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Research article

Weak dual generalized inverse of a dual matrix and its applications

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ABSTRACT

Recently, the dual Moore-Penrose generalized inverse has been applied to study the linear dual equation when the dual Moore-Penrose generalized inverse of the coefficient matrix of the linear dual equation exists. Nevertheless, the dual Moore-Penrose generalized inverse exists only in partially dual matrices. In this paper, to study more general linear dual equation, we introduce the weak dual generalized inverse described by four dual equations, and is a dual Moore-Penrose generalized inverses for it when the latter exists. Any dual matrix has the weak dual generalized inverse and is unique. We obtain some basic properties and characterizations of the weak dual generalized inverse. Also, we investigate relationships among the weak dual generalized inverse, the Moore-Penrose dual generalized inverses and the dual Moore-Penrose generalized inverses, give the equivalent characterization and use some numerical examples to show that the three are different dual generalized inverse. Afterwards, by applying the weak dual generalized inverse we solve two special linear dual equations, one of which is consistent and the other is inconsistent. Neither of the coefficient matrices of the above two linear dual equations has dual Moore-Penrose generalized inverses.

1. Introduction

In 1873, Clifford [1] proposed the use of quantity with size, direction and position to describe spiral motion, in particular rigid body motion. In 1903, Study [2] presented the operation rules of this new quantity and called it *dual number*. As a powerful mathematical tool, dual number are widely used in kinematics and dynamics analysis of space mechanisms, such as rigid body motion ([4–7]), spatial displacement analysis ([8–11]), robot ([12–14]), etc. In these practical applications, a large number of linear dual equations (LDEs for short) need to be solved.

In this paper, two real numbers a, a_0 and the dual unit ε form a dual numbers $\widehat{a} = a + \varepsilon a_0$, where $\varepsilon \neq 0$, but $\varepsilon^2 = 0$, in addition to $0\varepsilon = \varepsilon 0$, $1\varepsilon = \varepsilon 1 = \varepsilon$. Let $\mathbb{D}$ and $\mathbb{D}^{m\times n}$ be the sets of dual numbers and $m \times n$ dual matrices respectively. Dual matrix $\widehat{A} \in \mathbb{D}^{m\times n}$ is displayed in the form $\widehat{A} = A + \varepsilon A_0$, in which $A \in \mathbb{R}^{m\times n}$ and $A_0 \in \mathbb{R}^{m\times n}$. Moreover we denote its transpose by $\widehat{A}^T = A^T + \varepsilon {A_0}^T$, and the $n \times n$ identity matrix as I_n.

Let two dual matrices $\widehat{A} \in \mathbb{D}^{n\times n}$ and $\widehat{X} \in \mathbb{D}^{n\times n}$, if $\widehat{A}\widehat{X} = \widehat{X}\widehat{A} = I_n$, then $\widehat{A}$ is invertible and $\widehat{X}$ is the dual inverse of $\widehat{A}$, denoted $\widehat{X} = \widehat{A}^{-1}$. But the reality is that the dual inverse matrix $\widehat{X}$ does not always exist, so we need to seek some generalized inverses

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to solve LDEs. The *Moore-Penrose dual generalized inverse* (MPDGI for short) is introduced by Pennestrì *et al.* [15–17] through the expression of the inverse of dual matrix, denoted it by $\widehat{A}^{\mathrm{P}}$ and gave

$$\widehat{A}^{\mathrm{P}} = A^{\dagger} - \varepsilon A^{\dagger} A_0 A^{\dagger}. \tag{1.1}$$

It is obvious from the expression that any dual matrix has MPDGI. In [18], Falco *et al.* got some sufficient and necessary conditions for MPDGI to satisfy different types Penrose conditions, and applied MPDGI to solving LDEs of different kinematic problems.

Let the dual matrix $\widehat{A} \in \mathbb{D}^{m\times n}$ and $\widehat{X} \in \mathbb{D}^{n\times m}$, if the two dual matrices satisfy

$$(1)\ \widehat{A}\widehat{X}\widehat{A} = \widehat{A}\,,\ (2)\ \widehat{X}\widehat{A}\widehat{X} = \widehat{X}\,,\ (3)\ \left(\widehat{A}\widehat{X}\right)^{T} = \widehat{A}\widehat{X}\,,\ (4)\ \left(\widehat{X}\widehat{A}\right)^{T} = \widehat{X}\widehat{A}\,,$$

then call $\widehat{X}$ the *dual Moore-Penrose generalized inverse* (DMPGI for short) of $\widehat{A}$ [22], denoting $\widehat{X} = \widehat{A}^{\dagger}$. Obviously, if $\widehat{A}$ is invertible, then $\widehat{A}^{\dagger} = \widehat{A}^{-1}$. Unlike any matrix over the real field that has the Moore-Penrose generalized inverse, the conditions for the existence of the DMPGI are strict, so not all dual matrices have DMPGIs [22]. Udwadia [20] gave an equivalent characterization of the existence of the DMPGI for a dual matrix $\widehat{A}$, that is, $\widehat{A}$ satisfies $\widehat{A}\widehat{X}\widehat{A} = \widehat{A}$. Wang [19] obtained another equivalent condition for the existence of DMPGI. Udwadia [21] got some properties of dual generalized inverse, applied the inverse to solve consistent (inconsistent) LDEs, and obtained some equivalent characterization of the existence of solutions for LDEs.

It should be noted that not all dual matrices have DMPGIs, that is, the DMPGI can be used to study LDE only when the DMPGI of coefficient matrix of LDE exists. In this paper, a new class of dual generalized inverse is introduced to us, which is a generalized DMPGI, call it *weak dual generalized inverse* (WDGI for short), and discuss its properties, characterizations and applications.

2. Preliminary results

This section gives several known results for subsequent research.

Lemma 2.1 *([3], SVD). Let $A \in \mathbb{R}^{m\times n}$ and $\operatorname{rank}(A) = r$. Then there are two unitary matrices, $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$, with*

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^{T},$$

where $\Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_r)$, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are the nonzero singular value of A.

Lemma 2.2 *([19], Theorem 2.1). Let $\widehat{A} = A + \varepsilon A_0 \in \mathbb{D}^{m\times n}$. Then there is the following equivalent condition:*

(a) *The DMPGI $\widehat{A}^{\dagger}$ of $\widehat{A}$ exists;*
(b) $\left(I_m - AA^{\dagger}\right) A_0 \left(I_n - A^{\dagger}A\right) = 0$;
(c) $\operatorname{rank} \begin{bmatrix} A_0 & A \\ A & 0 \end{bmatrix} = 2\operatorname{rank}(A)$.

If the DMPGI $\widehat{A}^{\dagger}$ of $\widehat{A}$ exists, then

$$\widehat{A}^{\dagger} = A^{\dagger} - \varepsilon R, \tag{2.1}$$

where $R = A^{\dagger}A_0A^{\dagger} - \left(A^{T}A\right)^{\dagger} A_0^{T} \left(I_m - AA^{\dagger}\right) - \left(I_n - A^{\dagger}A\right) A_0^{T} \left(AA^{T}\right)^{\dagger}$.

Furthermore, let the SVD of A be as shown in Lemma 2.1, then

$$A_0 = U \begin{bmatrix} A_1 & A_2 \\ A_3 & 0 \end{bmatrix} V^{T}, \tag{2.2}$$

$$\widehat{A}^{\dagger} = V \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^{T} - \varepsilon V \begin{bmatrix} \Sigma^{-1}A_1\Sigma^{-1} & -\Sigma^{-2}A_3^{T} \\ -A_2^{T}\Sigma^{-2} & 0 \end{bmatrix} U^{T},$$

in which $A_1 \in \mathbb{R}^{r\times r}$.

Lemma 2.3 *([20], Result 20). Let $\widehat{A} \in \mathbb{D}^{m\times n}$. If the DMPGI of $\widehat{A}$ exists, then*

$$\widehat{A}^{\dagger} = \left(\widehat{A}^{T}\widehat{A}\right)^{\dagger} \widehat{A}^{T} = \widehat{A}^{T} \left(\widehat{A}\widehat{A}^{T}\right)^{\dagger}.$$

Lemma 2.4 *([19], Theorem 2.3). Let $\widehat{A} = A + \varepsilon A_0 \in \mathbb{D}^{m\times n}$. The following equivalence conditions are obtained:*

(a) *The DMPGI $\widehat{A}^{\dagger}$ of $\widehat{A}$ exists, and $\widehat{A}^{\mathrm{P}} = \widehat{A}^{\dagger}$;*
(b) $\left(I_m - AA^{\dagger}\right) A_0 = 0$ *and* $A_0 \left(I_n - A^{\dagger}A\right) = 0$;
(c) $\operatorname{rank}\begin{bmatrix} A & A_0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A^{T} & A_0^{T} \end{bmatrix} = \operatorname{rank}(A)$;
(d) $\mathcal{R}\left(A_0\right) \subseteq \mathcal{R}(A)$ *and* $\mathcal{R}\left(A_0^{T}\right) \subseteq \mathcal{R}\left(A^{T}\right)$.

Lemma 2.5. *Let $\hat{A} = A + \varepsilon A_0 \in \mathbb{D}^{m\times n}$, the SVD of A be as in Lemma 2.1. Write*

$$A_0 = U \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} V^T,$$

in which $A_1 \in \mathbb{R}^{r\times r}$. Then DMPGIs of $\hat{A}^T\hat{A}$ and $\hat{A}\hat{A}^T$ exist, and

$$\left(\hat{A}^T\hat{A}\right)^{\dagger} = V \begin{bmatrix} \Sigma^{-2} & \\ & 0 \end{bmatrix} V^T - \varepsilon V \begin{bmatrix} \Sigma^{-1}\left(A_1\Sigma^{-1} + \Sigma^{-1}A_1^T\right)\Sigma^{-1} & -\Sigma^{-3}A_2 \\ -A_2^T\Sigma^{-3} & 0 \end{bmatrix} V^T, \tag{2.3}$$

$$\left(\hat{A}\hat{A}^T\right)^{\dagger} = U \begin{bmatrix} \Sigma^{-2} & \\ & 0 \end{bmatrix} U^T - \varepsilon U \begin{bmatrix} \Sigma^{-1}\left(A_1^T\Sigma^{-1} + \Sigma^{-1}A_1\right)\Sigma^{-1} & -\Sigma^{-3}A_3^T \\ -A_3\Sigma^{-3} & 0 \end{bmatrix} U^T \tag{2.4}$$

and

$$\left(\hat{A}^T\hat{A}\right)^{\dagger}\hat{A}^T = \hat{A}^T\left(\hat{A}\hat{A}^T\right)^{\dagger}.$$

Proof. By calculation we have

$$\hat{A}^T\hat{A} = A^TA + \varepsilon\left(A^TA_0 + A_0^TA\right) \quad \text{and} \quad \hat{A}\hat{A}^T = AA^T + \varepsilon\left(AA_0^T + A_0A^T\right). \tag{2.5}$$

For $\hat{A}^T\hat{A}$, using $\left(I_n - A^T\left(A^T\right)^{\dagger}\right)A^T = 0$ and $A\left(I_n - A^{\dagger}A\right) = 0$, we obtain

$$\left(I_n - A^TA\left(A^TA\right)^{\dagger}\right)\left(A^TA_0 + A_0^TA\right)\left(I_n - \left(A^TA\right)^{\dagger}A^TA\right) = 0.$$

Therefore, according to Lemma 2.2, $\left(\hat{A}^T\hat{A}\right)^{\dagger}$ must exists. Next, by Lemma 2.2 and (2.5) we get (2.3).

Similarly, we get (2.4). □

3. Weak dual generalized inverse

It is well known that DMPGI has good properties, which can be used to study some problems of dual linear systems. However, it should be noted that the prerequisite for us to apply DMPGI to solving the problem is the existence of DMPGI. In this section, to study more problems of dual linear systems, we introduce a generalized DMPGI, and discuss characterizations and basic properties of the inverse.

Theorem 3.1. *Let $\hat{A} \in \mathbb{D}^{m\times n}$, then the solution $\hat{X} \in \mathbb{D}^{n\times m}$ to*

$$(1')\ \hat{A}^T\hat{A}\hat{X}\hat{A}\hat{A}^T = \hat{A}^T\hat{A}\hat{A}^T\ ,\ (2)\ \hat{X}\hat{A}\hat{X} = \hat{X}\ ,\ (3)\ \left(\hat{A}\hat{X}\right)^T = \hat{A}\hat{X}\ ,\ (4)\ \left(\hat{X}\hat{A}\right)^T = \hat{X}\hat{A} \tag{3.1}$$

is unique and

$$\hat{X} = \left(\hat{A}^T\hat{A}\right)^{\dagger}\hat{A}^T = \hat{A}^T\left(\hat{A}\hat{A}^T\right)^{\dagger}. \tag{3.2}$$

Proof. Applying Lemma 2.5 it is easy to check that $\left(\hat{A}^T\hat{A}\right)^{\dagger}\hat{A}^T = \hat{A}^T\left(\hat{A}\hat{A}^T\right)^{\dagger}$.

By applying (3.2) to (3.1), we get

$$\hat{A}^T\hat{A}\hat{X}\hat{A}\hat{A}^T = \hat{A}^T\hat{A}\left(\hat{A}^T\hat{A}\right)^{\dagger}\hat{A}^T\hat{A}\hat{A}^T = \hat{A}^T\hat{A}\hat{A}^T,$$

$$\hat{X}\hat{A}\hat{X} = \left(\hat{A}^T\hat{A}\right)^{\dagger}\hat{A}^T\hat{A}\left(\hat{A}^T\hat{A}\right)^{\dagger}\hat{A}^T = \left(\hat{A}^T\hat{A}\right)^{\dagger}\hat{A}^T = \hat{X},$$

$$\left(\hat{A}\hat{X}\right)^T = \left(\hat{A}\left(\hat{A}^T\hat{A}\right)^{\dagger}\hat{A}^T\right)^T = \hat{A}\left(\hat{A}^T\hat{A}\right)^{\dagger}\hat{A}^T = \hat{A}\hat{X},$$

$$\left(\hat{X}\hat{A}\right)^T = \left(\hat{A}^T\left(\hat{A}\hat{A}^T\right)^{\dagger}\hat{A}\right)^T = \hat{A}^T\left(\hat{A}\hat{A}^T\right)^{\dagger}\hat{A} = \hat{X}\hat{A}.$$

Then $\hat{X}$ given in (3.2) satisfies the four conditional equations in (3.1).

Let both $\hat{X}_1$ and $\hat{X}_2$ satisfy the equations (3.1). Then applying $\hat{X}_1\hat{A}\hat{X}_1 = \hat{X}_1$ and $\left(\hat{A}\hat{X}_1\right)^T = \hat{A}\hat{X}_1$ gives

$$\begin{aligned}\hat{X}_1 &= \hat{X}_1\hat{A}\hat{X}_1 = \hat{A}^T\hat{X}_1^T\hat{X}_1 = \hat{A}^T\hat{X}_1^T\hat{A}^T\hat{X}_1^T\hat{X}_1 = \hat{A}^T\hat{A}\hat{X}_1\hat{X}_1^T\hat{X}_1 \\ &= \hat{A}^T\hat{A}\hat{X}_1\hat{A}\hat{X}_1\hat{X}_1^T\hat{X}_1 = \hat{A}^T\hat{A}\hat{X}_1\hat{A}\hat{X}_1\hat{A}\hat{X}_1\hat{X}_1^T\hat{X}_1 = \hat{A}^T\hat{A}\hat{X}_1\hat{A}\hat{A}^T\hat{X}_1^T\hat{X}_1\hat{X}_1^T\hat{X}_1.\end{aligned}$$

Applying $(1')$ and (3), we get

$$\widehat{A}^T\widehat{A}\widehat{X}_1\widehat{A}\widehat{A}^T\widehat{X}_1^T\widehat{X}_1\widehat{X}_1^T\widehat{X}_1 = \widehat{A}^T\widehat{A}\widehat{X}_2\widehat{A}\widehat{A}^T\widehat{X}_1^T\widehat{X}_1\widehat{X}_1^T\widehat{X}_1 = \widehat{A}^T\widehat{X}_2^T\widehat{A}^T\widehat{A}\widehat{X}_1\widehat{A}\widehat{X}_1\widehat{X}_1^T\widehat{X}_1$$
$$= \widehat{X}_2\widehat{A}\widehat{A}^T\widehat{X}_1^T\widehat{X}_1 = \widehat{X}_2\widehat{A}\widehat{X}_1\widehat{A}\widehat{X}_1 = \widehat{X}_2\widehat{A}\widehat{X}_1.$$

Furthermore, applying (3.1) we get

$$\widehat{X}_2\widehat{A}\widehat{X}_1 = \widehat{X}_2\widehat{X}_2^T\widehat{A}^T\widehat{X}_2^T\widehat{A}^T\widehat{A}\widehat{X}_1 = \widehat{X}_2\widehat{X}_2^T\widehat{X}_2\widehat{A}\widehat{A}^T\widehat{A}\widehat{X}_1 = \widehat{X}_2\widehat{X}_2^T\widehat{X}_2\widehat{X}_2^T\widehat{A}^T\widehat{A}\widehat{X}_1\widehat{A}\widehat{A}^T$$
$$= \widehat{X}_2\widehat{X}_2^T\widehat{X}_2\widehat{A}\widehat{X}_2\widehat{A}\widehat{X}_2\widehat{A}\widehat{A}^T = \widehat{X}_2\widehat{X}_2^T\widehat{A}^T\widehat{X}_2^T\widehat{A}^T = \widehat{X}_2\widehat{X}_2^T\widehat{A}^T = \widehat{X}_2\widehat{A}\widehat{X}_2 = \widehat{X}_2.$$

Therefore, we get that the solution to (3.1) is unique. □

Definition 3.1. Let $\widehat{A} \in \mathbb{D}^{m\times n}$. Then the unique dual matrix that satisfies (3.1) is called the weak dual generalized inverse (WDGI for short), denoted by $\widehat{A}^{\mathrm{W}\dagger}$.

Theorem 3.2. *Let $\widehat{A} = A + \varepsilon A_0 \in \mathbb{D}^{m\times n}$. Then*

$$\widehat{A}^{\mathrm{W}\dagger} = A^\dagger - \varepsilon R^{\mathrm{W}}, \tag{3.3}$$

where $R^{\mathrm{W}} = A^\dagger A_0 A^\dagger - \left(A^T A\right)^\dagger A_0^T\left(I_m - AA^\dagger\right) - \left(I_n - A^\dagger A\right)A_0^T\left(AA^T\right)^\dagger$. Furthermore, write

$$A_0 = U\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}V^T,$$

in which $A_1 \in \mathbb{R}^{r\times r}$, then

$$\widehat{A}^{\mathrm{W}\dagger} = V\begin{bmatrix} \Sigma^{-1} & \\ & 0 \end{bmatrix}U^T - \varepsilon V\begin{bmatrix} \Sigma^{-1}A_1\Sigma^{-1} & -\Sigma^{-2}A_3^T \\ -A_2^T\Sigma^{-2} & 0 \end{bmatrix}U^T, \tag{3.4}$$

where U, V and Σ are given by Lemma 2.2.

Proof. By Lemma 2.5 we know that the DMPGI of $\widehat{A}^T\widehat{A}$ must exist for $\widehat{A}$. By (2.5) and (2.1) we obtain

$$\left(\widehat{A}^T\widehat{A}\right)^\dagger = \left(A^TA\right)^\dagger - \varepsilon\Big\{A^\dagger A_0\left(A^TA\right)^\dagger + \left(A^TA\right)^\dagger A_0^T\left(A^T\right)^\dagger - \left(A^TAA^TA\right)^\dagger\Delta\left(I_n - A^T\left(A^T\right)^\dagger\right)$$
$$- \left(I_n - A^\dagger A\right)A_0^T\left(A^T\right)^\dagger\left(A^TA\right)^\dagger\Big\},$$

and

$$\widehat{A}^{\mathrm{W}\dagger} = \left(\widehat{A}^T\widehat{A}\right)^\dagger\widehat{A}^T$$
$$= \Big\{\left(A^TA\right)^\dagger - \varepsilon\Big\{A^\dagger A_0\left(A^TA\right)^\dagger + \left(A^TA\right)^\dagger A_0^T\left(A^T\right)^\dagger - \left(A^TAA^TA\right)^\dagger\Delta\left(I_n - A^T\left(A^T\right)^\dagger\right)$$
$$- \left(I_n - A^\dagger A\right)A_0^T\left(A^T\right)^\dagger\left(A^TA\right)^\dagger\Big\}\Big\}\left(A^T + \varepsilon A_0^T\right)$$
$$= A^\dagger - \varepsilon\Big\{A^\dagger A_0A^\dagger - \left(A^TA\right)^\dagger A_0^T\left(I_m - AA^\dagger\right) - \left(I_n - A^\dagger A\right)A_0^T\left(AA^T\right)^\dagger\Big\},$$

where $\Delta = A^TA_0 + A_0^TA$. Therefore, we have (3.3). Moreover, by (2.2), (2.3) and (2.4) we get (3.4). □

Theorem 3.3. *Let $\widehat{A} \in \mathbb{D}^{m\times n}$. When the DMPGI of $\widehat{A}$ exists, we have $\widehat{A}^\dagger = \widehat{A}^{\mathrm{W}\dagger}$.*

Proof. Theorem 3.3 follows from Lemma 2.3 and Theorem 3.1. □

Remark 3.1. By Lemma 2.2 and Theorem 3.2, we observe that DMPGI has the same explicit expression as WDGI. By Theorem 3.3, we know that WDGI is equal to DMPGI when DMPGI of dual matrix exists. On the other hand, from the Lemma 2.3, $\widehat{A}^\dagger = \left(\widehat{A}^T\widehat{A}\right)^\dagger\widehat{A}^T$ when DMPGI of A exists, substituting $\widehat{A}^\dagger$ into (3.1) (1′) has $\widehat{A}^T\widehat{A}\widehat{A}^\dagger\widehat{A}\widehat{A}^T = \widehat{A}^T\widehat{A}\left(\widehat{A}^T\widehat{A}\right)^\dagger\widehat{A}^T\widehat{A}\widehat{A}^T = \widehat{A}^T\widehat{A}\widehat{A}^T$. This also indicates that WDGI is one generalized DMPGI. Of course, the difference between the two is that DMPGI needs to satisfy certain conditions to exist, while WDGI exists for any dual matrix. In conclusion, WDGI and DMPGI are two different types of dual generalized inverse. WDGI includes DMPGI and WDGI is more general.

According to the above theorems, WDGI is a generalization of DMPGI. We use the following example to illustrate that WDGI is more general.

Example 3.1 *([22], Equation (46)).* Let $\widehat{Z} = Z + \varepsilon Z_0 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -3 & 3 \end{bmatrix} + \varepsilon \begin{bmatrix} 1 & -3 & 3 \\ -1 & 6 & -13 \\ -1 & 1 & -1 \end{bmatrix}$, then

$$Z^{\dagger} = \begin{bmatrix} \frac{3}{2} & 0 & -\frac{1}{2} \\ \frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix},\ Z^{\dagger} Z_0 Z^{\dagger} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{3}{2} \\ \frac{1}{4} & 0 & \frac{1}{4} \\ -\frac{1}{4} & 0 & -\frac{1}{4} \end{bmatrix},\ \left(I_3 - ZZ^{\dagger}\right) Z_0 \left(I_3 - Z^{\dagger} Z\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{7}{2} & -\frac{7}{2} \\ 0 & 0 & 0 \end{bmatrix} \neq 0,$$

$$\left(Z^T Z\right)^{\dagger} Z_0^T \left(I_3 - ZZ^{\dagger}\right) = \begin{bmatrix} 0 & 7 & 0 \\ 0 & \frac{15}{8} & 0 \\ 0 & -\frac{15}{8} & 0 \end{bmatrix} \text{and } \left(I_3 - Z^{\dagger} Z\right) Z_0^T \left(ZZ^T\right)^{\dagger} = 0.$$

By applying $\left(I_3 - ZZ^{\dagger}\right) Z_0 \left(I_3 - Z^{\dagger} Z\right) \neq 0$ and Lemma 2.2, we get that the DMPGI of $\widehat{Z}$ does not exist. By applying Theorem 3.2 we get

$$\widehat{Z}^{\mathrm{W}\dagger} = \begin{bmatrix} \frac{3}{2} & 0 & -\frac{1}{2} \\ \frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix} - \varepsilon \begin{bmatrix} \frac{1}{2} & -7 & \frac{3}{2} \\ \frac{1}{4} & -\frac{15}{8} & \frac{1}{4} \\ -\frac{1}{4} & \frac{15}{8} & -\frac{1}{4} \end{bmatrix}.$$

Next, we give some basic properties of the weak dual generalized inverse.

Theorem 3.4. *Let $\widehat{A} \in \mathbb{D}^{m \times n}$. Then*

(a) $\left(\widehat{A}^T\right)^{\mathrm{W}\dagger} = \left(\widehat{A}^{\mathrm{W}\dagger}\right)^T$;
(b) $\left(\widehat{A}^{\mathrm{W}\dagger}\right)^{\mathrm{W}\dagger} = \widehat{A}\widehat{A}^{\mathrm{W}\dagger}\widehat{A}$;
(c) $\left(\widehat{A}\widehat{A}^T\right)^{\mathrm{W}\dagger} = \left(\widehat{A}^T\right)^{\mathrm{W}\dagger}\widehat{A}^{\mathrm{W}\dagger}$; $\left(\widehat{A}^T\widehat{A}\right)^{\mathrm{W}\dagger} = \widehat{A}^{\mathrm{W}\dagger}\left(\widehat{A}^T\right)^{\mathrm{W}\dagger}$;
(d) $\left(\widehat{\lambda}\widehat{A}\right)^{\mathrm{W}\dagger} = \widehat{\lambda}^{\dagger}\widehat{A}^{\mathrm{W}\dagger}$, *where* $\widehat{\lambda} \in \mathbb{D}$ *and* $\widehat{\lambda}^{\dagger} = \begin{cases} \widehat{\lambda}^{-1}, & \text{the real part of } \widehat{\lambda} \text{ is not zero} \\ 0, & \text{the real part of } \widehat{\lambda} \text{ is zero} \end{cases}$;
(e) $\widehat{A}\widehat{A}^{\mathrm{W}\dagger}\widehat{A}\widehat{A}^T = \widehat{A}\widehat{A}^T$; $\widehat{A}^T\widehat{A}\widehat{A}^{\mathrm{W}\dagger}\widehat{A} = \widehat{A}^T\widehat{A}$.

Proof. (a): By (3.2) we know $\widehat{A}^{\mathrm{W}\dagger} = \left(\widehat{A}^T\widehat{A}\right)^{\dagger}\widehat{A}^T = \widehat{A}^T\left(\widehat{A}\widehat{A}^T\right)^{\dagger}$. Then by applying $\left(\widehat{A}^{\mathrm{W}\dagger}\right)^T$ to (3.1) we get

$$(\widehat{A}^T)^T\widehat{A}^T\left(\widehat{A}^{\mathrm{W}\dagger}\right)^T\widehat{A}^T(\widehat{A}^T)^T = \widehat{A}\widehat{A}^T\widehat{A}\left(\widehat{A}^T\widehat{A}\right)^{\dagger}\widehat{A}^T\widehat{A} = \widehat{A}\widehat{A}^T\widehat{A} = (\widehat{A}^T)^T\widehat{A}^T(\widehat{A}^T)^T,$$

$$\left(\widehat{A}^{\mathrm{W}\dagger}\right)^T\widehat{A}^T\left(\widehat{A}^{\mathrm{W}\dagger}\right)^T = \widehat{A}\left(\widehat{A}^T\widehat{A}\right)^{\dagger}\widehat{A}^T\widehat{A}\left(\widehat{A}^T\widehat{A}\right)^{\dagger} = \widehat{A}\left(\widehat{A}^T\widehat{A}\right)^{\dagger} = \left(\widehat{A}^{\mathrm{W}\dagger}\right)^T,$$

$$\left(\widehat{A}^T\left(\widehat{A}^{\mathrm{W}\dagger}\right)^T\right)^T = \widehat{A}^{\mathrm{W}\dagger}\widehat{A} = \widehat{A}^T\left(\widehat{A}\widehat{A}^T\right)^{\dagger}\widehat{A} = \widehat{A}^T\left(\widehat{A}^{\mathrm{W}\dagger}\right)^T,$$

$$\left(\left(\widehat{A}^{\mathrm{W}\dagger}\right)^T\widehat{A}^T\right)^T = \widehat{A}\widehat{A}^{\mathrm{W}\dagger} = \widehat{A}\left(\widehat{A}^T\widehat{A}\right)^{\dagger}\widehat{A}^T = \left(\widehat{A}^{\mathrm{W}\dagger}\right)^T\widehat{A}^T.$$

By applying Theorem 3.1 we get that $\left(\widehat{A}^{\mathrm{W}\dagger}\right)^T$ is the WDGI of $\widehat{A}^T$.

The proof of (b), (c), (d) is analogous to (a).

(e): Substituting (3.2) in (e) gives $\widehat{A}\widehat{A}^{\mathrm{W}\dagger}\widehat{A}\widehat{A}^T = \widehat{A}\widehat{A}^T\left(\widehat{A}\widehat{A}^T\right)^{\dagger}\widehat{A}\widehat{A}^T = \widehat{A}\widehat{A}^T$ and $\widehat{A}^T\widehat{A}\widehat{A}^{\mathrm{W}\dagger}\widehat{A} = \widehat{A}^T\widehat{A}\left(\widehat{A}^T\widehat{A}\right)^{\dagger}\widehat{A}^T\widehat{A} = \widehat{A}^T\widehat{A}$. □

4. Relationships among WDGI, DMPGI and MPDGI

In this section, we discuss that WDGI is different from DMPGI and MPDGI, and further investigate relationships among WDGI, DMPGI and MPDGI.

Theorem 4.1. *Let $\widehat{A} = A + \varepsilon A_0 \in \mathbb{D}^{m \times n}$. Then there is the following equivalent condition:*

(a) *The DMPGI $\widehat{A}^{\dagger}$ of $\widehat{A}$ exists, and $\widehat{A}^{\dagger} = \widehat{A}^{\mathrm{W}\dagger}$;*
(b) *The DMPGI $\widehat{A}^{\dagger}$ of $\widehat{A}$ exists;*
(c) $\left(I_m - AA^{\dagger}\right)A_0\left(I_n - A^{\dagger}A\right) = 0$;
(d) $\operatorname{rank}\begin{bmatrix} A_0 & A \\ A & 0 \end{bmatrix} = 2\operatorname{rank}(A)$.

Proof. The equivalence among the above four conditions follows from Lemma 2.2, Lemma 2.3 and Theorem 3.3. □

The following examples illustrate relationships among WDGI, MPDGI and DMPGI.

Example 4.1. Let $\widehat{A} = A + \varepsilon A_0 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$, then

$$A^\dagger = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & 0 \\ -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A^\dagger A_0 A^\dagger = \begin{bmatrix} -\frac{3}{25} & \frac{6}{25} & 0 \\ \frac{14}{25} & -\frac{3}{25} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\left(I_3 - AA^\dagger\right) A_0 \left(I_3 - A^\dagger A\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \neq 0,$$

$$\left(A^T A\right)^\dagger A_0^T \left(I_3 - AA^\dagger\right) = \begin{bmatrix} 0 & 0 & \frac{3}{5} \\ 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \left(I_3 - A^\dagger A\right) A_0^T \left(AA^T\right)^\dagger = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{3}{5} & \frac{3}{5} & 0 \end{bmatrix}.$$

So by Lemma 2.2 we know that the DMPGI of $\widehat{A}$ does not exist, but $\widehat{A}^{W\dagger}$ does. By (1.1) and (3.3) we have

$$\widehat{A}^{P} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & 0 \\ -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \varepsilon \begin{bmatrix} -\frac{3}{25} & \frac{6}{25} & 0 \\ \frac{14}{25} & -\frac{3}{25} & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \widehat{A}^{W\dagger} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & 0 \\ -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \varepsilon \begin{bmatrix} -\frac{3}{25} & \frac{6}{25} & -\frac{3}{5} \\ \frac{14}{25} & -\frac{3}{25} & -\frac{1}{5} \\ -\frac{3}{5} & -\frac{3}{5} & 0 \end{bmatrix}.$$

Now, the DMPGI of $\widehat{A}$ does not exist and $\widehat{A}^{P} \neq \widehat{A}^{W\dagger}$.

Example 4.2. Let $\widehat{A} = A + \varepsilon A_0 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ -4 & -5 & 0 \end{bmatrix}$, then

$$A^\dagger = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & 0 \\ -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A^\dagger A_0 A^\dagger = \begin{bmatrix} -\frac{4}{25} & -\frac{7}{25} & 0 \\ \frac{2}{25} & -\frac{9}{25} & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \left(I_3 - AA^\dagger\right) A_0 \left(I_3 - A^\dagger A\right) = 0,$$

$$\left(A^T A\right)^\dagger A_0^T \left(I_3 - AA^\dagger\right) = \begin{bmatrix} 0 & 0 & -\frac{4}{5} \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \left(I_3 - A^\dagger A\right) A_0^T \left(AA^T\right)^\dagger = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{3}{5} & \frac{2}{5} & 0 \end{bmatrix}.$$

So the DMPGI of $\widehat{A}$ exists. By (1.1), (2.1) and (3.3) we have

$$\widehat{A}^{P} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & 0 \\ -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \varepsilon \begin{bmatrix} -\frac{4}{25} & -\frac{7}{25} & 0 \\ \frac{2}{25} & -\frac{9}{25} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\widehat{A}^{\dagger} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & 0 \\ -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \varepsilon \begin{bmatrix} -\frac{4}{25} & -\frac{7}{25} & \frac{4}{5} \\ \frac{2}{25} & -\frac{9}{25} & 1 \\ -\frac{3}{5} & -\frac{2}{5} & 0 \end{bmatrix} \text{ and } \widehat{A}^{W\dagger} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & 0 \\ -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \varepsilon \begin{bmatrix} -\frac{4}{25} & -\frac{7}{25} & \frac{4}{5} \\ \frac{2}{25} & -\frac{9}{25} & 1 \\ -\frac{3}{5} & -\frac{2}{5} & 0 \end{bmatrix}.$$

Now, the DMPGI of $\widehat{A}$ exists, $\widehat{A}^{P} \neq \widehat{A}^{\dagger}$ and $\widehat{A}^{\dagger} = \widehat{A}^{W\dagger}$.

Remark 4.1. From Examples 4.1–4.2 above, we know that MPDGI and WDGI could be unequal whether or not DMPGI exists. In other words, MPDGI and WDGI are two different types of dual generalized inverses.

Theorem 4.2. *Let $\widehat{A} = A + \varepsilon A_0 \in \mathbb{D}^{m\times n}$. Then there is the following equivalent condition:*

(a) $\widehat{A}^{P} = \widehat{A}^{W\dagger}$;
(b) $\left(I_m - AA^\dagger\right) A_0 A^\dagger = 0$ *and* $A^\dagger A_0 \left(I_n - A^\dagger A\right) = 0$.

Proof. "⇒" If $\widehat{A}^{P} = \widehat{A}^{W\dagger}$, then by (1.1) and (3.3) we have

$$\left(A^T A\right)^\dagger A_0^T \left(I_m - AA^\dagger\right) + \left(I_n - A^\dagger A\right) A_0^T \left(AA^T\right)^\dagger = 0. \tag{4.1}$$

Pre-multiplying A on (4.1) gives

$$0 = A\left(A^T A\right)^{\dagger} A_0^T \left(I_m - AA^{\dagger}\right) + A\left(I_n - A^{\dagger}A\right) A_0^T \left(AA^T\right)^{\dagger} = \left(A^{\dagger}\right)^T A_0^T \left(I_m - AA^{\dagger}\right).$$

Furthermore, by taking transposes of both sides, we get $\left(I_m - AA^{\dagger}\right) A_0 A^{\dagger} = 0$. Similarly, post-multiplying A on (4.1) gives $A^{\dagger} A_0 \left(I_n - A^{\dagger}A\right) = 0$.

"⇐" If $\left(I_m - AA^{\dagger}\right) A_0 A^{\dagger} = 0$ and $A^{\dagger} A_0 \left(I_n - A^{\dagger}A\right) = 0$, then

$$\begin{aligned} R^{\mathrm{W}} &= A^{\dagger}A_0A^{\dagger} - \left(A^T A\right)^{\dagger} A_0^T \left(I_m - AA^{\dagger}\right) - \left(I_n - A^{\dagger}A\right) A_0^T \left(AA^T\right)^{\dagger} \\ &= A^{\dagger}A_0A^{\dagger} - A^{\dagger}\left((I_m - AA^{\dagger})A_0A^{\dagger}\right)^T - \left(A^{\dagger}A_0\left(I_n - A^{\dagger}A\right)\right)^T A^{\dagger} \\ &= A^{\dagger}A_0A^{\dagger}. \end{aligned}$$

Therefore, we have $\widehat{A}^{\mathrm{P}} = \widehat{A}^{\mathrm{W}\dagger}$. □

Example 4.3. Let $\widehat{A} = A + \varepsilon A_0 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 3 & 1 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, then

$$A^{\dagger} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & 0 \\ -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A^{\dagger}A_0A^{\dagger} = \begin{bmatrix} -\frac{7}{25} & -\frac{21}{25} & 0 \\ \frac{16}{25} & -\frac{2}{25} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\left(I_3 - AA^{\dagger}\right) A_0 \left(I_3 - A^{\dagger}A\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \neq 0,$$

$$\left(A^T A\right)^{\dagger} A_0^T \left(I_3 - AA^{\dagger}\right) = 0 \ \ and \ \ \left(I_3 - A^{\dagger}A\right) A_0^T \left(AA^T\right)^{\dagger} = 0.$$

So the DMPGI of $\widehat{A}$ does not exist, but

$$\widehat{A}^{\mathrm{P}} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & 0 \\ -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \varepsilon \begin{bmatrix} -\frac{7}{25} & -\frac{21}{25} & 0 \\ \frac{16}{25} & -\frac{2}{25} & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \widehat{A}^{\mathrm{W}\dagger} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & 0 \\ -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \varepsilon \begin{bmatrix} -\frac{7}{25} & -\frac{21}{25} & 0 \\ \frac{16}{25} & -\frac{2}{25} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, the DMPGI of $\widehat{A}$ does not exist, but interestingly enough,

$$\widehat{A}^{\mathrm{P}} = \widehat{A}^{\mathrm{W}\dagger}.$$

Example 4.4. Let $\widehat{A} = A + \varepsilon A_0 = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 3 & 1 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then

$$A^{\dagger} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A^{\dagger}A_0A^{\dagger} = \begin{bmatrix} \frac{14}{25} & -\frac{12}{25} & 0 \\ \frac{7}{25} & \frac{19}{25} & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \left(I_3 - AA^{\dagger}\right) A_0 \left(I_3 - A^{\dagger}A\right) = 0,$$

$$\left(A^T A\right)^{\dagger} A_0^T \left(I_3 - AA^{\dagger}\right) = 0 \ \ and \ \ \left(I_3 - A^{\dagger}A\right) A_0^T \left(AA^T\right)^{\dagger} = 0.$$

Therefore, the DMPGI of $\widehat{A}$ exists and $\widehat{A}^{\dagger} = \widehat{A}^{\mathrm{W}\dagger}$. By (1.1) and (2.1) we have

$$\widehat{A}^{\mathrm{P}} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \varepsilon \begin{bmatrix} \frac{14}{25} & -\frac{12}{25} & 0 \\ \frac{7}{25} & \frac{19}{25} & 0 \\ 0 & 0 & 0 \end{bmatrix} \ and \ \widehat{A}^{\dagger} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \varepsilon \begin{bmatrix} \frac{14}{25} & -\frac{12}{25} & 0 \\ \frac{7}{25} & \frac{19}{25} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now,

$$\widehat{A}^{\mathrm{P}} = \widehat{A}^{\dagger} = \widehat{A}^{\mathrm{W}\dagger}.$$

Remark 4.2. From Examples 4.3–4.4 above, we know that MPDGI and WDGI could be equal whether or not DMPGI exists, and the above four examples show that when the DMPGI of dual matrix $\widehat{A}$ exists, then $\widehat{A}^{\dagger} = \widehat{A}^{\mathrm{W}\dagger}$. On the other hand, it shows that WDGI is a dual generalized inverse different from MPDGI and DMPGI, and more general than DMPGI.

Corollary 4.3. *Let $\widehat{A} = A + \varepsilon A_0 \in \mathbb{D}^{m\times n}$. If the DMPGI $\widehat{A}^{\dagger}$ of $\widehat{A}$ exists, then there is the following equivalent condition:*

(a) $\widehat{A}^{\mathrm{P}} = \widehat{A}^{\dagger} = \widehat{A}^{\mathrm{W}\dagger}$;
(b) $\widehat{A}^{\mathrm{P}} = \widehat{A}^{\dagger}$;
(c) $\left(I_m - AA^{\dagger}\right) A_0 = 0$ *and* $A_0\left(I_n - A^{\dagger}A\right) = 0$;
(d) $\operatorname{rank}\begin{bmatrix} A & A_0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A^T & A_0^T \end{bmatrix} = \operatorname{rank}(A)$;
(e) $\mathcal{R}\left(A_0\right) \subseteq \mathcal{R}(A)$ *and* $\mathcal{R}\left(A_0^T\right) \subseteq \mathcal{R}\left(A^T\right)$.

Proof. The equivalence among the above five conditions follows from Lemma 2.4 and Theorem 3.3. □

5. Applications of WDGI

Let the dual vector $\widehat{u} = p + \varepsilon q$. Then denote dual vector norm

$$\langle \widehat{u} \rangle = \langle p + \varepsilon q \rangle = \|p\| + \|q\|,$$

where $\|\cdot\|$ is the Euclidean norm of real vector [21].

DMPGI not only provides a new tool for studying some consistent linear dual equations (LDE), but also provides a tool for getting analogue of the least-squares solution to some inconsistent LDEs [21]. WDGI is one generalization of DMPGI. When DMPGI exists, WDGI is equal to DMPGI, which means that WDGI can handle any problem that DMPGI can handle. However, the conditions for the existence of DMPGI are hard to meet, which means that DMPGI has great limitations in practical applications. Here we give two special LDEs; one is a consistent LDE and the other is an inconsistent LDE. The DMPGIs of coefficient matrices of the two special LDEs do not exist, that is, the two special LDEs cannot be treated with DMPGI.

Example 5.1. Let the dual equation $\widehat{A}\widehat{x} = \widehat{b}$ be

$$\begin{bmatrix} 1+\varepsilon & 2+\varepsilon 3 & \varepsilon \\ -2-\varepsilon 2 & 1+\varepsilon 4 & 0 \\ \varepsilon 3 & 0 & -\varepsilon 2 \end{bmatrix} \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \end{bmatrix} = \begin{bmatrix} 5+\varepsilon \\ \varepsilon 2 \\ \varepsilon 3 \end{bmatrix}. \tag{5.1}$$

By Lemma 2.2 we have

$$\left(I_3 - AA^{\dagger}\right) A_0 \left(I_3 - A^{\dagger}A\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \neq 0.$$

In this case, the DMPGI of $\widehat{A}$ does not exist. By applying $\widehat{A}$ to (3.2), we get

$$\widehat{A}^{\mathrm{W}\dagger} = \begin{bmatrix} \frac{1}{5}+\varepsilon\frac{1}{5} & -\frac{2}{5}+\varepsilon\frac{3}{5} & \varepsilon\frac{3}{5} \\ \frac{2}{5}-\varepsilon\frac{4}{5} & \frac{1}{5}-\varepsilon\frac{2}{5} & 0 \\ \varepsilon\frac{1}{5} & 0 & 0 \end{bmatrix} \text{ and } \widehat{A}^{\mathrm{W}\dagger}\widehat{b} = \begin{bmatrix} 1+\varepsilon\frac{2}{5} \\ 2-\varepsilon\frac{16}{5} \\ \varepsilon \end{bmatrix}.$$

It's easy to check that

$$\widehat{A}\widehat{A}^{\mathrm{W}\dagger}\widehat{b} = \begin{bmatrix} 1+\varepsilon & 2+\varepsilon 3 & \varepsilon \\ -2-\varepsilon 2 & 1+\varepsilon 4 & 0 \\ \varepsilon 3 & 0 & -\varepsilon 2 \end{bmatrix} \begin{bmatrix} 1+\varepsilon\frac{2}{5} \\ 2-\varepsilon\frac{16}{5} \\ \varepsilon \end{bmatrix} = \begin{bmatrix} 5+\varepsilon \\ \varepsilon 2 \\ \varepsilon 3 \end{bmatrix}.$$

Therefore, $\widehat{A}^{\mathrm{W}\dagger}\widehat{b}$ is one solution to (5.1).

Example 5.2. Let the dual equation $\widehat{A}\widehat{y} = \widehat{d}$ be

$$\begin{bmatrix} 2+\varepsilon 3 & 0 & \varepsilon 2 \\ 0 & \varepsilon 4 & -\varepsilon \\ \varepsilon & 0 & -\varepsilon 2 \end{bmatrix} \begin{bmatrix} \widehat{y}_1 \\ \widehat{y}_2 \\ \widehat{y}_3 \end{bmatrix} = \begin{bmatrix} \varepsilon \\ 1 \\ 0 \end{bmatrix}. \tag{5.2}$$

It is easy to check that (5.2) is inconsistent, the DMPGI of $\widehat{A}$ does not exist, and

$$\widehat{A}^{\mathrm{W}\dagger} = \begin{bmatrix} \frac{1}{2}-\varepsilon\frac{3}{4} & 0 & \varepsilon\frac{1}{4} \\ 0 & 0 & 0 \\ \varepsilon\frac{1}{2} & 0 & 0 \end{bmatrix} \text{ and } \widehat{A}^{\mathrm{W}\dagger}\widehat{d} = \begin{bmatrix} \varepsilon\frac{1}{2} \\ 0 \\ 0 \end{bmatrix}. \tag{5.3}$$

Let $\begin{bmatrix} \widehat{y}_1 & \widehat{y}_2 & \widehat{y}_3 \end{bmatrix}^T = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T + \varepsilon \begin{bmatrix} y_{10} & y_{20} & y_{30} \end{bmatrix}^T$, which y_1, y_2, y_3, y_{10}, y_{20} and y_{30} are all real numbers. Since

$$\left\langle \widehat{A}\widehat{y} - \widehat{d} \right\rangle = \left\langle \begin{bmatrix} 2+\varepsilon 3 & 0 & \varepsilon 2 \\ 0 & \varepsilon 4 & -\varepsilon \\ \varepsilon & 0 & -\varepsilon 2 \end{bmatrix} \begin{bmatrix} \widehat{y}_1 \\ \widehat{y}_2 \\ \widehat{y}_3 \end{bmatrix} - \begin{bmatrix} \varepsilon \\ 1 \\ 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 2y_1 + \varepsilon\left(2y_{10} + 3y_1 + 2y_3 - 1\right) \\ -1 + \varepsilon\left(4y_2 - y_3\right) \\ \varepsilon\left(y_1 - 2y_3\right) \end{bmatrix} \right\rangle$$

$$= \left\| \begin{bmatrix} 2y_1 \\ -1 \\ 0 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 2y_{10} + 3y_1 + 2y_3 - 1 \\ 4y_2 - y_3 \\ y_1 - 2y_3 \end{bmatrix} \right\|,$$

$$\min_{y_1} \left\| \begin{bmatrix} 2y_1 \\ -1 \\ 0 \end{bmatrix} \right\| = 1 \ \ \textit{and} \ \ \min_{y_1,y_2,y_3,y_{10}} \left\| \begin{bmatrix} 2y_{10} + 3y_1 + 2y_3 - 1 \\ 4y_2 - y_3 \\ y_1 - 2y_3 \end{bmatrix} \right\| = 0,$$

then

$$\min_{\hat{y}_1,\hat{y}_2,\hat{y}_3} \left\langle \hat{A} \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \end{bmatrix} - \hat{d} \right\rangle = 1.$$

Furthermore, applying (5.3) gives

$$\left\langle \hat{A}\hat{A}^{W\dagger}\hat{d} - \hat{d} \right\rangle = \left\langle \begin{bmatrix} 2+\varepsilon 3 & 0 & \varepsilon 2 \\ 0 & \varepsilon 4 & -\varepsilon \\ \varepsilon & 0 & -\varepsilon 2 \end{bmatrix} \begin{bmatrix} \varepsilon \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \varepsilon \\ 1 \\ 0 \end{bmatrix} \right\rangle = 1.$$

Therefore, $\hat{A}^{W\dagger}\hat{b}$ is the least-squares solution to (5.2).

From the above examples, we see that WDGI can solve **some** problems that DMPGI cannot. We also note that it is difficult to give a general solution, and we will continue to explore it in the subsequent research.

6. Conclusions

This paper focuses on the weak dual generalized inverse (WDGI), which is one generalized DMPGI. This idea comes from the fact that any complex matrix in the complex field has the Moore-Penrose generalized inverse. Naturally, consider whether it is possible to find a class of dual generalized inverses in the dual ring that exist for any dual matrix. The WDGI is different from the DMPGI in that any dual matrix has WDGI. The weak dual generalized inverse definition is given and described by four dual equations. It exists and is unique for any dual matrix. The explicit expression and properties of WDGI are obtained. Furthermore, we discuss the relationship among WDGI, MPDGI and DMPGI, and give the equivalent characterization and illustrate with numerical examples. It is proved that the weak dual generalized inverse is different from the existing dual generalized inverse. Interestingly, DMPGI is equal to WDGI when the DMPGI of the dual matrix exists. In other words, if the problem can be solved by DMPGI, the same result can be obtained by using WDGI, and there is no need to judge the existence of WDGI, because WDGI exists for any dual matrix. DMPGI exists under strong conditions, and not all problems can be handled by DMPGI. WDGI can solve some problems that DMPGI cannot solve. Finally, we give two examples where the DMPGI does not exist, but the WDGI gives good results.

In this paper, the weak dual generalized inverse and its explicit expressions are given in the hope that they will be useful for practical problems in science and engineering, especially in dealing with systems of equations in problems such as robotics and rigid body motion.

CRediT authorship contribution statement

Hong Li, Hongxing Wang: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data included in article/supplementary material/referenced in article.

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References

[1] M.A. Clifford, Preliminary sketch of biquaternions, Proc. Lond. Math. Soc. s1–4 (1) (1871) 381–395.
[2] E. Study, Geometrie der Dynamen, Teubner, Leipzig, 1903.
[3] A. Ben-Israel, T.N. Greville, Generalized Inverses: Theory and Applications, Springer, New York, 2003.
[4] V. Brodsky, M. Shoham, Dual numbers representation of rigid body dynamics, Mech. Mach. Theory 34 (5) (1999) 693–718.
[5] D. Condurache, A. Burlacu, Dual Lie algebra representations of the rigid body motion, in: AIAA/AAS Astrodynamics Specialist Conference, San Diego, 2014.
[6] G. Vukovich, H.C. Gui, Robust adaptive tracking of rigid-body motion with applications to asteroid proximity operations, IEEE Trans. Aerosp. Electron. Syst. 53 (1) (2017) 419–430.
[7] D. Condurache, A Davenport dual angles approach for minimal parameterization of the rigid body displacement and motion, Mech. Mach. Theory 140 (2019) 104–122.
[8] A.T. Yang, F. Freudenstein, Application of dual-numbers quaternion algebra to the analysis of spatial mechanisms, J. Appl. Mech. 31 (2) (1964) 300–308.
[9] H.Y. Lee, C.G. Liang, Displacement analysis of the general spatial 7-link 7R mechanism, Mech. Mach. Theory 23 (3) (1988) 219–226.
[10] I.S. Fischer, Numerical analysis of displacements in spatial mechanisms with ball joints, Mech. Mach. Theory 35 (11) (2000) 1623–1640.
[11] I.S. Fischer, Dual-Number Methods in Kinematics, Statics and Dynamics, Routledge, 2017.
[12] G.R. Pennock, A.T. Yang, Application of dual-number matrices to the inverse kinematics problem of robot manipulators, ASME. J. Mech. Trans. Autom. 107 (2) (1985) 201–208.
[13] Y.L. Gu, J. Luh, Dual-number transformation and its applications to robotics, IEEE J. Robot. Autom. 3 (6) (1987) 615–623.
[14] K. Daniilidis, Hand-eye calibration using dual quaternions, Int. J. Robot. Res. 18 (3) (1999) 286–298.
[15] E. Pennestrì, R. Stefanelli, Linear algebra and numerical algorithms using dual numbers, Multibody Syst. Dyn. 18 (2007) 323–344.
[16] E. Pennestrì, P.P. Valentini, Linear dual algebra algorithms and their application to kinematics, in: C.L. Bottasso (Ed.), Multibody Dynamics, Computational Methods and Applications, Springer, Dordrecht, 2009, pp. 207–229.
[17] E. Pennestrì, P.P. Valentini, D. De Falco, The Moore-Penrose dual generalized inverse matrix with application to kinematic synthesis of spatial linkages, J. Mech. Des. 140 (10) (2018) 102303.
[18] D. De Falco, E. Pennestrì, F.E. Udwadia, On generalized inverses of dual matrices, Mech. Mach. Theory 123 (2018) 89–106.
[19] H.X. Wang, Characterizations and properties of the MPDGI and DMPGI, Mech. Mach. Theory 158 (2021) 104212.
[20] F.E. Udwadia, When does a dual matrix have a dual generalized inverse?, Symmetry 13 (8) (2021) 1386.
[21] F.E. Udwadia, Dual generalized inverses and their use in solving systems of linear dual equations, Mech. Mach. Theory 156 (2021) 104158.
[22] F.E. Udwadia, E. Pennestrì, D. De Falco, Do all dual matrices have dual Moore-Penrose generalized inverses?, Mech. Mach. Theory 151 (2020) 103878.