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Confidence interval estimation of the common mean of several gamma populations

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Abstract

Gamma distributions are widely used in applied fields due to its flexibility of accommodating right-skewed data. Although inference methods for a single gamma mean have been well studied, research on the common mean of several gamma populations are sparse. This paper addresses the problem of confidence interval estimation of the common mean of several gamma populations using the concept of generalized inference and the method of variance estimates recovery (MOVER). Simulation studies demonstrate that several proposed approaches can provide confidence intervals with satisfying coverage probabilities even at small sample sizes. The proposed methods are illustrated using two examples.

Introduction

Due to its flexibility of accommodating right-skewed data, the standard two-parameter Gamma distribution has been widely used in many applied fields such as meteorology, reliability, medical science, engineering and quality control [1–4]. Under many circumstances, the research interest lies in making inference about the mean. There exit abundant research regarding making inference about gamma mean(s). For example, Fraser et al. [5] investigated inference methods for gamma mean based on asymptotic approximation, and Krishnamoorthy and León-Novelo [6] investigated small sample inference for gamma parameters for one-sample and two-sample problems. Recently, several fiducial methods [7–9] constructed approximate generalized pivotal quantities for a single gamma mean in different ways. Wang et al. [10] extended a fiducial approach [7] for a single gamma mean to construct a fiducial confidence interval for the difference between two independent gamma means.

There also exist some research on testing equality of several gamma means. For example, Chang et al. [11] proposed a parametric bootstrap method for comparing several gamma means, and Krishnamoorthy et al. [12] presented likelihood ratio test for comparing several gamma distributions. When testing equality of several gamma means concludes the null hypothesis (i.e. all the means are equal) can not be rejected, naturally, making inference about the common mean is of interest. Despite the fact that inference procedures about the common gamma means are of practical and theoretical importance, there has not yet been a well-developed approach for this purpose at small sample sizes except some traditional large sample methods. Therefore, the goal of this paper is to present accurate small sample inference

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methods for confidence interval estimation for the common gamma mean derived from several independent samples.

The rest of this paper is organized as follows. We will first present preliminaries including notations and existing methods for confidence interval estimation of single gamma mean. Then we will propose several methods for constructing confidence intervals for common gamma mean. Simulation results are presented to evaluate the performance of the proposed methods and examples are analyzed using the proposed methods. Finally, summary and discussion are given.

Preliminaries

The setting

Consider K independent gamma populations. Let $Y_{i1}, Y_{i2}, \ldots, Y_{in_i}$ be a random sample from the ith gamma population as $Y_{ij} \sim gamma(\alpha_i, \beta_i)$ where α_i is shape parameter and β_i is rate parameter; i.e. the corresponding probability density function for Y_{ij} is

$$f(y_{ij}; \alpha_i, \beta_i) = \frac{y_{ij}^{\alpha_i - 1} e^{-\beta_i y_{ij}} \beta_i^{\alpha_i}}{\Gamma(\alpha_i)}$$

for $y_{ij} > 0$, $\alpha_i, \beta_i > 0$. Let μ_i denote the population mean for ith sample. Then $\mu_i = \alpha_i/\beta_i$ for $i = 1, 2, \ldots, K$. We assume that $\mu_1 = \mu_2 = \ldots = \mu_K$ and let μ denote the common mean. The goal of this paper is to present procedures for confidence interval estimation of μ at small to medium sample sizes.

Let $\hat{\alpha}_i$ and $\hat{\beta}_i$ stand for the maximum likelihood estimates for α_i and β_i, respectively. The maximum likelihood estimate of μ_i is $\hat{\mu}_i = \bar{Y}_i = \hat{\alpha}_i/\hat{\beta}_i$ where the large sample variance for $\hat{\mu}_i$ is [5]

$$var(\hat{\mu}_i) = \frac{\mu_i^2}{n_i \alpha_i}, \tag{1}$$

and its estimate is

$$\widehat{var}(\hat{\mu}_i) = \frac{\hat{\mu}_i^2}{n_i \hat{\alpha}_i}. \tag{2}$$

The common gamma mean can be estimated as a pooled estimate of sample means defined as

$$\hat{\mu} = \sum_{i=1}^{K} \frac{\hat{\mu}_i}{\widehat{var}(\hat{\mu}_i)} \Big/ \sum_{i=1}^{K} \frac{1}{\widehat{var}(\hat{\mu}_i)} \tag{3}$$

Using standard large sample theory, we have

$$\frac{\hat{\mu} - \mu}{\sqrt{var(\hat{\mu})}} \sim N(0, 1)$$

asymptotically. Hence, a simple large sample solution for confidence interval estimation for common μ is

$$(\hat{\mu} - z_{1-\alpha/2}\sqrt{1/\sum_{i=1}^{K}1/\widehat{var}(\hat{\mu}_i)}, \quad \hat{\mu} + z_{1-\alpha/2}\sqrt{1/\sum_{i=1}^{K}1/\widehat{var}(\hat{\mu}_i)} \quad). \tag{4}$$

Of course, we also can obtain a large sample confidence interval using standard maximum likelihood theory. However, these large sample solutions do not have good performance at small sample sizes. Hence in this paper, we will present some procedures with satisfactory performance.

Existing methods for confidence interval estimation for single gamma mean

In the following, we will review several existing methods for confidence interval estimation for single gamma mean. These methods are known to have reasonable performance at small to medium sample sizes, and will be used in the following to present our new procedures for confidence interval estimation for common gamma mean.

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a gamma population $gamma(\alpha, \beta)$. The population mean $\mu = \alpha/\beta$. Let $\bar{Y}$ and $\tilde{Y}$ denote the arithmetic mean and geometric mean, respectively. The maximum likelihood estimate of μ is $\hat{\mu} = \bar{Y} = \hat{\alpha}/\hat{\beta}$ where $\hat{\alpha}$ and $\hat{\beta}$ are the maximum likelihood estimates for α and β, respectively.

Methods based on generalized inference. The generalized variables and generalized pivots were introduced by Tsui and Weerahandi [13] and Weerahandi [14]. More details can be found in the book by Weerahandi [15]. The concepts of generalized pivotal quantity and generalized confidence interval have been successfully applied to a variety of practical problems where standard exact solutions do not exist and it has been shown that generalized inference method generally have good performance, even at small sample sizes; see e.g. [16–21]. Recently, Hannig et al. [22] demonstrated generalized confidence intervals coincide with fiducial confidence intervals. In the following, we review three existing methods for constructing generalized pivotal quantity for single gamma mean.

Krishnamoorthy and Wang's method: [7, 23] This method is based on the fact that $X_j = Y_j^{\frac{1}{3}}$ ($j = 1, 2, \ldots, n$) follows $N(\mu, \sigma^2)$ approximately for gamma distribution. Let $\bar{x}$ and s_i^2 be the observed sample mean and sample variance based on the transformed data $X_i's$. The generalized pivotal quantities for μ and σ^2 can be obtained as [24]:

$$R_\mu = \bar{x} - \frac{Z}{\sqrt{U}}\sqrt{\frac{(n-1)s^2}{n}}, R_{\sigma^2} = \frac{(n-1)s^2}{U} \sim \frac{(n-1)s^2}{\chi^2_{n-1}}$$

where $Z \sim N(0, 1)$, $U \sim \chi^2_{n-1}$, and Z and U are independent. Furthermore, the generalized pivotal quantities for for α and β can be written as:

$$R_\alpha = \frac{1}{9}\left\{\left(1 + 0.5\frac{R_\mu^2}{R_{\sigma^2}}\right) + \left[\left(1 + 0.5\frac{R_\mu^2}{R_{\sigma^2}}\right)^2 - 1\right]^{\frac{1}{2}}\right\},$$
$$R_\beta = \frac{1}{27(R_\alpha)^{\frac{1}{2}}(R_{\sigma^2})^{\frac{3}{2}}}. \tag{5}$$

Chen and Ye's method: [8, 25] Note that $V_1 = 2n\alpha \log(\bar{Y}/\tilde{Y}) \sim c\chi^2_v$ approximately, where

$v = 2E^2(V_1)/var(V_1)$, $c = E(V_1)/v$, $E(V_1) = 2n\alpha(\psi(n\alpha) - \psi(\alpha) - log(n))$ and $var(V_1) = 4n^2 \alpha^2(\psi'(\alpha)/n - \psi'(\alpha))$ with ψ and ψ' being the digamma and trigamma functions respectively. Then $\hat{c}$ and $\hat{v}$ can be obtained by substituting α with its point estimate $\hat{\alpha}$. An approximate generalized pivotal quantity (GPQ) for α is:

$$R_\alpha = \frac{V_1}{2n\log(\bar{y}/\tilde{y})}$$

where $V_1 \sim \hat{c}\chi^2_{\hat{v}}$, $\bar{y}$ and $\tilde{y}$ are observed values of $\bar{Y}$ and $\tilde{Y}$. Furthermore, as $2n\beta\bar{Y} \sim \chi^2_{2n\alpha}$, a GPQ for β can be constructed as:

$$R_\beta = \frac{V_2}{2n\bar{y}}, \tag{6}$$

where $V_2 \sim \chi^2_{2nR_\alpha}$.

Wang and Wu's method [9]: This method is based on Cornish-Fisher approximation. Let $T = \log(\tilde{Y}/\bar{Y})$ and $F(.)$ be the c.d.f. of T. Note that $U = F(T) \sim U(0, 1)$. Using the Cornish-Fisher expansion, the Uth percentile of T can be approximated by $g_1(\alpha) + [g_2(\alpha)]^{1/2} Q(\alpha, U)$, where $g_i(\alpha)$ is the ith cumulant of T and $Q(\alpha, U)$ is a function of $g_i(\alpha)$'s. Detailed formula can be found in [9]. Let t denote the observed value of T. Solving $t = g_1(\alpha) + [g_2(\alpha)]^{1/2}Q(\alpha, U)$ for α, we obtain the approximate R_α. The GPQ for β can be obtained similarly as in (6):

$$R_\beta = \frac{V_3}{2n\bar{y}}, \tag{7}$$

where $V_3 \sim \chi^2_{2nR_\alpha}$. This method improves Chen and Ye's method and can work well even when the shape parameter α is small.

The three aforementioned methods for generating R_α and R_β lead to three generalized pivots of a single gamma mean:

$$R_\mu = R_\alpha/R_\beta. \tag{8}$$

Via simulation, we can obtain an array of R_μ's and the estimated confidence interval for μ is $(R_\mu(\alpha/2), R_\mu(1 - \alpha/2))$ where $R_\mu(\alpha)$ is the 100αth percentile of R_μ's.

A parametric bootstrap method [6]. Krishnamoorthy and León-Novelo [6] presented a method based on parametric bootstrapping for confidence interval estimation using the following pivotal quantity:

$$Q = \frac{\bar{Y}^* - \bar{Y}}{\bar{Y}^*/\sqrt{n\hat{\alpha}^*}} \tag{9}$$

where $\hat{\alpha}^*$ and $\bar{Y}^*$ are based on a bootstrap sample from $Gamma(\hat{\alpha}, \hat{\beta})$ distribution. A two-sided $100(1 - p)\%$ confidence interval (l, u) for μ is:

$$\left(\bar{Y} - Q_{1-p/2}\frac{\bar{Y}}{\sqrt{n\hat{\alpha}}}, \bar{Y} - Q_{p/2}\frac{\bar{Y}}{\sqrt{n\hat{\alpha}}}\right), \tag{10}$$

where Q_p as the $100p$th percentile of Q defined in (9).

The proposed methods for confidence interval estimation of common gamma mean

The methods based on generalized inference

For the ith ($i = 1, 2, \ldots, K$) sample, we can obtain the generalized pivotal quantities R_{μ_i} using one of the three methods reviewed above (i.e. Krishnamoorthy and Wang's method [7, 23] Chen and Ye's method [8, 25], and Wang and Wu's method [9]). Replacing μ_i and α_i with R_{μ_i} and R_{α_i} in (1), the generalized pivotal quantity for $var(\hat{\mu}_i)$ can be written as

$$R_{var(\hat{\mu}_i)} = \frac{R^2_{\mu_i}}{n_i R_{\alpha_i}}. \tag{11}$$

The generalized pivotal quantity we propose for the common gamma mean μ is a weighted average of the generalized pivot R_{μ_i}'s based on K individual samples, i.e.

$$R_\mu = \frac{\sum_{i=1}^K R_{w_i} R_{\mu_i}}{\sum_{i=1}^K R_{w_i}} \tag{12}$$

where $R_{w_i} = 1/R_{var(\hat{\mu}_i)}$.

It is easy to see that R_μ satisfies the two conditions to be an approximate bona fide generalized pivotal quantity: 1) the distributions of R_μ is independent of any unknown parameters; and 2) the observed value of R_μ equals to the common gamma μ approximately. This way of constructing generalized pivots for common mean has been widely used in literature. For example, Krishnamoorthy and Lu [17] studied inferences on the common mean of several normal populations based on the generalized variable method; and Tian and Wu [26] studied common mean of several log-normal populations.

Computing algorithms. Consider a given data set Y_{ij}'s ($i = 1, 2, \ldots, K, j = 1, 2, \ldots, n_i$) where the ith sample $Y_{i1}, Y_{i2}, \ldots, Y_{in_i}$ is from $gamma(\alpha_i, \beta_i)$. We assume $\mu_i = \mu$ for all $i = 1, 2, \ldots, K$. The generalize confidence intervals for the common mean μ can be computed by the following steps:

1. Using one of the three methods presented above, generate R_{α_i} and R_{β_i}, then calculate generalized pivot R_{μ_i} for μ_i following (12) for $i = 1, 2, \ldots, K$.
2. Repeat steps 1, generate R_{α_i} and R_{β_i} and calculate R_{μ_i}. Using R_{α_i} and R_{μ_i}, calculate $R_{var(\hat{\mu}_i)}$ following (11) for $i = 1, 2, \ldots, K$.
3. Using R_{μ_i} obtained in step 1 and $R_{var(\hat{\mu}_i)}$ in step 2 for $i = 1, 2, \ldots, K$, calculate the generalized pivot of the common mean R_μ from (12).
4. Repeat Steps 1-3 a total B ($B = 2000$) times and obtain an array of R_μ's.
5. Rank this array of R_μ's from small to large.

The 100αth percentile of R_μ's, $R_\mu(\alpha)$, is an estimate of the lower bound of the one-sided 100(1 − α)% confidence interval and ($R_\mu(\alpha/2)$, $R_\theta(1 - \alpha/2)$) is a two-sided 100(1 − α)% confidence interval.

Remark 2.1: In computing algorithm, we used different sets of random variables for R_{μ_i} and $R_{var(\hat{\mu}_i)}$. Our simulation shows that the generalized pivotal quantity based on the same set of random variables for R_{μ_i} and R_{w_i} produces confidence intervals which are too liberal. Similar conclusions have been stated in [17, 26].

We refer these three methods based on the generalized pivots of common gamma mean as $\mathbf{GV_K}$, $\mathbf{GV_C}$, $\mathbf{GV_W}$, corresponding to the methods used for confidence interval estimation of a single gamma mean, i.e. Krishnamoorthy and Wang's method [7, 23] Chen and Ye's method [8, 25], and Wang and Wu's method [9].

The MOVER-type methods

The method of variance estimates recovery (MOVER) is a useful technique for obtaining a closed-form approximate confidence interval for a linear combination of parameters based on the confidence intervals of the individual parameters [27, 28]. In this section, using the methods for estimating confidence intervals for a single gamma mean reviewed above, the MOVER method is applied for confidence interval estimation of the common gamma mean.

Let l_i and u_i be the lower and upper limits of an approximate two-sided $100(1-p)\%$ confidence interval (l_i, u_i) for the gamma mean based only on ith sample. A MOVER $100(1-p)\%$ confidence interval (L, U) of the common gamma mean is given by [27, 28]:

$$\begin{aligned} L &= \sum_{i=1}^{K} \hat{w}_i \hat{\mu}_i - \sqrt{\sum_{i=1}^{K} \hat{w}_i^2 (\hat{\mu}_i - l_i)^2} \\ U &= \sum_{i=1}^{K} \hat{w}_i \hat{\mu}_i + \sqrt{\sum_{i=1}^{K} \hat{w}_i^2 (\hat{\mu}_i - u_i)^2}, \end{aligned} \tag{13}$$

where $\hat{w}_i = (1/\widehat{var}(\hat{\mu}_i))/\sum_{i=1}^{K}(1/\widehat{var}(\hat{\mu}_i))$, $\widehat{var}(\hat{\mu}_i)$ is defined in (2), and $\hat{\alpha}_i$ and $\hat{\mu}_i = \bar{Y}_i$ are the maximum likelihood estimates for α_i and μ_i, respectively.

For calculating confidence intervals (l_k, u_k) for the single gamma mean μ_i $(i = 1, \ldots, K)$, we will use the three generalized inference methods (i.e. Krishnamoorthy and Wang's method [7, 23], Chen and Ye's method [8, 25], and Wang and Wu's method [9]) as well as the parametric bootstrap method by Krishnamoorthy and León-Novelo [6] reviewed above. Each method provides an approximate confidence interval (l_i, u_i) for the ith single gamma mean μ_i $(i = 1, 2, \ldots K)$.

Substituting (l_k, u_k) in (13), we obtain confidence interval estimation of common mean μ. We refer these MOVER-type methods as $\mathbf{MOVER_K}$, $\mathbf{MOVER_C}$, $\mathbf{MOVER_W}$, $\mathbf{MOVER_{boot}}$ corresponding to the methods used for single gamma mean, i.e. Krishnamoorthy and Wang's method [7, 23] Chen and Ye's method [8, 25], Wang and Wu's method [9], and the parametric bootstrap method by Krishnamoorthy and León-Novelo's method [6], respectively.

Simulation studies

In previous section, we presented several methods for confidence interval estimation of common gamma mean: three methods based on the generalized pivots (i.e. $\mathbf{GV_K}$, $\mathbf{GV_C}$, $\mathbf{GV_W}$); and four MOVER-type methods (i.e. $\mathbf{MOVER_K}$, $\mathbf{MOVER_C}$, $\mathbf{MOVER_W}$, $\mathbf{MOVER_{boot}}$).

Simulation studies are carried out to evaluate the performances of proposed methods in terms of coverage probabilities and average lengths of proposed confidence intervals. The number of samples K is set as 2 and 5, and a variety of sample sizes from small (5) to large (50) including balanced and unbalanced settings are used. The parameter settings are as follows: 1) common mean μ is set as 1 and 5; 2) shape parameter for each sample varies from 0.5 or 1 to 5 or 10, and the differences among K shape parameters varies from small to large. For each parameter setting, 2,000 random samples are generated. For the confidence interval based on generalized pivots (i.e. $\mathbf{GV_K}$, $\mathbf{GV_C}$, $\mathbf{GV_W}$), $\mathbf{MOVER_K}$, $\mathbf{MOVER_C}$, $\mathbf{MOVER_W}$), 2000 values of generalized pivots are obtained for each random sample. For the confidence interval based on parametric bootstrapping ($\mathbf{MOVER_{boot}}$), 2000 bootstrap samples are generated for each random sample. The performances of each method is assessed by coverage probability and average lengths of proposed confidence intervals. The simulation results are presented in Tables 1 and 2.

Table 1 presents simulated coverage probabilities (CP) and confidence interval lengths (CI) for $K = 2$. Overall speaking, three methods based on the generalized pivots (i.e. $\mathbf{GV_K}$, $\mathbf{GV_C}$, $\mathbf{GV_W}$) maintains satisfactory coverage probabilities for all settings except that they might be slightly conservative at small sizes and $\mathbf{GV_K}$ was slightly liberal when $(\alpha_1, \alpha_2) = (1, 2)$ at sample sizes (50, 50) and $(\alpha_1, \alpha_2) = (0.5, 1)$ at sample sizes (20, 20). Among MOVER-type methods (i.e. $\mathbf{MOVER_K}$, $\mathbf{MOVER_C}$, $\mathbf{MOVER_W}$, $\mathbf{MOVER_{boot}}$), $\mathbf{MOVER_C}$ performs the best while all of them are generally liberal when sample sizes are from (5, 5) to (20, 20). When sample sizes reach 50, all the proposed methods perform satisfactorily. The $\mathbf{GV_K}$ method provides shortest confidence intervals among three generalized pivots based methods, followed by $\mathbf{GV_W}$. As sample sizes reach 20, all three methods (i.e. $\mathbf{GV_K}$, $\mathbf{GV_C}$, $\mathbf{GV_W}$) are generally comparable. $\mathbf{MOVER_K}$ and $\mathbf{MOVER_{boot}}$ provides shortest confidence intervals among MOVER-type methods. As sample sizes reach 20, all MOVER-type methods (i.e. $\mathbf{MOVER_K}$, $\mathbf{MOVER_C}$, $\mathbf{MOVER_W}$, $\mathbf{MOVER_{boot}}$) are comparable in terms of length. When sample sizes reach 50, all the proposed methods generate confidence intervals with comparable length.

Table 2 presents simulated coverage probabilities (CP) and confidence interval lengths (CI) for $K = 5$. The three generalized pivots based methods methods generally maintains satisfactory coverage probabilities for all settings except that they tend to be slightly conservative at small sizes and $\mathbf{GV_K}$ is liberal at $(\alpha_1, \ldots, \alpha_5) = (0.5, 0.5, 0.75, 0.75, 1)$ with sizes (50, 50, 50, 50, 50). Among MOVER-type methods (i.e. $\mathbf{MOVER_K}$, $\mathbf{MOVER_C}$, $\mathbf{MOVER_W}$, $\mathbf{MOVER_{boot}}$), $\mathbf{MOVER_C}$ performs the best while they are generally liberal when sample sizes are from (5, 5, 5, 5, 5) to (20, 20, 20, 20). When sample sizes reach 50, all methods perform satisfactorily. The $\mathbf{GV_K}$ method provides shortest confidence intervals among three generalized pivots based methods, followed by $\mathbf{GV_W}$. As sample sizes reach 20, all three methods (i.e. $\mathbf{GV_K}$, $\mathbf{GV_C}$, $\mathbf{GV_W}$) are comparable. $\mathbf{MOVER_K}$ provides shortest confidence intervals among three MOVER-type methods, followed by $\mathbf{MOVER_b}$ **oot**. As sample sizes reach 20, all four methods (i.e. $\mathbf{MOVER_K}$, $\mathbf{MOVER_C}$, $\mathbf{MOVER_W}$, $\mathbf{MOVER_{boot}}$) are comparable. When sample sizes reach 50, all methods are generally comparable in terms of length.

In summary, generally we recommend $\mathbf{GV_K}$, $\mathbf{GV_C}$, $\mathbf{GV_W}$ methods over MOVER-type methods due to the fact that they can generate confidence intervals with satisfactory coverage probabilities even at smaller sizes. The MOVER-type methods are not recommended unless sample sizes are greater than or equal to 50. The large sample approach in (4) can severely underestimate the coverage probabilities, hence its results are not presented.

Data examples

In this section, we illustrate the proposed methods using two examples. Both datasets was analyzed in [11] for testing equality of gamma means, and it was concluded that the null hypothesis (equality of gamma means) can not be rejected. Therefore, in this paper, we use these two datasets to illustrate our proposed methods for estimating confidence intervals of the common gamma mean.

Example 1. Wright [29] reported ground water yield from two types of wells in southwestern Virginia. Table 3 presents this dataset which includes ground water yield data from 12 wells from valley underlain by unfractured rocks, and 13 wells by fractured rocks. It has been argued that gamma distribution is appropriate to fit the data in each sample, and the test for equality of means [11] concluded that the means of water yields from two types of wells are the

Table 1. Coverage probabilities (CP) and length of confidence interval (CI) of proposed 95% confidence intervals for the common gamma mean (2000 simulations) with two independent samples ($K = 2$).

(α_1, α_2)	Sizes*	GV_K		GV_C		GV_W		$MOVER_K$		$MOVER_C$		$MOVER_W$		$MOVER_{boot}$	
		CP	CI	CP	CI	CP	CI	CP	CI	CP	CI	CP	CI	CP	CI
$\mu = 1$															
(0.5,1)	I	0.947	3.798	0.977	31.948	0.962	9.138	0.910	2.927	0.950	28.028	0.927	8.921	0.903	2.670
	II	0.952	2.131	0.976	10.748	0.966	4.693	0.917	2.411	0.952	35.248	0.941	9.745	0.917	2.443
	III	0.955	1.458	0.969	1.896	0.964	1.722	0.916	1.260	0.940	1.628	0.936	1.496	0.927	1.337
	IV	0.947	0.827	0.961	0.933	0.959	0.909	0.921	0.756	0.941	0.847	0.936	0.828	0.935	0.813
	V	0.941	0.465	0.954	0.505	0.951	0.499	0.922	0.446	0.942	0.482	0.939	0.479	0.943	0.477
(1,2)	I	0.965	2.274	0.974	4.504	0.966	2.656	0.927	1.804	0.942	3.353	0.930	2.126	0.917	1.441
	II	0.957	1.319	0.969	2.027	0.960	1.483	0.931	1.464	0.948	2.957	0.936	1.810	0.922	1.202
	III	0.958	0.971	0.966	1.050	0.960	0.992	0.926	0.860	0.933	0.922	0.931	0.879	0.924	0.839
	IV	0.954	0.584	0.960	0.605	0.959	0.593	0.934	0.545	0.945	0.562	0.941	0.554	0.941	0.549
	V	0.945	0.333	0.948	0.341	0.948	0.338	0.933	0.323	0.940	0.330	0.939	0.329	0.938	0.328
(1,10)	I	0.959	1.035	0.964	1.532	0.960	1.108	0.933	0.825	0.946	1.326	0.932	0.927	0.911	0.725
	II	0.967	0.579	0.971	0.798	0.969	0.629	0.948	0.643	0.954	1.187	0.946	0.771	0.934	0.551
	III	0.957	0.478	0.962	0.494	0.961	0.481	0.941	0.434	0.945	0.445	0.941	0.436	0.940	0.427
	IV	0.956	0.293	0.959	0.296	0.959	0.294	0.947	0.281	0.946	0.283	0.947	0.282	0.947	0.281
	V	0.948	0.172	0.951	0.172	0.950	0.172	0.946	0.169	0.947	0.170	0.946	0.170	0.946	0.170
(2,10)	I	0.960	0.891	0.963	1.023	0.953	0.862	0.931	0.727	0.933	0.812	0.927	0.709	0.917	0.647
	II	0.954	0.526	0.957	0.576	0.953	0.525	0.933	0.562	0.940	0.653	0.934	0.560	0.921	0.491
	III	0.964	0.446	0.964	0.453	0.961	0.444	0.943	0.408	0.939	0.411	0.943	0.405	0.935	0.400
	IV	0.949	0.279	0.949	0.280	0.949	0.278	0.940	0.267	0.938	0.268	0.939	0.267	0.935	0.266
	V	0.952	0.166	0.953	0.167	0.953	0.166	0.950	0.164	0.947	0.164	0.950	0.164	0.949	0.164
(5,10)	I	0.964	0.732	0.969	0.762	0.961	0.690	0.932	0.612	0.931	0.626	0.925	0.586	0.920	0.561
	II	0.960	0.478	0.963	0.489	0.960	0.467	0.944	0.497	0.946	0.510	0.936	0.480	0.935	0.456
	III	0.956	0.390	0.956	0.392	0.957	0.387	0.934	0.358	0.936	0.358	0.934	0.355	0.936	0.352
	IV	0.945	0.246	0.949	0.246	0.945	0.245	0.929	0.235	0.931	0.235	0.927	0.235	0.932	0.234
	V	0.957	0.148	0.955	0.148	0.957	0.148	0.949	0.145	0.949	0.145	0.949	0.145	0.948	0.145
$\mu = 5$															
(0.5,1)	I	0.956	18.934	0.977	152.736	0.965	46.391	0.923	14.473	0.953	163.612	0.936	46.134	0.907	13.224
	II	0.958	10.803	0.977	49.275	0.969	22.734	0.924	12.269	0.952	184.086	0.936	46.080	0.924	12.116
	III	0.956	7.174	0.974	9.363	0.966	8.501	0.917	6.177	0.941	8.038	0.931	7.393	0.927	6.588
	IV	0.932	4.122	0.950	4.651	0.948	4.523	0.906	3.782	0.928	4.226	0.925	4.133	0.923	4.059
	V	0.944	2.304	0.963	2.498	0.961	2.473	0.928	2.210	0.950	2.385	0.948	2.369	0.946	2.363
(1,2)	I	0.957	11.242	0.975	21.977	0.961	13.098	0.921	8.943	0.942	16.556	0.925	10.592	0.912	7.177
	II	0.961	6.637	0.976	10.351	0.966	7.517	0.938	7.429	0.954	15.540	0.944	9.356	0.924	6.099
	III	0.965	4.859	0.975	5.258	0.967	4.971	0.934	4.306	0.943	4.618	0.936	4.407	0.930	4.198
	IV	0.946	2.896	0.955	2.999	0.949	2.940	0.928	2.702	0.934	2.787	0.936	2.747	0.934	2.724
	V	0.952	1.673	0.953	1.710	0.956	1.699	0.945	1.624	0.950	1.660	0.950	1.651	0.948	1.650
(1,10)	I	0.963	5.098	0.969	7.328	0.967	5.373	0.935	4.086	0.947	6.156	0.933	4.451	0.919	3.588
	II	0.969	2.857	0.976	3.868	0.969	3.095	0.955	3.201	0.963	5.952	0.952	3.868	0.939	2.746
	III	0.958	2.398	0.960	2.482	0.956	2.411	0.938	2.170	0.940	2.229	0.940	2.183	0.933	2.138
	IV	0.952	1.455	0.954	1.470	0.951	1.458	0.940	1.393	0.945	1.405	0.948	1.397	0.946	1.393
	V	0.947	0.866	0.949	0.870	0.948	0.869	0.943	0.854	0.946	0.858	0.943	0.857	0.943	0.857

(*Continued*)

Table 1. (Continued)

(α_1, α_2)	Sizes*	GV_K		GV_C		GV_W		$MOVER_K$		$MOVER_C$		$MOVER_W$		$MOVER_{boot}$	
		CP	CI	CP	CI	CP	CI	CP	CI	CP	CI	CP	CI	CP	CI
(2,10)	I	0.960	4.540	0.965	5.190	0.956	4.367	0.926	3.701	0.931	4.105	0.923	3.602	0.917	3.287
	II	0.954	2.639	0.957	2.890	0.953	2.635	0.935	2.822	0.937	3.278	0.934	2.815	0.925	2.471
	III	0.961	2.215	0.963	2.246	0.959	2.199	0.945	2.025	0.948	2.043	0.946	2.012	0.943	1.986
	IV	0.952	1.396	0.955	1.403	0.954	1.394	0.946	1.339	0.944	1.343	0.946	1.338	0.946	1.334
	V	0.946	0.829	0.949	0.831	0.949	0.829	0.945	0.815	0.945	0.817	0.945	0.817	0.946	0.816
(5,10)	I	0.951	3.645	0.960	3.794	0.953	3.439	0.911	3.054	0.919	3.123	0.909	2.923	0.910	2.798
	II	0.962	2.383	0.966	2.432	0.961	2.319	0.942	2.473	0.944	2.536	0.934	2.385	0.936	2.267
	III	0.963	1.962	0.963	1.971	0.960	1.944	0.943	1.801	0.943	1.804	0.942	1.787	0.942	1.772
	IV	0.951	1.245	0.952	1.247	0.953	1.240	0.944	1.190	0.943	1.190	0.944	1.187	0.943	1.185
	V	0.942	0.737	0.942	0.738	0.945	0.736	0.942	0.723	0.941	0.724	0.941	0.723	0.941	0.723

* I:(5,5), II:(5,10), III:(10,10), IV:(20,20), V: (50,50)

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same. The estimated parameters are: $\alpha_1 = 0.4342$, $\hat{\beta}_1 = 2.2824$, $\alpha_2 = 1.1854$, $\hat{\beta}_2 = 3.7707$. The estimated 95% confidence intervals for the common gamma mean by all the proposed methods are presented in Table 4. Our simulation study demonstrate that MOVER-type methods could be liberal at sample sizes (10, 10). Give the sample sizes as 12 and 13 in this application, the confidence intervals by $\mathbf{GV_K}$, $\mathbf{GV_C}$, $\mathbf{GV_W}$ methods are recommended, and among them the $\mathbf{GV_K}$ method has the shortest length.

Example 2. Table 5 presents a dataset of chloride concentration in spring water samples from two types of rocks in Sierra Nevada, California and Nevada [30]. It has been argued that gamma distribution is appropriate to fit the data in each sample, and testing equality of means [11] concluded that the means of chloride concentration from two types of rocks are the same. The estimated parameters are: $\alpha_1 = 0.7594$, $\hat{\beta}_1 = 0.3616$, $\alpha_2 = 1.1359$, $\hat{\beta}_2 = 1.6092$. The estimated 95% confidence intervals for the common gamma mean by all the proposed methods are presented in Table 6. Given sample sizes 18 and 17 and parameter estimates, the confidence interval estimated by $\mathbf{GV_K}$ is most recommenced in practice.

Summary and discussion

Gamma distribution plays an important role in practice. When the result of testing equality of several gamma means is not significant, it is customary that we need to make inference about the common gamma mean. While the standard large sample methods exist, small sample inference for the common gamma mean has not been explored. In this article, we focus on accurate confidence interval estimation for the common gamma mean based on several independent gamma samples using the concepts of generalized pivots and the method of MOVER. Via a comprehensive simulation study, we discovered that the proposed methods based on generalized pivots can generally provide satisfactory confidence intervals with consistent performance despite parameter settings and sample sizes. The MOVER-type methods can be liberal for certain scenarios, especially when sample sizes are small.

The proposed methods are easy to implement. The R program is available at li.yan@roswellpark.org.

Due to the popularity of gamma distribution in applied fields, we expect the proposed methods have wide applicability in practice where right-skewed data are often observed.

Table 2. Coverage probabilities (CP) and length of confidence interval (CI) of proposed 95% confidence intervals for the common gamma mean (2000 simulations) with two independent samples (K = 5).

(α_1, α_2)	Sizes*	GV$_K$		GV$_C$		GV$_W$		MOVER$_K$		MOVER$_C$		MOVER$_W$		MOVER$_{boot}$	
		CP	CI	CP	CI	CP	CI	CP	CI	CP	CI	CP	CI	CP	CI
μ = 1															
(0.5,0.5,0.75,0.75,1)	VI	0.962	2.618	0.995	89.847	0.979	8.961	0.879	1.590	0.959	19.057	0.926	5.422	0.830	1.471
	VII	0.955	0.978	0.980	1.316	0.966	1.165	0.857	0.741	0.922	0.959	0.901	0.877	0.869	0.785
	VIII	0.931	0.561	0.961	0.639	0.956	0.620	0.874	0.475	0.913	0.533	0.908	0.520	0.898	0.510
	IX	0.959	0.687	0.984	2.241	0.975	1.184	0.927	1.196	0.966	15.958	0.949	4.277	0.913	1.213
	X	0.923	0.309	0.955	0.338	0.951	0.335	0.901	0.287	0.930	0.311	0.927	0.309	0.927	0.308
(0.5,1,2,5,10)	VI	0.970	0.903	0.987	3.417	0.976	1.300	0.901	0.609	0.944	2.858	0.922	1.146	0.863	0.551
	VII	0.961	0.396	0.969	0.425	0.964	0.407	0.906	0.322	0.924	0.346	0.916	0.335	0.899	0.322
	VIII	0.950	0.235	0.952	0.240	0.949	0.236	0.922	0.213	0.925	0.217	0.926	0.215	0.919	0.214
	IX	0.966	0.210	0.978	0.427	0.970	0.276	0.946	0.420	0.969	4.520	0.958	1.434	0.919	0.412
	X	0.956	0.135	0.960	0.137	0.957	0.136	0.947	0.130	0.950	0.132	0.948	0.132	0.947	0.131
(0.5,2,2,5,5)	VI	0.966	0.977	0.983	3.781	0.972	1.435	0.895	0.670	0.936	5.867	0.911	1.594	0.864	0.609
	VII	0.962	0.443	0.973	0.474	0.965	0.453	0.905	0.363	0.923	0.389	0.916	0.375	0.900	0.361
	VIII	0.945	0.268	0.951	0.274	0.950	0.270	0.916	0.241	0.923	0.246	0.922	0.244	0.917	0.243
	IX	0.957	0.276	0.976	0.602	0.965	0.368	0.944	0.504	0.966	5.257	0.954	1.571	0.922	0.495
	X	0.955	0.153	0.959	0.155	0.958	0.154	0.943	0.147	0.943	0.149	0.942	0.148	0.944	0.148
(1,2,2,5,5)	VI	0.971	0.916	0.983	1.551	0.972	0.954	0.904	0.629	0.924	0.887	0.901	0.654	0.864	0.537
	VII	0.960	0.426	0.966	0.442	0.958	0.425	0.915	0.354	0.918	0.363	0.911	0.354	0.901	0.346
	VIII	0.955	0.260	0.962	0.264	0.958	0.260	0.932	0.236	0.933	0.239	0.929	0.237	0.929	0.236
	IX	0.964	0.252	0.972	0.299	0.965	0.263	0.945	0.444	0.954	0.796	0.949	0.522	0.920	0.377
	X	0.945	0.151	0.948	0.152	0.945	0.151	0.939	0.145	0.941	0.146	0.938	0.146	0.940	0.146
(2,2,5,5,10)	VI	0.968	0.654	0.979	0.820	0.970	0.633	0.898	0.465	0.908	0.513	0.892	0.452	0.864	0.413
	VII	0.961	0.325	0.961	0.330	0.959	0.322	0.914	0.276	0.916	0.278	0.910	0.274	0.902	0.271
	VIII	0.966	0.202	0.967	0.204	0.964	0.202	0.941	0.186	0.940	0.187	0.941	0.186	0.936	0.186
	IX	0.961	0.180	0.967	0.188	0.959	0.180	0.936	0.302	0.948	0.345	0.932	0.300	0.915	0.267
	X	0.964	0.118	0.958	0.119	0.961	0.118	0.954	0.114	0.953	0.115	0.953	0.114	0.953	0.114
μ = 5															
(0.5,0.5,0.75,0.75,1)	VI	0.968	12.835	0.994	396.747	0.982	43.412	0.882	7.805	0.964	126.406	0.924	32.435	0.835	7.311
	VII	0.953	4.907	0.979	6.617	0.973	5.857	0.860	3.728	0.924	4.794	0.903	4.393	0.867	3.940
	VIII	0.947	2.813	0.975	3.203	0.970	3.107	0.881	2.384	0.920	2.676	0.910	2.615	0.905	2.563
	IX	0.957	3.430	0.985	10.460	0.976	5.677	0.918	5.834	0.962	63.516	0.947	20.451	0.906	5.913
	X	0.928	1.552	0.961	1.695	0.960	1.679	0.899	1.437	0.930	1.557	0.931	1.546	0.928	1.542
(0.5,1,2,5,10)	VI	0.972	4.566	0.987	19.559	0.979	7.006	0.897	3.095	0.946	18.131	0.917	6.738	0.858	2.842
	VII	0.965	1.971	0.973	2.117	0.967	2.025	0.915	1.611	0.934	1.730	0.926	1.670	0.907	1.609
	VIII	0.949	1.177	0.954	1.202	0.953	1.188	0.923	1.067	0.927	1.089	0.926	1.080	0.928	1.074
	IX	0.957	1.047	0.976	2.291	0.968	1.384	0.939	2.091	0.959	24.026	0.949	7.508	0.919	2.071
	X	0.950	0.675	0.953	0.681	0.950	0.679	0.939	0.650	0.945	0.657	0.944	0.656	0.944	0.656
(0.5,2,2,5,5)	VI	0.966	4.907	0.984	16.560	0.974	6.850	0.894	3.356	0.936	19.666	0.912	6.848	0.860	3.047
	VII	0.960	2.229	0.973	2.381	0.968	2.276	0.897	1.822	0.918	1.953	0.906	1.882	0.892	1.812
	VIII	0.948	1.337	0.952	1.364	0.950	1.345	0.911	1.200	0.919	1.225	0.912	1.213	0.911	1.208
	IX	0.959	1.391	0.973	3.026	0.968	1.882	0.934	2.512	0.959	25.382	0.947	8.083	0.913	2.437
	X	0.952	0.766	0.952	0.775	0.949	0.772	0.939	0.735	0.943	0.744	0.942	0.742	0.943	0.741
(1,2,2,5,5)	VI	0.966	4.606	0.979	7.967	0.966	4.825	0.901	3.162	0.923	4.510	0.897	3.319	0.858	2.697
	VII	0.968	2.151	0.971	2.234	0.965	2.148	0.916	1.791	0.920	1.837	0.914	1.788	0.901	1.746
	VIII	0.969	1.304	0.972	1.322	0.967	1.305	0.941	1.186	0.941	1.197	0.940	1.187	0.935	1.183
	IX	0.956	1.267	0.972	1.499	0.961	1.321	0.933	2.208	0.948	4.063	0.936	2.631	0.913	1.882
	X	0.958	0.751	0.955	0.757	0.952	0.753	0.946	0.722	0.947	0.727	0.945	0.725	0.946	0.725

(*Continued*)

Table 2. (Continued)

(α_1, α_2)	**Sizes***	**GV_K**		**GV_C**		**GV_W**		**$MOVER_K$**		**$MOVER_C$**		**$MOVER_W$**		**$MOVER_{boot}$**	
		CP	**CI**	**CP**	**CI**	**CP**	**CI**	**CP**	**CI**	**CP**	**CI**	**CP**	**CI**	**CP**	**CI**
(2,2,5,5,10)	VI	0.965	3.276	0.970	4.063	0.960	3.143	0.901	2.312	0.912	2.527	0.891	2.243	0.870	2.063
	VII	0.964	1.626	0.964	1.648	0.960	1.610	0.923	1.379	0.922	1.389	0.919	1.368	0.916	1.351
	VIII	0.949	1.007	0.952	1.014	0.947	1.004	0.928	0.927	0.927	0.930	0.927	0.925	0.925	0.923
	IX	0.958	0.896	0.964	0.937	0.956	0.899	0.939	1.514	0.947	1.740	0.938	1.506	0.922	1.337
	X	0.951	0.591	0.953	0.593	0.950	0.591	0.941	0.572	0.942	0.573	0.942	0.572	0.941	0.573

* Sizes are VI:(5,5,5,5,5), VII:(10,10,10,10,10), VIII:(20,20,20,20,20), IX: (5,10,10,20,50), X: (50,50,50,50,50)

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Table 3. Virginia ground water well yields data (in gal/min/ft) [29].

Without fractures	**with fractures**
0.001, 0.003, 0.007, 0.020	0.020, 0.031, 0.085, 0.013
0.030, 0.040, 0.041, 0.077	0.160, 0.160, 0.180, 0.300
0.100, 0.454, 0.490, 1.020	0.400, 0.440, 0.510, 0.720, 0.950

https://doi.org/10.1371/journal.pone.0269971.t003

Table 4. Estimated confidence interval for Virginia ground water well yields data (in gal/min/ft).

method	**lower**	**upper**	**LCI**
GV_K	0.140	0.491	0.351
GV_C	0.159	0.576	0.417
GV_W	0.153	0.574	0.421
$MOVER_K$	0.169	0.459	0.289
$MOVER_C$	0.178	0.548	0.370
$MOVER_W$	0.175	0.530	0.355
$MOVER_{boot}$	0.175	0.490	0.314

https://doi.org/10.1371/journal.pone.0269971.t004

Table 5. Chloride concentration (in mg/litre) in water data. [30].

Granodiorite	**Quartz Monzonite**
6.0, 0.5, 0.4, 0.7, 0.8, 6.0, 5.0, 0.6, 1.2	1.0, 0.2, 1.2, 1.0, 0.3, 0.1, 0.1, 0.4, 3.2
1.0, 0.2, 1.2, 1.0, 0.3, 0.1, 0.1, 0.4, 3.2	0.3, 0.4, 1.8, 0.9, 0.1, 0.2, 0.3, 0.5

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Table 6. Estimated confidence intervals and lengths for the common mean for Chloride concentration (in mg/litre) in water.

method	**lower**	**upper**	**length**
GV_K	0.524	1.366	0.842
GV_C	0.555	1.482	0.928
GV_W	0.547	1.455	0.908
$MOVER_K$	0.543	1.251	0.707
$MOVER_C$	0.569	1.317	0.749
$MOVER_W$	0.571	1.317	0.746
$MOVER_{boot}$	0.567	1.310	0.743

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Author Contributions

Writing – original draft: Li Yan.

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