Heliyon 9 (2023) e12748

Contents lists available at ScienceDirect

Heliyon

journal homepage: www.cell.com/heliyon

Research article

The second and third Hankel determinants for starlike and convex functions associated with Three-Leaf function

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ARTICLE INFO

Keywords:
Univalent function
Starlike function
Convex function
Three-leaf function
Hankel determinants

ABSTRACT

In this paper, we give sharp bounds of the Hankel determinant $H_2(3)(f)$ for the coefficients of functions in the class of starlike functions related to a domain that is like three leaves. We also give sharp bounds for the Hankel determinants $H_3(1)(f)$ and $H_2(3)(f)$ for the coefficients of functions in the class of convex functions related to the three-leaf-like domain.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f which are analytic in the open unit disc $\mathbb{D} := \{z : |z| < 1,\ z \in \mathbb{C}\}$ with Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D} \tag{1.1}$$

and let $\mathcal{S}$ be a subclass of $\mathcal{A}$ which contains univalent functions in $\mathbb{D}$.

An analytic function f in $\mathbb{D}$ is said to be subordinated by an analytic function g in $\mathbb{D}$, written as $f \prec g$ if there exists a self map w in $\mathbb{D}$ which is analytic with $w(0) = 0$ such that $f(z) = g(w(z))$. If g is univalent and $f(0) = g(0)$, then $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

Let $\varphi(z) = 1 + \frac{4}{5}z + \frac{1}{5}z^4$ be an analytic function which maps $\mathbb{D}$ onto a domain, which is like a three leaf. Gandhi [6] introduced a class $\mathcal{S}^*_{3\mathcal{L}}$ associated with a three-leaf function by using the notion of subordination and the function φ. Then he defined $\mathcal{S}^*_{3\mathcal{L}}$ as follows:

Definition 1.1. Let $f \in \mathcal{A}$. Then $f \in \mathcal{S}^*_{3\mathcal{L}}$ if and only if

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4.$$

Similarly, $f \in \mathcal{C}_{3\mathcal{L}}$ if and only if

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https://doi.org/10.1016/j.heliyon.2022.e12748
Received 17 September 2022; Received in revised form 23 November 2022; Accepted 27 December 2022

Available online 10 January 2023

$$1+\frac{zf''(z)}{f'(z)} \prec 1+\frac{4}{5}z+\frac{1}{5}z^4.$$

Since $\mathrm{Re}(\varphi(z)) > \left(3-\sqrt{5}\right)/4$ in $\mathbb{D}$, then $\mathcal{S}^*_{3\mathcal{L}}$ and $C_{3\mathcal{L}}$ are subclasses of $\mathcal{S}^*$ and C of starlike and convex univalent functions in $\mathbb{D}$, respectively which are defined as

$$\mathcal{S}^* := \left\{ f \in \mathcal{A} : Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D} \right\},$$

and

$$C = \left\{ f \in \mathcal{A} : Re\left(1+\frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D} \right\}.$$

Pommerenke [16] was the first who introduced the qth Hankel determinant for the coefficients of analytic functions f in $\mathcal{A}$ given by

$$H_q(n)(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix},$$

where $q \geq 1$ and $n \geq 1$. We note that

$$\left|H_2(3)(f)\right| = \left|a_3a_5 - a_4^2\right| \tag{1.2}$$

and

$$\left|H_3(1)(f)\right| = \left|2a_2a_3a_4 - a_3^3 - a_4^2 + a_3a_5 - a_2^2a_5\right|. \tag{1.3}$$

The non-sharp upper bound on Hankel determinant $H_3(1)(f)$ was first studied by Babalola [2] and then Raza and Malik [18] studied it for a subclass of starlike functions. The determinant $H_3(1)(f)$ for different subclass of $\mathcal{S}$ has been extensively studied in the literature. However, sharp bounds on $H_3(1)(f)$ have been obtained recently for different subclass of univalent functions by using a result in [12]. We refer [1,3–5,8–11,13,14,17,19–21,23] for sharp results for the completeness. The Hankel determinant $|H_2(3)(f)|$ was first studied by Zaprawa [24] and further examined in [7,19].

Relevant to this paper, we see that Shi et al. [22] obtained a non-sharp bound for $|H_3(1)(f)|$ for the functions in class $\mathcal{S}^*_{3\mathcal{L}}$. A sharp bound was investigated by Arif et al. [1] for the same problem in $\mathcal{S}^*_{3\mathcal{L}}$. Motivated by these works, we find sharp bound on $|H_2(3)(f)|$ for the class $\mathcal{S}^*_{3\mathcal{L}}$ and also obtain sharp bounds on the Hankel determinants $|H_3(1)(f)|$ and $|H_2(3)(f)|$ for the class $C_{3\mathcal{L}}$.

Denote by $\mathcal{P}$ a subclass of analytic functions p in $\mathbb{D}$ given by

$$p(z) = 1 + \sum_{n=1}^{\infty} \varsigma_n z^n \tag{1.4}$$

such that $Rep(z) > 0$ in $\mathbb{D}$.

We make use of the following result about functions in the class $\mathcal{P}$.

Lemma 1.2. *[12,15] Let $p \in \mathcal{P}$, and be given by (1.4), then*

$$2\varsigma_2 = \varsigma_1^2 + \delta(4-\varsigma_1^2), \tag{1.5}$$

$$4\varsigma_3 = \varsigma_1^3 + 2(4-\varsigma_1^2)\varsigma_1\delta - (4-\varsigma_1^2)\varsigma_1\delta^2 + 2(4-\varsigma_1^2)(1-|\delta|^2)\eta, \tag{1.6}$$

$$\begin{aligned} 8\varsigma_4 = {} & \varsigma_1^4 + (4-\varsigma_1^2)\delta(\varsigma_1^2(\delta^2-3\delta+3)+4\delta) - 4(4-\varsigma_1^2)(1-|\delta|^2)(\varsigma_1(\delta-1)\eta \\ & +\bar{\delta}\eta^2 - (1-|\eta|^2)\rho), \end{aligned} \tag{1.7}$$

where $|\rho| \leq 1$, $|\delta| \leq 1$ and $|\eta| \leq 1$.

2. $H_2(3)(f)$ for the class $\mathcal{S}^*_{3\mathcal{L}}$

Theorem 2.1. *Let $f \in \mathcal{S}^*_{3\mathcal{L}}$ and be given by (1.1). Then*

$$|H_2(3)(f)| \leq \frac{16}{225}. \tag{2.1}$$

The inequality is sharp for the function f_0 defined by

$$f_0(z) = z\exp\left(\frac{4z^3}{15} + \frac{z^{12}}{60}\right) = z + \frac{4}{15}z^4 + \cdots. \tag{2.2}$$

Proof. Let $f \in \mathcal{S}^*_{3\mathcal{L}}$, then by using the definition of subordination there exists a self map w in $\mathbb{D}$ with $w(0)=0$ such that

$$\frac{zf'(z)}{f(z)} = 1 + \frac{4}{5}w(z) + \frac{1}{5}w^4(z). \tag{2.3}$$

Let $p \in \mathcal{P}$. Then by using the definition of subordination, we write

$$w(z) = \frac{p(z)-1}{p(z)+1}, \qquad p \in \mathcal{P}. \tag{2.4}$$

From (2.3) and (2.4), equating coefficients we obtain

$$a_2 = \frac{2}{5}\varsigma_1, \tag{2.5}$$

$$a_3 = \frac{1}{5}\varsigma_2 - \frac{1}{50}\varsigma_1^2, \tag{2.6}$$

$$a_4 = \frac{1}{250}\varsigma_1^3 - \frac{4}{75}\varsigma_1\varsigma_2 + \frac{2}{15}\varsigma_3, \tag{2.7}$$

$$a_5 = \frac{81}{40,000}\varsigma_1^4 + \frac{53}{3000}\varsigma_2\varsigma_1^2 - \frac{7}{150}\varsigma_1\varsigma_3 - \frac{3}{100}\varsigma_2^2 + \frac{1}{10}\varsigma_4. \tag{2.8}$$

Let $f \in \mathcal{S}^*_{3\mathcal{L}}$. We see that the class $\mathcal{S}^*_{3\mathcal{L}}$ and $H_2(3)(f)$ are invariant under the rotation, we may suppose that $\varsigma_1 \in [0,2]$. With $\varsigma := \varsigma_1$, substituting (2.5)-(2.8) into (1.2), we obtain

$$H_2(3)(f) = \frac{1}{18000000}\left[\begin{array}{l} -1017\varsigma^6 + 8610\varsigma^4\varsigma_2 - 2400\varsigma^3\varsigma_3 - 36000\varsigma^2\varsigma_4 + 23200\varsigma_2^2\varsigma^2 \\ +88000\varsigma\varsigma_2\varsigma_3 + 360000\varsigma_2\varsigma_4 - 320000\varsigma_3^2 - 108000\varsigma_2^3 \end{array}\right].$$

Using (1.5)-(1.7) after some computations, we can write

$$H_2(3)(f) = \frac{1}{18000000}\left(v_1(\varsigma,\delta) + v_2(\varsigma,\delta)\eta + v_3(\varsigma,\delta)\eta^2 + \Psi(\varsigma,\delta,\eta)\rho\right),$$

where $\rho,\eta,\delta \in \overline{\mathbb{D}}$,

$$\begin{aligned} v_1(\varsigma,\delta) :=& 3988\varsigma^6 + (4-\varsigma^2)[(4-\varsigma^2)(2500\delta^4\varsigma^2 - 25200\varsigma^2\delta^2 + 15000\delta^3\varsigma^2 \\ &+ 36000\delta^3) + 72000\varsigma^2\delta^2 + 18000\varsigma^4\delta^3 - 24400\varsigma^4\delta^2 + 3705\varsigma^4\delta], \\ v_2(\varsigma,\delta) :=& -400\varsigma(4-\varsigma^2)(1-|\delta|^2)[4(45\delta-8)\varsigma^2 + 5(4-\varsigma^2)(24\delta+5\delta^2)], \\ v_3(\varsigma,\delta) :=& -2000(4-\varsigma^2)(1-|\delta|^2)(5(4-\varsigma^2)(8+|\delta|^2) + 36\varsigma^2\bar{\delta}), \\ \Psi(\varsigma,\delta,\eta) :=& 18000(4-\varsigma^2)(1-|\delta|^2)(1-|\eta|^2)(5(4-\varsigma^2)\delta + 4\varsigma^2). \end{aligned}$$

Let $u := |\delta|$, $t := |\eta|$ and utilizing $|\rho| \le 1$, we obtain

$$\begin{aligned} |H_2(3)(f)| &\le \frac{1}{18000000}\left(|v_1(\varsigma,\delta)| + |v_2(\varsigma,\delta)|t + |v_3(\varsigma,\delta)|t^2 + |\Psi(\varsigma,\delta,\eta)|\right) \\ &\le J(\varsigma,u,t), \end{aligned}$$

where

$$J(\varsigma,u,t) := \frac{1}{18000000}\left(j_1(\varsigma,u) + j_2(\varsigma,u)t + j_3(\varsigma,u)t^2 + j_4(\varsigma,u)(1-t^2)\right),$$

with

$$\begin{aligned} j_1(\varsigma,u) :=& 3988\varsigma^6 + (4-\varsigma^2)[(4-\varsigma^2)(2500u^4\varsigma^2 + 25200\varsigma^2u^2 + 15000u^3\varsigma^2 \\ &+ 36000u^3) + 72000\varsigma^2u^2 + 18000\varsigma^4u^3 + 24400\varsigma^4u^2 + 3705\varsigma^4u], \\ j_2(\varsigma,u) :=& 400\varsigma(4-\varsigma^2)(1-u^2)[4(45u+8)\varsigma^2 + 5(4-\varsigma^2)(24u+5u^2)], \\ j_3(\varsigma,u) :=& 2000(4-\varsigma^2)(1-u^2)(5(4-\varsigma^2)(8+u^2) + 36\varsigma^2u), \\ j_4(\varsigma,u) :=& 18000(4-\varsigma^2)(1-u^2)(5(4-\varsigma^2)u + 4\varsigma^2). \end{aligned}$$

Now we are to maximize $J(\varsigma,u,t)$ on the cuboid $\Lambda : [0,2]\times[0,1]\times[0,1]$. For this, we obtain the critical values on the twelve edges, in the interior of the six faces and in the interior of Λ.

I. We first show that there are no critical point in the interior of Λ.

Let $(\varsigma,u,t) \in (0,2)\times(0,1)\times(0,1)$. Differentiating $J(\varsigma,u,t)$ with respect to t, we get after simple computations

$$\begin{aligned} \frac{\partial J}{\partial t} =& \frac{1}{45000}(4-\varsigma^2)(1-u^2)[10t(u-1)(5(4-\varsigma^2)(u-8) + 36\varsigma^2) \\ &+ \varsigma(5u(4-\varsigma^2)(24+5u) + 4\varsigma^2(45u+8))]. \end{aligned}$$

So that $\frac{\partial J}{\partial t}=0$ when

$$t=\frac{\varsigma(5u(4-\varsigma^2)(24+5u)+4\varsigma^2(45u+8))}{10(1-u)(5(4-\varsigma^2)(u-8)+36\varsigma^2)}:=t_0.$$

For t_0 to be critical point, it should belong to the interval $(0,1)$, which implies that

$$4\varsigma^3(45u+8)+5\varsigma u(24+5u)(4-\varsigma^2)+50(u-1)(u-8)(4-\varsigma^2)<360(1-u)\varsigma^2 \tag{2.9}$$

and

$$\varsigma^2>\frac{20(u-8)}{5u-76}. \tag{2.10}$$

Thus for the existence of the critical points we must have solutions which satisfy both inequalities (2.9) and (2.10).

Suppose $j(u):=20(8-u)/(76-5u)$. Now $j'(u)<0$ for $(0,1)$. This shows that the function $j(u)$ is a decreasing in $(0,1)$. Hence $\varsigma^2>140/71$. A calculation shows that the equation (2.9) is satisfied for $\varsigma>1.499030727$ and $u<\frac{37}{90}$. Now we show that $J(\varsigma,u,t)<\frac{16}{225}$ in $(1.499030727,2)\times(0,\frac{37}{90})\times(0,1)$. From the above discussion, we see that $1-u^2<1$ for $u<\frac{37}{90}$, we may wite

$$\begin{aligned}
j_1(\varsigma,u)&\le 3988\varsigma^6+\left(4-\varsigma^2\right)\left(\frac{40021901}{26244}\varsigma^4+\frac{204431401}{6561}\varsigma^2+\frac{810448}{81}\right)=\phi_1(\varsigma)\\
j_2(\varsigma,u)&\le 400\varsigma(4-\varsigma^2)\left(\frac{16991}{324}\varsigma^2+\frac{17353}{81}\right):=\phi_2(\varsigma),\\
j_3(\varsigma,u)&\le 2000(4-\varsigma^2)\left(\frac{66169}{405}-\frac{42193}{1620}\varsigma^2\right):=\phi_3(\varsigma),\\
j_4(\varsigma,u)&\le 18000(4-\varsigma^2)\left(\frac{74}{9}+\frac{35}{18}\varsigma^2\right):=\phi_4(\varsigma).
\end{aligned}$$

Therefore, we have

$$J(\varsigma,u,t)\le\frac{1}{18000000}\left[\phi_1(\varsigma)+\phi_4(\varsigma)+\phi_2(\varsigma)t+\left[\phi_3(\varsigma)-\phi_4(\varsigma)\right]t^2\right]:=\Xi(\varsigma,t).$$

Obviously, it can be seen that

$$\frac{\partial\Xi}{\partial t}=\frac{1}{18000000}\left[\phi_2(\varsigma)+2\left(\phi_3(\varsigma)-\phi_4(\varsigma)\right)t\right]$$

and

$$\frac{\partial^2\Xi}{\partial t^2}=\frac{1}{9000000}\left[\phi_3(\varsigma)-\phi_4(\varsigma)\right].$$

Since $\phi_3(\varsigma)-\phi_4(\varsigma)\le 0$ for $\varsigma\in(1.499030727,2)$, we obtain that $\frac{\partial^2\Xi}{\partial t^2}\le 0$ for $t\in(0,1)$ and thus it follows that

$$\frac{\partial\Xi}{\partial t}\ge\frac{\partial\Xi}{\partial t}|_{t=1}=\frac{1}{18000000}\left[\phi_2(\varsigma)+2\left(\phi_3(\varsigma)-\phi_4(\varsigma)\right)\right]\ge 0,\quad t\in(0,1).$$

Therefore, we have

$$\Xi(\varsigma,t)\le\Xi(\varsigma,1)=\frac{1}{18000000}\left(\phi_1(\varsigma)+\phi_2(\varsigma)+\phi_3(\varsigma)\right):=\imath(\varsigma).$$

A computation shows that $\imath(\varsigma)$ has maximum value 0.05086953611 at $\varsigma\approx 1.499030727$. Thus, we have

$$J(\varsigma,u,t)<\frac{16}{225}\approx 0.071111,\quad (\varsigma,u,t)\in(1.499030727,2)\times(0,\frac{37}{90})\times(0,1).$$

Hence $J(\varsigma,u,t)<\frac{16}{225}$. This implies that J has no critical points in the interior of Λ.

II. We next consider the case for interior of the six faces of Λ.

On $\varsigma=0$, $J(\varsigma,u,t)$ takes the form

$$l_1(u,t):=J(0,u,t)=\frac{2[5(1-u^2)(u-1)(u-8)t^2-9u(3u^2-5)]}{1125},\ u,\ t\in(0,1).$$

l_1 has no critical point in $(0,1)\times(0,1)$ since

$$\frac{\partial l_1}{\partial t}=\frac{4(1-u^2)(u-1)(u-8)t}{225}\neq 0,\ u,\ t\in(0,1).$$

On $\varsigma=2$, $J(\varsigma,u,t)$ reduces to

$$J(2,u,t)=\frac{1994}{140625},\ u,\ t\in(0,1).$$

On $u=0$, $J(\varsigma,u,t)$ reduces to $J(\varsigma,0,t)$, given by

$$l_2(\varsigma,t) := \frac{2000(4-\varsigma^2)(40-19\varsigma^2)t^2 + 3200\varsigma^3(4-\varsigma^2)t + \varsigma^2(997\varsigma^4 - 18000\varsigma^2 + 72000)}{4500000},$$

where $\varsigma \in (0,2)$ and $t \in (0,1)$. We now solve the equations $\frac{\partial l_2}{\partial t} = 0$ and $\frac{\partial l_2}{\partial \varsigma} = 0$ to obtain possible points of maxima. On solving $\frac{\partial l_2}{\partial t} = 0$, we get

$$t = \frac{4\varsigma^3}{5(19\varsigma^2 - 40)} =: t_1. \tag{2.11}$$

For t, t_1 to be in $(0,1)$, it is possible only if $\varsigma > \varsigma_0$, $\varsigma_0 \approx 1.45095$. Also $\frac{\partial l_2}{\partial \varsigma} = 0$ implies

$$4000(-58 + 19\varsigma^2)t^2 + 1600\varsigma(12 - 5\varsigma^2)t + 2991\varsigma^4 - 36000\varsigma^2 + 72000 = 0. \tag{2.12}$$

By substituting (2.11) in (2.12) and simplifying, we get

$$335597\varsigma^8 - 5714320\varsigma^6 + 28294400\varsigma^4 - 55680000\varsigma^2 + 38400000 = 0. \tag{2.13}$$

We see that the equation (2.13) has solution in $(0,2)$ that is $\varsigma \approx 1.35402$. Thus, l_2 has no point of maxima in $(0,2)\times(0,1)$.

On $u = 1$, $J(\varsigma,u,t)$ reduces to

$$l_3(\varsigma,t) := J(\varsigma,1,t) = \frac{583\varsigma^6 - 193180\varsigma^4 + 683200\varsigma^2 + 576000}{18000000}, \quad \varsigma \in (0,2).$$

Solving $\frac{\partial l_3}{\partial \varsigma} = 0$, we obtain $\varsigma =: \varsigma_0 = 0$ and $\varsigma =: \varsigma_1 = \frac{2\sqrt{84467955 - 8745\sqrt{90308989}}}{1749} \approx 1.33517$ as critical points. Thus, l_3 achieves its maxima $\frac{90308989\sqrt{90308989} - 857537025929}{10324128375} \approx 0.06574$ at ς_1.

On $t = 0$, $J(\varsigma,u,t)$ reduces to

$$l_4(\varsigma,u) := J(\varsigma,u,0) = \frac{1}{18000000}\begin{pmatrix} 3988\varsigma^6 + (4-\varsigma^2)((4-\varsigma^2)(2500u^4\varsigma^2 + 15000u^3\varsigma^2 \\ -54000u^3 + 90000u + 25200\varsigma^2u^2) + 18000\varsigma^4u^3 \\ +24400\varsigma^4u^2 + 3705\varsigma^4u + 72000\varsigma^2) \end{pmatrix}.$$

A numerical method reveals that the system of equations $\frac{\partial l_4}{\partial u} = 0$ and $\frac{\partial l_4}{\partial \varsigma} = 0$ has no solution in $(0,2)\times(0,1)$.

On $t = 1$, $J(\varsigma,u,t)$ reduces to

$$l_5(\varsigma,u) := J(\varsigma,u,1) = \frac{1}{18000000}\begin{pmatrix} 3988\varsigma^6 + (4-\varsigma^2)((4-\varsigma^2)(15000u^3\varsigma^2 + 36000u^3 \\ +25200\varsigma^2u^2 - 10000u^4\varsigma + 48000u\varsigma + 2500u^4\varsigma^2 \\ +10000\varsigma u^2 - 70000u^2 - 10000u^4 - 48000u^3\varsigma \\ +80000) + 18000\varsigma^4u^3 + 24400\varsigma^4u^2 + 3705\varsigma^4u \\ +72000\varsigma^2u^2 + 12800\varsigma^3 - 12800\varsigma^3u^2 + 72000\varsigma^3u \\ -72000\varsigma^3u^3 - 72000u^3\varsigma^2 + 72000u\varsigma^2) \end{pmatrix},$$

and a similar calculation to that above shows that there is a unique solution $(\varsigma,u) \in (0.75758, 0.53964)$ to the system of equations $\frac{\partial l_5}{\partial u} = 0$ and $\frac{\partial l_5}{\partial \varsigma} = 0$ in $(0,2)\times(0,1)$. Thus, $l_5(\varsigma,u) \le 0.06493$.

III. On the vertices of Λ, we have

$$J(0,0,0) = 0, \quad J(0,0,1) = \frac{16}{225}, \quad J(0,1,0) = \frac{4}{125}, \quad J(0,1,1) = \frac{4}{125},$$
$$J(2,0,0) = J(2,0,1) = J(2,1,0) = J(2,1,1) = \frac{1994}{140625}.$$

IV. Lastly, we discuss the maxima of $J(\varsigma,u,t)$ on the 12 edges of Λ.

$$J(\varsigma,0,0) = \frac{997\varsigma^6 - 18000\varsigma^4 + 72000\varsigma^2}{4500000} \le J(\lambda_1,0,0)$$
$$= \frac{16096}{74550675}\sqrt{7545} - \frac{288}{994009} \approx 0.01846, \quad \varsigma \in (0,2),$$

where

$$\varsigma =: \lambda_1 = \frac{2}{997}\sqrt{1495500 - 9970\sqrt{7545}} \approx 1.59158.$$

$$J(\varsigma,0,1) = \frac{997\varsigma^6 - 3200\varsigma^5 + 20000\varsigma^4 + 12800\varsigma^3 - 160000\varsigma^2 + 320000}{4500000} \le J(0,0,1)$$
$$= \frac{16}{225} \approx 0.07111, \quad \varsigma \in (0,2).$$

$$J(\varsigma,1,0) = \frac{583\varsigma^6 - 193180\varsigma^4 + 683200\varsigma^2 + 576000}{18000000} \le J(\lambda_2,1,0)$$
$$= \frac{90308989\sqrt{90308989} - 857537025929}{10324128375} \approx 0.06574, \quad \varsigma \in (0,2),$$

where

$$\varsigma := \lambda_2 = \frac{2}{1749}\sqrt{84467955 - 8745\sqrt{90308989}} \approx 1.33517.$$

$$J(0,u,0) = \frac{2u(5-3u^2)}{125} \le J(0,\frac{\sqrt{5}}{3},0) = \frac{4\sqrt{5}}{225} \approx 0.03975, \quad u \in (0,1)$$

$$J(0,u,1) = \frac{2(-5u^4+18u^3-35u^2+40)}{1125} \le J(0,0,1) = \frac{16}{225}, \quad u \in (0,1).$$

$$J(2,u,0) = \frac{1994}{140625}, \quad u \in (0,1).$$

$$J(2,u,1) = \frac{1994}{140625}, \quad u \in (0,1).$$

$$J(0,0,t) = \frac{16}{225}t^2 \le \frac{16}{225}, \quad t \in (0,1).$$

$$J(0,1,t) = \frac{4}{125} \approx 0.03200, \quad t \in (0,1).$$

$$J(2,0,t) = \frac{1994}{140625}, \quad t \in (0,1).$$

$$J(2,1,t) = \frac{1994}{140625}, \quad t \in (0,1).$$

Since all cases have been dealt with, (2.1) holds. To see that (2.1) is sharp, consider f_0 given in (2.2), which is equivalent to choosing $a_3 = a_5 = 0$ and $a_4 = \frac{4}{15}$, which from (1.2) gives $|H_2(3)(f)| = \frac{16}{225}$. This completes the proof. □

3. $H_3(1)(f)$ for the class $C_{3\mathcal{L}}$

Theorem 3.1. *Let $f \in C_{3\mathcal{L}}$ and be of the form (1.1). Then*

$$\left|H_3(1)(f)\right| \le \frac{1}{225}. \tag{3.1}$$

This inequality is sharp for f_1 given by

$$f_1(z) = \int_0^z \left(\exp\left(\frac{4u^3}{15} + \frac{u^{12}}{60}\right)\right) du = z + \frac{1}{15}z^4 + \cdots. \tag{3.2}$$

Proof. Let $f \in C_{3\mathcal{L}}$. Then using the definition of subordination, we have

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{4}{5}w(z) + \frac{1}{5}w^4(z), \tag{3.3}$$

where w is analytic with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{D}$. Let p be given by (1.4). Using (3.3) and (2.4), we obtain

$$a_2 = \frac{1}{5}\varsigma_1, \tag{3.4}$$

$$a_3 = \frac{1}{15}\varsigma_2 - \frac{1}{150}\varsigma_1^2, \tag{3.5}$$

$$a_4 = \frac{1}{1000}\varsigma_1^3 - \frac{1}{75}\varsigma_1\varsigma_2 + \frac{1}{30}\varsigma_3, \tag{3.6}$$

$$a_5 = \frac{81}{200,000}\varsigma_1^4 + \frac{53}{15000}\varsigma_2\varsigma_1^2 - \frac{7}{750}\varsigma_1\varsigma_3 - \frac{3}{500}\varsigma_2^2 + \frac{1}{50}\varsigma_4. \tag{3.7}$$

Since the class $C_{3\mathcal{L}}$ is invariant under the rotation, we again assume that $\varsigma := \varsigma_1 \in [0,2]$ and substituting from (3.4)-(3.7) into (1.3), we obtain

$$H_3(1)(f) = \frac{1}{9990000000}\begin{bmatrix} -222481\varsigma^6 - 578310\varsigma^4\varsigma_2 + 2797200\varsigma^3\varsigma_3 - 9324000\varsigma^2\varsigma_4 + 710400\varsigma_2^2\varsigma^2 \\ +11544000\varsigma\varsigma_2\varsigma_3 + 13320000\varsigma_2\varsigma_4 - 11100000\varsigma_3^2 - 6956000\varsigma_2^3 \end{bmatrix}.$$

Using the equalities (1.5)-(1.7) and after some simple computations, we get

$$H_3(1)(f) = \frac{1}{9990000000}\left(v_1(\varsigma,\delta) + v_2(\varsigma,\delta)\eta + v_3(\varsigma,\delta)\eta^2 + \Psi(\varsigma,\delta,\eta)\rho\right),$$

where $\rho, \eta, \delta \in \overline{\mathbb{D}}$,

$$\begin{aligned} v_1(\varsigma,\delta) :=& -87986\varsigma^6 + (4-\varsigma^2)[(4-\varsigma^2)(177600\varsigma^2\delta^2 - 148000\delta^3 + 138750\delta^4\varsigma^2 \\ &- 296000\delta^3\varsigma^2) - 1332000\varsigma^2\delta^2 - 333000\varsigma^4\delta^3 + 244200\varsigma^4\delta^2 + 243645\varsigma^4\delta], \\ v_2(\varsigma,\delta) :=& -22200\varsigma(4-\varsigma^2)(1-|\delta|^2)[-4(15\delta+2)\varsigma^2 - 5(4-\varsigma^2)(6\delta-5\delta^2)], \end{aligned}$$

$$\upsilon_3(\varsigma,\delta) := 111000(4-\varsigma^2)(1-|\delta|^2)(-5(4-\varsigma^2)(5+|\delta|^2)+12\varsigma^2\bar{\delta}),$$
$$\Psi(\varsigma,\delta,\eta) := 666000(4-\varsigma^2)(1-|\delta|^2)(1-|\eta|^2)(5(4-\varsigma^2)\delta-2\varsigma^2).$$

Choosing $u := |\delta|$, $t := |\eta|$ and utilizing $|\rho| \le 1$, we can write

$$\begin{aligned}|H_3(1)(f)| &\le \frac{1}{9990000000}\left(|\upsilon_1(\varsigma,\delta)|+|\upsilon_2(\varsigma,\delta)|t+|\upsilon_3(\varsigma,\delta)|t^2+|\Psi(\varsigma,\delta,\eta)|\right)\\ &\le L(\varsigma,u,t),\end{aligned}$$

where

$$L(\varsigma,u,t) := \frac{1}{9990000000}\left(g_1(\varsigma,u)+g_2(\varsigma,u)t+g_3(\varsigma,u)t^2+g_4(\varsigma,u)(1-t^2)\right),$$

with

$$\begin{aligned}g_1(\varsigma,u) :=& 87986\varsigma^6+(4-\varsigma^2)[(4-\varsigma^2)(177600\varsigma^2u^2+148000u^3+138750u^4\varsigma^2\\ &+296000u^3\varsigma^2)+1332000\varsigma^2u^2+333000\varsigma^4u^3+244200\varsigma^4u^2+243645\varsigma^4u],\\ g_2(\varsigma,u) :=& 22200\varsigma(4-\varsigma^2)(1-u^2)[4(15u+2)\varsigma^2+5(4-\varsigma^2)(6u+5u^2)],\\ g_3(\varsigma,u) :=& 111000(4-\varsigma^2)(1-u^2)(5(4-\varsigma^2)(5+u^2)+12\varsigma^2u),\\ g_4(\varsigma,u) :=& 666000(4-\varsigma^2)(1-u^2)(5(4-\varsigma^2)u+2\varsigma^2).\end{aligned}$$

Now we maximize $L(\varsigma,u,t)$ on the cuboid $\Lambda : [0,2]\times[0,1]\times[0,1]$. For this, we find the maximum value of Λ, on the twelve edges and in the interior of the six faces of Λ.

I. We first show that there are no critical point in the interior of Λ.

Let $(\varsigma,u,t)\in(0,2)\times(0,1)\times(0,1)$. Differentiating $L(\varsigma,u,t)$ with respect to t, we get

$$\begin{aligned}\frac{\partial L}{\partial t} =& \frac{1}{450000}(4-\varsigma^2)(1-u^2)[10t(u-1)(5(4-\varsigma^2)(u-5)+12\varsigma^2)\\ &+\varsigma(5u(4-\varsigma^2)(6+5u)+4\varsigma^2(15u+2))].\end{aligned}$$

So that $\frac{\partial L}{\partial t}=0$ when

$$t=\frac{\varsigma(5u(4-\varsigma^2)(6+5u)+4\varsigma^2(15u+2))}{10(1-u)(5(4-\varsigma^2)(u-5)+12\varsigma^2)} := t_0.$$

If t_0 is a critical point inside Λ, then $t_0\in(0,1)$, which is possible only if

$$4\varsigma^3(15u+2)+5\varsigma u(6+5u)(4-\varsigma^2)+50(u-1)(u-5)(4-\varsigma^2) < 120(1-u)\varsigma^2 \tag{3.8}$$

and

$$\varsigma^2 > \frac{20(u-5)}{5u-37}. \tag{3.9}$$

Thus for the existence of the critical points we must have solutions which satisfy both inequalities (3.8) and (3.9).

Suppose $g(u) := 20(5-u)/(37-5u)$. Now $g'(u)<0$ for $(0,1)$. This shows that the function $g(u)$ is a decreasing in $(0,1)$. Hence $\varsigma^2 > 5/2$. A calculation shows that the equation (3.8) is satisfied for $\varsigma > 1.674585441$ and $u < \frac{13}{30}$. Now we show that $L(\varsigma,u,t) < \frac{1}{225}$ in $(1.674585441,2)\times(0,\frac{13}{30})\times(0,1)$. From the above discussion, we see that $1-u^2<1$ for $u<\frac{13}{30}$, we may wite

$$\begin{aligned}l_1(\varsigma,u) &\le 87986\varsigma^6+(4-\varsigma^2)\left(\frac{25100023}{216}\varsigma^4+\frac{8772959}{18}\varsigma^2+\frac{1300624}{27}\right)=\phi_1(\varsigma)\\ l_2(\varsigma,u) &\le 22200\varsigma(4-\varsigma^2)\left(\frac{587}{36}\varsigma^2+\frac{637}{9}\right) := \phi_2(\varsigma),\\ l_3(\varsigma,u) &\le 111000(4-\varsigma^2)\left(\frac{4669}{45}-\frac{3733}{180}\varsigma^2\right) := \phi_3(\varsigma),\\ l_4(\varsigma,u) &\le 666000(4-\varsigma^2)\left(\frac{26}{3}-\frac{1}{6}\varsigma^2\right) := \phi_4(\varsigma).\end{aligned}$$

Therefore, we have

$$L(\varsigma,u,t) \le \frac{1}{9990000000}\left[\phi_1(\varsigma)+\phi_4(\varsigma)+\phi_2(\varsigma)t+\left[\phi_3(\varsigma)-\phi_4(\varsigma)\right]t^2\right] := \mathcal{L}(\varsigma,t).$$

Obviously, it can be seen that

$$\frac{\partial\mathcal{L}}{\partial t}=\frac{1}{9990000000}\left[\phi_2(\varsigma)+2\left(\phi_3(\varsigma)-\phi_4(\varsigma)\right)t\right]$$

and

$$\frac{\partial^2 \mathcal{L}}{\partial t^2} = \frac{2}{9990000000}\left[\phi_3(\varsigma) - \phi_4(\varsigma)\right].$$

Since $\phi_3(\varsigma) - \phi_4(\varsigma) \le 0$ for $\varsigma \in (1.674585441, 2)$, we obtain that $\frac{\partial^2 \mathcal{L}}{\partial t^2} \le 0$ for $t \in (0,1)$ and thus it follows that

$$\frac{\partial \mathcal{L}}{\partial t} \ge \frac{\partial \mathcal{L}}{\partial t}|_{t=1} = \frac{1}{9990000000}\left[\phi_2(\varsigma) + 2\left(\phi_3(\varsigma) - \phi_4(\varsigma)\right)\right] \ge 0, \quad t \in (0,1).$$

Therefore, we have

$$\mathcal{L}(\varsigma, t) \le \mathcal{L}(\varsigma, 1) = \frac{1}{9990000000}\left(\phi_1(\varsigma) + \phi_2(\varsigma) + \phi_3(\varsigma)\right) := l(\varsigma).$$

It follows that the function $l(\varsigma)$ has maximum value 0.001597206860 at $\varsigma \approx 1.674585441$. Thus, we have

$$L(\varsigma, u, t) < \frac{1}{225} \approx 0.004444444444, \quad (\varsigma, u, t) \in (1.674585441, 2) \times (0, \frac{13}{30}) \times (0, 1).$$

Hence $L(\varsigma, u, t) < \frac{1}{225}$. This implies that L has no point of maxima interior of Λ.

II. We next consider the interior of the six faces of the cuboid Λ.

On the face $\varsigma = 0$, $L(\varsigma, u, t)$ reduces to

$$k_1(u, t) := L(0, u, t) = \frac{15(1-u^2)(u-1)(u-5)t^2 - 2u(43u^2 - 45)}{16875}, \; u, \; t \in (0,1).$$

k_1 has no point of maxima in $(0,1) \times (0,1)$ since

$$\frac{\partial k_1}{\partial t} = \frac{2(1-u^2)(u-1)(u-5)t}{1125} \ne 0, \; u, \; t \in (0,1).$$

On the face $\varsigma = 2$, $L(\varsigma, u, t)$ takes the form

$$L(2, u, t) = \frac{1189}{2109375}, \; u, \; t \in (0,1).$$

On the face $u = 0$, $L(\varsigma, u, t)$ reduces to $L(\varsigma, 0, t)$, given by

$$k_2(\varsigma, t) := \frac{1500(4-\varsigma^2)(100 - 37\varsigma^2)t^2 + 2400\varsigma^3(4-\varsigma^2)t + \varsigma^2(1189\varsigma^4 - 18000\varsigma^2 + 72000)}{135000000},$$

where $\varsigma \in (0,2)$ and $t \in (0,1)$. We solve $\frac{\partial k_2}{\partial t} = 0$ and $\frac{\partial k_2}{\partial \varsigma} = 0$ to obtain the possible point of maxima. The equation $\frac{\partial k_2}{\partial t} = 0$ gives

$$t = \frac{4\varsigma^3}{5(37\varsigma^2 - 100)} =: t_1. \tag{3.10}$$

For t, t_1 to be in $(0,1)$, it is possible only if $\varsigma > \varsigma_0$, $\varsigma_0 \approx 1.64399$. Also $\frac{\partial k_2}{\partial \varsigma} = 0$ implies

$$1000(37\varsigma^2 - 124)t^2 - 400\varsigma(5\varsigma^2 - 12)t + 1189\varsigma^4 - 12000\varsigma^2 + 24000 = 0. \tag{3.11}$$

By putting (3.10) in (3.11) and simplifying, we get

$$1592221\varsigma^8 - 25003880\varsigma^6 + 133162000\varsigma^4 - 297600000\varsigma^2 + 240000000 = 0. \tag{3.12}$$

We see that the equation (3.12) has solution in $(0,2)$ that is $\varsigma \approx 1.54247$. Thus, k_2 has no point of maxima in $(0,2) \times (0,1)$.

On the face $u = 1$, $L(\varsigma, u, t)$ reduces to

$$k_3(\varsigma, t) := L(\varsigma, 1, t) = \frac{-3257\varsigma^6 - 75660\varsigma^4 + 376800\varsigma^2 + 64000}{270000000}, \quad \varsigma \in (0,2).$$

Solving $\frac{\partial k_3}{\partial \varsigma} = 0$, we get $\varsigma =: \varsigma_0 = 0$ and $\varsigma =: \varsigma_1 = \frac{2\sqrt{16285\sqrt{2612819} - 20535385}}{3257} \approx 1.47733$ as critical points. Thus, k_3 achieves its maxima $\frac{2612819\sqrt{2612819}}{179010826875} - \frac{1299047884}{59670275625} \approx 0.00182$ at ς_1.

On the face $t = 0$, $L(\varsigma, u, t)$ takes the form

$$k_4(\varsigma, u) := L(\varsigma, u, 0) = \frac{1}{9990000000}\begin{pmatrix} 87986\varsigma^6 + (4-\varsigma^2)((4-\varsigma^2)(296000u^3\varsigma^2 + 177600u^2\varsigma^2 \\ +138750u^4\varsigma^2 + 3330000u - 3182000u^3) + 333000\varsigma^4 u^3 \\ +243645\varsigma^4 u + 244200\varsigma^4 u^2 + 1332000\varsigma^2) \end{pmatrix}.$$

A numerical approach indicates the system of equations $\frac{\partial k_4}{\partial u} = 0$ and $\frac{\partial k_4}{\partial \varsigma} = 0$ in $(0,2) \times (0,1)$ has no unique solution for $(\varsigma, u) \in (1.28377, 0.86651)$. Thus, $k_4(\varsigma, u) \le 0.00177$.

On the face $t = 1$, $L(\varsigma, u, t)$ reduces to

$$k_5(\varsigma,u) := L(\varsigma,u,1) = \frac{1}{9990000000}\left(\begin{array}{l} 87986\varsigma^6 + (4-\varsigma^2)((4-\varsigma^2)(555000\varsigma u^2 - 666000\varsigma u^3 \\ +666000\varsigma u + 148000u^3 + 177600u^2\varsigma^2 - 555000\varsigma u^4 \\ +296000\varsigma^2 u^3 - 2220000u^2 - 555000u^4 + 138750u^4\varsigma^2 \\ +2775000) - 1332000\varsigma^3 u^3 + 177600\varsigma^3 - 177600\varsigma^3 u^2 \\ +1332000\varsigma^3 u + 333000\varsigma^4 u^3 + 243645\varsigma^4 u + 244200\varsigma^4 u^2 \\ +1332000u^2\varsigma^2 - 1332000\varsigma^2 u^3 + 1332000\varsigma^2 u) \end{array}\right),$$

and a similar approach shows that the system of equations $\frac{\partial k_5}{\partial u} = 0$ and $\frac{\partial k_5}{\partial \varsigma} = 0$ has no solution in $(0,2)\times(0,1)$.
III. On the vertices of Λ, we have

$$L(0,0,0) = 0, \quad L(0,0,1) = \frac{1}{225}, \quad L(0,1,0) = \frac{4}{16875}, \quad L(0,1,1) = \frac{4}{16875},$$
$$L(2,0,0) = L(2,0,1) = L(2,1,0) = L(2,1,1) = \frac{1189}{2109375}.$$

IV. Finally we obtain possible points of maxima of $L(\varsigma, u, t)$ on the 12 edges of Λ.

$$\begin{aligned} L(\varsigma,0,0) &= \frac{1189\varsigma^6 - 18000\varsigma^4 + 72000\varsigma^2}{135000000} \le L(\lambda_1,0,0) \\ &= \frac{24288}{1570009} + \frac{7904}{117750675}\sqrt{3705} \approx 0.00064, \quad \varsigma \in (0,2). \end{aligned}$$

where

$$\varsigma =: \lambda_1 = \frac{2}{1253}\sqrt{1879500 - 12530\sqrt{3705}} \approx 1.65786.$$
$$\begin{aligned} L(\varsigma,0,1) &= \frac{1189\varsigma^6 - 2400\varsigma^5 + 37500\varsigma^4 + 9600\varsigma^3 - 300000\varsigma^2 + 600000}{135000000} \le L(0,0,1) \\ &= \frac{1}{225} \approx 0.00444, \quad \varsigma \in (0,2). \\ L(\varsigma,1,0) &= \frac{-3257\varsigma^6 - 75660\varsigma^4 + 376800\varsigma^2 + 64000}{270000000} \le L(\lambda_2,1,0) \\ &= \frac{2612819\sqrt{2612819}}{179010826875} - \frac{1299047884}{59670275625} \approx 0.00182, \quad \varsigma \in (0,2), \end{aligned}$$

where

$$\varsigma := \lambda_2 = \frac{2\sqrt{16285\sqrt{2612819} - 20535385}}{3257} \approx 1.47733.$$
$$L(0,u,0) = \frac{2u(45-43u^2)}{16875} \le L(0, \frac{\sqrt{645}}{43}, 0) = \frac{4\sqrt{645}}{48375} \approx 0.00209, \quad u \in (0,1)$$
$$L(0,u,1) = \frac{-15u^4 + 4u^3 - 60u^2 + 75}{16875} \le L(0,0,1) = \frac{1}{225}, \quad u \in (0,1).$$
$$L(2,u,0) = \frac{1189}{2109375}, \quad u \in (0,1).$$
$$L(2,u,1) = \frac{1189}{2109375}, \quad u \in (0,1).$$
$$L(0,0,t) = \frac{1}{225}t^2 \le \frac{1}{225}, \quad t \in (0,1).$$
$$L(0,1,t) = \frac{4}{16875} \approx 0.00024, \quad t \in (0,1).$$
$$L(2,0,t) = \frac{1189}{2109375}, \quad t \in (0,1).$$
$$L(2,1,t) = \frac{1189}{2109375}, \quad t \in (0,1).$$

Since all cases have been dealt with, (3.1) holds. To see that (3.1) is sharp, consider f_0 given in (3.2), which is equivalent to choosing $a_2 = a_3 = a_5 = 0$ and $a_4 = \frac{1}{15}$, which from (1.3) gives $|H_3(1)(f)| = \frac{1}{225}$. This completes the proof. □

4. $H_2(3)(f)$ for the class $C_{3\mathcal{L}}$

Theorem 4.1. *Let $f \in C_{3\mathcal{L}}$ and be given by (1.1). Then*

$$|H_2(3)(f)| \le \frac{1}{225}. \tag{4.1}$$

The inequality is sharp for the function f_1 defined by (3.2).

Proof. We use the same method as in previous Section. Let $f \in \mathcal{C}_{3\mathcal{L}}$. Since the class $\mathcal{C}_{3\mathcal{L}}$ and the functional $H_2(3)(f)$ are invariant under the rotation, we can assume that ς_1 lies in the interval $[0,2]$. With $\varsigma := \varsigma_1$, substituting (3.4)-(3.7) into (1.2), we obtain

$$H_2(3)(f) = \frac{1}{90000000}\begin{bmatrix} -333\varsigma^6 + 2710\varsigma^4\varsigma_2 - 400\varsigma^3\varsigma_3 - 12000\varsigma^2\varsigma_4 + 8800\varsigma_2^2\varsigma^2 \\ +24000\varsigma\varsigma_2\varsigma_3 + 120000\varsigma_2\varsigma_4 - 100000\varsigma_3^2 - 36000\varsigma_2^3 \end{bmatrix}.$$

Using (1.5)-(1.7) after some computations, we obtain

$$H_2(3)(f) = \frac{1}{90000000}\left(v_1(\varsigma,\delta) + v_2(\varsigma,\delta)\eta + v_3(\varsigma,\delta)\eta^2 + \Psi(\varsigma,\delta,\eta)\rho\right),$$

where $\rho, \eta, \delta \in \overline{\mathbb{D}}$,

$$\begin{aligned} v_1(\varsigma,\delta) :=& 1372\varsigma^6 + (4-\varsigma^2)[(4-\varsigma^2)(1250\delta^4\varsigma^2 + 12000\delta^3 - 7800\delta^2\varsigma^2 \\ &+ 4000\delta^3\varsigma^2) + 24000\delta^2\varsigma^2 - 8400\varsigma^4\delta^2 + 6000\varsigma^4\delta^3 + 1555\varsigma^4\delta], \\ v_2(\varsigma,\delta) :=& -200\varsigma(4-\varsigma^2)(1-|\delta|^2)[24(5\delta-1)\varsigma^2 + 5(4-\varsigma^2)(14\delta+5\delta^2)], \\ v_3(\varsigma,\delta) :=& -1000(4-\varsigma^2)(1-|\delta|^2)(5(4-\varsigma^2)(5+|\delta|^2) + 24\varsigma^2\bar{\delta}), \\ \Psi(\varsigma,\delta,\eta) :=& 6000(4-\varsigma^2)(1-|\delta|^2)(1-|\eta|^2)(5(4-\varsigma^2)\delta + 4\varsigma^2). \end{aligned}$$

By choosing $u := |\delta|$, $t := |\eta|$ and utilizing $|\rho| \le 1$, we get

$$\begin{aligned} |H_2(3)(f)| &\le \frac{1}{90000000}\left(|v_1(\varsigma,\delta)| + |v_2(\varsigma,\delta)|t + |v_3(\varsigma,\delta)|t^2 + |\Psi(\varsigma,\delta,\eta)|\right) \\ &\le K(\varsigma,u,t), \end{aligned}$$

where

$$K(\varsigma,u,t) := \frac{1}{90000000}\left(k_1(\varsigma,u) + k_2(\varsigma,u)t + k_3(\varsigma,u)t^2 + k_4(\varsigma,u)(1-t^2)\right),$$

with

$$\begin{aligned} k_1(\varsigma,u) :=& 1372\varsigma^6 + (4-\varsigma^2)[(4-\varsigma^2)(1250u^4\varsigma^2 + 12000u^3 + 7800u^2\varsigma^2 \\ &+ 4000u^3\varsigma^2) + 24000u^2\varsigma^2 + 8400\varsigma^4u^2 + 6000\varsigma^4u^3 + 1555\varsigma^4u], \\ k_2(\varsigma,u) :=& 200\varsigma(4-\varsigma^2)(1-u^2)[24(5u+1)\varsigma^2 + 5(4-\varsigma^2)(14u+5u^2)], \\ k_3(\varsigma,u) :=& 1000(4-\varsigma^2)(1-u^2)(5(4-\varsigma^2)(5+u^2) + 24\varsigma^2u), \\ k_4(\varsigma,u) :=& 6000(4-\varsigma^2)(1-u^2)(5(4-\varsigma^2)u + 4\varsigma^2). \end{aligned}$$

We only need to maximize $K(\varsigma,u,t)$ on the cuboid $\Lambda : [0,2]\times[0,1]\times[0,1]$. For this, we obtain the maximum values in the interior of Λ, on the twelve edges and in the interior of the six faces of Λ.

I. We first show that there are no critical point in the interior of Λ.

Let $(\varsigma,u,t) \in (0,2)\times(0,1)\times(0,1)$. Differentiating $K(\varsigma,u,t)$ with respect to t, we obtain after some simplification

$$\begin{aligned} \frac{\partial K}{\partial t} =& \frac{1}{450000}(4-\varsigma^2)(1-u^2)[10t(u-1)(5(4-\varsigma^2)(u-5) + 24\varsigma^2) \\ &+ \varsigma(5u(4-\varsigma^2)(14+5u) + 24\varsigma^2(5u+1))]. \end{aligned}$$

So that $\frac{\partial K}{\partial t} = 0$ when

$$t = \frac{\varsigma(5u(4-\varsigma^2)(14+5u) + 24\varsigma^2(5u+1))}{10(1-u)(5(4-\varsigma^2)(u-5) + 24\varsigma^2)} := t_0.$$

If t_0 is a critical point inside Λ, then $t_0 \in (0,1)$, which is possible only if

$$24\varsigma^3(5u+1) + 5\varsigma u(14+5u)(4-\varsigma^2) + 50(u-1)(u-5)(4-\varsigma^2) < 240(1-u)\varsigma^2 \tag{4.2}$$

and

$$\varsigma^2 > \frac{20(u-5)}{5u-49}. \tag{4.3}$$

Thus for the existence of the critical points we must have solutions which satisfy both inequalities (4.2) and (4.3).

Suppose $h(u) := 20(5-u)/(49-5u)$. Now $h'(u) < 0$ for $(0,1)$. This shows that the function $h(u)$ is a decreasing in $(0,1)$. Hence $\varsigma^2 > 20/11$. A calculation shows that the equation (4.2) is satisfied for $\varsigma > 1.483482934$ and $u < \frac{2}{5}$. Now we show that $K(\varsigma,u,t) < \frac{1}{225}$ in $(1.483482934, 2) \times (0, \frac{2}{5}) \times (0,1)$. From the above discussion, we see that $1-u^2 < 1$ for $u < \frac{2}{5}$, we may wite

$$\begin{aligned}
&k_1(\varsigma,u) \le 1372\varsigma^6 + (4-\varsigma^2)\left(814\varsigma^4 + 9216\varsigma^2 + 3072\right) = \phi_1(\varsigma)\\
&k_2(\varsigma,u) \le 200\varsigma(4-\varsigma^2)\left(40\varsigma^2 + 128\right) := \phi_2(\varsigma),\\
&k_3(\varsigma,u) \le 1000(4-\varsigma^2)\left(\frac{516}{5} - \frac{81}{5}\varsigma^2\right) := \phi_3(\varsigma),\\
&k_4(\varsigma,u) \le 6000(4-\varsigma^2)\left(8 + 2\varsigma^2\right) := \phi_4(\varsigma).
\end{aligned}$$

Therefore, we have

$$K(\varsigma,u,t) \le \frac{1}{90000000}\left[\phi_1(\varsigma) + \phi_4(\varsigma) + \phi_2(\varsigma)t + \left[\phi_3(\varsigma) - \phi_4(\varsigma)\right]t^2\right] := \mathfrak{K}(\varsigma,t).$$

Obviously, it can be seen that

$$\frac{\partial \mathfrak{K}}{\partial t} = \frac{1}{90000000}\left[\phi_2(\varsigma) + 2\left(\phi_3(\varsigma) - \phi_4(\varsigma)\right)t\right]$$

and

$$\frac{\partial^2 \mathfrak{K}}{\partial t^2} = \frac{2}{90000000}\left[\phi_3(\varsigma) - \phi_4(\varsigma)\right].$$

Since $\phi_3(\varsigma) - \phi_4(\varsigma) \le 0$ for $\varsigma \in (1.483482934, 2)$, we obtain that $\frac{\partial^2 \mathfrak{K}}{\partial t^2} \le 0$ for $t \in (0,1)$ and thus it follows that

$$\frac{\partial \mathfrak{K}}{\partial t} \ge \frac{\partial \mathfrak{K}}{\partial t}\Big|_{t=1} = \frac{1}{90000000}\left[\phi_2(\varsigma) + 2\left(\phi_3(\varsigma) - \phi_4(\varsigma)\right)\right] \ge 0, \quad t \in (0,1).$$

Therefore, we have

$$\mathfrak{K}(\varsigma,t) \le \mathfrak{K}(\varsigma,1) = \frac{1}{90000000}\left(\phi_1(\varsigma) + \phi_2(\varsigma) + \phi_3(\varsigma)\right) := k(\varsigma).$$

We see that the function $k(\varsigma)$ has maximum value 0.003339998897 at $\varsigma \approx 1.483482934$. Thus, we have

$$K(\varsigma,u,t) < \frac{1}{225} \approx 0.004444444444, \quad (\varsigma,u,t) \in (1.483482934, 2) \times (0, \frac{2}{5}) \times (0,1).$$

Hence $K(\varsigma,u,t) < \frac{1}{225}$. This implies that K has no critical points in the interior of Λ.

II. We next consider the interior of the six faces of the cuboid Λ.

On $\varsigma = 0$, $K(\varsigma,u,t)$ takes the form

$$m_1(u,t) := K(0,u,t) = \frac{5(1-u^2)(u-1)(u-5)t^2 - 6u(3u^2-5)}{5625}, \ u, \ t \in (0,1).$$

m_1 has no critical point in $(0,1) \times (0,1)$ since

$$\frac{\partial m_1}{\partial t} = \frac{2(1-u^2)(u-1)(u-5)t}{1125} \ne 0, \ u, \ t \in (0,1).$$

On $\varsigma = 2$, $K(\varsigma,u,t)$ reduces to

$$K(2,u,t) = \frac{686}{703125}, \ u, \ t \in (0,1).$$

On $u = 0$, $K(\varsigma,u,t)$ reduces to $K(\varsigma,0,t)$, given by

$$m_2(\varsigma,t) := \frac{250(4-\varsigma^2)(100-49\varsigma^2)t^2 + 1200\varsigma^3(4-\varsigma^2)t + \varsigma^2(343\varsigma^4 - 6000\varsigma^2 + 24000)}{22500000},$$

where $\varsigma \in (0,2)$ and $t \in (0,1)$. To find the points of maxima, we solve $\frac{\partial m_2}{\partial t} = 0$ and $\frac{\partial m_2}{\partial \varsigma} = 0$. From $\frac{\partial m_2}{\partial t} = 0$, we get

$$t = \frac{12\varsigma^3}{5(49\varsigma^2 - 100)} =: t_1. \tag{4.4}$$

For t, t_1 to be in $(0,1)$, it is possible only if $\varsigma > \varsigma_0$, $\varsigma_0 \approx 1.42857$. A calculation shows that $\frac{\partial m_2}{\partial \varsigma} = 0$ implies

$$500(-148 + 49\varsigma^2)t^2 + 600\varsigma(12 - 5\varsigma^2)t + 1029\varsigma^4 - 12000\varsigma^2 + 24000 = 0. \tag{4.5}$$

By putting (4.4) in (4.5) and simplifying, we obtain

$$752983\varsigma^8 - 12585240\varsigma^6 + 61262000\varsigma^4 - 118400000\varsigma^2 + 80000000 = 0. \tag{4.6}$$

The equation (4.6) has solution in $(0,2)$ that is $\varsigma \approx 1.32599$. Thus, m_2 has no point of maxima in $(0,2)\times(0,1)$.

On $u=1$, $K(\varsigma,u,t)$ reduces to

$$m_3(\varsigma,t) := K(\varsigma,1,t) = \frac{-1533\varsigma^6 - 52580\varsigma^4 + 208800\varsigma^2 + 192000}{90000000}, \quad \varsigma \in (0,2).$$

Solving $\frac{\partial m_3}{\partial \varsigma}=0$, we obtain critical points at $\varsigma =: \varsigma_0 = 0$ and $\varsigma =: \varsigma_1 = \frac{2\sqrt{2555\sqrt{9312319}-6717095}}{1533} \approx 1.35567$. Thus, m_3 achieves its maxima $\frac{9312319\sqrt{9312319}-26876349046}{356919766875} \approx 0.00432$ at ς_1.

On $t=0$, $K(\varsigma,u,t)$ reduces to

$$m_4(\varsigma,u) := K(\varsigma,u,0) = \frac{1}{90000000}\begin{pmatrix} 1372\varsigma^6 + (4-\varsigma^2)((4-\varsigma^2)(1250u^4\varsigma^2 + 7800\varsigma^2u^2 \\ +4000u^3\varsigma^2 + 30000u - 18000u^3) + 8400\varsigma^4u^2 \\ +6000\varsigma^4u^3 + 1555\varsigma^4u + 24000\varsigma^2) \end{pmatrix}.$$

A numerical approach shows that the system of equations $\frac{\partial m_4}{\partial u}=0$ and $\frac{\partial m_4}{\partial \varsigma}=0$ has no solution in $(0,2)\times(0,1)$.

On $t=1$, $K(\varsigma,u,t)$ reduces to

$$m_5(\varsigma,u) := K(\varsigma,u,1) = \frac{1}{90000000}\begin{pmatrix} 1372\varsigma^6 + (4-\varsigma^2)((4-\varsigma^2)(12000u^3 - 14000\varsigma u^3 \\ +14000\varsigma u - 5000\varsigma u^4 + 7800\varsigma^2u^2 + 1250u^4\varsigma^2 \\ +5000\varsigma u^2 - 20000u^2 - 5000u^4 + 4000\varsigma^2u^3 \\ +25000) + 8400\varsigma^4u^2 + 4800\varsigma^3 - 4800\varsigma^3u^2 \\ +24000\varsigma^2u^2 - 24000\varsigma^2u^3 + 6000\varsigma^4u^3 + 1555\varsigma^4u \\ -24000\varsigma^3u^3 + 24000\varsigma^3u + 24000\varsigma^2u) \end{pmatrix},$$

and a similar calculation to that above shows that there is a unique solution $(\varsigma,u)\in(0.64779, 0.48837)$ to the system of equations $\frac{\partial m_5}{\partial u}=0$ and $\frac{\partial m_5}{\partial \varsigma}=0$ in $(0,2)\times(0,1)$. Thus, $m_5(\varsigma,u)\le 0.00415$.

III. On the vertices of Λ, we have

$$K(0,0,0)=0, \quad K(0,0,1)=\frac{1}{225}, \quad K(0,1,0)=\frac{4}{1875}, \quad K(0,1,1)=\frac{4}{1875},$$
$$K(2,0,0)=K(2,0,1)=K(2,1,0)=K(2,1,1)=\frac{686}{703125}.$$

IV. Finally we find the points of maxima of $J(\varsigma,u,t)$ on the 12 edges of Λ.

$$\begin{aligned} K(\varsigma,0,0) &= \frac{343\varsigma^6 - 6000\varsigma^4 + 24000\varsigma^2}{22500000} \le K(\lambda_1,0,0) \\ &= \frac{5024\sqrt{785}}{132355125} + \frac{928}{5294205} \approx 0.00124, \quad \varsigma\in(0,2). \end{aligned}$$

where

$$\begin{aligned} \varsigma &=: \lambda_1 = \frac{2}{49}\sqrt{3500 - 70\sqrt{785}} \approx 1.601098. \\ K(\varsigma,0,1) &= \frac{343\varsigma^6 - 1200\varsigma^5 + 6250\varsigma^4 + 4800\varsigma^3 - 50000\varsigma^2 + 100000}{22500000} \le K(0,0,1) \\ &= \frac{1}{225} \approx 0.00444, \quad \varsigma\in(0,2). \\ K(\varsigma,1,0) &= \frac{-1533\varsigma^6 - 52580\varsigma^4 + 208800\varsigma^2 + 192000}{90000000} \le K(\lambda_2,1,0) \\ &= \frac{9312319\sqrt{9312319} - 26876349046}{356919766875} \approx 0.00432, \quad \varsigma\in(0,2), \end{aligned}$$

where

$$\begin{aligned} \varsigma &:= \lambda_2 = \frac{2\sqrt{2555\sqrt{9312319} - 6717095}}{1533} \approx 1.35567. \\ K(0,u,0) &= \frac{2u(5-3u^2)}{1875} \le K(0,\frac{\sqrt{5}}{3},0) = \frac{4\sqrt{5}}{3375} \approx 0.00265, \quad u\in(0,1) \\ K(0,u,1) &= \frac{-5u^4 + 12u^3 - 20u^2 + 25}{5625} \le K(0,0,1) = \frac{1}{225}, \quad u\in(0,1). \\ K(2,u,0) &= \frac{686}{703125}, \quad u\in(0,1). \\ K(2,u,1) &= \frac{686}{703125}, \quad u\in(0,1). \end{aligned}$$

$$K(0,0,t) = \frac{1}{225}t^2 \leq \frac{1}{225}, \quad t \in (0,1).$$

$$K(0,1,t) = \frac{4}{1875} \approx 0.00213, \quad t \in (0,1).$$

$$K(2,0,t) = \frac{686}{703125}, \quad t \in (0,1).$$

$$K(2,1,t) = \frac{686}{703125}, \quad t \in (0,1).$$

Since all cases have been dealt with, (4.1) holds. To see that (4.1) is sharp, consider f_1 given in (3.2), which is equivalent to choosing $a_3 = a_5 = 0$ and $a_4 = \frac{1}{15}$, which from (1.2) gives $|H_2(3)(f)| = \frac{1}{225}$. This completes the proof. □

5. Conclusion

In this paper we studied the sharp bounds of Hankel determinants $|H_3(1)(f)|$ and $|H_2(3)(f)|$ for the subclasses $\mathcal{S}^*_{3\mathcal{L}}$ and $\mathcal{C}_{3\mathcal{L}}$ of the starlike and convex functions associated with three leaf like domain, respectively.

The sharp bounds on Hankel determinant $|H_2(3)(f)|$ have not been studied more extensively for subclasses of univalent functions. Thus, our results provide motivation for researchers to study it for different subclasses of univalent functions.

Furthermore, invariance of the functional $|H_3(1)(f)|$ and $|H_2(3)(f)|$ for the subclass $\mathcal{C}_{3\mathcal{L}}$ of convex functions associated with three leaf like domain can be discussed.

Ethical approval

Not applicable.

Funding

Not applicable.

CRediT authorship contribution statement

The main idea of this paper was proposed by A.R. and M.R., developed by A.S., A.R. and M.R., A.R. and M.A.B. prepared the manuscripts. All authors checked the steps and arguments in the proof, read and approved the final manuscripts.

Declaration of competing interest

The authors declare no conflict of interest.

Availability of data and materials

Not applicable.

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