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Statistical inference for Nadarajah-Haghighi distribution under unified hybrid censored competing risks data

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ABSTRACT

Nadarajah and Haghighi distribution (NHD) inferences problem has been discussed under unified hybrid censoring scheme (UHCS) in the existence of competing risks model. Competing risks model is defined by time-to-failure under more than one cause of failure, which can be dependent or independent. This study focuses on discussing the case of failure partially observed causes of failure competing risks model. We obtain various inferences: we first obtain the MLE, in addition, we construct approximate confidence intervals (ACIs). Second, we obtain the Bayes estimator via SELF and related highest posterior density (HPD) using Markov Chain Monte Carlo (MCMC). Finally, an electrical appliances data set and simulation studies have been analyzed for further illustrations.

1. Introduction

In survival analysis, most of the study uses censored data due to the time and cost circumstances. In the statistical literature, the two most famous schemes are: type-I (T-I) and type-II (T-II) censoring schemes. Hybrid censoring scheme (HCS) is a combination of T-I and T-II censoring schemes; which is divided to: T-I HCS and T-II HCS, see [1]. The HCS can be characterized statistically by: let $X_{m:n}$ be denoted as the m^{th} failure time in which n items are employed in a lifetime and the prescribed test termination time presented by T. Under T-I HCS, the test is completed at a random time $T^* = \min\{X_{m:n}, T\}$: $T \in (0, \infty)$ and $1 \le m \le n$. However, T-II HCS satisfied fixed number of failures. Thus, in T-II HCS random completed time of the test is $T^* = \max\{X_{m:n}, T\}$, to satisfy that, at least m failures are observed.

There is more information about T-I HCS presented by [2,3]. Also, [4] has some considerable discussion on T-II HCS. However, T-I and T-II HCS both have some drawbacks, absence of elasticity test in a small period of time and getting a large number of failures are the foremost disadvantages of them. Thus, we are driven straight forward to the range of generalized HCS (GHCS), see [5].

Therefore, the concept of GHCS emerged. In generalized T-I HCS (GT-I HCS), let n be the independent units on the experiment and ϱ the placed object number that should be observed. The prior integers (ϱ, m), satisfy that $1 < \varrho < m \le n$. When the failure time $X_\varrho < T$, the test is completed at $\min(X_m, T)$. Also, if $X_\varrho > T$, the test is completed at X_ϱ. Then, (T^*, R) is defined by

Abbreviations: NHD, Nadarajah-Haghighi distribution; T^*, The termination point of the experiment; GHCS, Generalized Hybrid censoring scheme; UHCS, Unified hybrid censoring schemes; MLE, Maximum likelihood estimator; HCS, Hybrid censoring scheme; S(.), Survival function; h(.), Hazard rate function; HPD, Highest posterior density; CI, Confidence intervals; SND, standard normal distribution; pdf, Probability density function; SELF, Squared error loss functions; df, density function; cdf, Cumulative distribution function; ACIs, Approximate confidence intervals; MCMC, Marcov Chain Monto Carlo; LINEX, linear exponential loss functions.

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$$(T^*, R) = \begin{cases} (X_\varrho, \varrho), \text{ if } X_\varrho > T, \\ (T, R), \ \varrho \le R < m, \text{ if } X_\varrho < T < X_m, \\ (X_m, m), \text{ if } X_\varrho < X_m < T. \end{cases}$$

Where R is the observed number of failure times, T^* is the experiment completed time and T is the ideal test time.

The concept of generalized T-II HCS (GT-II HCS) is demonstrated as follows: consider n independent units are put in the test where the integer $m \in \{1, 2, ..., n\}$, and the two prior times $0 < T_1 < T_2 < \infty$. The time to failure X_i is recorded until the time T_1 appears. When the time to failure $X_m < T_1$, the experiment is terminated at T_1. But if $T_1 < X_m < T_2$, then the experiment is terminated at X_m. Covered by this scheme, if $T_1 < T_2 < X_m$, then the experiment is terminated at T_2. Accordingly, (T^*, R) is given by:

$$(T^*, R) = \begin{cases} (T_2, R), \ 1 < R \le m \text{ if } T_1 < T_2 < X_m, \\ (X_m, m), \text{ if } T_1 < X_m < T_2, \\ (T_1, R), \ m \le R \le n, \text{ if } X_m < T_1. \end{cases}$$

Both systems GT-I HCS and GT-II HCS have some drawbacks, where, in a GT-I HCS, we may not have the m_{th} failure due to the prefixed time, and in GT-II HCS, it may take a lot of time to get a sufficient number of effective samples.

Unified hybrid censoring schemes (UHCS) have been suggested by [6] to overcome the drawbacks of the above two schemes by combining them. In this scheme, let m be the predetermined observation number and ϱ be the object number that must be observed, it is assumed that the two prior fixed integers $\varrho, m \in 1, 2, ..., n$ satisfy $1 < \varrho < m \le n$. $T_1, T_2 \in (0, \infty)$ are denoted as the predetermined experiment completed time and the extended experiment completed time respectively, where $T_1 < T_2$. R_i $(i = 1, 2)$ is the number of observed objects until T_i $(i = 1, 2)$. In accordance with the end point and observed number, UHCS is divided into six cases

$$\begin{aligned} &\text{Case I: For } 0 < X_{\varrho:n} < X_{m:n} < T_1 < T_2, \text{ we get } T^* = T_1, \\ &\text{Case II: For } 0 < X_{\varrho:n} < T_1 < X_{m:n} < T_2, \text{ we get } T^* = X_{m:n}, \\ &\text{Case III: For } 0 < X_{\varrho:n} < T_1 < T_2 < X_{m:n}, \text{ we get } T^* = T_2, \\ &\text{Case IV: For } 0 < T_1 < X_{\varrho:n} < X_{m:n} < T_2, \text{ we get } T^* = X_{m:n}, \\ &\text{Case V: For } 0 < T_1 < X_{\varrho:n} < T_2 < X_{m:n}, \text{ we get } T^* = T_2, \\ &\text{Case VI: For } 0 < T_1 < T_2 < X_{\varrho:n} < X_{m:n}, \text{ we get } T^* = X_{\varrho:n}. \end{aligned} \tag{1.1}$$

When the $(\varrho - th)$ observation happens before T_1, the test will be completed at $min(max(X_{r:n}, T_1), T_2)$ and when the $(\varrho - th)$ observation happens between T_1 and T_2, the test will be terminated at $min(X_{r:n}, T_2)$. If it happens after T_2, the test will be terminated at $X_{\varrho:n}$.

Additionally, when $T_1 \to 0$, this scheme becomes GT-I HCS. When $\varrho \to 0$, it tends to GT-II HCS. In the case of if both $T_1 \to 0$ and $\varrho \to 0$, it becomes T-I HCS. Finally, when $T_2 \to \infty$ and $\varrho \to n$, UHCS becomes T-II HCS. Thus, UHCS provides a sufficient number of failures in between a fixed period of time. In recent years, statistical inference based on UHCS has been studied by various researchers. For more details one may be referred to [7], [8], [9], [10], [11]. [12], [13], [14], [15].

The Nadarajah-Haghighi distribution (NHD) is a generalization of the exponential distribution first suggested by [16]. [16] demonstrated that the PDF of the NHD may have a decreasing value, and that unimodal forms, as well as the HRF of NHD, can be decreasing, increasing, or constant shape, which is analogous to the generalized exponential distributions, Gamma, and Weibull. Parameter estimations for NHD have been the subject of research by several writers in the last few years. One may be referred to [17], [18], [19], [20]. [16] indicate that NHD has the fascinating feature of always having the zero mode. For $\theta = 1$, NHD reduces to the exponential distribution. This distribution in some literature is called as extended exponential distribution (EED) where the pdf and cdf are expressed as

$$f_j(x, \theta, \beta_j) = \theta \beta_j (1 + \beta_j x)^{\theta - 1} e^{1 - (1 + \beta_j x)^\theta}, x > 0, \theta > 0, \beta_j > 0, j = 1, 2, \tag{1.2}$$

$$F_j(x, \theta, \beta_j) = 1 - e^{1 - (1 + \beta_j x)^\theta}$$

$y(x, \theta, \beta_j) = (1 + \beta_j x)^\theta$, then the parameters of life, $h(.)$ and $S(.)$ of the NHD are given, respectively, by

$$h_j(t, \theta, \beta_j) = \theta \beta_j (1 + \beta_j t)^{\theta - 1}, \tag{1.3}$$

and

$$S_j(t, \theta, \beta_j) = e^{1 - y(x, \theta, \beta_j)}, \theta > 0, \beta_j > 0. \tag{1.4}$$

The second direction involves the use of competing risks data under UHCS to estimate the parameters β_1, β_2 and θ. For more details on recent studies using partially observed competing risks data, see [21], [22], [23], [24], [25], [26], and [27].

From the statistical literature, it has been seen that the studies of NHD mainly utilize complete, T-I or T-II and PCS. Even in competing risks model, NHD distribution has been considered based on T-II censoring. In addition to the significance of the UHCS, we got motivation to obtain estimators for NH distribution based on competing risks data under UHCS. Furthermore, no study has been mentioned in the literature about this issue, as far as we are aware. Thus, the study plays an important role in helping to close this gap.

The following is the outline of the paper. The MLE and the ACIs have been constructed and are shown in sect.2. Sect.3 investigates the Bayesian analysis under two loss functions. The hypothesis testing has been discussed in sect.4. A simulation study has been presented in sect.5. Further, we examine a real data set to demonstrate the estimating methods presented in this research in sect.6. Finally, concluding remarks end the paper in sect.7.

2. Model construction and MLE

Let's inspect n independently and identically distributed (i.i.d.) units proposed on a life-testing experiment with belief ordered failure times $X_1, \cdots, X_n$. Suppose the random failure time X_{ji}, $i = 1, 2, 3, \cdots, R$ under two competing causes of failures $j = 1, 2$, satisfies

$$X_i = \min\{X_{1i},\ X_{2i}\}, i = 1,\ 2,\ ...,\ n.$$

The notation η_i, $i = 1,\ 2,\ \cdots,\ R$ is value indicating whether failure cause for ith unified censored time is detected or not, i.e.

$$\eta_i = \begin{cases} 1, & \text{if unit fails due to the first cause;} \\ 0, & \text{if unit fails due to the second cause.} \end{cases}$$

The study of inference problems associated with the case of partially observed causes of failure competing risks model, where failure causes cannot be detected clearly, has recently gained practical significance. $R_1 = \sum_{i=1}^{R} I(\eta_i = 1)$ and $R_2 = \sum_{i=1}^{R} I(\eta_i = 0)$ indicate sums of failures attributable to causes 1 and 2, respectively. Here $R_1 + R_2 = R$. In addition, assume that $X = \{X_{1:n}, X_{2:n}, ..., X_{R:n}\}$ represent UHCS available from the distribution given in (1.2). As a result, note that for the corresponding competing risks data we have the six cases using (1.1) under this UHCS:

Case I: $(X_{1:n}, \eta_1), ..., (X_{m_1:n}, \eta_{m_1})$ If $0 < X_{\varrho:n} < X_{m:n} < T_1 < T_2$, then $T^* = T_1$,

Case II: $(X_{1:n}, \eta_1), ..., (X_{m:n}, \eta_m)$ If $0 < X_{\varrho:n} < T_1 < X_{m:n} < T_2$, then $T^* = X_{m:n}$,

Case III: $(X_{1:n}, \eta_1), ..., (X_{m_2:n}, \eta_{m_2})$ If $0 < X_{\varrho:n} < T_1 < T_2 < X_{m:n}$, then $T^* = T_2$,

Case IV: $(X_{1:n}, \eta_1), ..., (X_{m:n}, \eta_m)$ If $0 < T_1 < X_{\varrho:n} < X_{m:n} < T_2$, then $T^* = X_{m:n}$,

Case V: $(X_{1:n}, \eta_1), ..., (X_{m_2:n}, \eta_{m_2})$ If $0 < T_1 < X_{\varrho:n} < T_2 < X_{m:n}$, then $T^* = T_2$,

Case VI: $(X_{1:n}, \eta_1), ..., (X_{\varrho:n}, \eta_\varrho)$ If $0 < T_1 < T_2 < X_{\varrho:n} < X_{m:n}$, then $T^* = X_{\varrho:n}$.

Quantities T^* are denoting the experiment termination point. The two positive integers m_1 and m_2 are defined by $X_{m_1:n} < T_1 < X_{m_1+1:n}$ and $X_{m_2:n} < T_2 < X_{m_2+1:n}$.

The joint probability function of UHCS competing risk data $\underline{x} = \{x_{1:n}, ..., x_{R:n}\}$, with partially observed causes of failures obtained by combining the above mentioned cases, is given by

$$L(\underline{\mathbf{x}}) \propto \prod_{i=1}^{R} (h_1(x_i))^{R_1} (h_2(x_i))^{R_2} \left[S_1(x_i) S_2(x_i)\right] \left[S_1(T^*) S_2(T^*)\right]^{n-R}, \tag{2.1}$$

where R refers to the total number of failures up to time T^* and $X_{i:n} = X_i$.

The values of R and T^* in (2.1) are given by

$$(T^*, R) = \begin{cases} (T_1, m_1), & \text{case I }, \\ (X_{m:n}, m), & \text{case II and case IV}, \\ (T_2, m_2), & \text{case III and case V}, \\ (X_{\varrho:n}, \varrho), & \text{case VI}. \end{cases}$$

Following m_1 and m_2 represent the number of failures prior to T_1 and T_2. For the failure times, follow the two parameters of NHD with $S_j(x)$ and $h_j(x)$ given by (1.3) and (1.4). Thus, the likelihood function (LF) (2.1) can be re-written as

$$\begin{aligned} L(\theta, \beta_1, \beta_2) &\propto \beta_1^{R_1}\, \beta_2^{R_2}\, \theta^R\, exp[(n-R)[(1 - y(T^*, \theta, \beta_1)) + (1 - y(T^*, \theta, \beta_2))] \\ &exp[(\theta - 1)[\sum_{i=1}^{R_1} log(1 + \beta_1 x_i) + \sum_{i=1}^{R_2} log(1 + \beta_2 x_i)] exp[-\sum_{j=1}^{2}\sum_{i=1}^{R} y(x, \theta, \beta_j)]. \end{aligned} \tag{2.2}$$

Taking the natural log, LF of (2.2) is now given as

$$log L \propto R\log\theta + R_1\log\beta_1 + R_2\log\beta_2 - (n-R)[y(T^*,\theta,\beta_1) + y(T^*,\theta,\beta_2)] - \sum_{j=1}^{2}\sum_{i=1}^{R} y(x,\theta,\beta_j) + (\theta-1)[\sum_{i=1}^{R_1} log(1+\beta_1 x_i) + \sum_{i=1}^{R_2} log(1+\beta_2 x_i]. \quad (2.3)$$

Following are the first derivative of Eq. (2.3) with respect to β_1, β_2 and θ and solution to the corresponding LE given as

$$\frac{\partial L(\theta,\beta_1,\beta_2)}{\partial\beta_j} = \frac{R_j}{\beta_j} - (n-R)\sum_{j=1}^{2} y'_{\beta_j}(T^*,\theta,\beta_j) - \sum_{i=1}^{R} y'_{\beta_j}(x_i,\theta,\beta_j)) + (\theta-1)\sum_{i=1}^{R_j}\frac{x_i}{1+\beta_j x_i} = 0, j = 1,2 \quad (2.4)$$

where $\frac{\partial y(x_i,\theta,\psi(\beta_j))}{\partial\beta_j} = y'_{\beta_j}(x_i,\theta,\beta_j) = \theta x_i(1+\beta_j x_i)^{\theta-1}$, $j = 1,2,\ i = 1,2,...,R.$

$$\frac{\partial L(\theta,\beta_1,\beta_2)}{\partial\theta} \propto \frac{R}{\theta} - (n-R)\sum_{j=1}^{2} y'_\theta(T^*,\theta,\beta_j) - \sum_{j=1}^{2}\sum_{i=1}^{R} y'_\theta(x_i,\theta,\beta_j) + \sum_{i=1}^{R_1} log(1+\beta_1 x_i) + \sum_{i=1}^{R_2} log(1+\beta_2 x_i) = 0, \quad (2.5)$$

where $\frac{\partial y(x_i,\theta,\beta_j)}{\partial\theta} = y'_\theta(x_i,\theta,\beta_j) = y(x_i,\theta,\beta_j)\log(1+\beta_j x_i)$, $i = 1,2,...,R,\ j = 1,2.$

By solving the non-linear equations (2.4) and (2.5), the MLEs of the unknown model parameters β_1, β_2 and θ can be obtained. But these equations can't be solved explicitly. Hence, iteration techniques like the Newton's method are applied to solve these equations (2.4) and (2.5) and calculate MLEs as $\hat{\beta}_1$, $\hat{\beta}_2$, and $\hat{\theta}$.

2.1. Asymptotic confidence interval

The MLEs for β_1, β_2 and θ can't be gotten in analytic form, respectively, their actual distributions can't be obtained. We consider the asymptotic distribution of the MLE to express CIs for β_1, β_2 and θ.

Here, the ACIs of $\beta_1 > 0$, $\beta_2 > 0$ and $\theta > 0$ are derived by employing the asymptotic normality property of MLEs.

The asymptotic distribution of MLE $(\hat{\beta}_1,\hat{\beta}_2,\hat{\theta})$ is approximately bivariate normal such that $(\hat{\beta}_1,\hat{\beta}_2,\hat{\theta})$ - $(\beta_1,\beta_2,\theta) \sim N(0, I^{-1}(\hat{\beta}_1,\hat{\beta}_2,\hat{\theta}))$

Then, the inverting of the observed information matrix is

$$I_0^{-1}(\hat{\beta}_1,\hat{\beta}_2,\hat{\theta}) = \begin{pmatrix} -\frac{\partial^2\log l}{\partial\beta_1{}^2} & -\frac{\partial^2\log l}{\partial\beta_1\partial\beta_2} & -\frac{\partial^2\log l}{\partial\beta_1\partial\theta} \\ -\frac{\partial^2\log l}{\partial\beta_2\partial\beta_1} & -\frac{\partial^2\log l}{\partial\beta_2{}^2} & -\frac{\partial^2\log l}{\partial\beta_2\partial\theta} \\ -\frac{\partial^2\log l}{\partial\theta\partial\beta_1} & -\frac{\partial^2\log l}{\partial\theta\partial\beta_2} & -\frac{\partial^2\log l}{\partial\beta^2} \end{pmatrix}_{(\hat{\beta}_1,\hat{\beta}_2,\hat{\theta})} = \begin{pmatrix} var(\hat{\beta}_1) & cov(\hat{\beta}_1,\hat{\beta}_2) & cov(\hat{\beta}_1,\hat{\theta}) \\ cov(\hat{\beta}_2,\hat{\beta}_1) & var(\hat{\beta}_2) & cov(\hat{\beta}_2,\hat{\theta}) \\ cov(\hat{\theta},\hat{\beta}_1) & cov(\hat{\theta},\hat{\beta}_2) & var(\hat{\theta}) \end{pmatrix}$$

Consequently, the pivotal quantities $\frac{\hat{\beta}_1-\beta_1}{\sqrt{Var(\hat{\beta}_1)}}$, $\frac{\hat{\beta}_2-\beta_2}{\sqrt{Var(\hat{\beta}_2)}}$, and $\frac{\hat{\theta}-\theta}{\sqrt{Var(\hat{\theta})}}$ are approximately distributed as standard normal. Thus, the $100(1-\psi)\%,\ 0 < \psi < 1$, asymptotic CI for β_1, β_2 and θ are given by

$(\hat{\beta}_1 \pm Z_{\psi/2}\sqrt{Var(\hat{\beta}_1)})$, $(\hat{\beta}_2 \pm Z_{\psi/2}\sqrt{Var(\hat{\beta}_2)})$, and $(\hat{\theta} \pm Z_{\psi/2}\sqrt{Var(\hat{\theta})})$

where $Z_{\psi/2}$ denotes the upper $(\psi/2)^{th}$ quantile of the SND.

To estimate unknown parameters, we shall provide an alternative approach, such as the Bayesian method. It has been claimed that a useful approach for estimating unknown parameters is Bayesian analysis. Bayesian inference offers several advantages when compared to other forms of reasoning.

3. Bayesian estimation

Bayesian estimators for β_1, β_2, and θ using SELF and LINEX based on UHCS competing risks have been discussed in this section. Let $\tilde{\upsilon}$ be an estimator of the parameter and υ the parameter to be estimated, the SELF is given by

$$L(\upsilon,\tilde{\upsilon}) = (\tilde{\upsilon}-\upsilon)^2.$$

Hence, the Bayes estimator in case of SELF is the posterior mean $\tilde{\upsilon}$ of υ.

LINEX loss function takes the following forms

$$L(\mu,\tilde{\upsilon}) = e^{p(\tilde{\upsilon}-\upsilon)} - p(\tilde{\upsilon}-\upsilon) - 1, p \neq 0.$$

To compute Bayesian estimates of υ the following expectation is proposed

$$\hat{\upsilon}_{SE} = E_\upsilon(\upsilon \mid \underline{x}),$$

$$\hat{\upsilon}_{LI} = -\frac{1}{p} log E_\upsilon(e^{p\upsilon} \mid \underline{x}), p \neq 0.$$

3.1. Prior information and Bayes estimators

The prior knowledge is incorporated in Bayesian analysis and there is no obvious method to select an appropriate prior. Where all the elements of the corresponding expected Fisher information matrix are not of closed forms, so, Jeffrey's prior can't be defined. Furthermore, it is hard to obtain joint conjugate prior distribution for the current estimation problem. The gamma distribution is multilateral for detecting several shapes of the df. It has a logconcave df in the interval $(0,\infty)$. As special case of the gamma prior we can observe the Jeffery's prior. Here, we assume conjugate prior for the class of distribution having gamma distribution. The independent gamma density for the parameters β_1, β_2 and θ given by $\theta \sim Gamma(a_3, b_3)$, $\beta_1 \sim Gamma(a_1, b_1)$ and $\beta_2 \sim Gamma(a_2, b_2)$.

The choice of gamma priors hyperparameters $a_i > 0, b_i > 0, i = 1,2,3$ is inherently subjective and reflects the prior knowledge. Thus, the joint prior density is expressed by

$$h(\theta,\beta_1,\beta_2) \propto \left(\prod_{i=1}^{2} \beta_i^{a_i-1} e^{-b_i\beta_i}\right) \theta^{a_3-1} e^{-b_3\theta}, \quad a_i > 0, b_i > 0, i = 1,2,3\ \beta_1,\beta_2,\theta > 0. \tag{3.1}$$

Consequently, the posterior distribution of θ, β_1 and β_2 can be written as

$$\pi(\theta,\beta_1,\beta_2|\,\underline{x}) = \frac{h(\theta,\beta_1,\beta_2)L(\theta,\beta_1,\beta_2)}{\int_0^\infty\int_0^\infty\int_0^\infty h(\theta,\beta_1,\beta_2)L(\theta,\beta_1,\beta_2)\ d\theta\, d\beta_1\, d\beta_2}. \tag{3.2}$$

The Bayes estimate with respect to SELF and LLF for the function $g(\beta_1,\beta_2,\theta)$ are given by

$$\tilde{g}_{SE}(\beta_1,\beta_2,\theta) = E(g(\beta_1,\beta_2,\theta)|\underline{x}) = \frac{\int_0^\infty\int_0^\infty\int_0^\infty g(\beta_1,\beta_2,\theta)\ h(\theta,\beta_1,\beta_2)L(\theta,\beta_1,\beta_2)\ d\theta\, d\beta_1\, d\beta_2}{\int_0^\infty\int_0^\infty\int_0^\infty h(\theta,\beta_1,\beta_2)L(\theta,\beta_1,\beta_2)\ d\theta\, d\beta_1\, d\beta_2}, \tag{3.3}$$

and

$$\tilde{g}_{LI}(\beta_1,\beta_2,\theta) = -(\frac{1}{p})log[\frac{\int_0^\infty\int_0^\infty\int_0^\infty exp(-pg(\beta_1,\beta_2,\theta))\ h(\theta,\beta_1,\beta_2)L(\theta,\beta_1,\beta_2)\ d\theta\, d\beta_1\, d\beta_2}{\int_0^\infty\int_0^\infty\int_0^\infty h(\theta,\beta_1,\beta_2)L(\theta,\beta_1,\beta_2)\ d\theta\, d\beta_1\, d\beta_2}]. \tag{3.4}$$

3.2. Posterior analysis and importance sampling method

We highlighted that solving equations (3.3) and (3.4) analytically is impossible due to the difficulty in obtaining closed forms for the marginal posterior distributions for β_1, β_2, and θ. Thus, the Bayes estimates of $g(\beta_1,\beta_2,\theta)$ require approximation technique, such as Lindely approximation, numerical integration and MCMC. In this case, the importance sampling method is the most appropriate method to compute the approximate results.

Using (2.2) and (3.1) in (3.2), the posterior distribution of β_1, β_2, and θ takes the following forms:

$$\begin{aligned}\pi(\beta_1,\beta_2,\theta|\,\underline{x}) \propto\ & \beta_1^{R_1+a_1-1}\beta_2^{R_2+a_2-1}\theta^{R+a_3-1}e^{-\beta_1 b_1-\beta_2 b_2-\theta b_3}exp[(n-R)[(1-y(T^*,\theta,\beta_1))\\ &+(1-y(T^*,\theta,\beta_2))]exp[(\theta-1)[\sum_{i=1}^{R_1} log(1+\beta_1 x_i)+\sum_{i=1}^{R_2} log(1+\beta_2 x_i)]exp[-\sum_{j=1}^{2}\sum_{i=1}^{R} y(x,\theta,\beta_j)]\end{aligned} \tag{3.5}$$

Then

$$\pi(\beta_1,\beta_2,\theta \mid \underline{x}) = \pi_1(\theta \mid \beta_1,\beta_2,\underline{x})\pi_2(\beta_1)\pi_3(\beta_2)Z(\theta,\beta_1,\beta_2). \tag{3.6}$$

Thus, using (3.5) and (3.6), the marginal posterior probability density function of β_1, β_2, and θ, can be written as

$$\pi_1(\theta \mid \beta_1,\beta_2,\underline{x}) \propto Gamma(R+a_3, b_1-(\Sigma_{i=1}^{R_1} log(1+\beta_1 x_i)+\Sigma_{i=1}^{R_2} log(1+\beta_2 x_i))) \tag{3.7}$$

where $\pi_2(\beta_1)$ and $\pi_3(\beta_2)$ are the proper density functions given by

$$\pi_2(\beta_1 \mid \underline{x}) \propto \beta_1^{R_1+a_1-1} e^{-\beta_1 b_1-\Sigma_{i=1}^{R_1} log(1+\beta_1 x_i)} \tag{3.8}$$

and

$$\pi_3(\beta_2 \mid \underline{x}) \propto \beta_2^{R_2+a_2-1} e^{-\beta_2 b_2-\Sigma_{i=1}^{R_2} log(1+\beta_2 x_i)}. \tag{3.9}$$

Also

$$Z(\theta,\beta_1,\beta_2) \propto e^{-(n-R)[(1+y(T^*,\theta,\beta_1))+(1+y(T^*,\theta,\beta_2))]}e^{-\sum_{j=1}^{2}\sum_{i=1}^{R} y(x,\theta,\beta_j)}, \tag{3.10}$$

and the Bayes estimate with respect to SELF in equation (3.3), with (3.8), (3.9) and (3.9) are given by

$$\tilde{g}_{SE}(\beta_1,\beta_2,\theta) \propto \int_0^\infty\int_0^\infty\int_0^\infty g(\beta_1,\beta_2,\theta)\pi_1(\theta \mid \beta_1,\beta_2,\underline{x})\pi_2(\beta_1)\pi_3(\beta_2)Z(\theta,\beta_1,\beta_2)\ d\theta d\beta_1 d\beta_2. \tag{3.11}$$

In this case we will use the importance sampling technique to draw MCMC (see [28]) as follows. The plots of the two functions $\pi_2(\beta_1 \mid \underline{x})$ and $\pi_3(\beta_2 \mid \underline{x})$ are as similar as normal distribution. So, this generates a sample of this distribution using MH method. Also we generate a sample from the Gamma distribution $\pi_1(\theta \mid \beta_1, \beta_2)$.

We used this algorithm for computing Bayes estimates and the HPD credible intervals of β_1, β_2, and θ, the description of the algorithm is as follows.

Step 1: Assume that $\kappa = 1$ and $M = burn - in$, begin with initial iteration with MLE and choose an initial guess of $(\beta_1, \beta_2, \theta)$ and call it $(\beta_1^{(0)}, \beta_2^{(0)}, \theta^{(0)})$.

Step 2: Generate θ^{κ} from $\pi_1(\theta \mid \beta_1^{\kappa-1}, \beta_2^{\kappa-1})$ using (3.7).

Step 3: By using MH algorithm, generate β_1^{κ} and β_2^{κ} from $\pi_2(\beta_1 \mid \underline{x})$ and $\pi_3(\beta_2 \mid \underline{x})$ given in (3.8) and (3.9), with normal distribution where the mean β_1^{κ} and β_2^{κ} and variance are obtained from $I_0^{-1}(\hat{\beta}_1, \hat{\beta}_2, \hat{\theta})$.

Step 4: With $\kappa = 2$ repeat the previous three steps to get the data $\varphi^i = (\theta^i, \beta_1^i, \beta_2^i), i = 1, 2, ..., N$.

Step 5: The Bayesian estimators of any function $\widetilde{\pi}_B(\beta_1, \beta_2, \theta)$ under SELF can be obtained as

$$\widetilde{\pi}_B(\beta_1, \beta_2, \theta) = \frac{\sum_{i=M+1}^{N} \pi_B(\varphi^i) Z(\varphi^i)}{\sum_{i=M+1}^{N} Z(\varphi^i)},$$

Step 6: Also, the posterior variance of $\pi_B(\beta_1, \beta_2, \theta)$ is obtained by

$$V(\pi_B(\beta_1, \beta_2, \theta)) = \frac{\sum_{i=M+1}^{N} (\pi_B(\varphi^i) - \widetilde{\pi}_B)^2 Z(\varphi^i)}{\sum_{i=M+1}^{N} Z(\varphi^i)}.$$

Taking into consideration $100(1-\mu)\%,\ 0 < \mu < 1$, HPD intervals for $\pi_B(\beta_1, \beta_2, \theta)$ can be created using an idea of [29] as follows

Step 1: Rearrange $\varphi^i = (\theta^i, \beta_1^i, \beta_2^i),\ i = 1, 2, ..., N - M$ in an increasing order.

Step 2: Compute the $100(1-\mu)\%,\ 0 < \mu < 1$, HPD credible intervals of $\varphi(\beta_1, \beta_2, \theta)$ using

$$(\varphi^{\frac{\ell}{N}}, \varphi^{(\ell + [(1-\mu)N]/N)}),\ for,\ \ell = 1, 2, ..., N - [(1-\mu)N],$$

where [.] represents the greatest integer value.

Step 3: The $100(1-\mu)\%,\ 0 < \mu < 1$ HPD interval is the smallest interval width among all credible intervals satisfying

$$\varphi^{(\ell + [(1-\mu)N]/N)} - \varphi^{\frac{\ell}{N}} = min(\varphi^{(\ell + [(1-\mu)N]/N)} - \varphi^{\frac{\ell}{N}}),\ for\, \ell = 1, 2, ..., N - [(1-\mu)N].$$

4. Testing problem

As we can see, the above sections discussed inference based on UHCS for NHD competing risks data with shape parameter $\theta_1 = \theta_2 = \theta$ and two scale parameters β_1 and β_2. It is impractical to test, whenever α_1 and α_2 are equal or not. Likewise, we also checked whether the scale parameters β_1 and β_2 are equal or not in practice. For the sake of investigating the tests, the hypotheses have been expressed by

(1) $H_0 : \theta_1 = \theta_2 = \theta$ vs. $H_1 : \theta_1 \neq \theta_2$,

(2) $H_0 : \beta_1 = \beta_2 = \beta$ vs. $H_1 : \beta_1 \neq \beta_2$.

Then, we can compute the likelihood ratio statistic (LRS) for test (1) as

$$T_\theta = \frac{max\ L(\theta, \beta_1, \beta_2)}{max\ L(\theta_1, \theta_2, \beta_1, \beta_2)},$$

where $L(\cdot)$ represents the LF. Here, the LRS for large n is calculated by

$$LRS_\theta = -2 \log T_\theta = -2\big(l(\widehat{\theta}, \widehat{\beta}_1, \widehat{\beta}_2) - l(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\beta}_1, \widehat{\beta}_2)\big) \sim \chi^2{}_{(1)}.$$

Likewise, for test (2),

$$T_\beta = \frac{max\ L(\theta_1, \theta_2, \beta)}{max\ L(\theta_1, \theta_2, \beta_1, \beta_2)},$$

and

$$LRS_\beta = -2 \log T_\beta = -2\big(l(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\beta}) - l(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\beta}_1, \widehat{\beta}_2)\big) \sim \chi^2{}_{(1)}.$$

In light of the asymptotic distribution of LRS, we calculate the LRS and we reject the null hypothesis H_0 when $LRS > c$, where c can be obtained in such a way that size of test $= P(\chi^2{}_{(1)} > c)$.

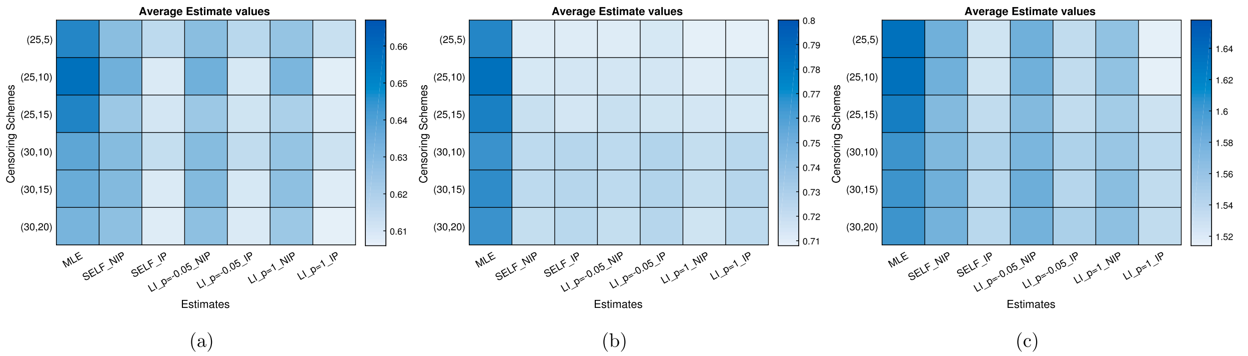


Fig. 1. AEs of (a) β_1, (b) β_2, and (c) θ when $n = 40$, $T_1 = 0.75$ and $T_2 = 1.5$ using a variety of values for (m, k).

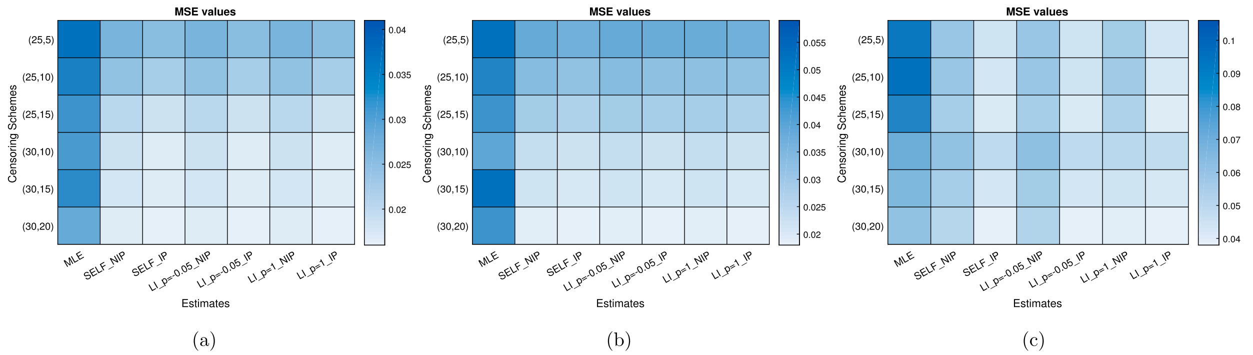


Fig. 2. MSEs for (a) β_1, (b) β_2, and (c) θ when $n = 40$, $T_1 = 0.75$ and $T_2 = 1.5$ using a variety of values for (m, k).

5. Simulation study

In this part, the Monte Carlo simulation study is used to examine the accuracy of parameter estimations based on UHCS. The mean squared error (MSE) criteria are used to evaluate the precision of each point estimate. Average lengths (AL) of confidence intervals and coverage probabilities (CP) are used to make a comparison between the interval estimators. Using the method described in [26], competing risks samples based on UPHC have been constructed.

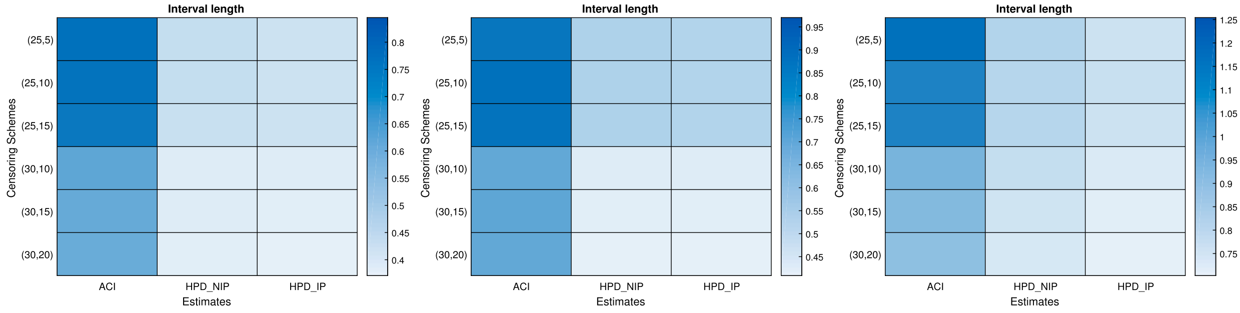
Within the scope of the simulation experiment, true values that should be supplied to the model parameters are $\beta_1 = 0.6$, $\beta_2 = 0.7$, and $\theta = 1.5$. Censoring schemes are evaluated for a variety of n, m, k, T_1, and T_2 values. Both the informative prior (IP) and the non-informative prior (NIP) are taken into consideration in this analysis so that Bayes estimates may be evaluated. For IP, the corresponding hyperparameters are considered as $a_1 = 3; a_2 = 3.5; a_3 = 3; b_1 = 5; b_2 = 5; b_3 = 2$, and under the NIP, the corresponding hyperparameters are considered as $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0$. Point and interval estimates of the simulated results are plotted in Figs. 1-15 based on 10,000 replications. From these figures, the following conclusions may be derived.

- For fixed time thresholds T_1 and T_2, the MSEs of the point estimates decrease when m and n have been increased.
- If n, m, and k are fixed, the MSEs of the point estimates decrease when T_1 and T_2 increase.
- The AEs of point estimates for the unknown parameters are very consistent with the actual values.
- When comparing MLEs and NIP Bayes estimates, the Bayes estimates under IP have lower MSEs.
- When the values of n, m, and k are increased, the AL of the confidence intervals becomes smaller.
- The values of AL are reduced whenever there is an increase in the values of T_1 and T_2 for fixed censoring scheme.
- HPD credible interval under IP yields superior results in terms of AL.
- CPs for HPD credible intervals under IP always stay closer to nominal value 0.95, for any censoring scheme.

From the above conclusions we can summarize that the Bayes estimates under IP outperform the other two point estimates in terms of AE and MSE. Similarly, interval estimates have been shown to have the same pattern of behaviour.

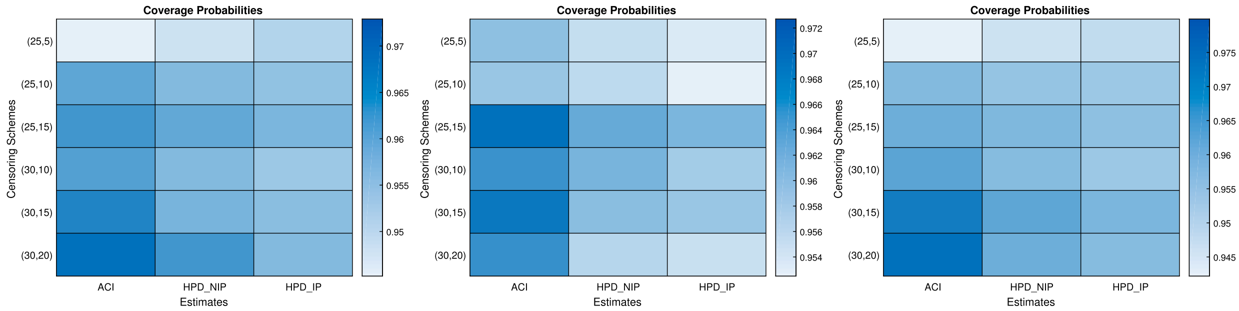
6. Applications to real life data set

A real data set consisting of failure times of electrical appliances, which is presented by [30], has been considered here. Out of the total of 18 unique failure modes in the dataset, 11 occurred several times. For this reason, mode 11 is regarded as cause 1, whereas all other modes are regarded as cause 2. The data set is presented as follows.



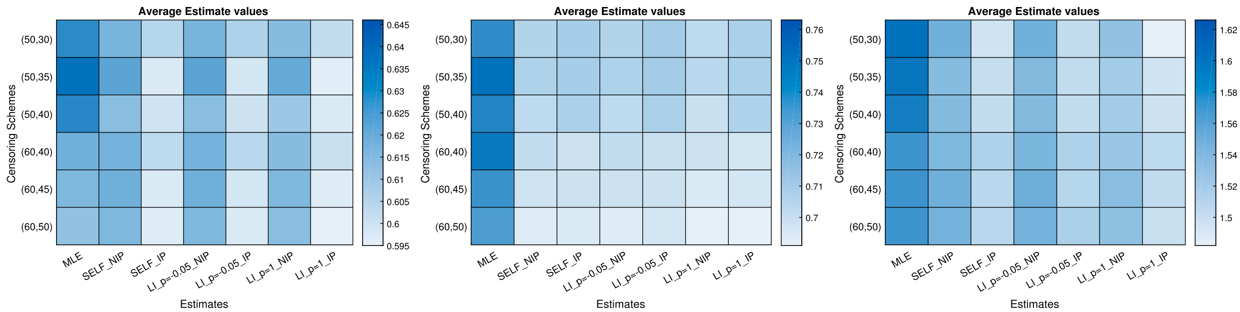
(a) (b) (c)

Fig. 3. ALs of the intervals estimates for (a) β_1, (b) β_2, and (c) θ when $n = 40$, $T_1 = 0.75$ and $T_2 = 1.5$ using a variety of values for (m, k).



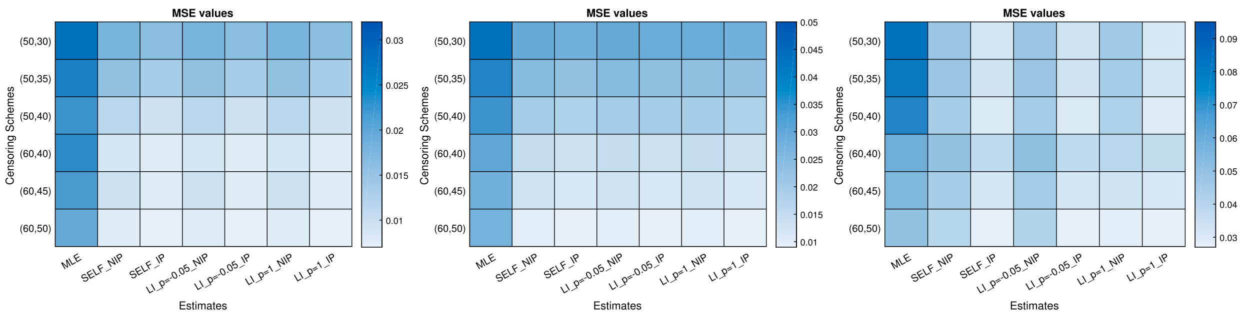
(a) (b) (c)

Fig. 4. CPs of the intervals estimates for (a) β_1, (b) β_2, and (c) θ when $n = 40$, $T_1 = 0.75$ and $T_2 = 1.5$ using a variety of values for (m, k).



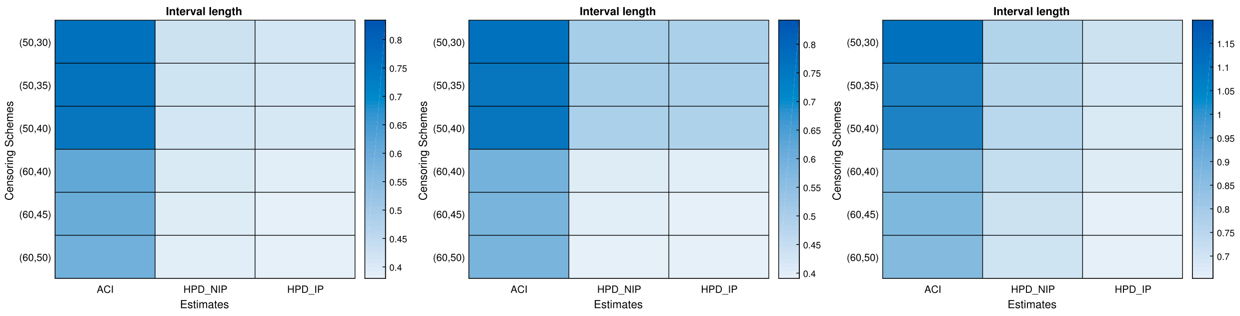
(a) (b) (c)

Fig. 5. AEs of (a) β_1, (b) β_2, and (c) θ when $n = 70$, $T_1 = 0.75$ and $T_2 = 1.5$ using a variety of values for (m, k).



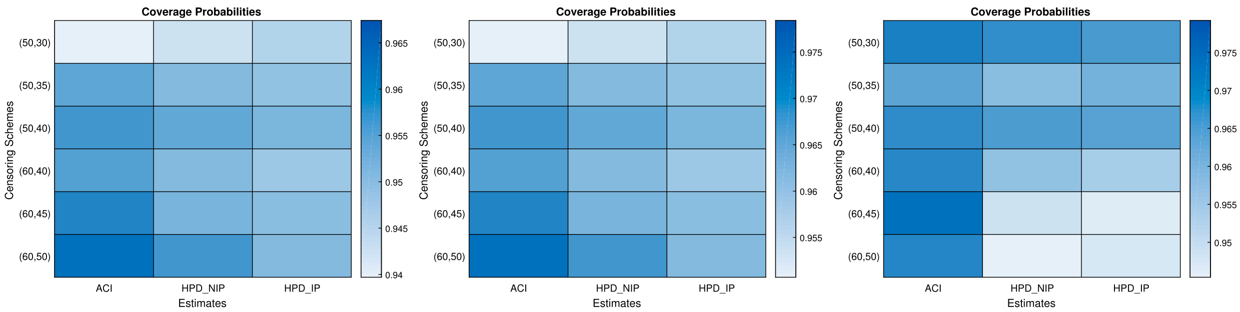
(a) (b) (c)

Fig. 6. MSEs for s (a) β_1, (b) β_2, and (c) θ when $n = 70$, $T_1 = 0.75$ and $T_2 = 1.5$ using a variety of values for (m, k).



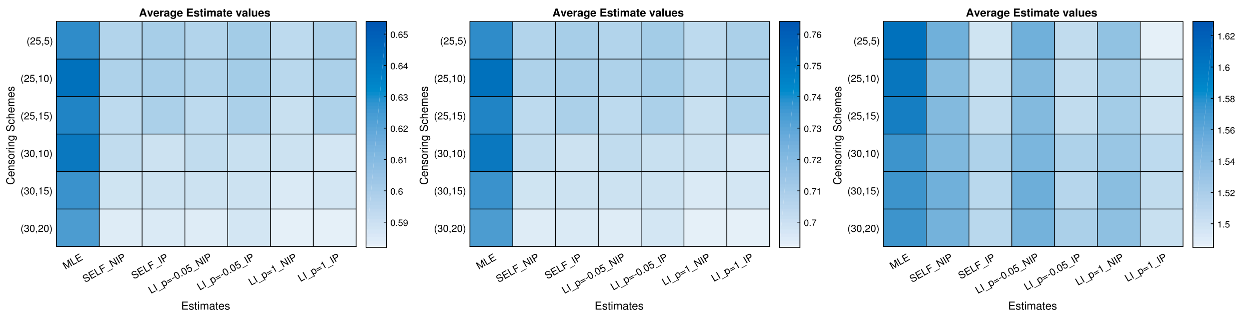
(a) (b) (c)

Fig. 7. ALs of the intervals estimates for (a) β_1, (b) β_2, and (c) θ when $n = 70$, $T_1 = 0.75$ and $T_2 = 1.5$ using a variety of values for (m, k).



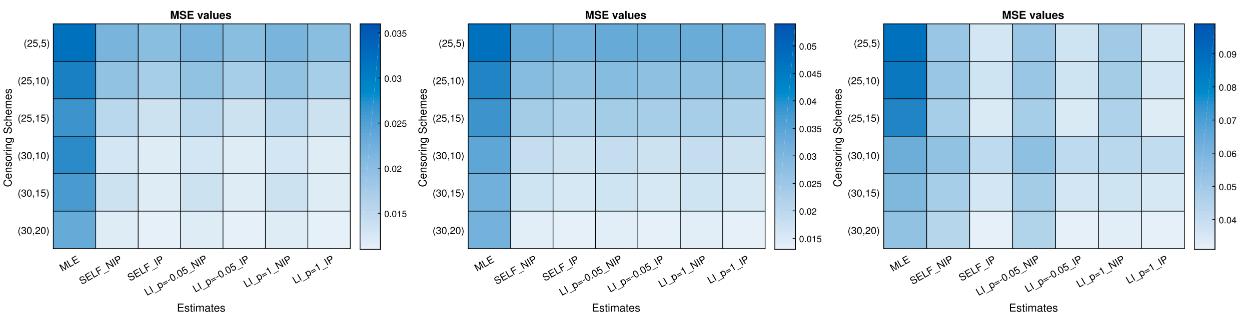
(a) (b) (c)

Fig. 8. CPs of the intervals estimates for (a) β_1, (b) β_2, and (c) θ when $n = 70$, $T_1 = 0.75$ and $T_2 = 1.5$ using a variety of values for (m, k).



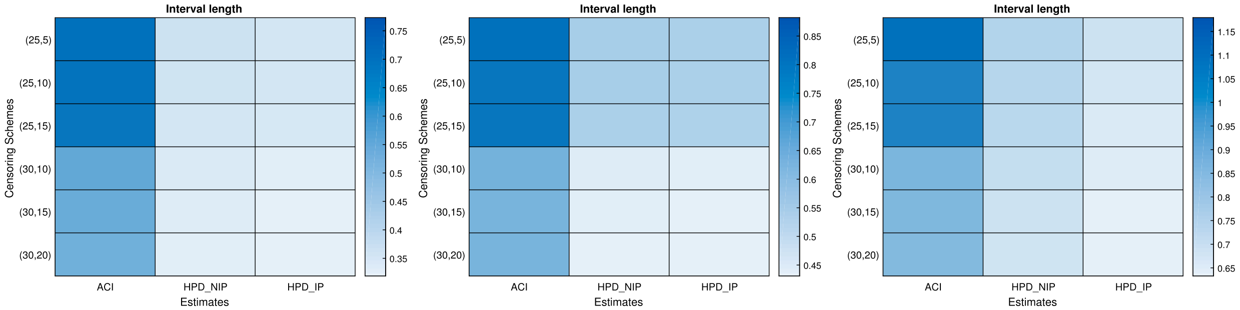
(a) (b) (c)

Fig. 9. AEs of (a) β_1, (b) β_2, and (c) θ when $n = 40$, $T_1 = 0.85$ and $T_2 = 1.7$ using a variety of values for (m, k).



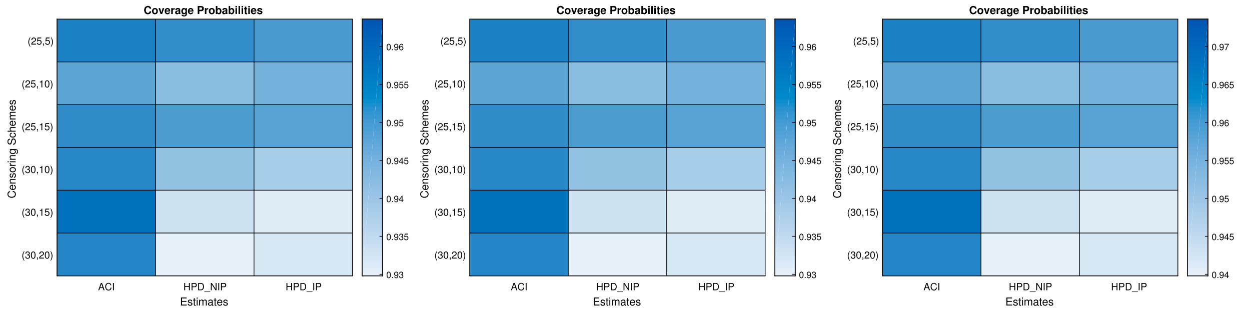
(a) (b) (c)

Fig. 10. MSEs for (a) β_1, (b) β_2, and (c) θ when $n = 40$, $T_1 = 0.85$ and $T_2 = 1.7$ using a variety of values for (m, k).



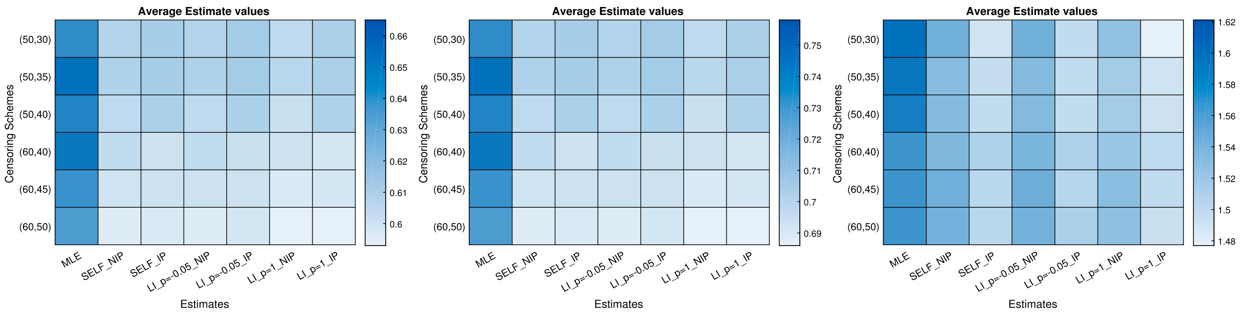
(a) (b) (c)

Fig. 11. ALs of the intervals estimates for (a) β_1, (b) β_2, and (c) θ when $n = 40$, $T_1 = 0.85$ and $T_2 = 1.7$ using a variety of values for (m, k).



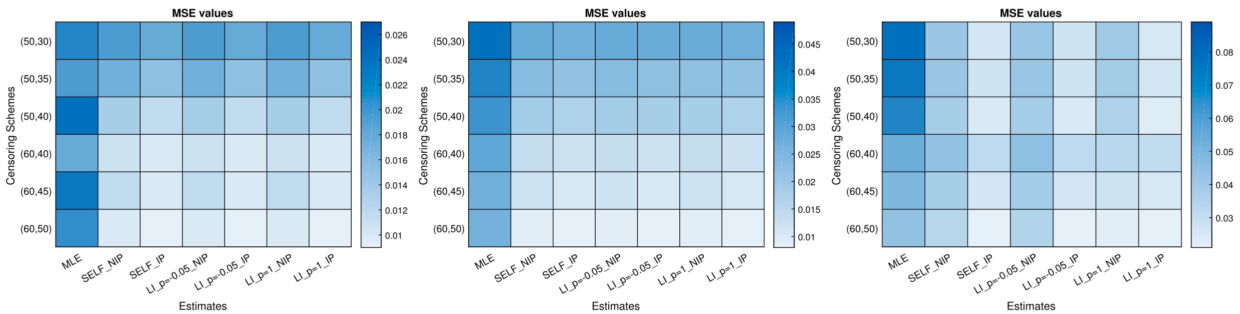
(a) (b) (c)

Fig. 12. CPs of the intervals estimates for (a) β_1, (b) β_2, and (c) θ when $n = 40$, $T_1 = 0.85$ and $T_2 = 1.7$ using a variety of values for (m, k).



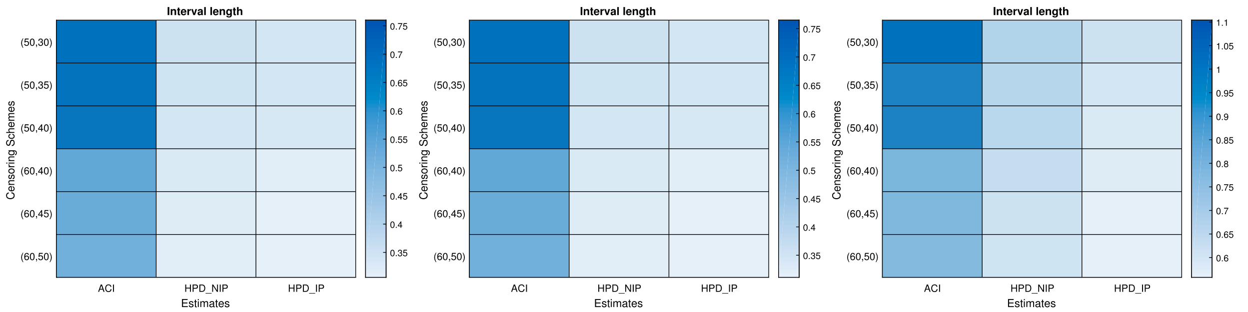
(a) (b) (c)

Fig. 13. AEs of (a) β_1, (b) β_2, and (c) θ when $n = 70$, $T_1 = 0.85$ and $T_2 = 1.7$ using a variety of values for (m, k).



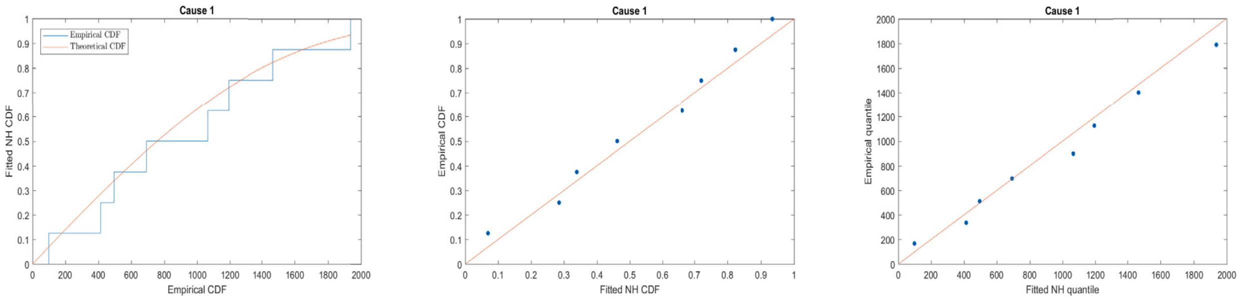
(a) (b) (c)

Fig. 14. MSEs for (a) β_1, (b) β_2, and (c) θ when $n = 70$, $T_1 = 0.85$ and $T_2 = 1.7$ using a variety of values for (m, k).



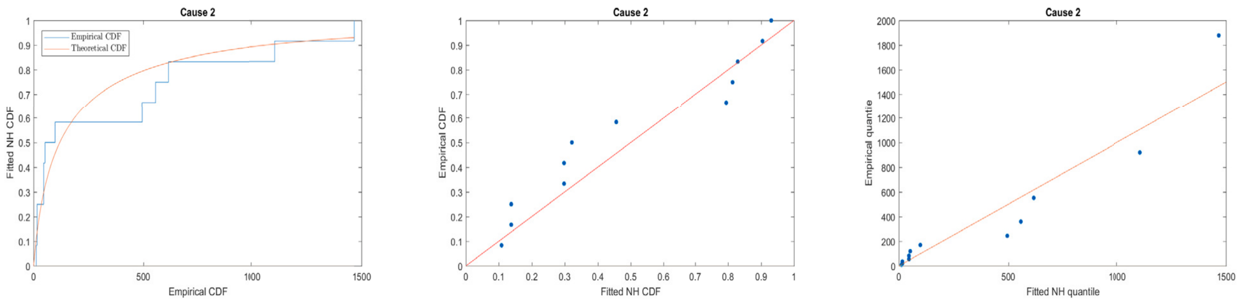
(a) (b) (c)

Fig. 15. ALs of the intervals estimates for (a) β_1, (b) β_2, and (c) θ when $n = 70$, $T_1 = 0.85$ and $T_2 = 1.7$ using a variety of values for (m, k).



(a) (b) (c)

Fig. 16. Plots of (a) ECDF, (b) P-P, and (c) Q-Q under cause 1 for given data.



(a) (b) (c)

Fig. 17. Plots of (a) ECDF, (b) P-P, and (c) Q-Q under cause 2 for given data.

{(12, 2), (16, 2), (16, 2), (46, 2), (46, 2), (52, 2), (98, 1), (98, 2), (270, 2), (413, 1), (495, 1), (495, 2), (557, 2), (616, 2), (692, 1), (1065, 1), (1107, 2), (1193, 1), (1467, 1), (1467, 2), (1937, 1)}.

A goodness-of-fit test is employed in this situation to examine whether or not the NHD is a good model match for the data that has been provided. Calculations are made to determine the K-S distances (p-values) that correlate with cause 1 and 2 as 0.1613 (0.9642) and 0.2114 (0.5855), respectively. If we check the goodness-of-fit test for this data set corresponding to inverted exponentiated exponential distribution, then the K-S distances (p-values) can be obtained as 0.2436 (0.6458) and 0.2249 (0.4786). As a result, the NHD is an appropriate model that may be applied to this particular data set. The LRS for θ and β, together with their associated p values (in brackets), have been derived as 1.4753 (0.1660) and 7.0253 (0.0453), respectively. Based on these findings, we may say that H_0 is accepted for test (1), yet fails to be accepted for the other one. As a result, it is reasonable to conclude that $\theta_1 = \theta_2 = \theta$ and $\beta_1 \neq \beta_2$ for this data set.

In order to provide more explanations, the empirical CDF (ECDF), probability-probability (P-P) and Quantile-Quantile (Q-Q) plots are shown in Figs. 16 and 17 based on cause 1 and 2, respectively. The conclusion that can be drawn from these plots is that the NHD is a good match for given data set.

Below is provided information on three distinct competing risks data set based on UHCS taken from the given data set at a point when n = 21 and m = 18.

Table 1
AEs (s.e) and confidence intervals of parameters for the real data set based on three distinct schemes.

Scheme	Θ	MLE	Bayes	ACI	BCI
I	β_1	0.0011 (0.0001)	0.0012 (0.0001)	(-0.0092,0.0114)	(-0.0047,0.0096)
	β_2	0.0023(0.0005)	0.0019 (0.0004)	(-0.0122,0.0168)	(-0.0083,0.0149)
	θ	0.1331 (0.0333)	0.1254(0.0296)	(0.0272,0.2390)	(0.0433,0.1808)
II	β_1	0.0013 (0.0002)	0.0014 (0.0002)	(-0.0085,0.0111)	(-0.0063,0.0097)
	β_2	0.0049(0.0006)	0.0038(0.0005)	(-0.0088,0.0186)	(-0.0053,0.0134)
	θ	0.1362(0.0331)	0.1285 (0.0304)	(0.0254,0.2470)	(0.0561,0.1953)
III	β_1	0.0012 (0.0001)	0.0022 (0.0002)	(-0.0076,0.0100)	(-0.0057,0.0084)
	β_2	0.0016 (0.0003)	0.0013 (0.0002)	(-0.0077,0.0109)	(-0.0060,0.0095)
	θ	0.1013 (0.0261)	0.0956 (0.0219)	(0.0356,0.1670)	(0.0407,0.1493)

Scheme I: $k = 13$, $T_1 = 500$, and $T_2 = 1100$.
(12, 2), (16, 2), (16, 2), (46, 2), (46, 2), (52, 2), (98, 1), (98, 2), (270, 2), (413,1), (495, 1), (495, 2), (557, 2), (616, 2), (692, 1), (1065, 1).
Scheme II: $k = 14$, $T_1 = 700$, and $T_2 = 1200$.
(12, 2), (16, 2), (16, 2), (46, 2), (46, 2), (52, 2), (98, 1), (98, 2), (270, 2), (413,1), (495, 1), (495, 2), (557, 2), (616, 2), (692, 1), (1065, 1), (1107, 2), (1193, 1).
Scheme III: $k = 15$, $T_1 = 450$, and $T_2 = 600$.
(12, 2), (16, 2), (16, 2), (46, 2), (46, 2), (52, 2), (98, 1), (98, 2), (270, 2), (413,1), (495, 1), (495, 2), (557, 2), (616, 2), (692, 1).

Table 1 displays point and interval estimates derived from the aforementioned three competing risks data sets using UHCS. From Table 1, it has been noticed that the point estimates are quite similar to one another. When comparing the standard error of MLE and Bayes estimates, the latter provides more accurate results. In comparison to ACIs, Bayesian credible intervals have superior performance based on length of intervals.

7. Conclusion

This paper deals with the statistical inference using UHCS for the NHD partially observed competing risks model. Point and interval estimates have been obtained along with both classical and Bayesian frameworks. A Monte Carlo simulation study is carried out to see how the estimations improve over time. When comparing point estimates, Bayes estimates under IP perform better than others. In terms of AL and CP, HPD credible intervals perform better than ACIs. Hypothesis testing is done after discussing two different likelihood functions in order to choose identical shape parameter. To further demonstrate the efficacy of the offered methodologies in the context of the competing risks model, a real-world data set has been analysed. Although the study focuses on two failure instances, the discussion may easily be generalized to include more.

CRediT authorship contribution statement

Tahani A. Abushal: Project administration, Methodology, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

The author declares that she has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

This specific manuscript has not been associated with publicly available repositories, since the real data and simulated data have been included in the main article. So, they are accessible within the publication.

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