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ORIGINAL ARTICLE

Research of population with impulsive perturbations based on dynamics of a neutral delay equation and ecological quality system

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Abstract This paper studies the global behaviors of a nonlinear autonomous neutral delay differential population model with impulsive perturbation. This model may be suitable for describing the dynamics of population with long larval and short adult phases. It is shown that the system may have global stability of the extinction and positive equilibria, or grow without being bounded under some conditions.

1. Introduction

Many evolutionary processes in nature are characterized by the fact that their states experience abrupt changes at certain moments, which can be described by impulsive systems. Moreover, the impulsive systems have much richer dynamics than the corresponding non-impulsive systems. This is the reason that this paper studies the neutral delay equation for an insect population with impulsive perturbations.

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In Stephen and Kuang (2009), we get

$$u'_m(t) = u_0(\tau - t)e^{-\mu t} - d(u_m(t))t \leqslant \tau \tag{1.1}$$

$$u'_m(t) = \big(b_2 u'_m(t-\tau) + b_2 d(u_m(t-\tau)) + b_0 u_i(\tau - t) + b_1 u_m(\tau - t)\big)e^{-\mu t} - d(u_m(t))t \geqslant \tau \tag{1.2}$$

Just like Stephen and Kuang (2009), let t and a denote time and age and let $u(t,a)$ be the density of individuals of age a at time t. It will be assumed that individuals take time τ to mature, so that the total numbers of mature and immature numbers u_m and u_i are given respectively by

$$u_m(t) = \int_\tau^\infty u(t,a)da, \quad u_i(t) = \int_0^\infty u(t,a)da$$

and with the initial condition $u_0 = u(0,a) \geqslant 0, \quad a \geqslant 0$ $u(t,0) = \int_0^\infty b(a)u(t,a)da$

Following Bocharov and Hadeler (2000), the birth rate function,

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$b(a) = b_0 + (b_1 - b_0)H(a-\tau) + b_2\delta(a-\tau)$

where $H(a)$ is Heaviside function and $\delta(a)$ the Dirac delta function. This choice for $b(a)$ implies that individuals age less than τ produce b_0 eggs per unit time, those of age greater than τ produce b_1 eggs per unit time, and additionally each individual lays b_2 eggs on reaching maturation age τ (the b_2 eggs all being laid at exactly that instant). In fact, we shall take $b_0 = 0$ for most of this paper, because most individuals do not lay eggs until they reach maturation age τ in nature.

In this paper, we will study the following system with impulsive perturbations, we will drop the subscript m for convenience,

$$\begin{cases} u'(t) = u_0(\tau - t)e^{-ut} - d(u(t)) \quad t \leqslant \tau \\ u'(t) = (b_2u'(\tau - t) + b_2d(u(t-\tau)) + b_1u(\tau - t))e^{-ut} - d(u(t)) \quad t \geqslant \tau \\ u(\tau_k^+) = (1 + c_k)u(\tau_k) \quad k = 1,2,3,\ldots \end{cases}$$

We will use the following hypotheses:

(H1) $0 = \tau_0 < \tau_1 < \tau_2 < \ldots$ are fixed impulsive points with $\lim_{k\to\infty} = \infty$, $\tau_k = k\tau$.
(H2) (c_k) is a real sequence and $c_k > -1$, $k = 1,2,3,\ldots$, $\prod_{0\leqslant\tau_k<t}(1 + c_k) < \infty$.
(H3) $d(\cdot)$ is a linear and continuous strictly monotonic increasing function of u satisfying $d(0) = 0$.

Here $(c_k), k = 1,2,3,\ldots$ are proportional coefficients. Impulsive reduction of the population is possible by catching or poisoning with chemicals used in agriculture $(-1 < c_k < 0)$, an impulsive increasing of the population is possible by artificial means by the population's impulsive immigration and introduction of natural enemies ($c_k \geqslant 0$). In this paper, we assume $\tau_k = k\tau$ which means the individuals lay eggs only once all their life.

The dynamic of the delay system (1.1) and (1.2) has been studied in Stephen and Kuang (2009), we could obtain the positivity and boundedness of solution by ad hoc methods and global stability of the extinction and positive equilibria by the method of iteration. We also know that if the time adjusted instantaneous birth rate at the time of maturation is greater than 1, then the population will grow without being bounded. Thus, it is interesting how the dynamics of (1.3) is affected by the impulsive perturbations.

2. Preliminary

From Stephen and Kuang (2009), we could apply (1.2) recursively and could get a non-neutral delay equation

$$u'(t) = b_2^n e^{-ut}u_0((n+1)\tau - t) + b_1e^{-u\tau}\sum_{j=0}^{n-1} b_2^j e^{-j\mu t}u(t-(j+1)\tau) + d(u(t)) \quad t \in (n\tau, (n+1)\tau) \tag{2.1}$$

Since $\tau_k = k\tau$ so (1.3) can be written as

$$\begin{cases} u'(t) = b_2^n e^{-ut}u_0(\tau_{n+1} - t) + b_1e^{-u\tau}\sum_{j=0}^{n-1} b_2^j e^{-j\mu t}u(t-(\tau_{j+1}) \\ \qquad -d(u(t)) \quad t \in (n\tau, (n+1)\tau) \\ u(\tau_k^+) = (1 + c_k)u(\tau_k) \quad k = 1,2,3,\ldots \end{cases} \tag{2.2}$$

Under the hypotheses (H1)–(H3), by a transformation $z(t) = \prod_{0\leqslant\tau_k<t}(1 + c_k)^{-1}u(t)$, we consider the nonimpulsive delay differential equation

$$z'(t) = b_2^n e^{-ut}\prod_{\tau_{n+1}-t\leqslant\tau_k<t}(1 + c_k)z_0(\tau_{n+1} - t) + b_1e^{-u\tau}\sum_{j=0}^{n-1} b_2^j e^{-j\mu t}\prod_{t-\tau_{j+1}-t\leqslant\tau_k<t}(1 + c_k)^{-1}z(t - \tau_{j+1}) - d(z(t)) \tag{2.3}$$

and $t \in (n\tau, (n+1)\tau)$, $n = 1,2,3,\ldots$, with the initial condition

$$z_0(a) = z(0,a) = \prod_{0\leqslant\tau_k<t}(1 + c_k)^{-1}u_0(a) \geqslant 0, \quad a \geqslant 0$$

Lemma 2.1. *Assume that (H1)–(H3) hold,*

(i) *If $z(t)$ is a solution of (2.3) on $(0,\infty)$, then $u(t) = \prod_{0\leqslant\tau_k<t}(1 + c_k)z(t)$ is a solution of (2.2) on $(0,\infty)$.*
(ii) *If $u(t)$ is a solution of (2.2) on $(0,\infty)$, then $z(t) = \prod_{0\leqslant\tau_k<t}(1 + c_k)^{-1}u(t)$ is a solution of (2.3) on $(0,\infty)$.*

Proof. First, we prove (i). It is easy to see that $z(t) = \prod_{0\leqslant\tau_k<t}(1 + c_k)^{-1}u(t)$ is absolutely continuous on each interval (τ_k, τ_{k+1}), $k = 1,2,3,\ldots$, On the other hand, for every τ_k.

$$u(\tau_k^+) = \lim_{t\to\tau_k^+}\prod_{0\leqslant\tau_k<t}(1 + c_j)z(t) = \prod_{0\leqslant\tau_k<t}(1 + c_j)z(t) \text{ and}$$
$$u(\tau_k) = \prod_{0\leqslant\tau_k<t}(1 + c_j)z(\tau_k)$$

Thus for every

$$k = 1,2,3,\ldots, \quad u(\tau_k^+) = (1 + c_k)u(\tau_k) \tag{2.4}$$

Now, one can easily check that $u(t) = \prod_{0\leqslant\tau_k<t}(1 + c_k)z(t)$ is a solution of (1.1) on $(0,\infty)$.

Next, we prove (ii), since $u(t)$ is absolutely continuous on each interval (τ_k, τ_{k+1}), $k = 1,2,3,\ldots$, and in view of (2.4), it follows that for any $k = 1,2,3,\ldots$.

$$z(\tau_k^+) = \prod_{0\leqslant\tau_k<t}(1 + c_j)^{-1}u(\tau_k^+) = \prod_{0\leqslant\tau_k<t}(1 + c_j)^{-1}u(\tau_k) = z(\tau_k)$$

and $z(\tau_k^-) = \prod_{0\leqslant\tau_k<t}(1 + c_j)^{-1}u(\tau_k^-) = \prod_{0\leqslant\tau_k<t}(1 + c_j)^{-1}u(\tau_k) = z(\tau_k)$. Which implies that $z(t)$ is continuous on $(0, \infty)$, it is easy to prove $z(t)$ is also absolutely continuous on $(0, \infty)$. Now one can easily check that $z(t) = \prod_{0\leqslant\tau_k<t}(1 + c_k)^{-1}u(t)$ is a solution of (2.3) on $(0, \infty)$. The proof of Lemma 2.1 is complete.

3. Main results

Theorem 3.1. *Assume that (H1)–(H3) hold, and $z_0(a) \geqslant 0$ for all $a \geqslant 0$. Then the solution of (2.3) satisfies $z(t) \geqslant 0$ for all $t \geqslant 0$. Furthermore, if $z_0(a) = 0$ on the interval $(0, \infty)$, then $z(t) > 0$ for all $t > 0$.*

Proof. On the interval $t \in (0, \tau]$, $z(t)$ satisfies (2.3), and so $z'(t)e^{-\mu t}z_0(\tau - t) \geqslant -d(z(t))\ 0 < t \leqslant \tau$

The initial value for $z(t)$ is $z(t) \geqslant 0$, also, since $d(0) = 0$ by Taylor expansions it follows that $d(z(t))$ has a factor of $z(t)$ and so from the standard argument it follows that $z(t) \geqslant 0$ for all $0 < t \leqslant \tau$.

Next we will prove non-negativity of $z(t)$ for $t \in (\tau, 2\tau]$. For such times, $t - \tau \leqslant \tau$, so that, from (2.3), $z'(t) = b_2^n e^{-ut}(1 + c_1)^{-1}z_0(2\tau - t) + b_1 e^{-u\tau}(1 + c_1)^{-1}z(t - \tau) - d(z(t)) \geqslant -d(z(t))$ because we already show non-negativity of $z(t)$ on the interval $t \in (0, \tau]$, using $d(0) = 0$ and non-negativity of $z(\tau)$, it follows that $z(t) \geqslant 0$ for all $t \in (\tau, 2\tau]$.

This argument can be continued to include all positive times and so non-negativity of $z(\tau)$ has been shown. If $z_0(a) = 0$ on the interval $(0, \infty)$, then $z(t) > 0$ for all $t > 0$, in this situation, inspection of the details of the above analysis shows that we can draw the conclusion that $z(t)$ is strictly positive for all positive times. The proof of Theorem 3.1 is complete.

Since $u(t) = \prod_{0 \leqslant \tau_k < t}(1 + c_k)z(t)$

Corollary 3.1. *Assume that (H1)–(H3) hold, and $u_0(a) \geqslant 0$ for all $a \geqslant 0$. Then the solution of (2.2) satisfies $u(t) \geqslant 0$ for all $t \geqslant 0$. Furthermore, if $u_0(a) = 0$ on the interval $(0, \infty)$, then $u(t) > 0$ for all $t > 0$.*

Here we get the results mainly used in the method Stephen and Kuang (2009), of course, by Theorem 1 in Stephen and Kuang (2009) and $c_k > -1$, $k = 1, 2, 3, \ldots$, we could get the same result directly.

Theorem 3.2. *Assume that (H1)–(H3) hold, $b_0 = b_1 = 0$ and $b_2 e^{-\mu t} < 1$. Then the solution of (2.3) satisfies $\lim_{t\to\infty} z(t) = 0$.*

Proof. Eq. (2.3) is for $t \in (n\tau, (n + 1)\tau]$, so t and n must go to infinity together. Since $b_1 = 0$, the term with summation is absent. Furthermore, the involvement of $z_0(\cdot)$ is for value of its argument between 0 and τ only, so $z_0((n + 1)\tau - t)$ can be bounded independently of n and t.

Since $0 < \prod_{0 \leqslant \tau_k \prec t}(1 + c_k) < \infty$, there exists $0 < A < B$, such that $A < \prod_{0 \leqslant \tau_k < t}(1 + c_k)^{-1} < B$.

Let $\varepsilon > 0$ be arbitrary, and choose N sufficiently large that $b_2^n e^{-\mu\tau}\sup_{a\in(0,\tau]} z_0(a) < \varepsilon$, whenever $n \geqslant N$.

Then it follows that, for $t > N\tau$, $z'(t) \leqslant \varepsilon B - d(z(t))$.

From a simple comparison argument, and using the sated properties of the function $d(\cdot)$ and also the positivity of z, it follows that $0 \leqslant \limsup_{t\to\infty} z(t) \leqslant d^{-1}(\varepsilon B)$.

This is true for any $\varepsilon > 0$ and therefore $\lim_{t\to\infty} z(t) = 0$. The proof is complete.

Since $u(t) = \prod_{0 \leqslant \tau_k < t}(1 + c_k)z(t)$, there exists $0 < A < B$, such that $A < \prod_{0 \leqslant \tau_k < t}(1 + c_k)^{-1} < B$, and by Theorem 3.2. We could get that.

Corollary 3.2. *Assume that (H1)–(H3) hold, $b_0 = b_1 = 0$ and $b_2 e^{-\mu\tau} < 1$. Then the solution of (2.2) satisfies $\lim_{t\to\infty} u(t) = 0$.*

Theorem 3.3. *Assume that (H1)–(H3) hold, $b_0 = 0$, $b_1 > 0$ and*

$$Bb_1 z e^{-\mu\tau} < d(z)(1 - b_2 e^{-\mu\tau}) \text{ for all } z \geqslant 0 \tag{3.1}$$

Then the solution of (2.3) satisfies $\lim_{t\to\infty} z(t) = 0$.

Proof. Noted that (3.1) forces $b_2 e^{-\mu\tau} < 1$. Firstly we shall establish that these solutions $z(t)$ are bounded.

Let $Z = \max(\max(z_0(a) :\ a \in (0, \tau]),\ \max(z(t) :\ t \in (0, \tau]))$

$$p = \prod_{\tau_{n+1} - t \leqslant \tau_k < t}(1 + c_k)^{-1} \text{ and } q_j = \prod_{t - \tau_{j+1} \leqslant \tau_k < t}(1 + c_k)^{-1}$$

absolutely $A < p < B$, $A < q_j < B$, of course

$$z'(t) = b_2^n e^{-\mu t} p z_0(\tau_{n+1} - t) + b_1 e^{-\mu\tau} \sum_{j=0}^{n-1} b_2^j e^{-j\mu\tau} q_j z(t - \tau_{j+1}) - d(z(t)) \tag{3.2}$$

and choose α sufficiently large that $\alpha > b_2/b_1$, we claim that

$$\limsup_{t\to\infty} z(t) \leqslant (\alpha + 1)Z \tag{3.3}$$

Suppose the contrary, then since the solution is bounded by Z for $t \in (0, \tau]$, there must exist $t_1 > \tau$, such that

$$Z(t_1) = (\alpha + 1)Z z(t) < (\alpha + 1)Z \text{ for all } t < t_1 \text{ and } z'(t_1) \geqslant 0 \tag{3.4}$$

and an integer k such that $t_1 \in (k\tau, (k + 1)\tau]$, but from (3.2)

$$\begin{aligned} z'(t) &= b_2^k e^{-\mu t_1} p z_0(\tau_{k+1} - t) + b_1 e^{-\mu\tau} \sum_{j=0}^{k-1} b_2^j e^{-j\mu\tau} q_j z(t_1 - \tau_{j+1}) \\ &\quad - d(z(t_1)) \leqslant b_2^k e^{-\mu k\tau} Z - Bb_1 e^{-\mu\tau} b_2^{k-1} e^{-(k-1)\mu\tau} \alpha Z \\ &\quad + Bb_1 e^{-\mu\tau} \sum_{j=0}^{k-1} b_2^j e^{-j\mu\tau}(\alpha + 1)Z - d((\alpha + 1)Z) \\ &\leqslant b_2^{k-1} e^{-\mu k\tau} Z(pb_2 - Bb_1 e\alpha) + Bb_1 e^{-\mu\tau}(\alpha + 1)Z/(1 - b_2 e^{-\mu\tau}) \\ &\quad - d((\alpha + 1)Z) < 0 \end{aligned} \tag{3.5}$$

This is contradictory (3.4) and there for $z(t)$ is bounded.

Let K be an upper bound for $z(t)$ and let $\eta > 0$ be arbitrary. As noted earlier, the nonautonomous term in (3.2) goes to zero as $t \to \infty$.

It follows that for a sufficiently large integer the nonautonomous term is bounded by η and therefore $Z'(t) \leqslant \eta + KBb_1 e^{-\mu\tau}/(1 - b_2 e^{-\mu\tau}) - d(z(t))$.

So $\limsup_{t\to\infty} Z(t) \leqslant d^{-1}(\eta + KBb_1 e^{-\mu\tau})/(1 - b_2 e^{-\mu\tau})$

This is true for all $\eta > 0$, and therefore $\limsup_{t\to\infty} Z(t) \leqslant d^{-1}(KBb_1 e^{-\mu\tau})/(1 - b_2 e^{-\mu\tau}) := z_1^*$.

That z_1^* is well defined follows from (3.1) and the other hypotheses on $d(\cdot)$.

In the subsequent steps of this analysis the nonautonomous term in (2.3) can be rigorously dealt with by introducing a small parameter which is later shrunk to zero as just described, and it is therefore sufficient to study the asymptotically autonomous from of (2.3), which is

$$z'(t) = b_1 e^{-\mu\tau} \sum_{j=0}^{n-1} b_2^j e^{-j\mu\tau} q_j z(t - \tau_{j+1}) - d(z(t)),$$
$$t \in (n\tau, (n+1)\tau] \quad n = 1, 2, 3, \ldots \tag{3.6}$$

Using Heaviside's function $H(t)$, (3.6) can be rewritten as

$$z'(t) = b_1 e^{-\mu\tau} \sum_{j=0}^{\infty} b_2^j e^{-j\mu\tau} H(t - \tau_{j+1}) q_j z(t - \tau_{j+1}) - d(z(t)),$$
$$t \in (n\tau, (n+1)\tau] \quad n = 1, 2, 3, \ldots \tag{3.7}$$

Let $\varepsilon \geqslant 0$, there exists $T > 0$ such that, for all $t > T$, $z(t) \leqslant z_1^* + \varepsilon$, and choose an integer N sufficiently large that $\sum_{j=N}^{\infty} b_2^j e^{-j\mu\tau} < \varepsilon$, which is possible because $b_2 e^{-\mu\tau} < 1$.

From (3.6), we find that, for $t > N\tau + T$

$$\begin{aligned} z'(t) &= b_1 e^{-\mu\tau} \sum_{j=0}^{\infty} b_2^j e^{-j\mu\tau} H(t - \tau_{j+1}) q_j z(t - \tau_{j+1}) - d(z(t)) \\ &\leqslant b_1 e^{-\mu\tau} B \left[\sum_{j=0}^{N-1} b_2^j e^{-j\mu\tau} z(t - \tau_{j+1}) \right. \\ &\quad \left. + \sum_{j=N}^{\infty} b_2^j e^{-j\mu\tau} H(t - \tau_{j+1}) z(t - \tau_{j+1}) \right] - d(z(t)) \\ &\leqslant b_1 e^{-\mu\tau} B \left[(z_1^* + \varepsilon) \sum_{j=0}^{N-1} b_2^j e^{-j\mu\tau} + K\varepsilon \right] - d(z(t)) \\ &< b_1 e^{-\mu\tau} B \left[(z_1^* + \varepsilon)/(1 - b_2 e^{-\mu\tau}) + K\varepsilon \right] - d(z(t)) \end{aligned} \tag{3.8}$$

From this, we deduce an ε-dependent upper bound for $\lim_{t\to\infty} z(t)$, and we may then shrink ε to zero to obtain $\limsup_{t\to\infty} Z(t) \leqslant d^{-1}(Bb_1 e^{-\mu\tau} z_1^*)/(1 - b_2 e^{-\mu\tau}) := z_2^*$.

By repeating the above procedure, we generate a sequence $z_n^* n = 1, 2, 3, \ldots$, of real numbers with the property that $\limsup_{t\to\infty} z(t) \leqslant z_n^*$ for each n and $d(z_{n+1}^*) = Bb_1 e^{-\mu\tau} z_n^* / (1 - b_2 e^{-\mu\tau})$ $n = 1, 2, 3, \ldots$.

From (3.1) it follows that $d(z_{n+1}^*) \leqslant d(z_n^*)$ and therefore, since $d(\cdot)$ is strictly monotonic increasing, $z_{n+1}^* \leqslant z_n^*$, therefore z_n^* approaches a limit $z_n^* \geqslant 0$ as $n \to \infty$, which satisfies $d(z^*) = Bb_1 e^{-\mu\tau} z_n^* / (1 - b_2 e^{-\mu\tau})$. By (3.1) limits z^* must be zero and therefore $\lim_{t\to\infty} z(t) = 0$.

The proof of the theorem is complete.

Corollary 3.3. *Assume that (H1)–(H3) hold,* $b_0 = 0$, $b_1 > 0$ *and* $Bb_1 u e^{-\mu\tau} < d(u)(1 - b_2 e^{-\mu\tau})$ for all $z \geqslant 0$.

Then the solution of (2.2) *satisfies* $\lim_{t\to\infty} u(t) = 0$.

Under the hypothesis of Theorems 3.2 and 3.3, if we don't protect the population, it will become extinct. So we must take measures to protect it, for instance, by the immigrating population artificially at every impulsive point $k\tau, k = 1, 2, 3ldots$ to make the population persistent existence. Of course, under this condition $\prod_{0 \leqslant \tau k < t} (1 + c_k) \to \infty$ as $k \to \infty$.

Theorem 3.4. *Assume that (H1)–(H3) hold,* $b_0 = 0$, $b_1 > 0$ *and* $b_2 e^{-\mu\tau} < 1$, $d(z) = o(z)$ *as* $d(z) = o(z) ast \to 0$ *and there exists* z^* *such that*

$$b_1 B u e^{-\mu\tau} < d(z)(1 - b_2 e^{-\mu\tau}) \text{ when } 0 < z < z^*$$

$$b_1 A e^{-\mu\tau} < d(z)(1 - b_2 e^{-\mu\tau}) \text{ when } z > z^*$$

Then if $z_0(a)$ *is continuous on the interval* $[0, \infty)$, $z_0(a) \geqslant 0$ *and* $z_0(a) = 0$, *then the solution of Eq.* (2.3) *satisfies* $z(t) \to z^*$ *as* $t \to \infty$.

Proof. By Theorem 3.1, we know that if $z_0(a) = 0$ on the interval $[\tau, \infty)$. Then $z(t) > 0$ for all $t > 0$. Without loss of generality, we assume that $z(t) > 0$. $t \in [0, \tau]$.

As noted previously, since $b_2 e^{-\mu\tau} < 1$, the asymptotic behavior of solution of (2.3) is the same as the asymptotic behavior of solution of

$$z'(t) = b_1 e^{-\mu\tau} \sum_{j=0}^{n-1} b_2^j e^{-j\mu\tau} q_j z(t - \tau_{j+1}) - d(z(t)) \;\; t \in (n\tau, (n+1)\tau]$$
$$n = 1, 2, 3, \ldots, \tag{3.9}$$

we shall consider (3.9) as an initial value problem starting at $t = \tau$, with the function $z(s)$, $s \in [0, \tau]$ treated as the initial data. From our comments above, we may assume that $\min_{s\in[0,\tau]} z(s) > 0$.

We claim that a comparison principle holds for (3.9), that is to say, if we take three sets of initial data ordered such as that $\underline{z}(s) \leqslant z(s) \leqslant \bar{z}(s)$ $s \in [0, \tau]$, then $\underline{z}(t) \leqslant z(t) \leqslant \bar{z}(t)$ for all $t > \tau$. Let $\delta > 0$ be small and let $\bar{z}^{\delta}(s)$ satisfy the equation

$$\partial_{\bar{z}^{\delta}} / \partial t = b_1 e^{-\mu\tau} \sum_{j=0}^{n-1} b_2^j e^{-j\mu\tau} q_j \bar{z}^{\delta}(t - \tau_{j+1}) - d(\bar{z}^{\delta}(t))$$
$$+ \delta \;\; t \in (n\tau, (n+1)\tau] \quad n = 1, 2, 3, \ldots, \tag{3.10}$$

and $\bar{z}^{\delta}(s) = \bar{z}(s) + \delta$ $s \in [0, \tau]$.

We claim that $z(t) < \bar{z}^{\delta}(t)$ for all $t > \tau$, shrinking δ to 0 then gives $z(t) \leqslant \bar{z}(t)$ certainly $z(\tau) \leqslant \bar{z}(\tau) = \bar{z}^{\delta}(\tau) - \delta < \bar{z}^{\delta}(\tau)$. So suppose that our claim is violated at the same time, i.e. suppose there exists $t^* > \tau$ such that $z(t^*) = \bar{z}^{\delta}(t^*)$ and $z(t) < \bar{z}^{\delta}(t)$ for all $t \in [\tau, t^*)$. Then for appropriate n,

$$\begin{aligned} \partial_{\bar{z}^{\delta}} / \partial t &= b_1 e^{-\mu\tau} \sum_{j=0}^{n-1} b_2^j e^{-j\mu\tau} q_j \bar{z}^{\delta}(t^* - \tau_{j+1}) - d(\bar{z}^{\delta}(t^*)) + \delta \\ &> b_1 e^{-\mu\tau} \sum_{j=0}^{n-1} b_2^j e^{-j\mu\tau} q_j z(t^* - \tau_{j+1}) - d(z(t^*)) \\ &= \partial z(t^*) / \partial t. \end{aligned} \tag{3.11}$$

Let $F(t) = \bar{z}^{\delta}(t) - z(t)$, then $F(t)$ has the following properties: $F(\tau) > 0$, $F(t^*) = 0$, $F(t) > 0$ on $[\tau, t^*)$ and $F'(t^*) > 0$. This is a contradiction. The proof that $\underline{z}(t) \leqslant z(t)$ is similar to show that $z(t) \to z^*$ it suffices to show that $\underline{z}(t) \to z^*$ and $\bar{z}(t) \to z^*$ as $t \to \infty$, where $\underline{z}(t)$ and $\bar{z}(t)$ are comparison functions that satisfy (3.9) subject to the initial conditions

$$\underline{z}(s) = \varepsilon \;\; s \in [0, \tau], \text{ where } 0 < \varepsilon < \min\left(z^*, \min_{\xi\in[0,\tau]} z(\xi)\right) \tag{3.12}$$

$$\underline{z}(s) = K \;\; s \in [0, \tau], \text{ where } K > \max\left(z^*, \min_{\xi\in[0,\tau]} z(\xi)\right). \tag{3.13}$$

We shall show that $\underline{z}(t)$ is monotonic increasing for all $t > \tau$, and this will be achieved via the methods of steps. Starting with $t \in (\tau, 2\tau)$. For a time $t \in (\tau, 2\tau)$, choose $h > 0$ sufficiently small that $t + h \in (\tau, 2\tau]$ and such that $\underline{z}(\tau + h) - \underline{z}(\tau) \geqslant 0$. The latter is possible because

$$z'(t) = b_1 e^{-\mu\tau} q_0 z(0) - d(z(t)) = b_1 e^{-\mu\tau}\varepsilon - d(\varepsilon) > 0 \tag{3.14}$$

for sufficiently small ε. Since $d(z) = o(z)$ as $t \to 0$. Let $w(t) = \underline{z}(t + h) - \underline{z}(t)$, then for $t \in (\tau, 2\tau)$

$$\begin{aligned}\omega'(t) &= b_1 e^{-\mu t}[\underline{z}(t + h - \tau) - \underline{z}(t - \tau)] - [d(\underline{z}(t + h)) - d(\underline{z}(t))] \\ &= \omega(t) d'(\theta(t - h))\end{aligned} \tag{3.15}$$

where $\theta(t, h)$ is some function arising from an application of mean value theorem. Also, $\omega(\tau) \geqslant 0$, thus, $\omega(\tau) \geqslant 0$ for all $t \in (\tau, 2\tau)$. Letting $h \to 0$, we deduce that $\underline{z}'(t) \geqslant 0$ for all $t \in (\tau, 2\tau)$, and this can be extended to $t \in (\tau, 2\tau]$ by continuous. For $t \in (2\tau, 3\tau)$

$$\begin{aligned}\omega'(t) &= b_1 b_2 e^{-2\mu t} q_1[\underline{z}(t + h - 2\tau) - \underline{z}(t - 2\tau)] \\ &\quad + b_1 e^{-\mu t}[\underline{z}(t + h - \tau) - \underline{z}(t - \tau)] - [d(\underline{z}(t + h)) - d(\underline{z}(t))] \\ &\geqslant -\omega(t) d'(\theta(t, h))\end{aligned} \tag{3.16}$$

Also, $\omega(2\tau) \geqslant 0$, therefore, $\omega(t) \geqslant 0$ for all $t \in (2\tau, 3\tau)$, and hence also $\underline{z}'(t) \geqslant 0$ on $(2\tau, 3\tau)$. This argument can be continued to deal with all intervals $(n\tau, (n + 1)\tau)$ and therefore all times $t > \tau$, and we conclude that $\underline{z}(t)$ is monotonic increasing for all $t > \tau$.

The proof that $\bar{z}(t)$ is monotonic decreasing is similar, and from the first step in the process it will became apparent that K has to be such that $b_1 e^{-\mu t} K < d(K)$. However, the theorem hypotheses assures us that this is automatically true for any K consistent with (3.13).

We have established that $\underline{z}(t)$ is monotonic increasing and bounded above (by K), and therefore must approach some $\lim \geqslant \varepsilon > 0$, while $\bar{z}(t)$ is monotonic decreasing and bounded below (by ε). Thus $\lim_{t\to\infty}\underline{z}(t) = z^*$. Hence $\lim_{t\to\infty} z(t) = z^*$. The proof is complete.

Since $0 < \prod_{0\leqslant\tau_k<t}(1 + c_k) < \infty$, $u(t) = \prod_{0\leqslant\tau_k<t}(1 + c_k) z(t)$, so the solution of (2.2) has the same result, only the time to reach the steady state is different. By Theorem 5, in Stephen and Kuang (2009), we have the next theorem:

Theorem 3.5. *Assume that (H1)–(H3) hold,* $b_0 = 0$, $b_1 > 0$, $b_2 e^{-\mu t} > 1$, $\prod_{0\leqslant\tau_k<t}(1 + c_k) \geqslant 1$. *Then, if* $u_0(a)$ *is continuous on the interval* $[0, \infty)$, $u_0(a) \geqslant 0$ *and* $u_0(a) = 0$, *then the solution of* (2.2) *grows without being bounded as* t *increases.*

Under the hypothesis of Theorem 3.5, the birth rate is high and the population will grow without being bounded, so we must control the rate at every impulsive point. For instance, we could use a combination of biological (natural enemies), agricultural (catching), and chemical (killing) tactics to control the population. Of course, at these times, $0 > c_k > -1$, $k = 1, 2, 3, \ldots$.

4. Conclusions

Organizational management research and management phenomenon can be described by differential equations, Impulsive effect of differential equations is caused by people's attention. The pulse phenomenon as a kind of instantaneous mutation and its mathematical model can often be attributed to an impulsive differential system. The most prominent feature of the pulse differential system is to give full consideration to the effect of transient mutation on the state. It can reveal the law of things more deeply. The enterprises are like the species in natural ecosystem. Every enterprise of the enterprise ecosystem has to share with the entire enterprise ecosystem in the end. The mutations of each enterprise management also affect the value of the entire enterprise population. So the pulse differential system can also give us inspiration.

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