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Research article

Characterizations of generalized Robertson-Walker spacetimes concerning gradient solitons

Krishnendu De [a], Mohammad Nazrul Islam Khan [b,*], Uday Chand De [c]

[a] Department of Mathematics, Kabi Sukanta Mahavidyalaya, The University of Burdwan. Bhadreswar, Hooghly, West Bengal, India
[b] Department of Computer Engineering, College of Computer, Qassim University, Buraydah, Saudi Arabia
[c] Department of Pure Mathematics, University of Calcutta, West Bengal, India

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ABSTRACT

In this article, we examine gradient type Ricci solitons and (m, τ)-quasi Einstein solitons in generalized Robertson-Walker (GRW) spacetimes. Besides, we demonstrate that in this scenario the GRW spacetime presents the Robertson-Walker (RW) spacetime and the perfect fluid (PF) spacetime presents the phantom era. Consequently, we show that if a GRW spacetime permits a gradient τ-Einstein solitons, then it also represents a PF spacetime under certain condition.

1. Introduction

Suppose $\mathcal{M}^n$ is a Lorentzian manifold of dimension n and g is a Lorentzian metric of signature $(+, +, ..., +, -)$. In 1995, the notion of GRW spacetimes was proposed by Alias et al. [1]. A GRW spacetime is a Lorentzian manifold M^n $(n \geq 4)$ which can be presented as $\mathcal{M} = -I \times f^2 M^*$, in which $I \subseteq \mathbb{R}$ (Real numbers set), M^* indicates the Riemannian manifold of dimension (n-1) and the smooth function $f > 0$ is termed as warping function or scale factor. If M^* is of dimension three and is of constant sectional curvature, then the above stated spacetime represents a RW spacetime. A comprehensive investigation of GRW spacetimes are presented in ([2–7]).

Definition 1.1. For a scalar function ψ and a 1-form ω_k (non vanishing), let the condition $\nabla_k u_h = \omega_k u_h + \psi g_{kh}$ be obeyed, the vector field u is then referred to as torse-forming.

The foregoing equation can be expressed as $\nabla_X u = \omega(X) u + \psi X$, ω being a 1-form. The following theorem has been demonstrated by Mantica and Molinari [5]:

* Corresponding author.
E-mail addresses: krishnendu.de@outlook.in (K. De), m.nazrul@qu.edu.sa (M.N.I. Khan), uc_de@yahoo.com (U.C. De).

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Theorem 1.1. *([5]) The Lorentzian manifold $\mathcal{M}^n$ $(n \geq 3)$ is a GRW spacetime iff the spacetime permits a unit torse-forming time-like vector field: $\nabla_j u_k = \psi(g_{ik} + u_k u_i)$, it is also an eigenvector of the Ricci tensor.*

The $\mathcal{M}$ is termed as a PF spacetime if for the non-vanishing Ricci tensor S, the spacetime fulfills

$$S = a_1 g + b_1 \eta \otimes \eta, \tag{1.1}$$

where a_1, b_1 are scalar fields and $g(U_1, \rho) = \eta(U_1)$ for any U_1 and $g(\rho, \rho) = -1$ in which ρ stands for a unit time-like vector field of the PF spacetime and η is a 1-form. Each and every RW spacetime presents a PF spacetime [8]. However, in the dimension 4, the GRW spacetime presents a PF spacetime iff the spacetime is RW [9].

In a PF spacetime the expression of the energy-momentum tensor T is described as

$$T = (v + p)\eta \otimes \eta + pg, \tag{1.2}$$

v denotes the energy density, p indicates the isotropic pressure [8].

In absence of the cosmological constant in the theory of general relativity, the Einstein's field equations which is a highly nonlinear equations, is written as

$$S - \frac{r}{2} g = k^2 T, \tag{1.3}$$

where $k = \sqrt{8\pi \mathrm{G}}$, G indicates Newton's gravitational constant and the scalar curvature is denoted by r.

Using differential equations (1.2) and (1.3), we reveal the equation (1.1), where

$$b_1 = k^2(p + v), \; a_1 = \frac{k^2(p - v)}{2 - n}. \tag{1.4}$$

Additionally, for a equation of state (EOS) parameter ω, v and p are interconnected by the equation $p = \omega v$. The EOS having the shape $p = p(v)$ is named isentropic. According to [10], if $p = 0$, $p = \frac{v}{3}$, and if $p + v = 0$, then the PF-spacetime is represented the dust matter, the radiation and the dark energy era, respectively. Furthermore, it includes the phantom era when $\omega < -1$. The physical implications are discussed in ([11–14]).

A self-reinforcing wave packet named as a soliton, also called a solitary wave, maintains its formation while traveling with a constant speed. It is created when nonlinear and dispersive effects in the medium are neutralized. Gradient is a common term in mathematics and physics to describe the direction and magnitude of a force acting on a particle. In other disciplines, such as chemistry and engineering, the gradient is also used to demonstrate how a substance's property changes in relation to other variables.

Hamilton [15] develops the novel idea of Ricci flow. It is referred to as a Ricci flow [15] if the partial differential equations $\frac{\partial}{\partial t} g_{ij}(t) = -2S_{ij}$ satisfies the metric of a Lorentzian manifold $\mathcal{M}$. The Ricci solitons (RS) are produced by the self-similar solutions to the Ricci flow. If a metric of $\mathcal{M}$ obeys the differential equations,

$$\mathfrak{L}_{W_1} g + 2S + 2\lambda_1 g = 0, \tag{1.5}$$

it is referred to as a RS [16], in which λ_1 indicates a real scalar. Also, $\mathfrak{L}_{W_1}$ stands for the Lie derivative operator and W_1 is the potential vector field. Equation (1.5) has the subsequent form

$$Hess\, f + S + \lambda_1 g = 0, \tag{1.6}$$

in which the Hessian is denoted by $Hess$ and D stands for the gradient operator of g if $W_1 = Df$, for a smooth function f. A gradient RS is a metric that fulfills the partial differential equation (1.6). The gradient RS is said to have the smooth function f as its potential function.

RSs have a significant impact in both physics and mathematics. In physics, metrics that obey (1.5) are attractive and helpful. In connection to string theory, theoretical physicists have also been investigating the RS equation. Friedan, who has done study on various features of RSs, has made the initial contribution to these studies [17]. In [18], Blaga has considered PF spacetime endowed with a torse-forming vector field to study η-RSs and η-Einstein solitons (ES) and deduced a poison equation from the soliton equation. Chen and Desmukh have characterized RSs with the help of concurrent potential fields and on Euclidean hypersurfaces, under certain restriction they classify shrinking RSs [19]. Also in [20], the authors investigated compact shrinking gradient RSs. Karaka and Ozgur have studied RSs of gradient type on multiply warped product manifolds [21] and obtained a necessary and sufficient condition for these manifolds to be gradient RSs. In [22], Wang established that an almost RS of gradient type on a $(k, \mu)'$ almost Kenmotsu manifold is a rigid gradient RSs.

In [23], the authors have obtained exact solution for the fractional differential equations and these are emerging from solitons theory. In [24], the authors have formulated plans that are useful in solving many different kinds of nonlinear partial differential equations arising in several areas of applied sciences. In [25], to acquire soliton solutions to the nonlocal integrable equations, the authors have developed a new formulation of solutions to Riemann-Hilbert problems with the identity jump matrix. Rezazadeh has found a new soliton solutions of the complex Ginzburg-Landau equation with Kerr law nonlinearity in [26]. Here, we may mention that zero curvature equations make the link between integrable models and geometry manifest, and the Kronecker product produces new zero curvature representations from old ones [27].

If there are λ_1, τ and m $(0 < m < \infty)$, three real constants which obeys the partial differential equation

$$\nabla^2 f + S - \frac{1}{m} df \otimes df = (\lambda_1 + \tau r)g = \beta_1 g, \tag{1.7}$$

then the semi-Riemannian metric g on the Lorentzian manifold $\mathcal{M}$ is known as a gradient (m,τ)-quasi Einstein soliton (QES), where $\otimes$ denotes tensor product. If the potential function f is constant, the soliton becomes trivial, which suggests that the manifold is Einstein. Additionally, the aforementioned relation turns into a gradient τ-ES when $m = \infty$. This idea was presented in [28], and Venkatesha et al. examined [29] τ-ES on almost Kenmotsu manifolds. More recently, in this same manifold we studied gradient (m,τ)-QES [30].

Many researchers recently examined various types of solitons in PF spacetimes, including RS ([18], [31]), gradient RSs ([31], [32]), Yamabe and gradient Yamabe solitons ([32], [33]), gradient m-QESs [32], gradient η-ESs [31], gradient Schouten solitons [31], Ricci-Yamabe solitons [34], respectively.

According to the information we have, there are many findings in the literature about PF spacetimes with solitons, but there are just a few results in GRW spacetimes. We want to fill this gap in this article and focus on characterizing the GRW spacetimes that satisfy gradient RS and gradient (m,τ)-QES.

In [5], it is established that a GRW spacetime with divergence free Weyl tensor is a PF spacetime. The foregoing result raises the question: Is the preceding result still valid if the condition divergence free Weyl tensor is substituted by a gradient Ricci soliton, or by a gradient (m,τ)-QES? Here, we provide evidence that the answer to this question is, in fact, 'yes' in both cases under certain conditions. Precisely, we prove the subsequent main theorems.

Theorem 1.2. *If a GRW spacetime admits a gradient RS with $\rho f = constant$, then it becomes a PF spacetime.*

Theorem 1.3. *If a GRW spacetime permits a gradient (m,τ)-QES with $\beta_1 = (n-1)\mu$ =constant and ρf = constant, then it becomes a PF spacetime.*

2. Preliminaries

Let $\mathcal{M}$ be a GRW spacetime and hence using Theorem 1.1, we acquire

$$\nabla_{U_1}\rho = \psi[U_1 + \eta(U_1)\rho] \tag{2.1}$$

and

$$S(U_1,\rho) = \xi\eta(U_1), \tag{2.2}$$

where ψ is a scalar and ξ is a non-zero eigenvector.

Lemma 2.1. *In a GRW spacetime, we have*

$$R(U_1,V_1)\rho = \mu[\eta(V_1)U_1 - \eta(U_1)V_1] \tag{2.3}$$

and

$$S(U_1,\rho) = (n-1)\mu\eta(U_1), \tag{2.4}$$

where we choose $\mu = (\rho\psi + \psi^2)$.

Proof. Differentiating covariantly equation (2.1), we obtain

$$\begin{aligned}\nabla_{V_1}\nabla_{U_1}\rho = {} & (V_1\psi)[U_1 + \eta(U_1)\rho] \\ & + \psi[\nabla_{V_1}U_1 + (\nabla_{V_1}\eta(U_1))\rho + \psi(V_1 + \eta(V_1)\rho)\eta(U_1)].\end{aligned} \tag{2.5}$$

Interchanging U_1 and V_1 yields

$$\begin{aligned}\nabla_{U_1}\nabla_{V_1}\rho = {} & (U_1\psi)[V_1 + \eta(V_1)\rho] \\ & + \psi[\nabla_{U_1}V_1 + (\nabla_X\eta(V_1))\rho + \psi(U_1 + \eta(U_1)\rho)\eta(V_1)].\end{aligned} \tag{2.6}$$

Also, we have

$$\nabla_{[U_1,V_1]}\rho = \psi\{[U_1,V_1] + \eta([U_1,V_1])\rho\}. \tag{2.7}$$

Equations (2.1), (2.5), (2.6) and (2.7) together implies

$$R(U_1, V_1)\rho = (U_1\psi)[V_1 + \eta(V_1)\rho] - (V_1\psi)[U_1 + \eta(U_1)\rho] + \psi^2[\eta(V_1)U_1 - \eta(U_1)V_1]. \tag{2.8}$$

Contracting V_1 from equation (2.8), we obtain

$$S(U_1, \rho) = (2-n)(U_1\psi) + (\rho\psi)\eta(U_1) + (n-1)\psi^2\eta(U_1). \tag{2.9}$$

Combining equations (2.2) and (2.9), we infer

$$\xi\eta(U_1) = (2-n)(U_1\psi) + (\rho\psi)\eta(U_1) + (n-1)\psi^2\eta(U_1). \tag{2.10}$$

Setting $U_1 = \rho$ in (2.10) entails that

$$\xi = (n-1)\mu, \tag{2.11}$$

where $\mu = (\rho\psi + \psi^2)$.

From the last two equations, we acquire

$$U_1\psi = -(\rho\psi)\eta(U_1). \tag{2.12}$$

Using equation (2.12) in equation (2.8), we get

$$R(U_1, V_1)\rho = \mu[\eta(V_1)U_1 - \eta(U_1)V_1].$$

In view of equations (2.2) and (2.11), we provide

$$S(U_1, \rho) = (n-1)\mu\eta(U_1).$$

This ends the proof.

Lemma 2.2. *In a GRW spacetime, we obtain*

$$\mu\{U_1 + \rho\eta(U_1)\} = 0. \tag{2.13}$$

Proof. From equation (2.3), we get

$$R(U_1, V_1)\rho = \mu[\eta(V_1)U_1 - \eta(U_1)V_1].$$

Now,

$$(\nabla_{W_1} R)(U_1, V_1)\rho = \nabla_{W_1} R(U_1, V_1)\rho - R(\nabla_{W_1} U_1, V_1)\rho - R(U_1, \nabla_{W_1} V_1)\rho - R(U_1, V_1)\nabla_{W_1}\rho. \tag{2.14}$$

Using equations (2.1) and (2.3) in equation (2.14) entails that

$$(\nabla_{W_1} R)(U_1, V_1)\rho = \{W_1\mu\}[\eta(V_1)U_1 - \eta(U_1)V_1] + \psi\mu[g(V_1, W_1)U_1 - g(U_1, W_1)V_1] - \psi R(U_1, V_1)W_1.$$

The well-known second Bianchi identity is given by

$$(\nabla_{W_1} R)(U_1, V_1)\rho + (\nabla_{U_1} R)(V_1, W_1)\rho + (\nabla_{V_1} R)(W_1, U_1)\rho = 0.$$

From the foregoing two equations, we infer

$$[\{W_1\mu\}\eta(V_1) - \{V_1\mu\}\eta(W_1)]U_1 + [\{U_1\mu\}\eta(W_1) - \{W_1\mu\}\eta(U_1)]V_1 + [\{V_1\mu\}\eta(U_1) - \{U_1\mu\}\eta(V_1)]W_1 - \psi[R(U_1, V_1)W_1 + R(V_1, W_1)U_1 + R(W_1, U_1)V_1] = 0.$$

Putting $W_1 = \rho$ in the previous equation gives

$$[\{\rho\mu\}\eta(V_1) + \{V_1\mu\}]U_1 - [\{U_1\mu\} + \{\rho\mu\}\eta(U_1)]V_1 \tag{2.15}$$

$$+[\{V_1\mu\}\eta(U_1) - \{U_1\mu\}\eta(V_1)]\rho$$
$$-\psi[R(U_1,V_1)\rho + R(V_1,\rho)U_1 + R(\rho,U_1)V_1] = 0.$$

From equation (2.3), we get

$$R(\rho,U_1)V_1 = \mu[g(U_1,V_1)\rho - \eta(V_1)U_1] \tag{2.16}$$

and

$$R(U_1,\rho)V_1 = \mu[\eta(V_1)U_1 - g(U_1,V_1)\rho]. \tag{2.17}$$

Using equations (2.3), (2.16) and (2.17) in equation (2.15) entails that

$$\begin{aligned}&\{\rho\mu\}[\eta(V_1)U_1 - \eta(U_1)V_1]\\&+\{V_1\mu\}[U_1 + \eta(U_1)\rho]\\&-\{U_1\mu\}[V_1 + \eta(V_1)\rho] = 0.\end{aligned} \tag{2.18}$$

Contracting V_1 from the equation (2.18), we infer

$$\mu\{U_1 + \rho\eta(U_1)\} = 0.$$

Hence the proof is completed.

Lemma 2.3. *In a GRW spacetime, we have*

$$g((\nabla_\rho Q)U_1 - (\nabla_{U_1}Q)\rho, \rho) = 0, \tag{2.19}$$

in which the Ricci operator Q is described by $g(QU_1,V_1) = S(U_1,V_1)$.

Proof. From equation (2.4), we get

$$Q\rho = (n-1)\mu\rho. \tag{2.20}$$

Differentiating equation (2.20), we acquire

$$\begin{aligned}(\nabla_{U_1}Q)\rho = {} & (n-1)\{U_1\mu\}\rho\\&+(n-1)\psi\mu[U_1 + \eta(U_1)\rho]\\&-\psi QU_1 - (n-1)\psi\mu\eta(U_1)\rho.\end{aligned} \tag{2.21}$$

Using equation (2.21) and Lemma 2.2, we easily acquire the desired result.

3. Proof of the prime theorems

Proof of the Theorem 1.2. Let us suppose that a GRW spacetime admit a gradient RS. Then the equation (1.6) may be written as

$$\nabla_{U_1}Df = -QU_1 - \lambda_1 U_1.$$

The foregoing equation and the following relation

$$R(U_1,V_1)Df = \nabla_{U_1}\nabla_{V_1}Df - \nabla_{V_1}\nabla_{U_1}Df - \nabla_{[U_1,V_1]}Df$$

give

$$R(U_1,V_1)Df = (\nabla_{V_1}Q)(U_1) - (\nabla_{U_1}Q)(V_1).$$

Taking inner product of the previous equation with ρ and making use of Lemma 2.3, we acquire

$$g(R(U_1,V_1)Df,\rho) = 0. \tag{3.1}$$

Again, from equation (2.3) we infer

$$g(R(U_1,V_1)\rho, Df) = \mu[\eta(V_1)(U_1f) - \eta(U_1)(V_1f)]. \tag{3.2}$$

Combining equations (3.1) and (3.2), we get

$$-\mu[\eta(V_1)(U_1f) - \eta(U_1)(V_1f)] = 0. \tag{3.3}$$

Replacing V_1 by ρ in equation (3.3), we obtain

$$\mu[(U_1 f) + \eta(U_1)(\rho f)] = 0. \tag{3.4}$$

This entails that either $\mu = 0$, or $\mu \neq 0$.

Case (i): If $\mu = 0$, then from equation (2.4) we have $S(U_1, \rho) = 0$. This reflects that the eigenvector ξ is zero, which contradicts the Theorem 1.1.

Case (ii): If $\mu \neq 0$, then from equation (3.4), we reveal

$$[(U_1 f) + (\rho f)\eta(U_1)] = 0,$$

which implies

$$Df = -(\rho f)\rho. \tag{3.5}$$

Differentiating the equation (3.5), we acquire

$$\nabla_{U_1} Df = -\{U_1(\rho f)\}\rho - \psi(\rho f)\{U_1 + \eta(U_1)\rho\}. \tag{3.6}$$

If we take $\rho f = c_1 =$ constant, then either $c_1 \neq 0$, or $c_1 = 0$.

Case (i): If $c_1 \neq 0$, then equation (3.6) implies

$$\nabla_{U_1} Df = -c_1 \psi\{U_1 + \eta(U_1)\rho\}. \tag{3.7}$$

Using equation (3.7) in equation (3.1) yields

$$QU_1 = c_1\psi\{U_1 + \eta(U_1)\rho\} - \lambda_1 U_1,$$

which implies

$$S(U_1, V_1) = \{c_1\psi - \lambda_1\}g(U_1, V_1) + c_1\psi\eta(U_1)\eta(V_1).$$

Therefore, the spacetime under consideration is a PF spacetime.

Case (ii): If $c_1 = 0$, then equation (3.5) gives $Df = 0$. Using this in equation (3.1) yields

$$S(U_1, V_1) = -\lambda_1 g(U_1, V_1). \tag{3.8}$$

We know that

$$(divC)(U_1, V_1)W_1 = \frac{n-3}{n-2}[\{(\nabla_{U_1} S)(V_1, W_1) - (\nabla_{V_1} S)(U_1, W_1)\} - \frac{1}{2(n-1)}\{g(V_1, W_1)dr(U_1) - g(U_1, W_1)dr(V_1)\}], \tag{3.9}$$

in which C stands for the Weyl conformal curvature tensor. Therefore using equation (3.8), from equation (3.9) we acquire $(divC)(U_1, V_1)W_1 = 0$.

Thus, the spacetime is a GRW spacetime with $divC = 0$ and hence, it is a PF spacetime [5].

Hence the proof is finished.

Since, in 4−dimension, a GRW spacetime is a PF spacetime iff the spacetime is a RW spacetime [9]. Therefore, from the above theorem, we arrive:

Corollary 3.1. *In* 4−*dimension, for* $\rho f = constant$, *a* GRW *spacetime admitting a gradient* RS *turns into a* RW *spacetime.*

Remark 1. For $n = 4$, comparing the equations (1.1) and (3.8), we have

$$a_1 g(U_1, V_1) + b_1\eta(U_1)\eta(V_1) = \{c_1\psi - \lambda_1\}g(U_1, V_1) + c_1\psi\eta(U_1)\eta(V_1).$$

Making use of equation (1.4), the foregoing equation yields

$$k^2(3p - \nu) = -\lambda_1,$$

which implies

$$(3p - \nu) = c \ (say).$$

Hence, the PF spacetime satisfies the EOS $\nu = -3p+$ constant.

If $c=0$, the above equation yields

$$\omega = \frac{p}{v} = -\frac{1}{3},$$

which entails that the PF spacetime presents the phantom era [12].

Proof of the Theorem 1.3. Let the GRW spacetime permit a (m,τ)-QES. Then the equation (1.7) may be expressed as

$$\nabla_{U_1} Df + QU_1 = \frac{1}{m} g(U_1, Df)Df + \beta_1 U_1. \tag{3.10}$$

Differentiating covariantly equation (3.10), we obtain

$$\nabla_{V_1}\nabla_{U_1} Df = -\nabla_{V_1} QU_1 + \frac{1}{m}\nabla_{V_1} g(U_1, Df)Df + \frac{1}{m} g(U_1, Df)\nabla_{V_1} Df + \beta_1 \nabla_{V_1} U_1 + (V_1\beta_1)U_1. \tag{3.11}$$

Interchanging U_1 and V_1 in the above equation, we get

$$\nabla_{U_1}\nabla_{V_1} Df = -\nabla_{U_1} QV_1 + \frac{1}{m}\nabla_{U_1} g(V_1, Df)Df + \frac{1}{m} g(V_1, Df)\nabla_{U_1} Df + \beta_1 \nabla_{U_1} V_1 + (U_1\beta_1)V_1 \tag{3.12}$$

and

$$\nabla_{[U_1,V_1]} Df = -Q[U_1,V_1] + \frac{1}{m} g([U_1,V_1], Df)Df + \beta_1 [U_1,V_1]. \tag{3.13}$$

From equations (3.10)-(3.13), we have

$$R(U_1,V_1)Df = (\nabla_{V_1} Q)U_1 - (\nabla_{U_1} Q)V_1 + \frac{\beta_1}{m}\{(V_1 f)U_1 - (U_1 f)V_1\} + \frac{1}{m}\{(U_1 f)QV_1 - (V_1 f)QU_1\} + \{(U_1\beta_1)V_1 - (V_1\beta_1)U_1\}. \tag{3.14}$$

Taking inner product of equation (3.14) with ρ and using Lemma 2.3, we infer

$$g(R(U_1,V_1)Df,\rho) = \frac{\beta_1}{m}\{(V_1 f)\eta(U_1) - (U_1 f)\eta(V_1)\} + \frac{1}{m}\{(U_1 f)\eta(QV_1) - (V_1 f)\eta(QU_1)\} + \{(U_1\beta_1)\eta(V_1) - (V_1\beta_1)\eta(U_1)\}. \tag{3.15}$$

Again, from equation (2.3) we acquire

$$g(R(U_1,V_1)\rho, Df) = \mu[\eta(V_1)(U_1 f) - \eta(U_1)(V_1 f)]. \tag{3.16}$$

Comparing equations (3.15) and (3.16), we obtain

$$-\mu[\eta(V_1)(U_1 f) - \eta(U_1)(V_1 f)] = \frac{\beta_1}{m}\{(V_1 f)\eta(U_1) - (U_1 f)\eta(V_1)\} + \frac{1}{m}\{(U_1 f)\eta(QV_1) - (V_1 f)\eta(QU_1)\} + \{(U_1\beta_1)\eta(V_1) - (V_1\beta_1)\eta(U_1)\}.$$

Replacing V_1 by ρ in the previous equation, we reveal

$$\{\mu - \frac{\beta_1}{m} + \frac{n-1}{m}\mu\}[(U_1 f) + \eta(U_1)(\rho f)] + \{(U_1\beta_1) + (\rho\beta_1)\eta(U_1)\} = 0. \tag{3.17}$$

If we take $\beta_1 = (n-1)\mu$ =constant (non zero), then from equation (3.17), we infer

$$[(U_1 f) + (\rho f)\eta(U_1)] = 0,$$

which implies

$$Df = -(\rho f)\rho. \tag{3.18}$$

Differentiating the equation (3.18), we acquire

$$\nabla_{U_1} Df = -\{U_1(\rho f)\}\rho - \psi(\rho f)\{U_1 + \eta(U_1)\rho\}. \tag{3.19}$$

If we take $\rho f = c_1 =$ constant, then either $c_1 \neq 0$, or $c_1 = 0$.

Case (i): If $c_1 \neq 0$, then equation (3.19) implies

$$\nabla_{U_1} Df = -c_1 \psi \{U_1 + \eta(U_1)\rho\}. \tag{3.20}$$

Using equation (3.20) in equation (3.10) gives

$$QU_1 = c_1 \psi \{U_1 + \eta(U_1)\rho\} + \beta_1 U_1,$$

which implies

$$S(U_1, V_1) = \{c_1 \psi + \beta_1\} g(U_1, V_1) + c_1 \psi \eta(U_1)\eta(V_1). \tag{3.21}$$

Hence, the spacetime taking into account is a PF spacetime.

Case (ii): If $c_1 = 0$, then equation (3.18) yields $Df = 0$. Using this in equation (3.10) gives

$$S(U_1, V_1) = \beta_1 g(U_1, V_1).$$

Using the foregoing equation in (3.9), we get $divC = 0$. Therefore, it is a GRW spacetime with $divC = 0$ and hence, it is a PF spacetime [5].

This ends the proof.

It is known that when $m = \infty$, a gradient (m, τ)-QES produces a gradient τ-ES. In (3.17), we put $m = \infty$ and easily acquire the equation (3.18). Therefore, we have:

Corollary 3.2. *If a GRW spacetime permits a gradient τ-ES, then the gradient of the τ-ES potential function is pointwise collinear with the potential vector field ρ.*

Similarly, as Corollary 3.1 we acquire:

Corollary 3.3. *In 4−dimension, a GRW spacetime admitting a gradient (m, τ)-QES with $\beta_1 = (n-1)\mu =$ constant and $\rho f = constant$ turns into a RW spacetime.*

Similarly, as above we can state:

Corollary 3.4. *If a GRW spacetime admits a gradient τ-ES with $\rho f =$ constant, then it becomes a PF spacetime.*

Remark 2. For $n = 4$, comparing the equations (1.1) and (3.21), we have

$$a_1 g(U_1, V_1) + b_1 \eta(U_1)\eta(V_1) = \{c_1 \psi + \beta_1\} g(U_1, V_1) + c_1 \psi \eta(U_1)\eta(V_1).$$

Using (1.4), the previous equation gives

$$k^2(3p - \nu) = \beta_1,$$

which implies

$$(3p - \nu) = c \ (say).$$

Therefore, the PF spacetime admitting a gradient (m, τ)-QES obeys the EOS $\nu = -3p+$ constant.

If $c = 0$, the above equation yields

$$\omega = \frac{p}{\nu} = -\frac{1}{3},$$

which implies that the PF spacetime represents the phantom era [12].

4. Discussion

The stage of the physical world's current modeling is spacetime, which is a torsion less, time oriented Lorentzian manifold. Albert Einstein first proposed the idea of general relativity theory in 1915, in which the matter content of the universe is stated by picking the suitable energy momentum tensor and is accepted to act like a perfect fluid spacetime in the cosmological models. GRW spacetimes, where large scale cosmology is staged, are a natural and extensive extension of RW spacetimes.

This article will be read not only by readers working in this field, but also by researchers from other engineering disciplines. In future, other researchers or we, will investigate others solitons in general relativity theory and cosmology.

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Additional information

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

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