms of the CDF of X, that is

$$F_Y(y) = F_X\left[\left(\frac{y}{k}\right)^{1/\gamma}\right] \tag{11}$$

Since the CDF of the exponential decaying random variable X is given by Eq. (4), the PDF of the resulting derived distribution is given by

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy}\left[1 - e^{-\alpha\left(\frac{y}{k}\right)^{1/\gamma}}\right] = \frac{\alpha\left(\frac{y}{k}\right)^{\frac{1}{\gamma}-1} e^{-\alpha\left(\frac{y}{k}\right)^{1/\gamma}}}{k\gamma} \tag{12}$$

We now want to investigate the similarity between equations (12) and (8) (with $\mu = 0$, for simplicity). Because our goal is to adjust their curves, it is reasonable to equal the variable γ of both equations. For this purpose, we establish an equation between the two PDFs, in α_E and α_P, respectively, solve the limits when $\gamma \to \infty$ at each side. We use α_E to refer the parameter we would obtain from experimentation or observation, and α_P to refer the parameter we would adjust in the Pareto distribution of Eq. (8). In the positive feedback experiment, it is reasonable to assume a large value of γ (i.e., that the iteration has been done many times). The condition $\gamma \to \infty$, see Fig. 1, reflects the fact of observing several generations of a process given by the transformation of Eq. (10). By eliminating common terms at both sides after some routine algebra, we arrive to the following expressions derived from equations (12) and (8) respectively

$$\lim_{\gamma\to\infty} e^{-\alpha_E\left(\frac{y}{k}\right)^{1/\gamma}} = \frac{\alpha_E}{e^{\alpha_E}} \tag{13}$$

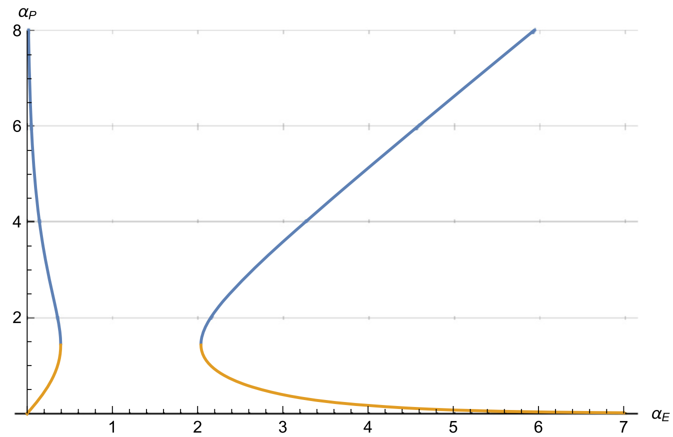

Fig. 2. Solutions space for X exponential.

$$\lim_{\gamma\to\infty}\left[\left(\frac{y}{k}\right)^{1/\gamma}+1\right]^{-(\alpha_P+1)}\alpha_P=\frac{\alpha_P}{2^{(\alpha_P+1)}} \tag{14}$$

By solving equations (14) and (13) for α_P given α_E, we get a Lambert's transcendental equation with solutions

$$\alpha_P=\begin{cases}-\frac{W_{-1}[-e^{-\alpha_E}\alpha_E\log(4)]}{\log(2)}\\ -\frac{W_0[-e^{-\alpha_E}\alpha_E\log(4)]}{\log(2)}\end{cases} \tag{15}$$

where $W_k(z)$ is the Lambert function which gives the kth solution for w in $z=we^w$. Fig. 2 shows the space of solutions given by equation (15). The blue line corresponds the upper branch solution whereas the brown line corresponds to the lower branch solution. Fig. 2 shows a range for which only complex solutions exists for α_P given α_E (approximately between 0.393201 and 2.039184). In this region, an acceptable approximation is given by the magnitude of the conjugate pairs complex solutions. Apart from this, there is an infinite number of solutions for which equation (12) is essentially equal to the Pareto probability distribution of equation (8). At the other hand, as Fig. 2 shows, for a given α_P there will always exist real solutions in α_E. Fig. 2 shows that there can exist two different values (two solutions) of α_P that produce the same Pareto distribution. This means, for instance, that researchers working with the same data might obtain two different values of α_P in the model of their Pareto distribution, being both valid. Equations (13) and (14) also show that the PDFs do not depend on k when γ is large, as expected since k is a constant and it can be interpreted simply as a scale parameter. For the special case $\alpha_P=\alpha_E$, one of the solutions is the transcendental number $\alpha=\frac{\log(2)}{1-\log(2)}$ with the right-hand side of equations (13) and (14) being equal to $2^{\frac{1}{\log(2)-1}}$. However, we must keep in mind the condition $\gamma\to\infty$ that in many cases this is not a too stringent condition (as we will see in the next example).

3.2. Would the power transformation converge to a LogNormal distribution for X exponential?

We could ask if the process kX^γ for X being an exponential distribution could converge to the LogNormal distribution, at least within some subspace, by means of the limit approach as we did with the Pareto distribution in subsection 3.1. Following the same approach, we arrive at the best to $\lim_{\sigma\to\infty}\frac{e^{-\frac{[\log(y)-\mu]^2}{2\sigma^2}}}{\sqrt{2\pi}}=\frac{1}{\sqrt{2\pi}}$ and $\lim_{\mu\to\infty,\sigma>0}\frac{e^{-\frac{[\log(y)-\mu]^2}{2\sigma^2}}}{\sqrt{2\pi}}=0$. We can see that there are no parameters left to adjust the LogNormal distribution to the distribution resulting from the transformation kX^γ, contrary to what occurs with the Pareto distribution. As shown in subsection 3.5 (or also Fig. 8), there can be instances where a Pareto distribution can be almost indistinguishable from a LogNormal distribution. The LogNormal distribution, in contrast, models very well the multiplicative Kolmogorov transformation, as detailed in section 4.

3.3. Power transformation for X having a normal distribution

Let us consider the power transformation of equation (10) where the random variable X has the normal distribution

$$f_X(x)=\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}\qquad \mu\in\mathbb{R},\ \sigma>0 \tag{16}$$

Since the CDF of X is given by

$$F_X(x)=\frac{1}{2}\operatorname{erfc}\left(\frac{\mu-x}{\sqrt{2}\sigma}\right)\qquad \mu\in\mathbb{R},\ \sigma>0 \tag{17}$$

where $\operatorname{erfc}(z)$ is the complementary error function given by $1-\operatorname{erf}(z)=1-\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$, using Eq. (11) we find the probability distribution of Y as

$$f_Y(y)=\frac{d}{dy}\left\{\frac{1}{2}\operatorname{erfc}\left[\frac{\mu-\left(\frac{y}{k}\right)^{1/\gamma}}{\sqrt{2}\sigma}\right]\right\}=\frac{\left(\frac{y}{k}\right)^{1/\gamma}e^{-\frac{\left[\mu-\left(\frac{y}{k}\right)^{1/\gamma}\right]^2}{2\sigma^2}}}{\sqrt{2\pi}\gamma\sigma y} \tag{18}$$

We now establish an equation between equations (18) and (8). By making the same analysis and assumptions for the value of γ as in subsection 3.1 and eliminating common terms at both sides of the equation, we arrive to the following limit derived from Eq. (18)

$$\lim_{\gamma\to\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{\left[\mu-\left(\frac{y}{k}\right)^{1/\gamma}\right]^2}{2\sigma^2}}=\frac{e^{-\frac{(\mu-1)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} \tag{19}$$

The limit of the Pareto distribution of Eq. (8) is the same as given by Eq. (14). As we mentioned earlier, in practice, the condition for γ being large can be easily fulfilled due to the power nature of the positive feedback transformation. Solving the equation $\frac{\alpha_P}{2^{(\alpha_P+1)}}=\frac{e^{-\frac{(\mu-1)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}$ for α_P we get

$$\alpha_P=\begin{cases}-\frac{1}{\log(2)}W_{-1}\left[-\frac{\sqrt{\frac{2}{\pi}}\log(2)e^{-\frac{(\mu-1)^2}{2\sigma^2}}}{\sigma}\right]\\ -\frac{1}{\log(2)}W_0\left[-\frac{\sqrt{\frac{2}{\pi}}\log(2)e^{-\frac{(\mu-1)^2}{2\sigma^2}}}{\sigma}\right]\end{cases} \tag{20}$$

Since the Lambert function $W_k(z)$ is real for $z>-1/e$, the space of solutions for real values for α_P imposes the constraint $-\frac{\sqrt{\frac{2}{\pi}}\log(2)e^{-\frac{1}{2}\left(\frac{\mu-1}{\sigma}\right)^2}}{\sigma}>-\frac{1}{e}$ which leads to the following constraint between μ and σ

$$\begin{array}{ll}1+b<\mu<1-b & 0<\sigma\le e\sqrt{\frac{2}{\pi}}\log(2)\\ -\infty<\mu<\infty & \sigma>e\sqrt{\frac{2}{\pi}}\log(2)\end{array} \tag{21}$$

where $b=2+\sigma\sqrt{\log\left[\frac{2\log^2(2)}{\pi\sigma^2}\right]}$. Fig. 3 shows an example of the space of solutions given by Eq. (20). The brown plane corresponds to the upper branch solution whereas the blue plane corresponds to the lower branch solution.

3.4. Power transformation for X having a uniform distribution

Let us consider the power transformation of equation (10) with X having the uniform distribution

$$f_X(x)=\begin{cases}\frac{1}{b-a} & a\le x\le b\\ 0 & \text{otherwise}\end{cases}\qquad \{a,b\in\mathbb{R},\ a<b\} \tag{22}$$

Since the CDF of X is given by

$$F_X(x)=\begin{cases}\frac{x-a}{b-a} & a\le x\le b\\ 1 & x>b\end{cases} \tag{23}$$

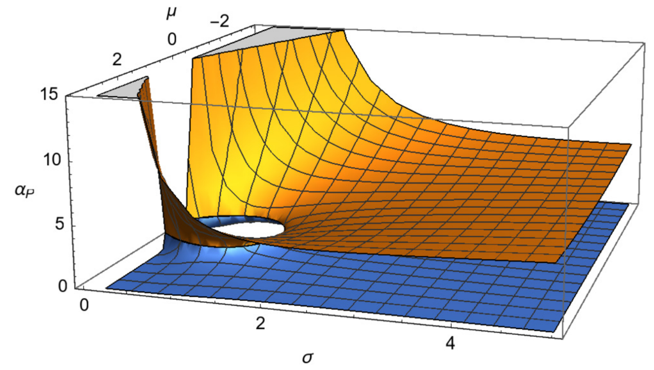

Fig. 3. Example of a solutions space for X normal in the power transformation.

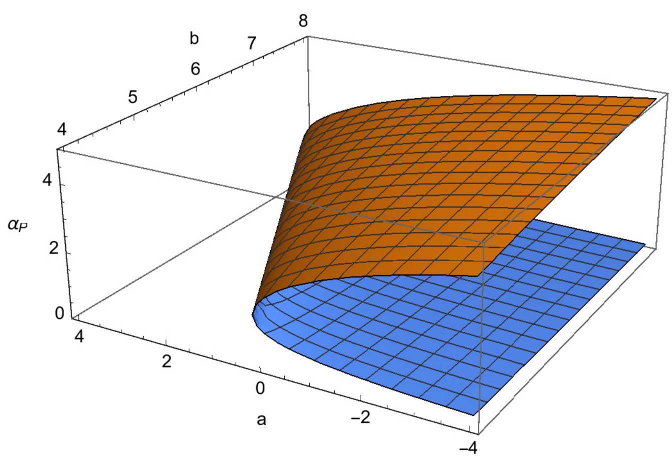

Fig. 4. Example of a solutions space for X uniform in the power transformation.

then by using Eq. (11) we find the probability distribution of Y as

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy}\left[\frac{\left(\frac{y}{k}\right)^{1/\gamma} - a}{b-a}\right] = \frac{\left(\frac{y}{k}\right)^{\frac{1}{\gamma}-1}}{\gamma k(b-a)} \tag{24}$$

Similarly to our work in the previous subsections, we establish an equation between equations (24) and (8). Making the same analysis and assumptions for the value of γ as in subsection 3.1 and eliminating common terms at both sides of the equation, the limit of the Pareto distribution of Eq. (8) being the same as given by Eq. (14), we arrive to the equation $\frac{\alpha_P}{2^{(\alpha_P+1)}} = \frac{1}{b-a}$. Solving for α_P we get

$$\alpha_P = \begin{cases} -\frac{1}{\log(2)} W_{-1}\left[\frac{2\log(2)}{a-b}\right] \\ -\frac{1}{\log(2)} W_0\left[\frac{2\log(2)}{a-b}\right] \end{cases} \tag{25}$$

Fig. 4 shows an example of the space of solutions given by Eq. (25). The brown plane corresponds to the upper branch solution whereas the blue plane corresponds to the lower branch solution.

3.5. Power transformation for X having a LogNormal distribution

Let's consider the power transformation of equation (10) when X has the log normal distribution given by

$$f_X(x) = \frac{e^{-\frac{[\log(x)-\mu]^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x} \qquad x > 0, \mu \in \mathbb{R}, \sigma > 0 \tag{26}$$

Since the CDF of X is given by

$$F_X(x) = \frac{1}{2}\mathrm{erfc}\left[\frac{\mu - \log(x)}{\sqrt{2}\sigma}\right] \qquad x \geq 0 \tag{27}$$

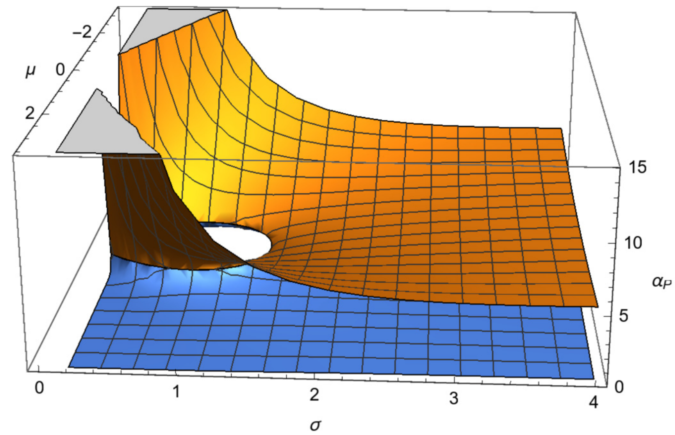

Fig. 5. Example of a solutions space for X LogNormal in the power transformation.

then by using Eq. (11) we find the probability distribution of Y as

$$f_Y(y) = \frac{d}{dy}\left\{\frac{1}{2}\mathrm{Erfc}\left[\frac{\mu - \log\left[\left(\frac{y}{k}\right)^{1/\gamma}\right]}{\sqrt{2}\sigma}\right]\right\} = \frac{e^{-\frac{\left\{\mu-\log\left[\left(\frac{y}{k}\right)^{1/\gamma}\right]\right\}^2}{2\sigma^2}}}{\sqrt{2\pi}\gamma\sigma y} \tag{28}$$

If we express Eq. (26) as $f_X(x;\mu,\sigma)$, then Eq. (28) can be expressed as $f_Y[y;\gamma\mu + \log(k), \gamma\sigma]$, which is in agreement with the fact that the multiplicative process of a LogNormal distribution follows also a LogNormal distribution [21, pp. 362]. We now establish an equation using equations (28) and (8), with $\mu = 0$ for the latter for simplicity. By making the same analysis and assumptions for the value of γ as in subsection 3.1 and eliminating common terms at both sides, we arrive to the following limits

$$\lim_{\gamma\to\infty} k^{-1/\gamma}\left[\left(\frac{y}{k}\right)^{1/\gamma} + 1\right]^{-(\alpha_P+1)} y^{1/\gamma}\alpha_P = \frac{\alpha_P}{2^{\alpha_P+1}} \tag{29}$$

$$\lim_{\gamma\to\infty} \frac{e^{-\frac{\left\{\mu-\log\left[\left(\frac{y}{k}\right)^{1/\gamma}\right]\right\}^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} = \frac{e^{-\frac{\mu^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} \tag{30}$$

Solving the equation $\frac{\alpha_P}{2^{\alpha_P+1}} = \frac{e^{-\frac{\mu^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}$ for α_P we get

$$\alpha_P = \begin{cases} -\frac{1}{\log(2)} W_{-1}\left[-\frac{\sqrt{\frac{2}{\pi}}\log(2)e^{-\frac{\mu^2}{2\sigma^2}}}{\sigma}\right] \\ -\frac{1}{\log(2)} W_0\left[-\frac{\sqrt{\frac{2}{\pi}}\log(2)e^{-\frac{\mu^2}{2\sigma^2}}}{\sigma}\right] \end{cases} \tag{31}$$

Similarly to the normal distribution case, since the Lambert function $W_k(z)$ is real for $z > -1/e$, the space of solutions for real values for α_P imposes the constraint $-\frac{\sqrt{\frac{2}{\pi}}\log(2)e^{-\frac{\mu^2}{2\sigma^2}}}{\sigma} > -\frac{1}{e}$ which leads to the following constraint between μ and σ

$$\begin{array}{ll} b < \mu < -b & 0 < \sigma \leq e\sqrt{\frac{2}{\pi}}\log(2) \\ -\infty < \mu < \infty & \sigma > e\sqrt{\frac{2}{\pi}}\log(2) \end{array} \tag{32}$$

where $b = \sigma\sqrt{\log\left[\frac{2\log^2(2)}{\pi\sigma^2}\right]}$.

Fig. 5 shows an example of the space of solutions given by Eq. (31). The brown plane corresponds to the upper branch solution whereas the blue plane corresponds to the lower branch solution.

3.6. Power transformation for X having a Laplace distribution

Let us consider the power transformation of equation (10) where X has the Laplace distribution given by

$$f_X(x) = \begin{cases} \frac{e^{-\frac{x-\mu}{\beta}}}{2\beta} & x \geq \mu \\ \frac{e^{-\frac{\mu-x}{\beta}}}{2\beta} & \text{Otherwise} \end{cases} \qquad \mu \in \mathbb{R}, \beta > 0,\ x \in \mathbb{R} \tag{33}$$

Since the CDF of X is given by

$$F_X(x) = \begin{cases} 1 - \frac{1}{2} e^{-\frac{x-\mu}{\beta}} & x \geq \mu \\ \frac{1}{2} e^{-\frac{\mu-x}{\beta}} & \text{Otherwise} \end{cases} \tag{34}$$

then by using Eq. (11) the probability distribution of Y is

$$f_Y(y) = \frac{d}{dy} \begin{cases} 1 - \frac{1}{2} e^{-\frac{\left(\frac{y}{k}\right)^{1/\gamma} - \mu}{\beta}} & y \geq k\mu^\gamma \\ \frac{1}{2} e^{-\frac{\mu - \left(\frac{y}{k}\right)^{1/\gamma}}{\beta}} & \text{Otherwise} \end{cases}$$
$$= \begin{cases} \frac{\left(\frac{y}{k}\right)^{\frac{1}{\gamma}-1} e^{-\frac{\left(\frac{y}{k}\right)^{1/\gamma} - \mu}{\beta}}}{2\beta\gamma k} & y \geq k\mu^\gamma \\ \frac{\left(\frac{y}{k}\right)^{\frac{1}{\gamma}-1} e^{-\frac{\mu - \left(\frac{y}{k}\right)^{1/\gamma}}{\beta}}}{2\beta\gamma k} & \text{Otherwise} \end{cases} \tag{35}$$

We now establish an equation using equations (35) and (8) with $\mu = 0$ for the latter for simplicity. Making the same analysis and assumptions for the value of γ as in subsection 3.1 and eliminating common terms at both sides we arrive to the following limit derived from Eq. (35) for the upper branch

$$\lim_{\gamma\to\infty} \frac{e^{-\frac{\left(\frac{y}{k}\right)^{1/\gamma} - \mu}{\beta}}}{2\beta} = \frac{e^{\frac{\mu-1}{\beta}}}{2\beta} \tag{36}$$

The limit of the Pareto distribution of Eq. (8) is the same as given by Eq. (14). By solving the equation $\frac{\alpha_P}{2^{(\alpha_P+1)}} = \frac{e^{\frac{\mu-1}{\beta}}}{2\beta}$ for α_P, we get

$$\alpha_P = \begin{cases} -\frac{1}{\log(2)} W_{-1}[-\frac{\log(2) e^{\frac{\mu-1}{\beta}}}{\beta}] \\ -\frac{1}{\log(2)} W_0[-\frac{\log(2) e^{\frac{\mu-1}{\beta}}}{\beta}] \end{cases} \tag{37}$$

Similarly to the normal distribution case, since the Lambert function $W_k(z)$ is real for $z > -1/e$, the space of solutions for real values for α_P imposes the constraint $-\frac{\log(2) e^{\frac{\mu-1}{\beta}}}{\beta} > -\frac{1}{e}$, which leads to the following constraint between μ and β

$$\mu < 1 + \beta \log\left[\frac{\beta}{e \log(2)}\right] \tag{38}$$

Fig. 6 shows an example of the space of solutions given by Eq. (37). The brown plane corresponds to the upper branch solution whereas the blue plane corresponds to the lower branch solution.

For the lower branch of equation (35), we solve the limit $\lim_{\gamma\to\infty} \frac{e^{-\frac{\mu - \left(\frac{y}{k}\right)^{1/\gamma}}{\beta}}}{2\beta} = \frac{e^{\frac{1-\mu}{\beta}}}{2\beta}$ and by solving the resulting equation $\frac{\alpha_P}{2^{\alpha_P+1}} = \frac{e^{\frac{1-\mu}{\beta}}}{2\beta}$ for α_P we get

$$\alpha_P = \begin{cases} -\frac{1}{\log(2)} W_{-1}[-\frac{\log(2) e^{\frac{1-\mu}{\beta}}}{\beta}] \\ -\frac{1}{\log(2)} W_0[-\frac{\log(2) e^{\frac{1-\mu}{\beta}}}{\beta}] \end{cases} \tag{39}$$

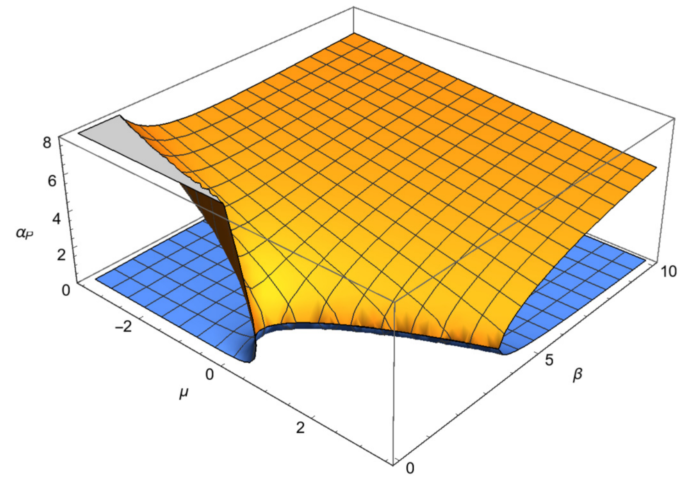

Fig. 6. Example of a solutions space for X Laplace in the power transformation.

Due to the condition $-\frac{\log(2) e^{\frac{1-\mu}{\beta}}}{\beta} > -\frac{1}{e}$ to get real solutions the following constraint between μ and β should be imposed

$$1 - \beta \log\left[\frac{\beta}{e \log(2)}\right] < \mu \tag{40}$$

In summary, as long as $1 - \beta \log\left[\frac{\beta}{e\log(2)}\right] < \mu < 1 + \beta \log\left[\frac{\beta}{e\log(2)}\right]$ there will be real solutions for α_P. Since Eq. (35) is composed of two segments, there is a set of solutions of α_P for each one, given by equations (37) and (39).

3.7. Power transformation for X having a Pareto distribution

Let us consider the Pareto IV distribution of equation (8) with $\mu = 0$, that is

$$f_X(x) = \frac{\alpha x^{\frac{1}{\gamma}-1} \left[\left(\frac{x}{k}\right)^{1/\gamma} + 1\right]^{-(\alpha+1)}}{\gamma k^{1/\gamma}} \qquad \gamma, k, x > 0 \tag{41}$$

We are interested in the distribution produced by the power transformation of equation (10) with X following the distribution of equation (41). Since the CDF of X of is given by

$$F_X(x) = 1 - \left[\left(\frac{x}{k}\right)^{1/\gamma} + 1\right]^{-\alpha} \qquad x > 0 \tag{42}$$

then we can find the probability distribution of the derived process in terms of y as

$$f_Y(y) = \frac{d}{dy}\left\{1 - \left[\left(\frac{\left[\frac{y}{k}\right]^{1/\gamma}}{k}\right)^{1/\gamma} + 1\right]^{-\alpha}\right\} = \frac{\alpha x^{\frac{1}{\gamma^2}-1} \left(k^{-\frac{\gamma+1}{\gamma^2}} x^{\frac{1}{\gamma^2}} + 1\right)^{-(\alpha+1)}}{k^{\frac{\gamma+1}{\gamma^2}} \gamma^2} \tag{43}$$

In this case it is not possible to find a space of solutions, as we did earlier, for the distribution produced by the transformation of Eq. (10). Thus, a Pareto distribution of the form given by Eq. (41) under the transformation of Eq. (10) does not produce a distribution that could be essentially indistinguishable from a Pareto distribution. This observation can also be easily verified graphically. Returning to our bank analogy, bank interests following a Pareto distribution would mean that most people will choose poor banks, and only a lucky few will choose very good banks. Therefore, when iterating over γ periods, only an extremely few people will see large benefits, while the rest will see extremely reduced benefits (compared to when the distribution of interest was uniform). Therefore, it is reasonable to think that the result is no longer fat-tail but it has a more abrupt descent.

4. Kolmogorov transformation

The Kolmogorov transformation has been studied by Kolmogorov [21, pp. 451] and others, where it is well-known that a multiplicative Kolmogorov transformation produces a LogNormal distribution. However, in this section, we want to emphasise the differences with the power transformation.

The central limit theorem for the sum of random variables is a well-known and established theorem in probability theory [18, pp. 273, ch. 5]. A related theorem is the Central Limit Theorem for the multiplication of random variables [22, pp. 220, ch. 8] which is usually stated as follows. Consider the random variable Y given by

$$Y = X_1 X_2 \dots X_n \qquad n \in \mathbb{N} \tag{44}$$

for large n the probability density function of Y is approximately the logNormal distribution

$$f_Y(y) = \frac{e^{-\frac{[\log(y)-\mu]^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma y} \qquad y > 0 \tag{45}$$

with mean and variance given by

$$\mu = \sum_{i=1}^{n} \overline{\log(X_i)} \qquad \sigma^2 = \sum_{i=1}^{n} \mathrm{Var}\left[\log(X_i)\right] \tag{46}$$

Now if we transform the random variable Y as $Z = \log(Y)$ then since

$$Z = \log(X_1) + \log(X_2) + \dots + \log(X_n) \tag{47}$$

according to the CLT for the sum of random variables, for large n, the right-hand side of equation (47) will be close to a normal distribution. Meanwhile, since in general the transformed distribution e^u where u is a normal distribution is a LogNormal distribution and noting that $Y = e^Z$, the conclusion is made that Y should be a LogNormal distribution.

In contrast, let us now consider the transformation given by Eq. (10) with $k = 1$ constant, that is

$$Y = X^n \qquad n \in \mathbb{R} \tag{48}$$

From the derived transformations theory, we obtain the exact general expression for the distribution of Y

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy}\left[F_X(y^{1/n})\right] \tag{49}$$

Example 1. We now compare equations (45) and (49) with a simple example involving the uniform distribution. Consider that the distribution X_i in equation (44) corresponds to a uniform distribution between 0 and 1. Then, by applying equation (45) we get the distribution

$$f_Y(y) = \frac{e^{-\frac{[\log(y)+n]^2}{2n}}}{\sqrt{2\pi n}y} \tag{50}$$

Meanwhile, the exact expression for the multiplication when the X_i in Eq. (44) comes from the uniform distribution in the interval $[0,1]$ is given by

$$f_Y(y) = \frac{(-1)^{n-1}\log^{n-1}(y)}{(n-1)!} \tag{51}$$

By applying equation (49), we get the derived distribution

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy}\left[y^{1/n}\right] = \frac{y^{\frac{1}{n}-1}}{n} \tag{52}$$

Fig. 7 shows a comparison of equations (50), (51) and (52) for $n = 20$. The green curve corresponds to expression (52), the blue curve corresponds to expression (50), and the orange curve corresponds to expression (51).

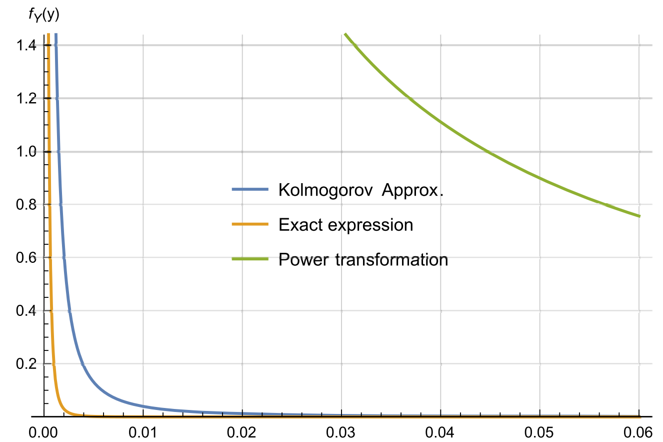

Fig. 7. Comparison of CLT for multiplication and power transformation.

Fig. 7 shows that there is a notorious discrepancy for the results produced by equations (45) and (49). This has to be so because we are dealing with two quite different models. The CLT considers the multiplication of n distinct values taken from independent identical distributions. The transformation X^n corresponds to the multiplication of the same value taken from a given distribution n times, the experiment being repeated many times.

As shown in section 3, when considering the power transformation (equation (48)), in general there is a space of solutions where the Pareto distribution is essentially indistinguishable to the result of equation (52). In the limit when $n \to \infty$, the subspace of solutions provided by the Pareto distribution is actually infinite. This is not the case for the LogNormal distribution, although, as we will discuss in Appendix B, for certain parameters a LogNormal distribution can look similar to Pareto.

5. Conclusions

Through this work we have been able to show that fat-tail distributions emerge as convergent distributions when multiplicative transformations are involved, for many input distributions, this being particularly the case for the exponential and the power transformation. The key aspect of the power transformation is that small initial differences, for instance as given by a uniform distribution, can lead to highly different results after γ iterations. The initial conditions do not seem to influence too much, that is, it does not seem to matter if we start from a uniform distribution or other kind of distribution, for an ample range of input distributions. The final result, after many iterations, can be well approximated by a Pareto distribution and, as showed in equation (25) for instance, it is possible to determine the space of solutions for which this is valid (for further details, see also Appendix C). Thus the power transformation suggests a behaviour opposite to chaotic phenomena whose result depends strongly on the initial conditions. Contrary to chaotic systems, the accumulation of successive multiplications shapes the result, quite independently of the initial conditions. It is worth noting that although we are assuming γ integer, mathematically there is no reason to be so, as indicated in equation (10), the values of γ can be real values.

The power multiplicative transformation suggests certain resemblance with the central limit theorem in which the Normal distribution arises when adding many independent distributions. In our work, as we have shown, fat-tail distributions arise, for many distributions, when there is an underlying multiplicative transformation. This result is quite general: a Pareto-like distribution can fit very well the result within a mathematically proven space of solutions. Even for the multiplicative Kolmogorov transformation, which produces a LogNormal distribution for a wide range of distributions, as shown in section 4, the result can be approximated by a Pareto distribution, as shown in subsection 3.5.

Finally, a very interesting concept for the sciences of complexity arises by observing that when X is a uniform distribution, the transformation X^γ leads to a Pareto distribution. This shows how easy complexity can emerge. A uniform distribution express the randomness concept par excellence, while a Pareto distribution is the complexity par excellence.

Declarations

Author contribution statement

F.G. Guerrero, A. Garcia-Baños: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

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Appendix A. Is Pareto IV a pure power-law?

Power-laws are characterised by exhibiting a straight line behaviour in a log-log plot. The equation of a straight line in a log-log plot is given by $\log(y) = \log(k) + \gamma\log(x)$ where γ is the slope of the straight line and k a constant, $k > 0$. Thus $y = e^{\log(k)+\gamma\log(x)} = e^{\log(k)}e^{\gamma\log(x)} = e^{\log(k)}\left[e^{\log(x)}\right]^\gamma = kx^\gamma$. Therefore, any expression of the form

$$f(x) = kx^\gamma \tag{A.1}$$

will draw a straight line in a log-log plot.

Does a Pareto IV distribution display a pure straight line in log-log plot? Let us consider an expression of the form of equation (8) in the variable x. For simplicity, let the location parameter $\mu = 0$. Taking the log of the expression we obtain

$$\log\left[f_X(x)\right] = \log\left(\frac{\alpha}{\gamma}\right) - \frac{\log(k)}{\gamma} - (\alpha+1)\log\left[\left(\frac{x}{k}\right)^{1/\gamma} + 1\right] + \left(\frac{1}{\gamma} - 1\right)\log(x) \tag{A.2}$$

As can be observed in equation (A.2), there are two terms of x. Taking the limit when $\gamma \to \infty$ for each term, we get

$$\lim_{\gamma\to\infty} -(\alpha+1)\log\left[\left(\frac{x}{k}\right)^{1/\gamma} + 1\right] = -(\alpha+1)\log(2) \tag{A.3}$$

$$\lim_{\gamma\to\infty}\left(\frac{1}{\gamma} - 1\right)\log(x) = -\log(x) \tag{A.4}$$

Equation (A.3) shows that such term is essentially constant over a large range of x. However, we are interested into seeing the value of the left-hand side of equation (A.3) when x is very large, that is

$$\lim_{x\to\infty} -(\alpha+1)\log\left[\left(\frac{x}{k}\right)^{1/\gamma} + 1\right] = -\infty \tag{A.5}$$

Thus, in the very long run the term of equation (A.3) tends to minus infinity, no matter the value of γ. In general for equation (A.2) $\lim_{x\to\infty}\left(\log\left[f_X(x)\right]\right) = -\infty$. Also $\lim_{x\to 0,\gamma>1}\left(\log\left[f_X(x)\right]\right) = \infty$. We can see thus that a Pareto IV distribution, as given by equation (8) is not a straight line in a pure mathematical sense in log-log plot. However, the component that makes it "impure" grows so slowly that, for a very large interval, it is almost indistinguishable from a constant. However, in the limit when x tends to infinity, this component also tends to infinity.

Example 2. Let us consider in equation (A.2) the term $g(x) = -(\alpha+1)\log\left[\left(\frac{x}{k}\right)^{1/\gamma} + 1\right]$ with the following parameters $\alpha = 2$, $\gamma = 2000$ and $k = 4$. The result of $g(x)$ evaluated for $x = 10$ and $x = 10^{10}$ is -2.08013 and -2.09572, respectively. Using equation (8) in x with $\mu = 0$ and given that the CDF of $f_X(x)$ is given by $F_X(x) = 1 - \left[\left(\frac{x}{k}\right)^{1/\gamma} + 1\right]^{-\alpha}$, we find that $P[X \le 10^{10}]$ is 75.27% of the total probability. We also see that the value of x for which $P[X \le x] = 0.99$, in this example, is $x = 1.22202 \times 10^{1909}$, but for this value of x the value of $g(x)$ is -6.90776, as suggested by equation (A.5). Thus, in this example, up to at least $x = 10^{10}$ the log-log plot of equation (A.5) will be almost indistinguishable from a pure straight line.

Appendix B. Does a LogNormal distribution display a pure straight line in a log-log plot?

Let us now return to the analysis of the LogNormal distribution and why it can be easily confused in log-log plot with the Pareto distribution. Let us take the log of the LogNormal distribution as given by Eq. (26), that is

$$\log\left[f_X(x)\right] = -\frac{[\log(x) - \mu]^2}{2\sigma^2} - \log\left(\sqrt{2\pi}\sigma x\right) \tag{B.1}$$

As explained in Appendix A, for Eq. (B.1) to be a pure straight line, the first term of the right-hand side must be a constant. One way in which this is possible is observing that $\lim_{\sigma\to\infty} -\frac{(\log(x)-\mu)^2}{2\sigma^2} = 0$. This means in practice that if σ is large, then Eq. (B.1) will plot an almost pure straight line. Also, if $\mu >> \log(x)$ then the term $-\frac{[\log(x)-\mu]^2}{2\sigma^2}$ will be dominated by μ, thus becoming almost constant, and in this case, Eq. (B.1) would also look like an almost pure straight line. Finally, if both μ and σ are quite large the first term of the right-hand side in Eq. (B.1) will be zero, that is, $\lim_{\mu\to\infty,\sigma\to\infty} -\frac{[\log(x)-\mu]^2}{2\sigma^2} = 0$. As explained also in Appendix A, since the Pareto distribution under certain parameters is able to display a plot very similar to a straight line in log-log plot, this reminds us how easy it i to confuse the Pareto and LogNormal distributions in a log-log plot. Mathematically speaking, as we see in Section 3.5, under certain assumptions, it is possible for a Pareto distribution to be mathematically similar to a LogNormal distribution. There is an infinite set of values of the Pareto distribution parameters (even though those values form a subspace of the universe of possible values) for which the Pareto distribution can be almost indistinguishable from the LogNormal distribution. By carefully selecting the parameters of a Pareto distribution, a curve very similar to the LogNormal distribution can be obtained. However, the LogNormal distribution is much more limited if we want to approximate it to a Pareto distribution. In this sense, the Pareto distribution can be regarded as a more universal distribution than the LogNormal distribution.

Example 3. Let us consider the LogNormal of Eq. (26) distribution with $\mu = 0.79$ and $\sigma = 3.2$ and the Pareto distribution of Eq. (8) with $\alpha = 1.06$, $\gamma = 2.05$, $k = 2$ and $\mu = 0$. Fig. 8 shows the two curves on the same plot, and as it can be observed, the two curves are indistinguishable at eye. The integral between 0.1 and 3.0 for the LogNormal distribution is equal to 0.371501 whereas the integral for the Pareto between the same limits is equal to 0.371971, as expected, in coincidence with their very similar look.

Appendix C. Power transformation with multiplication distribution as input

Let us consider a uniform distribution multiplied by a Pareto Distribution which has been obtained through a multiplicative process. Let V be a uniform distribution with probability distribution function given by

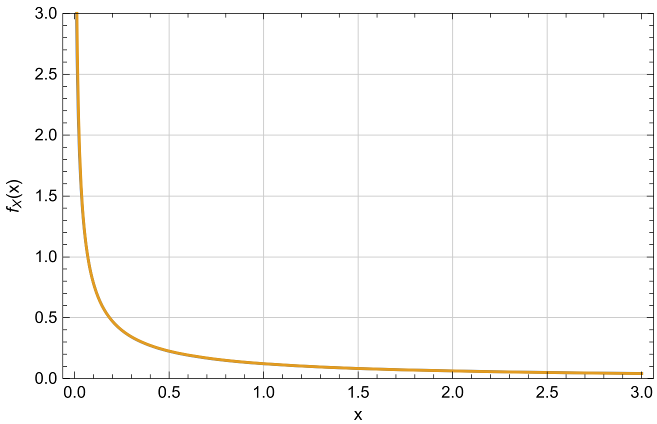

Fig. 8. Example of LogNormal and Pareto similarity.

$$f_V(v) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \tag{C.1}$$

with $a > 0, b > 0$.

The uniform distribution can be considered a reasonable distribution as one of the factors because it provides a sense of equity and fairness as the starting point for a population distribution. Let X be a Pareto IV distribution with location parameter zero with probability distribution given by equation (41). It can be shown the probability distribution function of the multiplicative transformation $U = VX$ is given by

$$f_U(u) = \frac{\alpha e^{-i\pi(\alpha+\gamma)}\left[B_{-\left(\frac{ak}{u}\right)^{1/\gamma}}(\alpha+\gamma,-\alpha) - B_{-\left(\frac{bk}{u}\right)^{1/\gamma}}(\alpha+\gamma,-\alpha)\right]}{k(a-b)} \quad u \geq 0 \tag{C.2}$$

where $B_z(a,b) = \int_0^z t^{a-1}(1-t)^{b-1}dt$ is the incomplete beta function. The probability distribution in Eq. (C.2) is quite similar to the original pdf in Eq. (41). For instance, the integral of equations (41) and (C.2) with $\alpha = 1$, $\gamma = 100$, $k = 1$, $a = 1$, $b = 2$ between 10^{-9} and 10^9 are 0.103247 and 0.103256, respectively. The resulting distribution is extremely similar to Pareto for a very large range of values. It is also interesting to note that when multiplying the uniform distribution by other distribution, in many cases, the result has a great similarity with the second one, in a way that resembles the neutral element of multiplication. For instance, let us consider the variable X a zero-mean normal distribution with probability distribution given by

$$f_X(x) = \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} \tag{C.3}$$

It can be shown that the probability distribution of the derived transformation $U = VX$ is given by

$$f_U(u) = \begin{cases} \left|\frac{\Gamma\left(0,\frac{u^2}{2a^2\sigma^2}\right)}{\sqrt{2\pi}(2a\sigma)}\right| & b = 0 \\ \left|\frac{\Gamma\left(0,\frac{u^2}{2b^2\sigma^2}\right)}{\sqrt{2\pi}(2b\sigma)}\right| & a = 0 \\ \left|\frac{\Gamma\left(0,\frac{u^2}{2a^2\sigma^2}\right)-\Gamma\left(0,\frac{u^2}{2b^2\sigma^2}\right)}{2\sqrt{2\pi}\sigma(a-b)}\right| & \text{Otherwise} \end{cases} \tag{C.4}$$

where $\Gamma(0,z)$ is the incomplete gamma function $\Gamma(0,z) = \int_z^\infty t^{-1}e^{-t}dt$.

This behaviour is also observed, for instance, when multiplying the uniform distribution by the exponential, LogNormal, Laplace and other distributions. However, this is not a mathematical rule because, for instance, multiplying two uniform distributions, as can be easily observed, gives us a result that is significantly different from a uniform distribution.

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