Heliyon 9 (2023) e23073

Contents lists available at ScienceDirect

Heliyon

journal homepage: www.cell.com/heliyon

Game of strokes: Optimal & conversion strategy algorithms with simulations & application

David Iheke Okorie [a,b,c,d,e,*], Joel Miworse Gnatchiglo [e]

[a] Hangzhou City University (HZCU), Hangzhou, China
[b] University of Waikato, Management School, School of Accounting, Finance and Economics, Hamilton, New Zealand
[c] Centre for the Study of the Economies of Africa (CSEA), Nigeria
[d] SD Consulting Agency (SCA), Nigeria
[e] Center for African Department Strategy (CFADS), Monrovia, Liberia

A R T I C L E I N F O

Keywords:
Sequential games
Continuity of rationality
Optimal strategy
Mover advantages conversion
Energy
Environmental resources

A B S T R A C T

Strategic decision-making for sequential move games requires rationality and continuity of rationality to guarantee maximum payoffs at all nodes/stages/levels. Rationality and continuity of rationality in a player's behaviour are not often observed and/or maintained thus, leading to less optimal outcomes. More so, the belief in an opponent's rationality, on the other hand, co-determines the level of effort a player employs while making strategic decisions. Given irrationality and discontinuity of rationality in a sequential move game with mover advantages, there are strategic steps (algorithms) to convert and/or maintain the mover advantages of an irrational player. In this paper, the conversion strategy algorithms, as well as the optimal strategy algorithms, are developed using the Beta Limit Sum (BLS) strategy model and the game of strokes. The simulation exercises confirm that the BLS strategy model is an optimal solution for the finite sequential game of strokes. One of the key applications of these strategies is that of resource economics like environmental resources (clean water, air & land). These are public goods, as such, the optimal strategy entails that the community cooperates (as one entity) and takes the same actions or strategy to maintain a healthy and clean state of the communal environmental resources.

1. Introduction

The theory of Rationalizability [1] is built on the Rational Choice Theory (RCT) which postulates that a player takes an action to maximize personal benefits [2–4]. A player is considered rational when they do what is best for them or maximize expected utility or payoff [5]. Both RCT and Rationalizability commonly assume that a player is first, able to identify what is best for them (i.e., the utility or payoff) before taking actions to achieve their goals. Generally, a player is rational when they first identify what is best for them (i.e., advantages in a game) and then, take strategic actions or decisions that maximize their advantages to achieve optimal payoffs. In finite sequential or repeated games, rationality at a decision node or stage remains vital [3,6]. However, this study shows that rationality at a decision node is only necessary but not sufficient to achieve optimal payoffs. Conversely, to achieve optimal payoffs in finite sequential games, a rational player needs to continually identify and take rational actions or decisions at all the decision nodes or stages.

* Corresponding author. Hangzhou City University (HZCU), Hangzhou, China.
E-mail address: okorie.davidiheke@gmail.com (D.I. Okorie).

https://doi.org/10.1016/j.heliyon.2023.e23073
Received 26 January 2022; Received in revised form 13 November 2023; Accepted 26 November 2023
Available online 30 November 2023
2405-8440/© 2023 The Authors. Published by Elsevier Ltd. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

Therefore, continuity of rationality suggests that at each decision node or stage, a rational player maintains the strategic path that guarantees the optimal payoff. Thus, a player needs to first identify what is best for them and maintain their best or optimal actions in a finite sequential game. This invariably means that there are chances of a player not identifying what is best for them. Therefore, a step further to just identifying and maintaining optimal decisions at each node in a finite sequential game is the ability of a rational player to convert (and maintain) an opponent's mover advantage to theirs, when their opponent fails to identify what is best for them at a decision node.

In contribution, this study is an algorithmic contribution to the field of Game Theory as it pertains to developing and applying optimal strategy paths to finite sequential games, using the Game of Strokes (GOS) as an example. Specifically, there are four major contributions of this study. firstly, this paper develops the optimal strategy algorithm that identifies and maintains mover-advantages for the finite sequential game of strokes. This optimal strategy is the strategic actions or path for a rational player at each decision node. Nonetheless, this strategy assumes that, at least, a player is rational in a finite sequential game of strokes. The optimal strategy algorithm ensures that a rational player identifies, initiates and maintains (continuity of rationality) their strategy to maximize their payoff, thus called the Optimal Strategic Path (OSP). Secondly, this study shows, using the idea of mover advantage conversion, that in finitely repeated games, identifying one's mover advantage does not guarantee an optimal outcome and rational players are capable of converting their opponents' mover advantages to their benefit in finite repeated or sequential games. Thirdly, over 10,000 simulations of the finite sequential GOS were played using the proposed Beta Limits Sum (BLS) strategy. The proposed BLS model strategy includes three algorithms. These are the Identification Algorithm, the Mover Advantage Maintenance Algorithm and the Mover Advantage Conversion Algorithm. Based on the simulation results, the performance of the proposed BLS model is second to none. Lastly, the application of these developed optimal algorithms in the area of Resource Economics is itemized and discussed. Generally, this paper builds on the nature of rational choices in strategic games [5] to validate the existence of rationality in the behaviour of a player when faced with a strategic decision-making scenario or game. By implication, this study reveals the player's rationality, as seen in the optimal strategic path of the sequential game of strokes. Secondly, this paper shows the need to maintain an optimal strategy i.e., continuity of rationality, to guarantee a superior outcome. Thirdly, this study presents evidence of mover advantages conversion in a repeated or sequential game. Lastly, this study shows the application of this optimal strategic decision-making in the field of Resource economics. In a nutshell, these contributions & implications highlight the relevance and significance of this study. They also buttress the novelty of this article in solving and providing the ever-first optimal solution for the finite sequential GOS.

The next vital point to illustrate is the relevance of the GOS. Every model is an abstraction of reality. Like models, strategic games like the GOS are abstractions of reality. The GOS is strategically designed to depict decision-making processes in a finite sequential game. GOS seeks to train players on how to identify and act on what is best for you or your team. In real-time, business corporations constantly engage in sequential pricing and output decision processes over time (finite or infinite). Like the GOS, models like the Cournot model, the Bertrand model and the Stackelberg model are all examples of reality abstractions that seek to equip decision-makers with the right skill set, knowledge and readiness to make simultaneous and sequential decisions that are optimal and maximize payoffs or benefits. The decision-making skills of the GOS are an essential part of high-level managers who are faced with other skilled competitors to maximize individual or group benefits or payoffs. An example would be making and taking sequential price and output decisions in an oligopoly market wherein every firm or company wants to maximize their market share, minimize costs, maximize profits, etc. Unlike Cournot, Bertrand and Stackelberg models which have existing optimal solution models, the GOS, until now, has no optimal solution model. This is the gap filled by this study. The proposed optimal solution model for the GOS, in this study, does more than just solve the GOS. In addition, it identifies and converts competitors' advantages to one's benefit whenever an irrational decision is identified.

To juggle our minds, Game of Strokes (GOS) is a popular sequential game wherein players (simplest case: 2 players) strike out β strokes at each sequential node out of ϑ strokes. β can be a number between a minimum of one and a maximum of β_{max} at each node. β *and* ϑ are determined at the beginning of the game. Whoever strikes out the last stroke(s) wins the game. In simple and clear terms, β *and* $\vartheta \in N^{++}$. For instance, in Fig. 1, $\vartheta = 6$ and say $\beta_{max} = 3$. This means that a player can strike out a minimum of 1 ($\beta_{min} = 1$) stroke and a maximum of 3 strokes ($\beta_{max} = 3$) in a turns or node. Interestingly, for $n-players$, there always exists a mover's advantage (α) for all β *and* ϑ i.e. $\alpha \ \varepsilon \ [0, \ldots, \beta_{max}] \ \forall \ \beta$ *and* ϑ. $\alpha > 0$ means only a first mover advantage exists at the beginning of the game, otherwise, a second mover advantage.[1] Therefore, considering Fig. 1 and setting $n = 2$, every rational player would easily observe that there is a first-mover advantage when $\beta_{max} = 3$ and a second mover advantage when $\beta_{max} = 2$. Also, this can be illustrated in a decision tree (see Fig. 2). The optimal strategy set for $\beta_{max} = 3$ that achieves $\alpha = 2$, first mover advantage, is $\{\{2,3\},\{2,2\},\{2,1\}\}$ while that of $\alpha = 0$, second mover advantage, given $\beta_{max} = 2$ is $\{\{2,2\},\{2,1\},\{1,2\},\{1,1\}\}$. Each element of the optimal strategy sets is an optimal strategy path that solves the finite sequential GOS $\forall \ \beta$ *and* ϑ. The optimal path $\{2,3\}$ in simple terms means, to win the game, the first player should strike out 2 strokes at the initial or first node and strike out 3 strokes (if the second player strikes out 1 stroke at the previous node). These optimal paths depend on the choice of strokes the second player chooses to strike out, however, a rational player needs to be consistent (continuity of rationality) with an optimal strategic path to assure victory. Deviations from an optimal strategy path present one's opponent an opportunity to convert existing mover advantage to their benefit.

It is important to note that an irrational player would most likely fail to identify these mover advantages. Secondly, starting with the optimal strategic path does not also guarantee winning the game as continuity on this optimal strategy path is vital in achieving success in this game. Using the example as illustrated in Figs. 1 and 2, when $\beta = 3$ and $\alpha = 1$ exists. The fact that the first player strikes

[1] It is necessary to point out that this mover advantage exists AT THE BEGINNING of the game. In other words, it could be converted to another player/mover's advantage during the course of the game.

out 2 strokes at the first decision node or turn still does not guarantee that they win even though this is a rational move at the first node but, the continuity of the rational moves given the opponent's action at all decision nodes guarantees the first player wins the game.

Taking a step further into this game, when ϑ is very large, an example is shown in Fig. 3, it becomes difficult for any player to ascertain a mover advantage. This difficulty, however, does not deny the existence of a mover advantage in the game of strokes. We could make a game of strokes more interesting and complex by varying one or more of the model parameters like $n, \beta, \sigma, \vartheta,$ etc. For the purpose of this paper, a player is said to behave irrationally when they cannot identify or/and consistently maintain the optimal strategy path that maximizes their payoff. Conversely, Discontinuity of rationality is the scenario when an irrational player does not continue taking the optimal strategy path when it exists. In other words, a rational player must exhibit the continuity of rationality while an irrational player must not.

2. Literature review

Interactive epistemology begins with the basic assumption of rationality and the common belief in rationality [7]. On this basis, taking Strategic actions when presented with scenarios that warrant it to maximize one's benefit or payoff is not often the outcome as expected [8,9], thus, creating the puzzle of why players do not take the optimal actions that maximize their payoffs. Empirically, it is clear that in strategic decision-making games, players do not often realize the existence of an optimal strategy or Nash Equilibrium(ria) which gives the best outcome or payoff (assuming common knowledge of the game and with or without prior information) [10–13]. Therefore, the assumption of rationality is not often validated in every game. In cases a player is not rational enough to maximize their payoff, what does a rational opponent do to convert and/or maintain their (opponent's) mover advantage to their benefit in a sequential move game?

Battigalli and Siniscalchi [14] studied epistemology in dynamic games as they provide and analyze the solution to dynamic games which involves the conditional belief of a player of the other player's rationality conditioned on the historical actions of that player. Their results show common certainty of the opponent's rationality conditional on an arbitrary collection of histories in multi-stage games with observed actions and incomplete information. However, in their 2002 paper on belief and forward induction reasoning, they noted that forward induction reasoning governs the players' belief revision process as they provide an epistemic analysis of forward induction solutions on complete and incomplete information games [15]. Brandenburger [16] presented the paradox of game reasoning as it has played significant roles in interactive epistemology which studies players' beliefs, knowledge, etc. which he called the paradox of Backward Induction and that of iterated weak dominance. These paradoxes are more or less, the assumptions that do not hold when the backward induction path is not played and the iterated deletion is on the assumption that they are not expected to occur when they can also occur given the rationality of the players.

Rubinstein [13] showed that in a theoretical game, common knowledge aids the rational behaviour of the players using a closely related game to the coordinated attack problem. The outcomes showed that a coordinated game with common knowledge is significantly different from a coordinated game with almost common knowledge [13,17]. Aumann, however, noted in their paper, agreeing to disagree, that the fact that each player has a common belief of the other player(s)' rationality suggests that these players cannot agree to disagree. In other words, rationality in the behaviour of players with some level of common knowledge will always agree to work together [12].

Payoffs in a strategic game depend on the strategic actions of the players, their belief in other players, what they think others believe about other players, etc. Beliefs are dependent on reality thus, there are always sequential equilibria in sequential games [18]. Rationality in the behaviour of the players entails the iterative elimination of weakly dominated strategy. This behaviour is based on the assumption that each player knows exactly the other player's payoff without doubts but when doubts exist, Dekel and Fudenberg [19] showed that the only action close to rationality is one round elimination of weakly dominated strategy followed by the strongly dominated strategy. Bornstein & Yaniv [20] investigated if rationality is seen more when individuals make independent decisions relative to making group decisions via experiments (Ultimatum game) and the result shows that rationality while working as a group is less relative to individual decision-making scenarios.

Sequential games are a key policy strategy and have been applied in many other aspects other than game theory. They include option contracts [21], carbon tax and emission [22], electricity market [23], robbers and cops [24], traffic monitoring and vehicular networks [25], labour union bargaining and entry [26,27], inter-generational resource sharing [28], nuclear fuel transition [29], fleet management [30], endowment and property rights [31], defence resource allocation [32–35], cluster analysis [36], climate policy [30, 37], stochastic programming [38], behaviour and time horizon [39–41], security cooperation [42], parental care [43], competitiveness and gender differences [44], health insurance choices [45], grid applications [46], candidate interviews [47], classroom quizzes [48], mental accounting [49], rumour propagation [50], desire thinking [51], kind and money transfer [52], social influence [53], football game [54], ambiguity attitude [55], etc. These are only a few of the exploits, in literature, that utilize the sequential game approach.

In emphasizing the significance of this study, other existing studies on sequential games have shown that irrational behaviours are inherent even though rationality is assumed. Therefore, in cases of irrationality (i.e., at least, a player behaves irrationally), firstly, how does a rational player identify the mover advantages of a finite sequential game like the GOS? Secondly, what does a rational player do in other to maintain the mover advantage to their benefit? Thirdly, in cases of discontinuity of rationality, what does a rational player do to convert the opponent's mover advantages to their benefit? Fourthly, is the developed model's strategy optimal? Hence, this study takes a step further into solving sequential games, like the Game of Strokes (GOS), by answering these key questions. In contribution, this paper builds on the fact that rationality and its (dis)continuity are inherent in sequential games such as the GOS to develop optimal strategic algorithms for identification, maintenance, and conversion of mover advantages. The relevance of this paper on strategic decision-makers is therefore summarized into the identification, maintenance, and conversion of mover advantages in sequential strategic decision-making. Furthermore, more discussions on the application of these strategies in other areas, such as resource and environmental economics, are provided.

3. Methods

The Optimal Strategic path algorithm is developed in this section. This is the model that identifies mover advantages (player's benefits), maintains the mover advantage at each decision node or stage, and converts the opponent's mover advantages when the opponent takes irrational actions or decisions.

3.1. The game of strokes (GOS)

The Game of Strokes (GOS) is a sequential game that will always have a mover advantage. These mover advantages are identified considering the model parameters explained in section 1.0. For better comprehension and simplicity's sake, the GOS is defined as.

1. $n = 2$ i.e. Two Players
2. Total strokes, ϑ, and maximum stricken strokes per node, β_{max} are determined.
3. At each node, players strike β strokes sequentially ($\beta \in N^{++}; \beta \geq 1;\ 1 \leq \beta \leq \beta_{max}$).
4. The player that strikes out the last stroke(s) wins.
5. $\{n, \beta, \vartheta, \beta_{max}\} \in N^{++}$.

3.2. Beta Limit Sum (BLS) strategy model

In any sequential game such as the game of strokes, there will always be sequential equilibria [18]. The sequential equilibria of the GOS coincide with the optimal strategic paths. To understand this strategy, throughout the remaining parts of this paper, let X_i be First player's choice (number of strokes cancelled) in the *ith* turn and Y_i be the Second player's choice in the *ith* turn. After identifying the player with a mover advantage, the optimal strategy path for the mover-advantaged player in the game of strokes hinges on leaving a multiple of ($\beta_{min} + \beta_{max}$) after each turn. For all versions of GOS, β_{min} equals 1, therefore, the mover-advantaged player needs to leave a multiple of (β+1) strokes after each turn to guarantee they win the game. In other words, for each turn, the sum of the strokes removed by you and your opponent must equal (β+1), this guarantees that the number of strokes left after a mover-advantaged player's turn will be a multiple of (β+1). The name of this strategy is, therefore, the *Beta Limits Sum (BLS)* Strategy.

To develop the Beta Limits Sum (BLS) strategy model, the following additional assumptions are made.

1. At least, a player is rational: A rational player wants to maximize their payoffs no matter the cost (i.e., the player is not altruistic). Specifically, a rational player identifies, maintains, and converts mover advantages to establish an optimal strategy path.
2. A rational player must exhibit continuity of rationality.
3. Perfect Information: each player knows the choice/action of the player that played before them as they make their decision (sequential move game), except for the first mover in the first round.
4. $\alpha \in [0, 1, 2, \ldots, \beta]\ \forall\ n, \beta, \sigma$ *and* ϑ; Mover advantage always exists
5. $\beta_{max} \neq \beta_{min}$: This implies that $\beta > 1$. However, if $\beta = 1$, then $\beta_{max} = \beta_{min} = 1$ then when ϑ *is even* the First-Mover always wins and when ϑ *is odd* the Second Mover always wins without effort or thinking (will of the gods).
6. $\vartheta > (\beta + 1)$.
7. $\sigma = \max\ (Mul(\beta+1)) \leq \vartheta\beta+1)) \leq \vartheta Mul$ i.e., σ is the highest multiple of (β+1) such that $\sigma \leq \vartheta$.

These underlining assumptions guide the formation of the games for all complexity versions of the GOS and guarantee that there will always exist a mover advantage in every game. The identification of a mover advantage becomes rather simple in this model given the model parameters. After the identification of a mover advantage, the next essential action is to establish and maintain the optimal strategic paths (optimal strategic set). Finally, not all players can identify, establish and maintain an optimal strategy path. In such an irrational decision scenario, a rational player needs to be able to convert the irrational player's advantages or benefits to themselves. These are the three (3) components of the BLS strategy model, proposed in this study.

3.2.1. BLS identification of a mover advantage

Recall that $\forall\ \delta, \beta$ and $\exists\ \alpha = i\ \exists\ i \in [0, 1, 2, \ldots, \beta]$. This fact only guarantees the existence of a mover advantage but does not say which player has or enjoys the mover advantage in the finite sequential game of strokes. To identify the player with the mover advantage given any δ and β, the 7th BLS model assumption is required. The player with the mover advantage can be identified in two steps: first, determine the value of $\alpha \in [0, \ldots, \beta]$ such that $\vartheta - \sigma = \alpha$ and next, there exists a second-mover advantage if $\alpha = 0$ while there exists a first-mover advantage if $\alpha \neq 0$.

3.2.2. BLS maintenance optimal strategy equilibria path

3.2.2.1. First mover advantage ($\boldsymbol{\alpha} > \mathbf{0}$) *optimal strategy algorithm.* As long as ϑ is not a multiple of $(\beta+1)$ there will always be a first-mover advantage, i.e. $\alpha \neq 0$. When this first-mover advantage exists, the first play has a unique first-step action to take and subsequently plays the Beta Limits Sum (BLS) strategy. The steps are as follows: Using the definitions aforementioned, $\boldsymbol{X}_i$ & $\boldsymbol{Y}_i$ and $\alpha = \vartheta - \sigma{:}\alpha \neq 0$. The optimal strategy path algorithm of the first mover becomes $\boldsymbol{X}_1 = \alpha$ strokes and subsequently play the BLS maintenance strategy, i.e. play

$$\boldsymbol{X}_i = (\beta+1) - \boldsymbol{Y}_{i-1} \text{ where } i \geq 2 \tag{1}$$

3.2.2.2. Second mover advantage ($\boldsymbol{\alpha} = \mathbf{0}$) *optimal strategy algorithm.* As long as ϑ is a multiple of $(\beta+1)$ there will always be a second-mover advantage. This guarantees that $\varepsilon = 0$ and such situations, however, implies that the first mover should cancel 0 strokes but this is not obtainable given that $\boldsymbol{X}_i, \boldsymbol{Y}_i \in [1, \beta]$. This, however, shifts the mover advantage to the second mover and whenever this is the case, the optimal strategy path algorithm that guarantees the second mover wins is the BLS strategy, i.e.

$$\boldsymbol{Y}_i = (\beta+1) - \boldsymbol{X}_i \text{ where } i \geq 1 \tag{2}$$

An illustration of the BLS strategy using the GOS shown in Fig. 3 i.e. Example 2. First, we define the model parameters, $\vartheta = 92$ and setting $\beta = \beta_{max} = 2$ gives $\varepsilon = 2$ implying a first mover advantage and setting $\beta = \beta_{max} = 3$ gives $\varepsilon = 0$ implying a second mover advantage. For $\beta = \beta_{max} = 3$, the second mover quickly adopts the BLS strategy to maintain their mover advantage and wins. On the other hand, when $\beta = \beta_{max} = 2$, the first mover must first strike out 2 strokes then, play the BLS to maintain their mover advantage and win. The optimal strategy sets developed for Example 1 adopt the BLS strategy. It is vital to emphasize that this rational strategy (BLS) is only a necessary condition for winning a GOS given one has a mover advantage while a sufficient condition is the continuity of rationality. This implies that for a first-mover advantaged player, the first action of striking out ε strokes and/or using the BLS strategy till an intermediate stage does not guarantee a win, vice versa the second mover advantaged player. Any deviation from the BLS strategy makes a mover-advantaged player prone to losing the game and if their opponent is a rational player, this is a perfect opportunity to convert the mover advantage to their benefit. Just a single period of deviation is enough to convert a mover advantage for a rational player.

3.2.3. BLS conversion algorithm

3.2.3.1. Irrational responses and belief of rationality. Irrational responses or actions from one's opponent inform a player about the rationality stance of their opponent and thus, explain the concept, and belief of rationality in a game [2]. The maximization of payoffs in a strategic decision-making game is a function of the choice(s) of action taken by a player. A player's strategic action further depends on their perception or belief in their opponent's rationality or K-levels of reasoning [56]. Intuitively, when a rational player is matched with another player whom they believe is irrational, the rational player can apply a necessary (but relatively lower) level of reasoning and win but if they believe their opponent is rational, to win, they have to apply a higher level of reasoning [57]. Therefore, the belief of rationality is not instantaneous or given but can be deduced from the actions taken by one's opponent. A player will automatically believe that their opponent is (ir)rational when they observe that their opponent had taken an(a) (less-)optimal action and vice versa. Hence, this belief in the opponent's rationality suggests the level of effort a player employs.

An irrational behaviour is feasible both at the beginning of the game or at an intermediate stage of the game of strokes. The identification of irrational behaviour is equally important especially when there are possibilities of mover advantage conversion. In the game of strokes, the conversion mechanism is somewhat different, depending on the round or stage (i.e., first or intermediate stage) the irrational action was taken by the mover-advantaged player.

3.2.3.2. Continuity and discontinuity of rationality. Based on the arguments of this study, it is not enough to start off making rational choices but its continuity. Discovering a mover advantage is a necessary condition while taking and maintaining the rational choice path is the sufficient condition towards winning a game of strokes. Therefore, that a player discovers they have a mover advantage is not informative of winning a game except in a situation where the player with the mover advantage is continuously making rational choices at each round or decision node of the game. Thus, identifying when one has a mover advantage is rational and this rational behaviour can be continuous or discontinuous. Moreover, a discontinuity on the optimal strategy path can be observed at the beginning (round 1) or any intermediate round of the game. Therefore, a rational player looks out for the continuity as well as discontinuity of rationality in the behaviour of their opponent. The question becomes, how does a rational player continue behaving rationally when they do not have a mover advantage? In other words, we are considering the conversion mechanism of mover advantages from an irrational player to a rational player. In a game of strokes, we have established that there is always either a first or a second-mover advantage and these advantages can as well be converted to a second or a first-mover advantage respectively when an advantaged player behaves irrationally at the beginning or intermediate round of the game. Therefore, this conversion is not only possible at the beginning of the game but can also take place at any point in the game aside from the terminal stage or round. Firstly, we examine the discontinuity of rationality at the beginning of the game via.

4. Discussions, results and application

4.1. Irrational advantaged first mover and a rational second mover

When there exists a first-mover advantage and the player does not take the optimal strategy path, they are said to be an irrationally Advantaged First Mover. To illustrate and discuss the conversion algorithm in this scenario, let's first establish that X_1 is the irrational action i.e., $X_1 \neq \alpha$. By definition, the first mover's irrational choice of action $X_1 : X_1 \in [1, \beta_{max}] \cap \alpha^c$ where $\alpha \in N^{++}$. The conversion algorithm of the first-mover advantage to a second-mover advantage requires the second mover to strikeout $Y_1{}^*$ strokes such that:

$$X_1 + Y_1{}^* = \alpha \text{ if } X_1 < \alpha \tag{3}$$

$$X_1 + Y_1{}^* = (\beta+1) + \alpha \text{ if } X_1 > \alpha \tag{4}$$

and subsequently, the second player adopts the BLS maintenance strategy path illustrated in equation (2). $Y_1{}^*$ is the unique choice that converts the first-mover advantage to a second-mover advantage. Next, we discuss the second possible case at the initial node of the finite sequential GOS.

4.2. Irrational advantaged second mover and a rational first mover

When there exists a second-mover advantage and the second-mover is irrational and does not take the optimal strategy path (behaves irrationally) i.e., their choice of action is $Y_1 : Y_1 \neq (\beta + 1) - X_1$. To convert this second-mover advantage to a first-mover advantage, the first-mover should choose $X_2{}^*$ such that satisfies the condition;

$$X_2{}^* + Y_1 + X_1 = Mul(\beta+1) \tag{5}$$

and subsequently, play the BLS maintenance strategy in equation (1). Where $Mul(\beta+1)$ represents any element in the set of multiples of $(\beta + 1)$. These unique algorithms can only guarantee the conversion of second-mover advantages from an irrational player to a rational player's first-mover advantage at the first node or round or stage. That is, the mover advantage would shift to the disadvantaged second mover from the advantaged first mover. It is important to reaffirm that these conversion algorithms are peculiar and unique to irrational choices at the beginning of the GOS. What if this irrationality occurs at an intermediate node and not at the initial node? The conversion strategy for an irrational decision of an advantaged mover at an intermediate node is equally developed and discussed in the next subsection.

4.3. Irrationality at an intermediate node or round or stage of a GOS

Sections 3.1 and 3.2 discuss cases of irrational decisions at the first node of the GOS. However, a player can only identify mover advantages but neither maintain nor convert an opponent's mover advantages to their benefit at this first node. Maintaining mover advantages and converting an opponent's mover advantage is only possible at an intermediate node or stage or round of the finite sequential GOS given an irrational (or suboptimal) action or decision. Thus, in a situation, where an opponent discovers they have the mover advantage and initiates the optimal strategy path at the first or earlier stage(s) of the game but does not continue to follow the optimal strategy equilibria path, the BLS model, how should a rational player convert their opponent's mover advantage to theirs? This is a case of discontinuity of rationality in an intermediate stage or round of a GOS i.e., X_1 and Y_1 are elements of the optimal strategy path. Generally, the path that guarantees winning given a mover advantage is that which ensures the cancellation of ρ *strokes* after *jth* player in the *ith* round or stage for *player* $- j$.

$$\rho = \alpha + i(\beta+1) \tag{6}$$

$j = 1$ *or* 2, i is the number of rounds. When $\alpha = 0$, $i \in [1, 2, \ldots]$ and when $\alpha \neq 0$, $i \in [0, 1, \ldots]$ i.e. a first-mover owns the advantage and quickly strikes out α strokes in their first choice and subsequently adopts the BLS strategy (if they are rational). In this case, the first action of this first mover is indexed as $i = 0$ i.e. $X_0 = \alpha$ (with a corresponding Y_0) while the subsequent action of the first mover becomes X_1 such that follows the BLS strategy i.e. satisfies equation (1) if they are a rational player. This adjustment on the indexation of i round is necessary to satisfy equation (6). The *jth* player has the mover advantage if they strike out ρ strokes and wins if they maintain striking out ρ strokes at the end of each round or stage. Recall, the optimal strategy path hinges on who completes the Beta Limits Sum $(\beta+1)$ cycle. In other words, the player that always completes the Beta Limits Sum cycle after their decision wins. Moreover, based on the BLS strategy, the player with the initial mover advantage completes the BLS cycle if they make rational choices at each node i.e., they are guaranteed to win if they maintain the BLS strategy path (continuity of rationality). However, when at an intermediate stage or round, a mover-advantaged player strategically misses taking a decision that completes the BLS strategy path or cycle, the mover-advantage conversion mechanism entails that the opponent completes the cycle of $(\beta + 1)$. That is, for a second (first) mover to convert a first (second) mover advantage to their benefit, once the initially advantaged player misses to complete the BLS cycle of $(\beta + 1)$, the rational second (first) player should strike out $Y_i(X_i)$ strokes, such that satisfies equation 2 (1) respectively.

To illustrate the conversion algorithm, we adopt Example 2 (Fig. 3) of = 92 , when $\beta = 2$ and $\alpha = 2$. Firstly, let the irrational decision of the advantaged first player be at the initial stage hence, let be $X_1 = 1$. Based on equation (3), $Y_1{}^* = 1$ and subsequent maintenance of the BLS strategy (in equation (2)) has converted the initial first-mover advantage to a second-mover advantage. Secondly, let the irrational decision of the advantaged first player be $X_1 = 3$. Based on equation (4), $Y_1^* = 2$ and subsequent maintenance of the BLS strategy (in equation (2)) has converted the initial first mover advantage to a second-mover advantage. Similar scenarios can be deduced for an initial second-mover advantage conversion to a first-mover advantage due to a first-stage irrational decision of an advantaged second-mover using equation (5). Secondly, equation (6) is used for the conversion of either an initial first or second-mover advantage to a second or first-mover advantage as a result of an intermediate irrational decision of an advantaged first or second mover in a GOS.

4.4. Cooperation, collusion to cheat and relaxing the BLS model assumptions

When there are more than two players, there are two possible resulting scenarios. First, two or more players can join and cooperate as a single group player. In such cases, the BLS strategy remains robust. However, for the second scenario, all players play as single individual units in the GOS but cooperate in terms of information sharing and general agreements or decisions on the number of strokes each team member strikes per turn. This is just like forming a cartel. Nonetheless, in cooperative sequential games, there are still arguments on what level of cooperation or transparency, from the cooperating players, is good or optimal [58]. However, in this scenario, at most, $n-1$ players can form a cartel or/and collude to cheat, i.e., cooperating so that one of them wins the game. This involves a reasonable amount of transparency expected on the part of the cooperating players. This, however, entails that, at each turn, the cooperating group decides the optimal number of strokes or tokens to cancel as a group and for each group member and leaves at least $m * (\beta+1)$ strokes or tokens after the last group member plays, where m is the number of unit players (individuals and groups playing as a unit player) aside from the group members. This guarantees that a member of the cooperating team wins. It is also important to note that, for the cooperating team or group, there are chances of one or more members not cooperating at the terminal stage as long as they can win the game for themselves. This would be a situation wherein the number of strokes or tokens left is less than β, hence, the team member or player would rather strike them all out than leave any stroke for the next group member or team player irrespective of the general agreement held in the cooperating group.

Relaxing assumption one (1) and given that it is one of the Beta Limits Sum (BLS) strategy model parameters, however, makes the optimal strategy of the game rather complex. However, the BLS can still be applied. Applying the BLS strategy when $n > 2 \ \forall \beta$ is to ensure that the number of strokes left after you take your decision (strike out stroke(s)) is always a multiple of the sum of your decision and other players' decisions. If any player can maintain this strategy, they are guaranteed to win the game. Maintaining this strategy all through the game is rather difficult since you have no control over other players' decisions thus, the total number of strokes left will not always be a multiple of the sum of your decision and other players. Therefore, playing the BLS strategy, in this case, entails some level of deviations from this optimal strategy path but a rational play is saddled with the responsibility of seeking out this path when available to win the game. When assumptions two (2) and three (3) are relaxed, predicting the player with the mover advantage and the winner of the game is very much random due to irrationality and discontinuity of rationality. Relaxing assumption four (4) suggests a simultaneous GOS. This is seemingly impossible to play the GOS as a simultaneous move game. Relaxing assumptions five (5) and eight (8) will change the basic principles of the GOS and possibly become another game entirely. Based on assumption one (1), assumption six (6) will always hold. However, when we relax assumption one (1) then assumption six (6) fails i.e., there will not always exist a mover advantage for all versions of the GOS when $n > 2$. Relaxing assumption seven (7) makes strategic decision-making useless in any GOS since the winner is already known from the beginning of the game (i.e. will of the gods) even when assumption one (1) is relaxed. Assumptions nine (9) and ten (10) are key in building the Beta Limits Sum (BLS) model, relaxing these assumptions will outrightly make the BLS strategy inefficient and ineffective.

4.5. Simulation experiment results

Computer-based simulation exercises or experiments were conducted to further strengthen the BLS strategy model in optimally solving the finite sequential GOS. 10,000 simulated GOS were solved using the BLS model. In these simulations, the total number of strokes, ϑ, and the maximum number of strokes per node, β_{max}, are randomly sampled. To include different scenarios, β_{max} is randomly simulated without replacement while ϑ is randomly simulated without replacement.[2] In these simulations, the parameters follow uniform distributions described as $\beta_{max} \in [1, 500]$ and $\vartheta \in [501, 2000]$.

4.5.1. Benchmark BLS model algorithm scenario

The benchmark BLS model algorithmic steps are itemized below.

1. Set $k = 1$
2. Simulate $\beta_{max} \sim U[1, 500]$ and $\vartheta \sim U[1, 500]$ to design a finite sequential GOS.
3. Let the opponent's decision, $Y_k \sim Bin(n = 1, p = 0.5, trials = \beta_{max})$.
4. Apply the BLS mover advantage identification algorithm.
5. Decide the first and second mover at the initial node based on the BLS result in step 4.

[2] Suffices to mention that the results are not affected, in anyway, when β_{max} is randomly simulated with replacement and ϑ is randomly simulated without replacement. This is also the case, when both ϑ and β_{max} are simulated randomly with and without replacements. In all these designs, the BLS strategy model solves the GOS perfectly.

6. Apply the BLS mover advantage maintenance algorithm until the GOS is solved (or won).
7. Repeat step 1 to step 6 for $k \in [2, 3, \ldots, 10000]$.

This is the benchmark scenario or business-*as*-usual scenario since the decision of which player moves first or second is strictly determined by the BLS model. As such, deviations from the optimal solutions of each GOS are expected to be zero. That is to say that, the BLS is expected to win for all the times the GOS is simulated. Based on the 10,000 simulated GOS, the BLS model solved and won every single simulated GOS. Some of the solved GOS simulated results are presented in Table 1. Table 1 shows: the total strokes, the maximum strokes per node, the advantaged mover, the initial node decision, and the BLS solutions. Based on the simulation results, the BLS solves and wins all the 10,000 simulated GOS. Only four (4), out of the 10,000 simulated GOS are reported in Table 1.

4.5.2. BLS model without conversion algorithm scenario

The benchmark BLS model in subsection 4.5.1does not utilize the conversion algorithm in all the 10,000 simulations because it optimally decides whether to be the first or second mover. Therefore, it always has the identified mover advantage and thus, follows the BLS optimal path. In this scenario, this assumption is relaxed. That is to say, the BLS identification algorithm is truncated to allow the BLS strategy model to solve each GOS without the choice of mover advantage. To further make this scenario interesting, the BLS conversion algorithm is also truncated. This makes the BLS strategy unable to identify the opponent's rationality discontinuity and take optimal actions, to convert the opponent's mover advantage. The algorithmic steps used in this scenario are as follows.

1. Set k = 1
2. Simulate $\beta_{max} \sim U[1, 500]$ and $\vartheta \sim U[1, 500]$ to design a finite sequential GOS.
3. Apply the BLS mover advantage identification algorithm
4. Let the opponent's decision, $\boldsymbol{Y}_k \sim Bin(n = 1, p = 0.5, \boldsymbol{trials} = \beta_{max})$.
5. Apply the BLS mover advantage maintenance algorithm until the GOS is solved (or won).
6. Repeat step 1 to step 5 for $k \in [2, 3, \ldots, 10000]$.

The different simulation results are presented in Table 2. It is important to mention that from steps 4 and step 5, the BLS model is only applied as either the first or second mover/player while the computer remains the second or first mover/player, respectively, for all the simulated GOS. In Table 2, category A shows the results when the BLS maintenance strategy is applied as the second mover/player while category B is when the BLS maintenance strategy is applied as the first mover/player. As expected and based on the results in Table 2, for all the simulated GOS, the BLS model optimally solved all the simulated games when second-mover advantaged GOS games were stimulated but failed to optimally solve the games when first-mover advantaged GOS games were simulated, shown in category A. This is mainly because of the truncation of the BLS conversion algorithm. The BLS maintenance algorithm, thus, becomes a response action to an opponent's decision. Thus, when second-mover advantages exist, the BLS maintenance algorithm performs optimally. Conversely, the same simulated GOS games are played the second time but alternating iteration or step 3 and iteration or step 4. Now, the BLS model optimally solved all the simulated games when a first-mover advantage exists but failed to solve the games with second-mover advantages, as shown in category B. This is mainly because only the BLS conversion algorithm is truncated, thereby, the applied BLS model still identified mover advantages. Therefore, the FM and SM advantaged simulation results in Table 2, category A, are exactly the opposite when step 4 and step 5 are interchanged, making the first mover player apply the BLS model while the computer becomes the second player, in Table 2, category B. In general, alternating the first and second mover of any simulated GOS game will be solved optimally by the BLS model (without conversion algorithms) as a first (second) mover or player when the first (second) mover advantage exists in the finite sequential GOS. Therefore, the results in Table 2 further confirm that the BLS strategy is a complete set algorithm. That is, winning a GOS is not guaranteed when the BLS strategy model is applied in parts. To apply the BLS strategy model, the identification, maintenance, and conversion algorithms must be applied simultaneously. This is further buttressed in the next subsection.

4.5.3. BLS model with conversion algorithm scenario

1. Set k = 1
2. Simulate $\beta_{max} \sim U[1, 500]$ and $\vartheta \sim U[1, 500]$ to design a finite sequential GOS.
3. Randomly decides the First and Second player
4. Apply the BLS mover advantage identification algorithm
5. Let the opponent's decision, $\boldsymbol{Y}_k \sim Bin(n = 1, p = 0.5, \boldsymbol{trials} = \beta_{max})$.
6. Apply the BLS mover advantage conversion algorithm
7. Apply the BLS mover advantage maintenance algorithm until the GOS is solved (or won).
8. Repeat step 1 to step 7 for $k \in [2, 3, \ldots, 10000]$.

This simulation scenario only added the BLS conversion algorithm, while introducing a random choice or decision of who plays first or second (step or iteration 3). The results for this scenario are synonymous with that of the benchmark scenario presented in Table 1. That is to say that, the BLS strategy model solves all the simulated finite sequential GOS irrespective of whether it is applied when a mover advantage exists or not. This is due to the application of the mover advantage conversion algorithm which converts the opponent's mover advantage immediately the opponent makes an irrational decision at any decision node. It is important to state that the computer, the opponent, always makes irrational decisions at every node since its choice or decision follows a binomial distribution with β_{max} trials. A couple of these BLS strategy model solutions are presented in Table 3.

4.6. Application in resource economics

The BLS model or strategy could be adopted and applied in several decision-making units and areas. These areas and units may include business or firm units, household units, environmental economics, resources economics, health economics, competitive markets, monopolistic competitions, duopoly, oligopoly, duopsony, oligopsony, etc. In any area where sequential decisions are made to optimize (maximize or minimize) payoffs, as a group or individually, the BLS strategy can be applied. This is because the BLS strategy deals with the identification, maintenance, and conversion of mover advantages in a sequential game, using the game of strokes as an example. The application of the BLS model only requires that there is, at least, a rational player or group of players acting as a unit in a sequential move game. As such, the BLS model identifies and shows the optimal path strategy algorithms towards optimizing payoffs given continuity or discontinuity of rationality. For an illustration of the application of the BLS model, resource economics scenarios are highlighted and discussed.

Resource economics deals with the optimal allocation of economic and natural resources amongst uses. This allocation is dynamic and time-variant as it requires adjustments to maintain the optimal path over time [59]. As such, one could view it as a sequential game played over periods. Generally, some good examples of economic resources are energy and environmental resources [60]. Environmental resources include water resources, land resources, and air resources. There are several uses for these resources such as energy sources, agriculture, building and construction, health, living well-being, etc. As such, the optimal allocation of these environmental or economic resources is crucial for the concerned authorities. Based on the developed BLS strategy, an optimal strategy for the allocation of these economic or environmental resources includes strategic actions like cooperation. As such, all the players act as a single unit to achieve and maximize one target, goal, benefit, or payoff. For instance, considering the climate change scenario, every economy is advised to take emission mitigating or abatement actions to reduce the overall level of emission in the economy over time [61–64]. Given this payoff or target or benefit or goal, the users of these environmental or economic resources (water, land, and air) are better off when they cooperate and work as a team in the use of the allocated resources to achieve reduced levels of emissions while using these resources [65].

Nonetheless, given a scenario where a player or group of players acting as a unit, decides to pursue other different targets, the mover advantage of such player or unit could be identified and converted to their detriment or loss, as detailed by the conversion algorithm of the Beta Limits Sum (BLS) strategy. This could be done in several ways which may include terminating or restricting the access of the player to the public economic (and environmental) goods or resources such as water, land, etc. Such actions are geared towards identifying an opponent's mover advantage and terminating or converting it to the opponent's detriment. As such, the general and common goal or payoff or objective can be achieved or actualized.

5. Conclusion

In conclusion, this work sets out to not only validate the existence of rationality in the behaviour of strategic decision-makers but to establish the extent of rationality in the players' decision behaviour and to show the chances and possibilities of mover advantage conversion in a sequential move game. This paper, therefore, provides answers and evidence to the existence of optimal strategic decisions when mover advantages are identified as well as strategic conversion mechanisms for a mover advantage when a supposed advantaged player fails to identify and maintain such advantages in a sequential move game (GOS). The proposed Beta Limits Sum (BLS) model or strategy, as developed in this paper, can also be applied in other fields and aspects of sequential move competitions such as competitive markets, monopolistic competitions, duopoly, oligopoly, duopsony, oligopsony, etc.

There were some limitations encountered throughout this paper. For instance, this study was initially designed to be implemented in an economic experimental laboratory where graduate and undergraduate students will be randomly sampled to play the finite sequential GOS. The finite sequential GOS experiment was designed using ZTree for the laboratory experimentations. These economic laboratory experiments of the BLS strategy model would allow for an empirical contribution of the proposed BLS model. This is because subjects' decisions at each node will be recorded and collected as primary data for analyses. From the analyses, we would be able to identify whether or not the subjects were able to identify their mover advantages at every node and whether they could convert an opponent's advantages to their benefits. Furthermore, statistical tests like the assumed rationality of the subjects at each stage would be conducted. In summary, the laboratory experiments aimed to first, ascertain whether or not, players are rational enough to identify their mover's advantages and secondly, whether or not, a rational player could convert their opponent's advantage to their benefit given the discontinuity of rationality. The intended subjects were both undergraduate and graduate students of Xiamen University taking Game Theory and Microeconomics courses. To incentivise the experiment, we designed the payoffs to be an increasing function of the total number of strokes in each GOS. As such, the subjects would prefer to play GOS with higher strokes which would reveal their rationality on identifying, maintaining, and converting mover advantages. Our hypotheses include: few subjects will be able to identify and maintain their mover advantages given lower ϑ, the ability to identify mover advantages decreases in ϑ, and the conversion of mover advantages is less likely irrespective of the size of ϑ.

Due to circumstances beyond the authors' control, this laboratory experiment was not implemented. One of the major reasons is lack of funding. Therefore, as a recommendation for future studies, researchers could use randomized and controlled laboratory experimental data to empirically test whether or not players can identify, maintain, and convert mover advantages given different levels or numbers of strokes in the GOS. This current paper is mainly an algorithmic and simulation contribution to the field of Game

Theory. A step further will include an empirical contribution and a theoretical contribution with formal proofs of the BLS model. Amongst other things, a theoretical contribution should include the probability distribution of the BLS model that describes the likelihood of irrational decisions of a player at every node of the finite sequential GOS (discontinuity of rationality) and the likelihood of mover advantage conversion. This theoretical contribution will support and strengthen both the algorithmic BLS model contribution and the empirical contribution of the BLS model.

Data availability statement

The data used in this study is not deposited in a publicly available repository and will be made available on request.

Ethics statement

- Review and/or approval by an ethics committee was not needed for this study because no laboratory experiments were conducted with the students (respondents) and no human experimental data was used for analysis in this study.
- Informed consent was not required for this study because no laboratory experiments were conducted with the students (respondents) and no human experimental data was used for analysis in this study.

CRediT authorship contribution statement

David Iheke Okorie: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Joel Miworse Gnatchiglo:** Writing – original draft, Resources.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendices.

Table 1
BLS Model Solutions

ϑ	β_{max}	AM	IND	BLS Model Solution		
				Nodes	FM - BLS	SM
1409	199	First Mover	9	1	9	100
				2	118	82
				3	109	91
				4	110	90
				5	105	95
				6	99	101
				7	94	106
				8	100	*
				Nodes	FM	SM - BLS
1645	234	Second Mover	0	1	119	116
				2	119	120
				3	115	119
				4	116	123
				5	112	128
				6	107	124
				7	111	116
				Nodes	FM - BLS	SM
1964	186	First Mover	94	1	94	93
				2	95	92
				3	87	100
				4	81	106
				5	110	77
				6	98	89
				7	101	86
				8	90	97
				9	88	99
				10	98	87
				11	94	*

(*continued on next page*)

Table 1 (*continued*)

ϑ	β_{max}	AM	IND	BLS Model Solution		
				Nodes	FM	SM - BLS
684	341	Second Mover	0	1	178	180
				2	162	164

The Advantaged Mover (AM) is identified by the BLS model. The Initial Node Decision (IND) follows the BLS model after the mover advantage has been identified.

Table 2

BLS Model Solutions without Identification and Conversion Algorithms

Simulations				FM Advantaged Simulations				SM Advantaged Simulations			
Category A – BLS Second Mover											
	Win	Loss	P	Total	Win	Loss	P	Total	Win	Loss	P
K = 500	7	493	1.4	493	0	493	0	7	7	0	100
K = 1000	13	987	1.3	987	0	987	0	13	13	0	100
K = 2000	24	1976	1.2	1976	0	1976	0	24	24	0	100
K = 5000	70	4930	1.4	4930	0	4930	0	70	70	0	100
K = 7000	72	6928	1.03	6928	0	6928	0	72	72	0	100
K = 10000	124	9876	1.24	9876	0	9876	0	124	124	0	100
Category B – BLS First Mover											
K = 500	493	7	98.6	493	493	0	100	7	0	7	0
K = 1000	987	13	98.9	987	987	0	100	13	0	13	0
K = 2000	1976	24	98.8	1976	1976	0	100	24	0	24	0
K = 5000	4930	70	98.6	4930	4930	0	100	70	0	70	0
K = 7000	6928	72	98.97	6928	6928	0	100	72	0	72	0
K = 10000	9876	124	98.76	9876	9876	0	100	124	0	124	0

The number of wins using BLS (Win), the number of losses using BLS (Loss), and the proportion of the times the BLS model won the simulated GOS games (P) are reported for the overall simulations, the first mover (FM) advantaged simulations, and second mover (SM) advantaged simulations. The overall simulations are grouped into the FM and SM-advantaged simulations. P is reported in percentages.

Table 3

BLS Model Solutions with Identification and Conversion Algorithms

ϑ	β_{max}	AM	IND	BLS Model Solution		
				Nodes	FM	SM - BLS
1347	408	First Mover	120	1	220	309
				2	198	211
				3	199	210
				Nodes	FM - BLS	SM
1320	329	Second Mover	0	1	167	163
				2	164	166
				3	167	163
				4	161	169
				Nodes	FM	SM - BLS
1351	220	First Mover	25	1	114	132
				2	101	120
				3	120	101
				4	105	116
				5	110	111
				6	124	97
				Nodes	FM – BLS	SM
1624	202	Second Mover	0	1	110	93
				2	102	101
				3	106	97
				4	95	108
				5	100	103
				6	112	91
				7	92	111
				8	99	104

The reported BLS strategy model solutions are for the cases where the simulated GOS has a first (second) mover advantage but the BLS model is applied for the randomly selected second (first) mover.

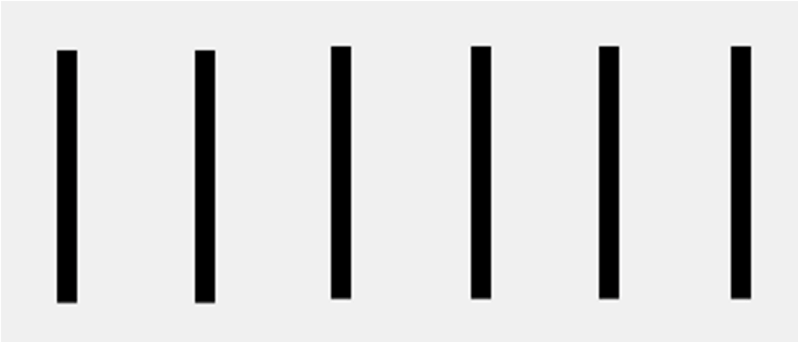

Fig. 1. Game of Strokes (GOS), Example 1.

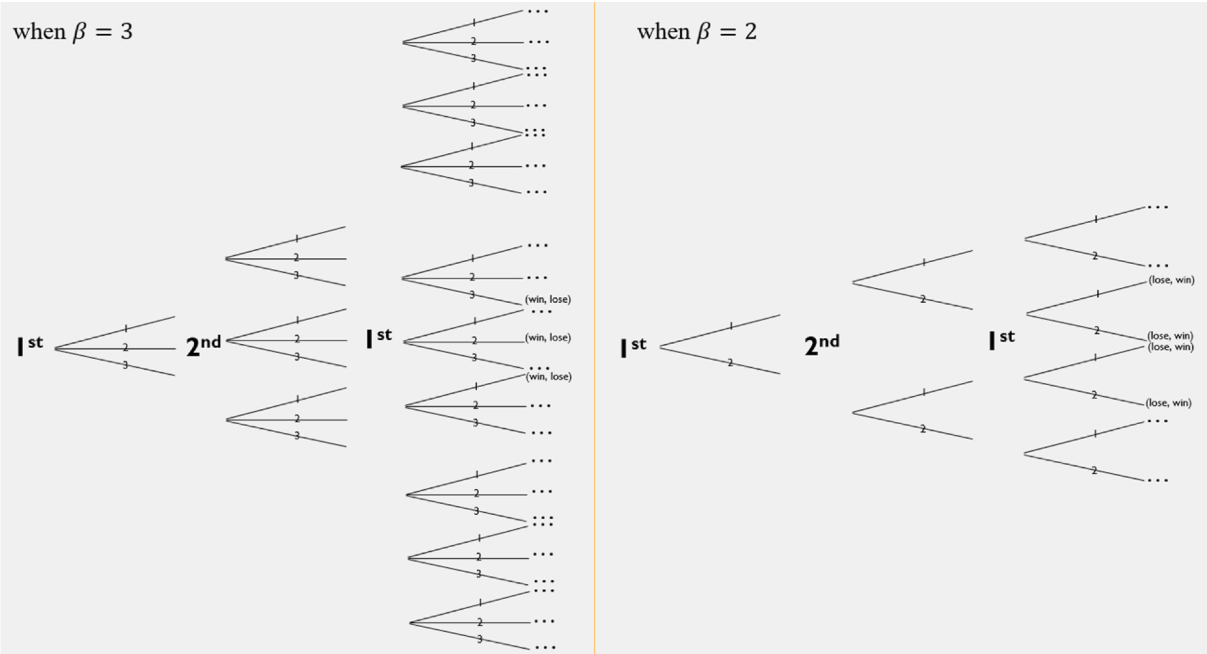

Fig. 2. Decision Trees for GOS, Example 1.

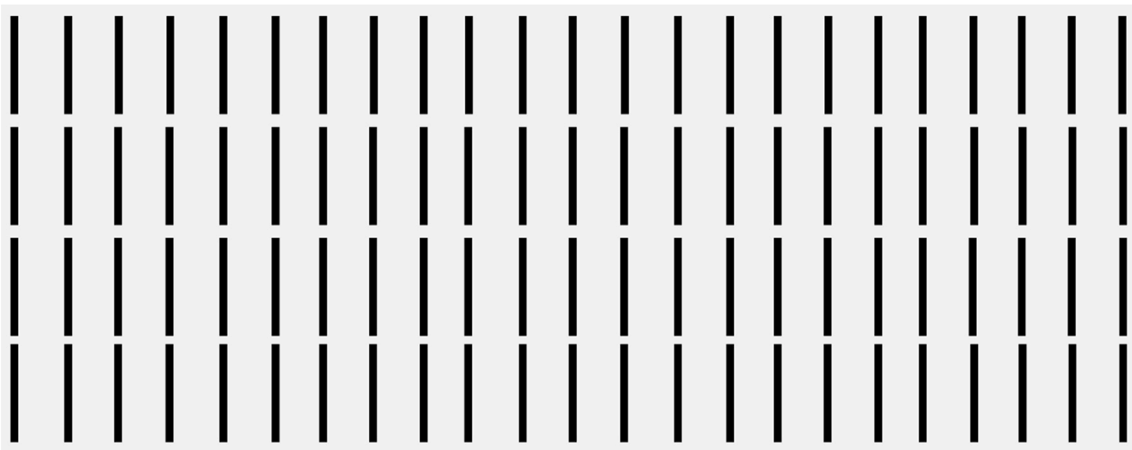

Fig. 3. Game of Strokes (GOS), Example 2.

References

[1] B.D. Bernheim, Rationalizable strategic behavior, Econometrica 52 (1984) 1007–1028.
[2] G. Askari, M.E. Gordji, C. Park, The behavioral model and game theory, Humanities and Social Science Communications 5 (2019) 57.
[3] Y. Chen, X. m Luo, C. Qu, Rationalizability in general situations, Econ. Theor. 61 (2016) 147–167.
[4] R. Boudon, Beyond rational choice theory, Annu. Rev. Sociol. 29 (1) (2003) 1–21.
[5] D.G. Pearce, Rationalizable strategic behavior and the problem of perfection, Econometrica 52 (4) (1984) 1029–1050.
[6] M.J. Osborne, A. Rubinstein, Games with procedurally rational players, Am. Econ. Rev. 88 (4) (1998) 834–847.
[7] P. Weirich, Introduction: interactive epistemology, Episteme 8 (3) (2012) 201–208. Cambridge University Press.
[8] R.J. Aumann, Interactive epistemology I: knowledge, Int. J. Game Theor. 28 (1999) 263–300.
[9] R.J. Aumann, Interactive epistemology II: probability, Int. J. Game Theor. 28 (1999) 301–314.
[10] J.C. Harasanyi, Games with incomplete information played by Bayesian players, Journal of Management Science 14 (3) (1967) 159–183.
[11] M. Schiffer, R. Jefferson, A System of Designating Behaviorally-Significant Provinces Problems, Arkansas Archeological Survey, Research Series, 1975, pp. 253–255.
[12] R.J. Aumann, Agreeing to disagree, Ann. Stat. 4 (6) (1976), 1336 - 1239.
[13] A. Rubinstein, The electronic mail game: strategic behavour under "almost common knowledge", Am. Econ. Rev. 79 (3) (1989) 385–391.
[14] P. Battigalli, M. Siniscalchi, Hierarchies of conditional beliefs and interactive epistemology in dynamic games, J. Econ. Theor. 88 (1999) 188–230.
[15] P. Battigalli, M. Siniscalchi, Strong belief and forward induction reasoning, J. Econ. Theor. 106 (2002) 356–391.
[16] A. Brandenburger, The power of paradox: some recent developments in interactive epistemology, Int. J. Game Theor. 35 (2) (2007) 465–492.
[17] S. Morris, Coordination, Communication and Common Knowledge: a retrospective on the electronic mail game, Oxf. Rev. Econ. Pol. 18 (4) (2002) 433–445.
[18] J. Geanakopolo, D. Pearce, E. Stacchetti, Psychological games and sequential rationality, Game. Econ. Behav. 1 (1) (1989) 60–79.
[19] E. Dekel, D. Fudenberg, Rational behavior with payoff uncertainty, J. Econ. Theor. 52 (2) (1990) 243–267.
[20] G. Bornstein, I. Yaniv, Individual and group behavour in the ultimatum game: are group more rational players? Journal of Experimental Economics 1 (1) (1998) 101–108.
[21] I. Shaban, F. Chan, S. Chung, T. Qu, A mixed wholesale-option-contract to fix the demand imbalance between substitutable air cargo routes: a cooperative game approach, Expert Syst. Appl. 182 (2021), 115300.
[22] S. Wei, A sequential game analysis on carbon tax policy choices in open economies: from the perspective of carbon emission responsibilities, J. Clean. Prod. 283 (2021), 124588.
[23] P. Wesseh, B. Lin, Bulk storage technologies in imperfect electricity markets under time-of-use pricing: implications for the environment and social welfare, Technol. Forecast. Soc. Change 171 (2021), 120942.
[24] A. Bonato, M.A. Huggan, T.G. Marbach, F.M. Inerney, The game of cops and eternal robbers, Theor. Comput. Sci. 874 (2021) 80–93.
[25] A. Alioua, H.-e. Djeghri, M.E. Cherif, S.-M. Senouci, H. Sedjelmaci, UAVs for traffic monitoring: a sequential game-based computation offloading/sharing approach, Comput. Network. 177 (2020), 107273.
[26] D. Buccella, L. Fanti, Do labour union recognition and bargaining deter entry in a network industry? A sequential game model, Util. Pol. 64 (2020), 101025.
[27] D.I. Okorie, The choice of wage rate and incentive for labour, Indian J. Lab. Econ. 62 (2019) 279–290.
[28] S. Wolf, C. Dron, The effect of an experimental veil of ignorance on intergenerational resource sharing: empirical evidence from a sequential multi-person dictator game, Ecol. Econ. 175 (2020), 106662.
[29] U. Phathanapirom, D. Haas, B. Leibowicz, A game-theoretic approach to nuclear fuel cycle transition analysis under uncertainty, Ann. Nucl. Energy 137 (2020), 107112.
[30] Z. Yang, Reevaluation and renegotiation of climate change coalitions—a sequential closed-loop game approach, J. Econ. Dynam. Control 27 (9) (2003) 1563–1594.
[31] Z. Dong, Y. Zhang, A sequential game of endowment effect and natural property rights, Econ. Lett. 149 (2016) 108–111.
[32] K. Hunt, P. Agarwal, J. Zhang, Technology adoption for airport security: modeling public disclosure and secrecy in an attacker-defender game, Reliab. Eng. Syst. Saf. 207 (2021), 107355.
[33] X. Zhang, S. Ding, B. Ge, B. Xia, Pedrycz, Resource allocation among multiple targets for a defender-attacker game with false targets consideration, Reliab. Eng. Syst. Saf. 211 (2021), 107617.
[34] X. Shan, J. Zhuang, Modeling cumulative defensive resource allocation against a strategic attacker in a multi-period multi-target sequential game, Reliab. Eng. Syst. Saf. 179 (2018) 12–26.
[35] X. Shan, J. Zhuang, Hybrid defensive resource allocations in the face of partially strategic attackers in a sequential defender–attacker game, Eur. J. Oper. Res. 228 (1) (2013) 262–272.
[36] I. Heloulou, M.S. Radjef, M.T. Kechadi, A multi-act sequential game-based multi-objective clustering approach for categorical data, Neurocomputing 267 (2017) 320–332.
[37] J.C. Ciscar, A. Soria, Prospective analysis of beyond Kyoto climate policy: a sequential game framework, Energy Pol. 30 (15) (2002) 1327–1335.
[38] N. Xu, A.F. Veinott, Sequential stochastic core of a cooperative stochastic programming game, Oper. Res. Lett. 41 (5) (2013) 430–435.
[39] H.-T. Hou, Integrating cluster and sequential analysis to explore learners' flow and behavioral patterns in a simulation game with situated-learning context for science courses: a video-based process exploration, Comput. Hum. Behav. 48 (2015) 424–435.
[40] W. Muller, F. Tan, Who acts more like a game theorist? Group and individual play in a sequential market game and the effect of the time horizon, Game. Econ. Behav. 82 (2013) 658–674.
[41] R. Avenhaus, M.J. Canty, Playing for time: a sequential inspection game, Eur. J. Oper. Res. 167 (2) (2005) 475–492.
[42] Y. Pavlova, G. Reniers, A sequential-move game for enhancing safety and security cooperation within chemical clusters, J. Hazard Mater. 186 (1) (2011) 401–406.
[43] C.-C. Lin, L.A. Dugatkin, H.-W. Yuan, P.-F. Lee, S.-F. Shen, A sequential collective action game and its applications to cooperative parental care in a songbird, Anim. Behav. 129 (2017) 151–159.
[44] M.-Y. Hwang, J.-C. Hong, H.-Y. Cheng, Y.-C. Peng, N.-C. Wu, Gender differences in cognitive load and competition anxiety affect 6th grade students' attitude toward playing and intention to play at a sequential or synchronous game, Comput. Educ. 60 (1) (2013) 254–263.
[45] Z. Wenhui, W. Qiang, Z. Ren-Qian, Choosing among hospitals in the subsidized health insurance system of China: a sequential game approach, Eur. J. Oper. Res. 257 (2) (2017) 568–585.
[46] R. Duan, R. Prodan, X. Li, A sequential cooperative game theoretic approach to scheduling multiple large-scale applications in grids, Future Generat. Comput. Syst. 30 (2014) 27–43.
[47] S. Alpern, S. Gal, E. Solan, A sequential selection game with vetoes, Game. Econ. Behav. 68 (1_) (2010) 1–14.
[48] J. Hellermann, The sequential and prosodic co-construction of a 'quiz game' activity in classroom talk, J. Pragmat. 37 (6) (2005) 919–944.
[49] G. Antonides, Mental accounting in a sequential Prisoner's Dilemma game, J. Econ. Psychol. 15 (2) (1994) 351–374.
[50] P. Agarwal, R.A. Aziz, J. Zhuang, Interplay of rumor propagation and clarification on social media during crisis events - a game-theoretic approach, Eur. J. Oper. Res. (2021). In Press.
[51] A. Brandtner, E. Wegmann, M. Brand, Desire thinking promotes decisions to game: the mediating role between gaming urges and everyday decision-making in recreational gamers, Addictive Behaviors Reports 12 (2020), 100295.
[52] T. Sandler, S. Vicary, Weakest-link public goods: giving in-kind or transferring money in a sequential game, Econ. Lett. 74 (1) (2001) 71–75.
[53] T.N. Cason, V.-L. Mui, Social influence in the sequential dictator game, J. Math. Psychol. 42 (2–3) (1998) 248–265.
[54] M. Shubik, S. Sahi, A model of a sudden-death field-goal football game as a sequential duel, Math. Soc. Sci. 15 (3) (1988) 205–215.

[55] P. Battigalli, E. Catonini, G. Lanzani, M. Marinacci, Ambiguity attitudes and self-confirming equilibrium in sequential games, Game. Econ. Behav. 115 (2019) 1–29.

[56] H. Zhou, Informed speculation with K-level reasoning, J. Econ. Theor. 200 (2022), 105384.

[57] X. Zhang, Level-k Reasoning with Heterogeneous Information Signals, SSRN papers, 2020, https://doi.org/10.2139/ssrn.3786735.

[58] J. Vyrastekova, Y. Funaki, Cooperation in a sequential dilemma game: how much transparency is good for cooperation? Journal of Behavioral and Experimental Economics 77 (2018) 88–95.

[59] Y.N. Kiselev, S.N. Avvakumov, M.V. Orlov, S.M. Orlov, Optimal resource allocation in a two-sector economy with an integral function: theoretical analysis and numerical experiments, Comput. Math. Model. 31 (2020) 190–227.

[60] V.A. Kneese, The economics of natural resources, Popul. Dev. Rev. 14 (1988) 281–309.

[61] D.I. Okorie, An input-output augmented Kaya identity and application: quantile regression approach, Social Sciences and Humanities Open 4 (1) (2021), 100214.

[62] D.I. Okorie, B. Lin, Emissions in agricultural-based developing economies: a case of Nigeria, J. Clean. Prod. 337 (2022), 130570.

[63] D.I. Okorie, B. Lin, Africa's Biofuel energy and emissions prospect: forward-looking into 2030, Sustain. Energy Technol. Assessments 53 (D) (2022), 102775.

[64] F. Zhang, X. Deng, F. Phillips, C. Feng, C. Wang, Impacts of industrial structure and technical progress on carbon emission intensity: evidence from 281 cities in China, Technol. Forecast. Soc. Change 154 (2020), 119949.

[65] D.I. Okorie, P.K. Wesseh, Climate agreements and carbon intensity: towards increased production efficiency and technical progress? Structural Change and Economic Dynamic 66 (2023) 300–313.