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Solving Pythagorean fuzzy partial fractional diffusion model using the Laplace and Fourier transforms

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Abstract

Many mathematical models describe the Corona virus disease 2019 (COVID-19) outbreak; however, they require advance mathematical skills. The need for this study is to determine the diffusion of the COVID-19 vaccine in humans. To this end, we first establish a Pythagorean fuzzy partial fractional differential equation using the Pythagorean fuzzy integral transforms to express the effects of COVID-19 vaccination on humans under the generalized Hukuhara partial differential conditions. We extract the analytical solution of the Pythagorean fuzzy partial fractional differential equation using the Pythagorean fuzzy Laplace transform under the generalized Hukuhara partial differential and the Pythagorean fuzzy Fourier transform using the Caputo generalized Hukuhara partial differential. Moreover, we present some essential postulates and results related to the Pythagorean fuzzy Laplace transform and the Pythagorean fuzzy Fourier transform. Furthermore, we develop the solution procedure to extract the solution of the Pythagorean fuzzy partial fractional differential equation. To grasp the considered approach, a mathematical model for the diffusion of the COVID-19 vaccination in the human body is provided and analyzed the behavior to visualize and support the proposed model. Our proposed method is efficient and has a great worth to discuss the bio-mathematical models in various fields of science and medicines.

Keywords Partial fractional differential equation · COVID-19 vaccination · Pythagorean fuzzy integral transforms · Caputo generalized Hukuhara partial differentiability

1 Introduction

Zadeh (1965) first proposed the idea of fuzzy set that helped the researchers to easily describe the vague information very clearly by a mathematical phenomena. Fuzzy set theory was examined by many scholars and they acquired a lot of achievements in different fields. In some areas, fuzzy sets were unable to deal uncertainty effectively, because they are associated to only the grade of appreciation or membership, and therefore, the grade of rejection or non-membership was neglected. To tackle this drawback in fuzzy sets, Atanassov (1986) gave the concept of Intuitionistic fuzzy set (IFS). IFS is associated to each element of the universe, taking into account both membership and non-membership values whose sum is less than or equal to 1. As a result, IFS can tackle uncertainty more precisely and effectively than fuzzy set. IFS handles a variety of practical issues. Moreover, in many real life applications, the sum of membership and non-membership values satisfying the parameters provided by experts may be greater than one, but the sum of the squares of their membership and non-membership degrees is less than or equal to one. Yager (2013a, b) gave the concept of the Pythagorean fuzzy sets (PFSs) in 2013 to bridge this gap. He gave a situation to illustrate this condition: an expert says his support for an object's membership is $\frac{\sqrt{3}}{2}$ and his support for non-membership is $\frac{1}{2}$. It is easy to see that $\frac{\sqrt{3}}{2}+\frac{1}{2}\geq 1$. Consequently, IFSs cannot explain this situation. On the other hand, PFSs took appropriate attention, because $(\frac{\sqrt{3}}{2})^2+(\frac{1}{2})^2\leq 1$. Clearly, PFS handles issues more efficiently than IFS in simulating vagueness in real-world decision-making problems. Some researchers (Naz et al.

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2018; Peng and Selvachandran 2019; Akram et al. 2019; Ezadi and Allahviranloo 2020; Akram and Ali 2020; Asif et al. 2020; Akram and Khan 2021; Akram and Shahzadi 2021; Peng and Luo 2021; Rahman 2022; Akram et al. 2022a, b) have used the concept of PFSs in various problems to get satisfactory results.

Fuzzy sets have many applications in the dynamical models that contain vague parameters instead of randomness. These models have made a route to describe the fuzzy differential equations (FDEs). They gained a remarkable attention due to their applications in the area of mathematics and related branches, such as artificial intelligence, diffusion processes, medical sciences and much more. Chang and Zadeh (1972) first introduced the idea of fuzzy derivative in 1972. The idea established by Chang and Zadeh was followed up by Dubios and Prade (1982) and they applied the theory of extension principle. FDE first discussed by Kaleva (1987) using Puri-Relescue Hukuhara derivative (*H*-derivative). Seikkala (1987) established the idea of the fuzzy derivative that was the generalization of *H*-derivative (Bede and Gal 2005) that was developed on the base of Hukuhara difference (*H*-difference) (Stefanini and Bede 2013). Song and Wu (2000) studied about FDEs, and they established the modification of the main consequences of Kaleva (1987). The derivative of a fuzzy-valued function (FVF) can be determined using *H*-differentiability. Bede and Gal (2005) had introduced the generalized Hukuhara differentiability (*gH*-differentiability) of FVFs based on g*H*-difference. Later, it was proved that *gH*-differentiability had more accuracy than *H*-differentiability. Therefore, fuzzy derivatives of FVFs were determined using *gH*-differentiability. Fractional calculus and fractional differential equations are upcoming stars in the modern era to deal many problems in both theoretical and applied science. They are concerned in the modeling of many physical and chemical processes Baleanu et al. 2012. A wide range of publications have been made to test the efficiency of the solutions of fractional differential equations. Kilbas et al. (2006) gave a valuable work on the theory and applications of fractional differential equations. A wide range of important contributions on FDEs in the fuzzy environment have been published by many researchers. Furthermore, several articles have been published discussing the solution of the fuzzy fractional differential equations (FFDEs). Agarwal et al. (2010) studied the analytical solution of fractional order differential equations with vagueness. Ahmad et al. (2021) discussed the computational approach of fuzzy fractional non-dimensional Fisher equation. (Allahviranloo et al. 2012, 2015), Allahviranloo et al. (2021) studied the fuzzy solution of fractional differential equations under *gH*-differentiability and generalized Caputo derivative. Ezadi and Allahviranloo (2020) used the artificial neural method to solve fuzzy fractional initial value problem under *gH*-differentiability. Khakrangin et al. (2021) gave the numerical solution of FFDE by haar wavelet. A valuable work on the fuzzy fractional wave equation was done by Melliani et al. (2021). Vu and Hoa (2019) solved uncertain fractional differential equations on a time scale under the concept of granular differentiability. The existence and uniqueness of a fuzzy solution to FFDE was established in Arshad and Lupulescu (2011). Salahshour et al. (2012) introduced some fractional theoretical concepts, such as Riemann-Liouville fractional differentiability and caputo *H*-differentiability for FVF. In Allahviranloo and Ahmadi (2010), Salahshour and Allahviranloo (2013), Salahshour and Allahviranloo (2013), FDEs were solved using the fuzzy Laplace transform. These publications outlined the fuzzy derivative of a FVF using the core idea of *H*-differentiability or strongly generalized differentiability. However, the FDEs demonstrated by these differentiability concepts do not have a unique solution. For this purpose, Allahviranloo et al. (2014) postulated the *gH*-Caputo fractional derivative of a FVF and demonstrated the existence and uniqueness of the solution for fuzzy initial value problem (FIVP) with a fuzzy initial condition. The delay fractional differential equations were investigated in Hoa (2015), Van Hoa (2015). Podlubny (1998) discussed the fractional derivatives, the methodology to extract the solution of FDEs and their applications.

Viet Long et al. (2017) investigated fuzzy partial fractional differential equations (FPFDEs). Based on *gH*-differentiability for fuzzy multivariate functions, they established the idea of fuzzy fractional integral and Caputo partial differentiability. FPFDEs are capable of modeling a wide range of natural phenomena in various sciences. In biological medicine, FPFDEs have been used to design the emergence of diseases and the proliferation of cancer cells and many other issues. However, our understanding of FPFDEs is limited. Fuzzy integral transforms such as the fuzzy Laplace transform (FLT) (Allahviranloo and Ahmadi 2010) and the fuzzy Fourier transform (FFT) (Gouyandeha et al. (2017)) are very efficient techniques to solve FPFDEs of fuzzy multivariable functions. Akram et al. (2022a) introduced the concepts of generalized Hukuhara fractional Caputo derivative of Pythagorean fuzzy valued function (PFVF) and the Pythagorean fuzzy Laplace transform (PFLT) to solve Pythagorean fuzzy fractional differential equations (PFFDEs). Akram et al. (2022a) studied the analytical solution of fourth-order FDE using the FLT. Akram et al. (2022a) discussed the fuzzy fractional Langevin differential equations in Caputo's derivative sense. This research article includes the concept of generalized Hukuhara partial differentiability ([*gH* - p] differentiability) and the Pythagorean fuzzy Fourier transform (PFFT) of PFVF to solve Pythagorean fuzzy partial fractional differential equation (PFPFDE). We investigate the Pythagorean fuzzy solution of the PFPFDEs with triangular Pythagorean fuzzy initial conditions in terms of triangular Pythagorean fuzzy numbers (TPFNs) under $[gH$ - p] differentiability using the Pythagorean fuzzy integral transforms. Then, we present some important theorems, such as the PFLT and the PFFT of the triangular Pythagorean fuzzy-valued function (TPFVF) with fractional-order derivative of order $0 < \varrho \leq 2$, Pythagorean fuzzy convolution theorem under type of $[gH$ - p] differentiability. Furthermore, we prove a theorem to solve the PFPFDE. Finally, the Pythagorean fuzzy solution of the PFPFDE as a mathematical model for the diffusion of COVID-19 vaccination in the human body is obtained. The COVID-19 disease was discovered in January 2020 by the Wuhan Health Commission of China (Maxmen (2021)). The World Health Organization (WHO) declared a global public health emergency. This COVID-19 was declared a global pandemic a few weeks later. It was difficult for health departments to mange the newly emerged COVID-19 pandemic at first. The fastly spreading disease encounters a significant task for academic institutions and the industries in developing the effective drug treatments and vaccination. For this, a lot of drugs have been tested on COVID-19 patients. As a result, scientists invented that these medicines have a little bit effect on the whole mortality. At the end, scientists discovered vaccines to control the attack of COVID-19 virus on the human body.

A brief summary of the contents is now as given. In Sect. 2, some preliminary concepts related to TPFNs and gH-differentiability are expressed. Section 3 presents some new theorems related to Pythagorean fuzzy calculus. Section 4 contains the PFLT, the PFFT and some new theorems and lemmas are proved for the Pythagorean fuzzy integral transforms. In Sect. 5, the Pythagorean fuzzy fundamental solution of the PFPFDE with triangular Pythagorean fuzzy initial conditions is obtained using the Pythagorean fuzzy integral transforms and it is followed up by solving several examples. Section 6 presents the Pythagorean fuzzy fundamental solution of the Pythagorean fuzzy partial fractional mathematical model for the diffusion of COVID-19 vaccination in the human body and some related examples with figures. Finally, conclusions are drawn in Sect. 7.

2 Preliminaries

In this section, we review some fundamental definitions of fuzzy operations in the Pythagorean fuzzy environment. Suppose that R_T denotes the set of TPFNs in the space of PFS R_P. The TPFN $u \in R_T$ is denoted by a triplet

$$u = (u_1, u_2, u_3; \tilde{u_1}, u_2, \tilde{u_3}), \tag{1}$$

where $\tilde{u_1} \leq u_1 \leq u_2 \leq u_3 \leq \tilde{u_3}$.

Definition 1 (Ullah et al. (2020)) A PFS R_P in X is an object of the form

$$R_P = \{ < \mathfrak{m}, \mu_{u_t}(\mathfrak{m}), \mu_{u_f}(\mathfrak{m}) > \ : \mathfrak{m} \in X \},$$

where $\mu_{u_t} : X \to [0,1]$ and $\mu_{u_f} : X \to [0,1]$ denote the degree of membership and non-membership, respectively, of the element $\mathfrak{m} \in X$ with the constraint $\mu^2_{u_t}(\mathfrak{m}) + \mu^2_{u_f}(\mathfrak{m}) \leq 1$. Furthermore, the term $v = \sqrt{1 - (\mu^2_{u_t}(\mathfrak{m}) + \mu^2_{u_t}(\mathfrak{m}))}$ is expressed as the hesitancy degree.

Definition 2 (Mondal and Roy (2015)) A Pythagorean fuzzy number (PFN) $u = (u_t, u_f)$ is a non-empty subset of X with the rule of membership grade $\mu_{u_t} : X \to [0,1]$ and non-membership grade $\mu_{u_f} : X \to [0,1]$. Firmly, u is convex, that is,

$$\mu_{u_t}(\ltimes \mathfrak{m} + (1 - \ltimes)\mathfrak{m}_1) \geq \min\{\mu_{u_t}(\mathfrak{m}), \mu_{u_t}(\mathfrak{m}_1)\},$$
$$\forall \ltimes, \mathfrak{m}, \mathfrak{m}_1 \ \text{ with } \ \ltimes \in [0,1] \text{ and } \mathfrak{m}, \mathfrak{m}_1 \in X$$

and concave, that is

$$\mu_{u_f}(\ltimes \mathfrak{m} + (1 - \ltimes)\mathfrak{m}_1) \geq \max\{\mu_{u_f}(\mathfrak{m}), \mu_{u_f}(\mathfrak{m}_1)\},$$
$$\forall \ltimes, \mathfrak{m}, \mathfrak{m}_1 \ \text{ with } \ \ltimes \in [0,1] \ \textit{and} \ \mathfrak{m}, \mathfrak{m}_1 \in X.$$

Also u is normal, because there exists $\mathfrak{m} \in X$ such that $\mu_{u_t} = 1$ and $\mu_{u_f} = 0$.

Definition 3 (Mondal et al. (2019)) A TPFN u is a subset of PFN in R with the following membership and non-membership functions:

$$\mu_{u_t}(\mathfrak{m}) = \begin{cases} \dfrac{\mathfrak{m} - u_1}{u_2 - u_1} & ; \ u_1 \leq \mathfrak{m} \leq u_2, \\ \dfrac{u_3 - \mathfrak{m}}{u_3 - u_2} & ; \ u_2 \leq \mathfrak{m} \leq u_3, \\ 0 & ; \ \text{elsewhere} \end{cases} \quad \text{and}$$

$$\mu_{u_f}(\mathfrak{m}) = \begin{cases} \dfrac{u_2 - \mathfrak{m}}{u_2 - \tilde{u_1}} & ; \ \tilde{u_1} \leq \mathfrak{m} \leq u_2, \\ \dfrac{\mathfrak{m} - u_2}{\tilde{u_3} - u_2} & ; \ u_2 \leq \mathfrak{m} \leq \tilde{u_3}, \\ 1 & ; \ \text{elsewhere}, \end{cases}$$

where $\tilde{u_1} \leq u_1 \leq u_2 \leq u_3 \leq \tilde{u_3}$ and a TPFN is denoted by $u = (u_1, u_2, u_3; \tilde{u_1}, u_2, \tilde{u_3})$.

Definition 4 (Bede and Stefanini (2013)) The gH-difference of two TPFNs $u, v \in R_T$ is the TPFN w, if exists, such that

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} u = v \oplus w \\ or \quad v = u \oplus (-1)w. \end{cases}$$

The condition for the existence of $w = u \ominus_{gH} v \in R_T$ is given in Bede and Stefanini (2013).

Remark 1 For two TPFNs $u = (u_1, u_2, u_3; \tilde{u_1}, u_2, \tilde{u_3})$ and $v = (v_1, v_2, v_3; \tilde{v_1}, v_2, \tilde{v_3})$, we have the following results regarding the gH-difference of u and v.

1.

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} w = (u_1 - v_1, u_2 - v_2, u_3 - v_3; \tilde{u_1} - \tilde{v_1}, u_2 - v_2, \tilde{u_3} - \tilde{v_3}) \\ or \quad w = (u_3 - v_3, u_2 - v_2, u_1 - v_1; \tilde{u_3} - \tilde{v_3}, u_2 - v_2, \tilde{u_1} - \tilde{v_1}). \end{cases}$$

2. We assume that $u \ominus_{gH} v \in R_T$ throughout this manuscript.
3. The following results are easy to prove:
   (i) $u \ominus (-1)v = (u_1 + v_3, u_2 + v_2, u_3 + v_1; \tilde{u_1} + \tilde{v_3}, u_2 + v_2, \tilde{u_3} + \tilde{v_1})$, such that $u \ominus (-1)v$ is a TPFN.
   (ii) $\ominus_{gH} u = (-1)u$, such that $(-1)u$ is a TPFN.
   (iii) $\ominus_{gH}(-1)u = u$.
   (iv) $u \ominus_{gH} (-1)v \neq u \oplus v$.

Assume that $\mathbb{D} \subseteq \mathbb{R}^2$ and $\mathbb{I} = [a, b] \subseteq \mathbb{R}$. Suppose that $C^P(\mathbb{D})$ denotes the space of continuous PFVFs on $\mathbb{D}$ and $L^P(\mathbb{D})$ denotes the space of Lebesgue integrable PFVFs defined on $\mathbb{D}$. Throughout this manuscript, we suppose that $\aleph(\mathfrak{h}) \in C^P(\mathbb{I}) \cap L^P(\mathbb{I})$ and $\Lambda(\mathfrak{u}, \mathfrak{h}) \in C^P(\mathbb{D}) \cap L^P(\mathbb{D})$ are two TPFVFs. The TPFVF $\aleph(\mathfrak{h})$ has the following representation:

$$\aleph(\mathfrak{h}) = \Big(\aleph_1(\mathfrak{h}), \aleph_2(\mathfrak{h}), \aleph_3(\mathfrak{h}); \tilde{\aleph_1}(\mathfrak{h}), \aleph_2(\mathfrak{h}), \tilde{\aleph_3}(\mathfrak{h})\Big). \tag{2}$$

The TPFVF $\Lambda(\mathfrak{u}, \mathfrak{h})$ in two variables is expressed as:

$$\Lambda(\mathfrak{u}, \mathfrak{h}) = \Big(\Lambda_1(\mathfrak{u}, \mathfrak{h}), \Lambda_2(\mathfrak{u}, \mathfrak{h}), \Lambda_3(\mathfrak{u}, \mathfrak{h}); \tilde{\Lambda}_1(\mathfrak{u}, \mathfrak{h}), \Lambda_2(\mathfrak{u}, \mathfrak{h}), \tilde{\Lambda}_3(\mathfrak{u}, \mathfrak{h})\Big). \tag{3}$$

Definition 5 (Akram et al. (2022a)) The TPFVF $\aleph(\mathfrak{h})$ is said to be gH-differentiable at $\mathfrak{h}_0 \in \mathbb{I}$ if there exists $\aleph'_{gH}(\mathfrak{h}_0) \in R_T$, such that for every ϵ positive, the expression $\aleph(\mathfrak{h}_0 + \epsilon) \ominus_{gH} \aleph(\mathfrak{h}_0)$ exists and the following constraint is satisfied:

$$\aleph'_{gH}(\mathfrak{h}_0) = \lim_{\epsilon \searrow 0} \frac{\aleph(\mathfrak{h}_0 + \epsilon) \ominus_{gH} \aleph(\mathfrak{h}_0)}{\epsilon}.$$

Using results in Akram et al. (2022a), we define the following definitions of first and second gH-differentiability of TPFVF $\aleph(\mathfrak{h})$.

Definition 6 Let $\aleph(\mathfrak{h}) \in C^P(\mathbb{I}) \cap L^P(\mathbb{I})$ be a TPFVF. Then, the first and second gH-differentiability of $\aleph(\mathfrak{h})$ is defined as follows:

1. $\aleph(\mathfrak{h})$ is said to be first gH-differentiable ($[(\imath) - gH]$-differentiable) if $\forall\ \mathfrak{h} \in \mathbb{I}$, the following expression defines a TPFN.

$$\aleph'_{(\imath)gH}(\mathfrak{h}) := \aleph'_{gH}(\mathfrak{h}) = \Big(\aleph'_1(\mathfrak{h}), \aleph'_2(\mathfrak{h}), \aleph'_3(\mathfrak{h}); \tilde{\aleph}'_1(\mathfrak{h}), \aleph'_2(\mathfrak{h}), \tilde{\aleph}'_3(\mathfrak{h})\Big).$$

2. $\aleph(\mathfrak{h})$ is said to be second gH-differentiable ($[(\imath\imath) - gH]$-differentiable) if $\forall\ \mathfrak{h} \in \mathbb{I}$, the following expression defines a TPFN.

$$\aleph'_{(\imath\imath)gH}(\mathfrak{h}) := \aleph'_{gH}(\mathfrak{h}) = \Big(\aleph'_3(\mathfrak{h}), \aleph'_2(\mathfrak{h}), \aleph'_1(\mathfrak{h}); \tilde{\aleph}'_3(\mathfrak{h}), \aleph'_2(\mathfrak{h}), \tilde{\aleph}'_1(\mathfrak{h})\Big).$$

Using Definition 6, the second-order derivative of TPFVF $\aleph(\mathfrak{h})$ is defined by the following definintion:

Definition 7 Let $\aleph(\mathfrak{h}) \in C^P(\mathbb{I}) \cap L^P(\mathbb{I})$ be a TPFVF and $\aleph'_{gH}(\mathfrak{h})$ be gH-differentiable PFVF. If the type of gH-differentiability of $\aleph(\mathfrak{h})$ and $\aleph'_{gH}(\mathfrak{h})$ are same, then $\aleph'_{gH}(\mathfrak{h})$ is $[(\imath) - gH]$-differentiable and is defined as:

$$\aleph''_{(\imath)gH}(\mathfrak{h}) := \aleph''_{gH}(\mathfrak{h}) = \Big(\aleph''_1(\mathfrak{h}), \aleph''_2(\mathfrak{h}), \aleph''_3(\mathfrak{h}); \tilde{\aleph}''_1(\mathfrak{h}), \aleph''_2(\mathfrak{h}), \tilde{\aleph}''_3(\mathfrak{h})\Big).$$

If the type of gH-differentiability of $\aleph(\mathfrak{h})$ and $\aleph'_{gH}(\mathfrak{h})$ are different, then $\aleph'_{gH}(\mathfrak{h})$ is $[(\imath\imath) - gH]$-differentiable and is defined as:

$$\aleph''_{(\imath\imath)gH}(\mathfrak{h}) := \aleph''_{gH}(\mathfrak{h}) = \Big(\aleph''_3(\mathfrak{h}), \aleph''_2(\mathfrak{h}), \aleph''_1(\mathfrak{h}); \tilde{\aleph}''_3(\mathfrak{h}), \aleph''_2(\mathfrak{h}), \tilde{\aleph}''_1(\mathfrak{h})\Big).$$

Definition 8 (Akram et al. 2022a) Let $\aleph(\mathfrak{h}) \in C^P(\mathbb{I}) \cap L^P(\mathbb{I})$. Then, the generalized Hukuhara fractional Caputo derivative (${}^{CF}[gH]-$differentiable for short) of PFVF $\aleph(\mathfrak{h})$ of order $\varrho \in \mathbb{C}$, $\mathrm{Re}(\varrho) \geq 0$ is defined as:

$${}^{C}_{gH}\mathscr{D}^{\varrho}_{a^+}\aleph(\mathfrak{h}) = \frac{1}{\Gamma(n-\varrho)} \int_a^{\mathfrak{h}} (\mathfrak{h} - s)^{n-\varrho-1} \aleph^{(n)}(s) ds, \quad \text{for } \mathfrak{h} > a, \tag{4}$$

where n is the natural number, such that $n - 1 \leq Re(\varrho) < n$.

In particular, if $\varrho \in (0,1)$ and $a=0$ then Equ. (4) takes the following representation:

$$ {}_{gH}^{C}\mathscr{D}_{0^+}^{\varrho}\aleph(\mathfrak{h}) = \frac{1}{\Gamma(1-\varrho)}\int_0^{\mathfrak{h}} \frac{\aleph'(s)ds}{(\mathfrak{h}-s)^{\varrho}}, \quad \text{for } \mathfrak{h}>0. \tag{5} $$

Definition 9 Let $\aleph(\mathfrak{h})$ be ${}^{CF}[gH]-$differentiable, then $\aleph(\mathfrak{h})$ is:

(i) ${}^{CF}[(\iota)-gH]-$differentiable if $\forall$ $\mathfrak{h}\in \mathbb{I}$

$$ {}_{(\iota).gH}^{C}\mathscr{D}_{a^+}^{\varrho}\aleph(\mathfrak{h}) = \Big({}^{C}\mathscr{D}_{a^+}^{\varrho}\aleph_1(\mathfrak{h}), {}^{C}\mathscr{D}_{a^+}^{\varrho}\aleph_2(\mathfrak{h}), {}^{C}\mathscr{D}_{a^+}^{\varrho}\aleph_3(\mathfrak{h}); {}^{C}\mathscr{D}_{a^+}^{\varrho}\tilde{\aleph_1}(\mathfrak{h}), {}^{C}\mathscr{D}_{a^+}^{\varrho}\aleph_2(\mathfrak{h}), {}^{C}\mathscr{D}_{a^+}^{\varrho}\tilde{\aleph_3}(\mathfrak{h})\Big), \forall \varrho\in(0,1). $$

(ii) ${}^{CF}[(u)-gH]-$differentiable if $\forall$ $\mathfrak{h}\in \mathbb{I}$

$$ {}_{(u).gH}^{C}\mathscr{D}_{a^+}^{\varrho}\aleph(\mathfrak{h}) = \Big({}^{C}\mathscr{D}_{a^+}^{\varrho}\aleph_3(\mathfrak{h}), {}^{C}\mathscr{D}_{a^+}^{\varrho}\aleph_2(\mathfrak{h}), {}^{C}\mathscr{D}_{a^+}^{\varrho}\aleph_1(\mathfrak{h}); {}^{C}\mathscr{D}_{a^+}^{\varrho}\tilde{\aleph_3}(\mathfrak{h}), {}^{C}\mathscr{D}_{a^+}^{\varrho}\aleph_2(\mathfrak{h}), {}^{C}\mathscr{D}_{a^+}^{\varrho}\tilde{\aleph_1}(\mathfrak{h})\Big), \quad \forall \varrho\in(0,1). $$

Definition 10 (Povstenko (2015)) The Mittag-Leffler function in one variable is one of the major functions used in this manuscript and it has the following representation:

$$ E_{\varrho}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\varrho k+1)}. $$

Other important function is the Minardi function defined as follows:

$$ M(\varrho;z) = W(-\varrho, 1-\varrho; -z) = \sum_{j=0}^{\infty}\frac{(-1)^j z^j}{j!\Gamma[-\varrho j+(1-\varrho)]}, $$

where $W(\varrho,\beta;z)$ is the Wright function.

3 Pythagorean fuzzy calculus

Definition 11 A TPFVF $\Lambda(\mathfrak{u},\mathfrak{h})$ having no switching point on $\mathbb{D}\subseteq\mathbb{R}^2$ is said to be first or second partial differentiable with respect to $\mathfrak{u}$ such that:

(i) $[(\iota)-p]-$differentiable with respect to $\mathfrak{u}$ at $(\mathfrak{u},\mathfrak{h})\in\mathbb{D}$ if

$$ \frac{\partial\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}} = \Big(\frac{\partial\Lambda_1(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}}, \frac{\partial\Lambda_2(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}}, \frac{\partial\Lambda_3(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}}; \frac{\partial\tilde{\Lambda}_1(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}}, \frac{\partial\Lambda_2(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}}, \frac{\partial\tilde{\Lambda}_3(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}}\Big). $$

(ii) $[(u)-p]-$differentiable with respect to $\mathfrak{u}$ at $(\mathfrak{u},\mathfrak{h})\in\mathbb{D}$ if

$$ \frac{\partial\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}} = \Big(\frac{\partial\Lambda_3(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}}, \frac{\partial\Lambda_2(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}}, \frac{\partial\Lambda_1(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}}; \frac{\partial\tilde{\Lambda}_3(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}}, \frac{\partial\Lambda_2(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}}, \frac{\partial\tilde{\Lambda}_1(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}}\Big). $$

Furthermore, if $\frac{\partial\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}}$ is $[gH-p]$-differentiable with respect to $\mathfrak{u}$ at $(\mathfrak{u},\mathfrak{h})\in\mathbb{D}$ having no switching point on $\mathbb{D}$ and

(i) If the type of $[gH-p]$-differentiability of both $\Lambda(\mathfrak{u},\mathfrak{h})$ and $\frac{\partial\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}}$ are same, then $\frac{\partial\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}}$ is $[(\iota)-p]$-differentiable with respect to $\mathfrak{u}$ and

$$ \frac{\partial^2\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^2} = \Big(\frac{\partial^2\Lambda_1(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^2}, \frac{\partial^2\Lambda_2(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^2}, \frac{\partial^2\Lambda_3(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^2}; \frac{\partial^2\tilde{\Lambda}_1(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^2}, \frac{\partial^2\Lambda_2(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^2}, \frac{\partial^2\tilde{\Lambda}_3(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^2}\Big). $$

(ii) If the type of $[gH-p]$-differentiability of both $\Lambda(\mathfrak{u},\mathfrak{h})$ and $\frac{\partial\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}}$ are different, then $\frac{\partial\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}}$ is $[(u)-p]$-differentiable with respect to $\mathfrak{u}$ and

$$ \frac{\partial^2\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^2} = \Big(\frac{\partial^2\Lambda_3(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^2}, \frac{\partial^2\Lambda_2(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^2}, \frac{\partial^2\Lambda_1(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^2}; \frac{\partial^2\tilde{\Lambda}_3(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^2}, \frac{\partial^2\Lambda_2(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^2}, \frac{\partial^2\tilde{\Lambda}_1(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^2}\Big). $$

Remark 2 Assume that $\aleph(\mathfrak{h})$ is a TPFVF defined on $\mathbb{I}$, such that the type of $gH-$differentiability remains unchanged $\forall\mathfrak{h}\in\mathbb{I}$, then

$$ \int_a^b \aleph(\mathfrak{h})d\mathfrak{h} = \Big(\min\{\int_a^b\aleph_1(\mathfrak{h})d\mathfrak{h}, \int_a^b\aleph_3(\mathfrak{h})d\mathfrak{h}\}, \int_a^b\aleph_2(\mathfrak{h})d\mathfrak{h}, \max\{\int_a^b\aleph_1(\mathfrak{h})d\mathfrak{h}, \int_a^b\aleph_3(\mathfrak{h})d\mathfrak{h}; \min\{\int_a^b\tilde{\aleph_1}(\mathfrak{h})d\mathfrak{h}, \int_a^b\tilde{\aleph_3}(\mathfrak{h})d\mathfrak{h}\}, \int_a^b\aleph_2(\mathfrak{h})d\mathfrak{h}, \max\{\int_a^b\tilde{\aleph_1}(\mathfrak{h})d\mathfrak{h}, \int_a^b\tilde{\aleph_3}(\mathfrak{h})d\mathfrak{h}\}\Big). $$

Consider a TPFVF

$$\aleph(\mathfrak{h}) = \left(-2.3e^{-\mathfrak{h}}, -1.2e^{-\mathfrak{h}}, 3e^{-\mathfrak{h}}; -3e^{-\mathfrak{h}}, -1.2e^{-\mathfrak{h}}, 4e^{-\mathfrak{h}} \right).$$

Then

$$\int \left(-2.3e^{-\mathfrak{h}}, -1.2e^{-\mathfrak{h}}, 3e^{-\mathfrak{h}}; -3e^{-\mathfrak{h}}, -1.2e^{-\mathfrak{h}}, 4e^{-\mathfrak{h}} \right) d\mathfrak{h} = \left(-3e^{-\mathfrak{h}}, 1.2e^{-\mathfrak{h}}, 2.3e^{-\mathfrak{h}}; -4e^{-\mathfrak{h}}, 1.2e^{-\mathfrak{h}}, 3e^{-\mathfrak{h}} \right).$$

Theorem 1 *Let $\aleph : \mathbb{I} \to \mathbb{R}_T$ be a TPFVF without any switching point in the interval $\mathbb{I}$. Then, the fundamental theorem of calculus in the Pythagorean fuzzy context is given by*:

1. $\int_a^b \aleph'_{(\imath)-gH}(\mathfrak{h})d\mathfrak{h} = \aleph(b) - \aleph(a)$, if $\aleph(\mathfrak{h})$ is $[(\imath) - gH]$-differentiable.
2. $\int_a^b \aleph'_{(\imath\imath)-gH}(\mathfrak{h})d\mathfrak{h} = (-1)\aleph(a) \ominus (-1)\aleph(b)$, if $\aleph(\mathfrak{h})$ is $[(\imath\imath) - gH]$-differentiable.

Recently, Chalco-Canoa et al. studied the $gH-$derivative for the product of a real-valued differentiable function and $gH-$differentiable interval valued functions (Chalco-Canoa et al. 2019). We extend these results for TPFVFs in the following.

Theorem 2 *Suppose that $\aleph : \mathbb{I} \to \mathbb{R}_T$ is a TPFVF on $\mathbb{I}$, such that the type of $gH-$differentiability remains unaltered on the closed interval $\mathbb{I}$ and $\flat(\mathfrak{h})$ is a monotonic real - valued continuous differentiable function in $\mathbb{I}$. Then, according to the type of $gH-$differentiability of $\aleph(\mathfrak{h})$ and monotonicity of $\flat(\mathfrak{h})$, we have following cases*:

Case 1. If $\aleph(\mathfrak{h})$ is $[(\imath) - gH]-$differentiable and

(i) $\flat(\mathfrak{h})$ is a positive and increasing function $\forall \mathfrak{h} \in \mathbb{I}$, then $\flat(\mathfrak{h}) \odot \aleph(\mathfrak{h})$ is $[(\imath) - gH]-$differentiable and

$$\left(\flat(\mathfrak{h}) \odot \aleph(\mathfrak{h}) \right)'_{(\imath).gH} = \flat'(\mathfrak{h}) \odot \aleph(\mathfrak{h}) \oplus \flat(\mathfrak{h}) \odot \aleph'_{(\imath).gH}(\mathfrak{h}).$$

Furthermore

$$\int_a^b \flat(\mathfrak{h}) \odot \aleph'_{(\imath).gH}(\mathfrak{h})d\mathfrak{h} = \flat(b) \odot \aleph(b) \ominus \flat(a) \odot \aleph(a) \ominus_{gH} \int_a^b \flat'(\mathfrak{h}) \odot \aleph(\mathfrak{h})d\mathfrak{h}.$$

(ii) $\flat(\mathfrak{h})$ is a positive and decreasing function $\forall \mathfrak{h} \in \mathbb{I}$, then $\flat(\mathfrak{h}) \odot \aleph(\mathfrak{h})$ is $[(\imath) - gH]-$differentiable and

$$\left(\flat(\mathfrak{h}) \odot \aleph(\mathfrak{h}) \right)'_{(\imath).gH} = \flat(\mathfrak{h}) \odot \aleph'_{(\imath).gH}(\mathfrak{h}) \ominus (-\flat'(\mathfrak{h})) \odot \aleph(\mathfrak{h}),$$

such that the H-difference exists. Furthermore

$$\int_a^b \flat(\mathfrak{h}) \odot \aleph'_{(\imath).gH}(\mathfrak{h})d\mathfrak{h} = \flat(b) \odot \aleph(b) \ominus \flat(a) \odot \aleph(a) \oplus \int_a^b (-\flat'(\mathfrak{h})) \odot \aleph(\mathfrak{h})d\mathfrak{h}.$$

(iii) $\flat(\mathfrak{h})$ is a negative and increasing function $\forall \mathfrak{h} \in \mathbb{I}$, then $\flat(\mathfrak{h}) \odot \aleph(\mathfrak{h})$ is $[(\imath\imath) - gH]-$differentiable and

$$\left(\flat(\mathfrak{h}) \odot \aleph(\mathfrak{h}) \right)'_{(\imath\imath).gH} = \flat(\mathfrak{h}) \odot \aleph'_{(\imath).gH}(\mathfrak{h}) \ominus (-\flat'(\mathfrak{h})) \odot \aleph(\mathfrak{h}),$$

given that H-difference exists. Moreover

$$\int_a^b \flat(\mathfrak{h}) \odot \aleph'_{(\imath).gH}(\mathfrak{h})d\mathfrak{h} = (-\flat(a)) \odot \aleph(a) \ominus (-\flat(b)) \odot \aleph(b) \oplus (-1) \int_a^b \flat'(\mathfrak{h}) \odot \aleph(\mathfrak{h})d\mathfrak{h}.$$

(iv) $\flat(\mathfrak{h})$ is a negative and decreasing function $\forall \mathfrak{h} \in \mathbb{I}$, then $\flat(\mathfrak{h}) \odot \aleph(\mathfrak{h})$ is $[(\imath\imath) - gH]-$differentiable and

$$\left(\flat(\mathfrak{h}) \odot \aleph(\mathfrak{h}) \right)'_{(\imath\imath).gH} = \flat'(\mathfrak{h}) \odot \aleph(\mathfrak{h}) \oplus \flat(\mathfrak{h}) \odot \aleph'_{(\imath).gH}(\mathfrak{h}),$$

provided that H-difference exists. Moreover

$$\int_a^b \flat(\mathfrak{h}) \odot \aleph'_{(\imath).gH}(\mathfrak{h})d\mathfrak{h} = -(\flat(a)) \odot \aleph(a) \ominus (-\flat(b)) \odot \aleph(b) \ominus_{gH} \int_a^b \flat'(\mathfrak{h}) \odot \aleph(\mathfrak{h})d\mathfrak{h}.$$

Case 2. If $\aleph(\mathfrak{h})$ is $[(\imath\imath) - gH]-$differentiable and

(i) $\flat(\mathfrak{h})$ is a positive and increasing function $\forall \mathfrak{h} \in \mathbb{I}$, then $\flat(\mathfrak{h}) \odot \aleph(\mathfrak{h})$ is $[(\imath\imath) - gH]-$differentiable and

$$\left(\flat(\mathfrak{h}) \odot \aleph(\mathfrak{h})\right)'_{(u).gH} = \flat(\mathfrak{h}) \odot \aleph'_{(u).gH}(\mathfrak{h}) \ominus (-1)\flat'(\mathfrak{h}) \odot \aleph(\mathfrak{h}),$$

given that H-difference exists. Furthermore

$$\int_a^b \flat(\mathfrak{h}) \odot \aleph'_{(u).gH}(\mathfrak{h})d\mathfrak{h} = (-\flat(a)) \odot \aleph(a) \ominus (-\flat(b)) \odot \aleph(b) \oplus (-1)\int_a^b \flat'(\mathfrak{h}) \odot \aleph(\mathfrak{h})d\mathfrak{h}.$$

(ii) $\flat(\mathfrak{h})$ is a positive and decreasing function $\forall \mathfrak{h} \in \mathbb{I}$, then $\flat(\mathfrak{h}) \odot \aleph(\mathfrak{h})$ is $[(u) - gH]-$differentiable and

$$\left(\flat(\mathfrak{h}) \odot \aleph(\mathfrak{h})\right)'_{(u).gH} = \flat(\mathfrak{h}) \odot \aleph'_{(u).gH}(\mathfrak{h}) \oplus \flat'(\mathfrak{h}) \odot \aleph(\mathfrak{h}).$$

Furthermore

$$\int_a^b \flat(\mathfrak{h}) \odot \aleph'_{(u).gH}(\mathfrak{h})d\mathfrak{h} = (-\flat(a)) \odot \aleph(a) \ominus (-\flat(b)) \odot \aleph(b) \ominus_{gH} \int_a^b \flat'(\mathfrak{h}) \odot \aleph(\mathfrak{h})d\mathfrak{h}.$$

(iii) $\flat(\mathfrak{h})$ is a negative and increasing function $\forall \mathfrak{h} \in \mathbb{I}$, then $\flat(\mathfrak{h}) \odot \aleph(\mathfrak{h})$ is $[(\imath) - gH]-$differentiable and

$$\left(\flat(\mathfrak{h}) \odot \aleph(\mathfrak{h})\right)'_{(\imath).gH} = \flat'(\mathfrak{h}) \odot \aleph(\mathfrak{h}) \oplus \flat(\mathfrak{h}) \odot \aleph'_{(u).gH}(\mathfrak{h}).$$

Moreover

$$\int_a^b \flat(\mathfrak{h}) \odot \aleph'_{(u).gH}(\mathfrak{h})d\mathfrak{h} = \flat(b) \odot \aleph(b) \ominus \flat(a) \odot \aleph(a) \ominus_{gH} \int_a^b \flat'(\mathfrak{h}) \odot \aleph(\mathfrak{h})d\mathfrak{h}.$$

(iv) $\flat(\mathfrak{h})$ is a negative and decreasing function $\forall \mathfrak{h} \in \mathbb{I}$, then $\flat(\mathfrak{h}) \odot \aleph(\mathfrak{h})$ is $[(\imath) - gH]-$differentiable and

$$\left(\flat(\mathfrak{h}) \odot \aleph(\mathfrak{h})\right)'_{(\imath).gH} = \flat(\mathfrak{h}) \odot \aleph'_{(u).gH}(\mathfrak{h}) \ominus (-1)\flat'(\mathfrak{h}) \odot \aleph(\mathfrak{h}),$$

such that H-difference exists and

$$\int_a^b \flat(\mathfrak{h}) \odot \aleph'_{(u).gH}(\mathfrak{h})d\mathfrak{h} = \flat(b) \odot \aleph(b) \ominus \flat(a) \odot \aleph(a) \oplus (-1)\int_a^b \flat'(\mathfrak{h}) \odot \aleph(\mathfrak{h})d\mathfrak{h}.$$

Proof We prove only **Case 1(ii)** and **Case 2(ii)**.

According to the supposition of **Case 1(ii)**, $\aleph(\mathfrak{h})$ is $[(\imath) - gH]-$differentiable and $\flat(\mathfrak{h})$ is a positive and decreasing function $\forall \mathfrak{h} \in \mathbb{I}$. Then, $\flat(\mathfrak{h}) > 0$ and $\flat'(\mathfrak{h}) < 0$,

$$\begin{aligned}
&\flat(\mathfrak{h}) \odot \aleph'_{(\imath).gH}(\mathfrak{h}) \ominus (-\flat'(\mathfrak{h})) \odot \aleph(\mathfrak{h}) \\
&= \flat(\mathfrak{h}) \odot \left(\aleph'_1(\mathfrak{h}), \aleph'_2(\mathfrak{h}), \aleph'_3(\mathfrak{h}); \tilde{\aleph}'_1(\mathfrak{h}), \aleph'_2(\mathfrak{h}), \tilde{\aleph}'_3(\mathfrak{h})\right) \\
&\ominus (-1)\flat'(\mathfrak{h}) \odot \left(\aleph_1(\mathfrak{h}), \aleph_2(\mathfrak{h}), \aleph_3(\mathfrak{h}); \tilde{\aleph}_1(\mathfrak{h}), \aleph_2(\mathfrak{h}), \tilde{\aleph}_3(\mathfrak{h})\right), \\
&= \left(\flat(\mathfrak{h})\aleph'_1(\mathfrak{h}), \flat(\mathfrak{h})\aleph'_2(\mathfrak{h}), \flat(\mathfrak{h})\aleph'_3(\mathfrak{h}); \flat(\mathfrak{h})\tilde{\aleph}'_1(\mathfrak{h}), \flat(\mathfrak{h})\aleph'_2(\mathfrak{h}), \flat(\mathfrak{h})\tilde{\aleph}'_3(\mathfrak{h})\right) \\
&\ominus (-1)\left(\flat'(\mathfrak{h})\aleph_3(\mathfrak{h}), \flat'(\mathfrak{h})\aleph_2(\mathfrak{h}), \flat'(\mathfrak{h})\aleph_1(\mathfrak{h}); \flat'(\mathfrak{h})\tilde{\aleph}_3(\mathfrak{h}), \flat'(\mathfrak{h})\aleph_2(\mathfrak{h}), \flat'(\mathfrak{h})\tilde{\aleph}_1(\mathfrak{h})\right), \\
&= \left(\flat(\mathfrak{h})\aleph'_1(\mathfrak{h}), \flat(\mathfrak{h})\aleph'_2(\mathfrak{h}), \flat(\mathfrak{h})\aleph'_3(\mathfrak{h}); \flat(\mathfrak{h})\tilde{\aleph}'_1(\mathfrak{h}), \flat(\mathfrak{h})\aleph'_2(\mathfrak{h}), \flat(\mathfrak{h})\tilde{\aleph}'_3(\mathfrak{h})\right) \\
&\ominus \Big(-\flat'(\mathfrak{h})\aleph_1(\mathfrak{h}), -\flat'(\mathfrak{h})\aleph_2(\mathfrak{h}), -\flat'(\mathfrak{h})\aleph_3(\mathfrak{h}); -\flat'(\mathfrak{h})\tilde{\aleph}_1(\mathfrak{h}), \\
&\quad -\flat'(\mathfrak{h})\aleph_2(\mathfrak{h}), -\flat'(\mathfrak{h})\tilde{\aleph}_3(\mathfrak{h})\Big), \\
&\flat(\mathfrak{h}) \odot \aleph'_{(\imath).gH}(\mathfrak{h}) \ominus (-\flat'(\mathfrak{h})) \odot \aleph(\mathfrak{h}) = \left(\flat(\mathfrak{h}) \odot \aleph(\mathfrak{h})\right)'_{(\imath).gH}.
\end{aligned}$$

Thus, $\flat(\mathfrak{h}) \odot \aleph(\mathfrak{h})$ is $[(\imath) - gH]$-differentiable. We obtain the following equation.

$$\left(\flat(\mathfrak{h}) \odot \aleph(\mathfrak{h})\right)'_{(\imath).gH} = \flat(\mathfrak{h}) \odot \aleph'_{(\imath).gH}(\mathfrak{h}) \ominus (-\flat'(\mathfrak{h})) \odot \aleph(\mathfrak{h}). \tag{6}$$

Taking integral on both side of Equ. (6), we have:

$$\int_a^b \left(\flat(\mathfrak{h}) \odot \aleph(\mathfrak{h})\right)'_{(\imath).gH} d\mathfrak{h} = \int_a^b \flat(\mathfrak{h}) \odot \aleph'_{(\imath).gH}(\mathfrak{h})d\mathfrak{h} \ominus \int_a^b (-\flat'(\mathfrak{h})) \odot \aleph(\mathfrak{h})d\mathfrak{h}.$$

Hence Theorem 1 implies that:

$$\int_a^b \flat(\mathfrak{h}) \odot \aleph'_{(\imath).gH}(\mathfrak{h})d\mathfrak{h} = \flat(b) \odot \aleph(b) \ominus \flat(a) \odot \aleph(a) \oplus \int_a^b (-\flat'(\mathfrak{h})) \odot \aleph(\mathfrak{h})d\mathfrak{h}.$$

This completes the proof. The remaining cases can be proved in the same way.

Now, according to the supposition of **Case 2(ii)**, $\aleph(\mathfrak{h})$ is $[(u) - gH]-$differentiable and $\flat(\mathfrak{h})$ is a positive and decreasing function $\forall \mathfrak{h} \in \mathbb{I}$. Then, $\flat(\mathfrak{h}) > 0$ and $\flat'(\mathfrak{h}) < 0$,

$$\flat(\mathfrak{h})\odot\aleph'_{(u).gH}(\mathfrak{h})\oplus\flat'(\mathfrak{h})\odot\aleph(\mathfrak{h})$$
$$=\flat(\mathfrak{h})\odot\left(\aleph'_3(\mathfrak{h}),\aleph'_2(\mathfrak{h}),\aleph'_1(\mathfrak{h});\tilde{\aleph'_3}(\mathfrak{h}),\aleph'_2(\mathfrak{h}),\tilde{\aleph'_1}(\mathfrak{h})\right)$$
$$\oplus\flat'(\mathfrak{h})\odot\left(\aleph_1(\mathfrak{h}),\aleph_2(\mathfrak{h}),\aleph_3(\mathfrak{h});\tilde{\aleph_1}(\mathfrak{h}),\aleph_2(\mathfrak{h}),\tilde{\aleph_3}(\mathfrak{h})\right),$$
$$=\left(\flat(\mathfrak{h})\aleph'_3(\mathfrak{h}),\flat(\mathfrak{h})\aleph'_2(\mathfrak{h}),\flat(\mathfrak{h})\aleph'_1(\mathfrak{h});\flat(\mathfrak{h})\tilde{\aleph'_3}(\mathfrak{h}),\flat(\mathfrak{h})\aleph'_2(\mathfrak{h}),\flat(\mathfrak{h})\tilde{\aleph'_1}(\mathfrak{h})\right)$$
$$\oplus\left(\flat'(\mathfrak{h})\aleph_3(\mathfrak{h}),\flat'(\mathfrak{h})\aleph_2(\mathfrak{h}),\flat'(\mathfrak{h})\aleph_1(\mathfrak{h});\flat'(\mathfrak{h})\tilde{\aleph_3}(\mathfrak{h}),\flat'(\mathfrak{h})\aleph_2(\mathfrak{h}),\flat'(\mathfrak{h})\tilde{\aleph_1}(\mathfrak{h})\right),$$
$$\flat(\mathfrak{h})\odot\aleph'_{(u).gH}(\mathfrak{h})\oplus\flat'(\mathfrak{h})\odot\aleph(\mathfrak{h})=\left(\flat(\mathfrak{h})\odot\aleph(\mathfrak{h})\right)'_{(u).gH}.$$

Thus $\flat(\mathfrak{h})\odot\aleph(\mathfrak{h})$ is $[(u)-gH]$-differentiable. Moreover, we obtain the following equation.

$$\left(\flat(\mathfrak{h})\odot\aleph(\mathfrak{h})\right)'_{(u).gH}=\flat(\mathfrak{h})\odot\aleph'_{(u).gH}(\mathfrak{h})\oplus\flat'(\mathfrak{h})\odot\aleph(\mathfrak{h}). \tag{7}$$

Taking integral on both side of Equ. (7), we have:

$$\int_a^b\left(\flat(\mathfrak{h})\odot\aleph(\mathfrak{h})\right)'_{(u).gH}d\mathfrak{h}$$
$$=\int_a^b\flat(\mathfrak{h})\odot\aleph'_{(u).gH}(\mathfrak{h})d\mathfrak{h}\oplus\int_a^b\flat'(\mathfrak{h})\odot\aleph(\mathfrak{h})d\mathfrak{h}.$$

From Theorem 1, it follows that:

$$\int_a^b\flat(\mathfrak{h})\odot\aleph'_{(u).gH}(\mathfrak{h})d\mathfrak{h}$$
$$=(-\flat(a))\odot\aleph(a)\ominus(-\flat(b))\odot\aleph(b)\ominus_{gH}\int_a^b\flat'(\mathfrak{h})\odot\aleph(\mathfrak{h})d\mathfrak{h}.$$

This completes the proof. The remaining cases can be proved in the similar manner. □

Theorem 3 *Let $\aleph(\mathfrak{h})$ and $\aleph'(\mathfrak{h})$ be $gH-$differentiable TPFVFs having no switching point on the closed interval $\mathbb{I}$ and $\flat(\mathfrak{h})$ be real-valued continuous differentiable function, such that $\flat(\mathfrak{h})>0$ and $\flat'(\mathfrak{h})<0$, $\forall\mathfrak{h}\in\mathbb{I}$. Then, we have the following results*:

1. If $\aleph(\mathfrak{h})$ and $\aleph'(\mathfrak{h})$ are $[(\imath)-gH]-$differentiable then $\flat(\mathfrak{h})\odot\aleph'(\mathfrak{h})$ is $[(\imath)-gH]-$differentiable and
$$\left(\flat(\mathfrak{h})\odot\aleph'(\mathfrak{h})\right)'_{(\imath).gH}$$
$$=\flat(\mathfrak{h})\odot\aleph''_{(\imath).gH}(\mathfrak{h})\ominus(-1)\flat'(\mathfrak{h})\odot\aleph'_{(\imath).gH}(\mathfrak{h}),$$
with the constraint that H-difference exists. Furthermore
$$\int_a^b\flat(\mathfrak{h})\odot\aleph''_{(\imath).gH}(\mathfrak{h})d\mathfrak{h}$$
$$=\flat(b)\odot\aleph'_{gH}(b)\ominus\flat(a)\odot\aleph'_{gH}(a)\oplus(-1)\int_a^b\flat'(\mathfrak{h})\odot\aleph'_{(\imath).gH}(\mathfrak{h})d\mathfrak{h}.$$
2. If $\aleph(\mathfrak{h})$ and $\aleph'(\mathfrak{h})$ are $[(u)-gH]-$differentiable then $\flat(\mathfrak{h})\odot\aleph'(\mathfrak{h})$ is $[(\imath)-gH]-$differentiable and
$$\left(\flat(\mathfrak{h})\odot\aleph'(\mathfrak{h})\right)'_{(\imath).gH}$$
$$=\flat(\mathfrak{h})\odot\aleph''_{(\imath).gH}(\mathfrak{h})\oplus\flat'(\mathfrak{h})\odot\aleph'_{(u).gH}(\mathfrak{h}).$$
Moreover
$$\int_a^b\flat(\mathfrak{h})\odot\aleph''_{(\imath).gH}(\mathfrak{h})d\mathfrak{h}$$
$$=\flat(b)\odot\aleph'_{gH}(b)\ominus\flat(a)\odot\aleph'_{gH}(a)\ominus_{gH}$$
$$\int_a^b\flat'(\mathfrak{h})\odot\aleph'_{(u).gH}(\mathfrak{h})d\mathfrak{h}.$$
3. If $\aleph(\mathfrak{h})$ is $[(\imath)-gH]-$differentiable and $\aleph'(\mathfrak{h})$ is $[(u)-gH]-$differentiable then $\flat(\mathfrak{h})\odot\aleph'(\mathfrak{h})$ is $[(u)-gH]-$differentiable and
$$\left(\flat(\mathfrak{h})\odot\aleph'(\mathfrak{h})\right)'_{(u).gH}$$
$$=\flat(\mathfrak{h})\odot\aleph''_{(u).gH}(\mathfrak{h})\oplus\flat'(\mathfrak{h})\odot\aleph'_{(\imath).gH}(\mathfrak{h})$$
and
$$\int_a^b\flat(\mathfrak{h})\odot\aleph''_{(u).gH}(\mathfrak{h})d\mathfrak{h}$$
$$=(-1)\flat(a)\odot\aleph'_{gH}(a)\ominus(-1)\flat(b)\odot\aleph'_{gH}(b)\ominus_{gH}$$
$$\int_a^b\flat'(\mathfrak{h})\odot\aleph'_{(\imath).gH}(\mathfrak{h})d\mathfrak{h}.$$
4. If $\aleph(\mathfrak{h})$ is $[(u)-gH]-$differentiable and $\aleph'(\mathfrak{h})$ is $[(\imath)-gH]-$differentiable then $\flat(\mathfrak{h})\odot\aleph'(\mathfrak{h})$ is $[(u)-gH]-$differentiable and
$$\left(\flat(\mathfrak{h})\odot\aleph'(\mathfrak{h})\right)'_{(u).gH}$$
$$=\flat(\mathfrak{h})\odot\aleph''_{(u).gH}(\mathfrak{h})\ominus(-1)\flat'(\mathfrak{h})\odot\aleph'_{(u).gH}(\mathfrak{h}).$$
Provided that H-difference exists. Thus, we have:
$$\int_a^b\flat(\mathfrak{h})\odot\aleph''_{(u).gH}(\mathfrak{h})d\mathfrak{h}$$
$$=(-1)\flat(a)\odot\aleph'_{gH}(a)\ominus(-1)\flat(b)\odot\aleph'_{gH}(b)\oplus(-1)$$
$$\int_a^b\flat'(\mathfrak{h})\odot\aleph'_{(u).gH}(\mathfrak{h})d\mathfrak{h}.$$

Proof We only give proofs of **1.** and **3.**

Suppose that $\aleph(\mathfrak{h})$ and $\aleph'(\mathfrak{h})$ are $[(\imath)-gH]-$differentiable. Given that $\flat(\mathfrak{h})>0$ and $\flat'(\mathfrak{h})<0$. Then

$$\flat(\mathfrak{h})\odot\aleph''_{(\imath).gH}(\mathfrak{h})\ominus(-1)\flat'(\mathfrak{h})\odot\aleph'_{(\imath).gH}(\mathfrak{h})$$
$$=\flat(\mathfrak{h})\odot\left(\aleph''_1(\mathfrak{h}),\aleph''_2(\mathfrak{h}),\aleph''_3(\mathfrak{h});\tilde{\aleph}''_1(\mathfrak{h}),\aleph''_2(\mathfrak{h}),\tilde{\aleph}''_3(\mathfrak{h})\right)$$
$$\ominus(-1)\flat'(\mathfrak{h})\odot\left(\aleph'_1(\mathfrak{h}),\aleph'_2(\mathfrak{h}),\aleph'_3(\mathfrak{h});\tilde{\aleph}'_1(\mathfrak{h}),\aleph'_2(\mathfrak{h}),\tilde{\aleph}'_3(\mathfrak{h})\right),$$
$$=\left(\flat(\mathfrak{h})\aleph''_1(\mathfrak{h}),\flat(\mathfrak{h})\aleph''_2(\mathfrak{h}),\flat(\mathfrak{h})\aleph''_3(\mathfrak{h});\flat(\mathfrak{h})\right.$$
$$\left.\tilde{\aleph}''_1(\mathfrak{h}),\flat(\mathfrak{h})\aleph''_2(\mathfrak{h}),\flat(\mathfrak{h})\tilde{\aleph}''_3(\mathfrak{h})\right)$$
$$\ominus(-1)\left(\flat'(\mathfrak{h})\aleph'_3(\mathfrak{h}),\flat'(\mathfrak{h})\aleph'_2(\mathfrak{h}),\flat'(\mathfrak{h})\aleph'_1(\mathfrak{h});\flat'(\mathfrak{h})\tilde{\aleph}'_3(\mathfrak{h}),\flat'(\mathfrak{h})\aleph'_2(\mathfrak{h}),\flat'(\mathfrak{h})\tilde{\aleph}'_1(\mathfrak{h})\right),$$
$$=\left(\flat(\mathfrak{h})\aleph''_1(\mathfrak{h}),\flat(\mathfrak{h})\aleph''_2(\mathfrak{h}),\flat(\mathfrak{h})\aleph''_3(\mathfrak{h});\flat(\mathfrak{h})\tilde{\aleph}''_1(\mathfrak{h}),\flat(\mathfrak{h})\aleph''_2(\mathfrak{h}),\flat(\mathfrak{h})\tilde{\aleph}''_3(\mathfrak{h})\right)\ominus$$
$$\left(-\flat'(\mathfrak{h})\aleph'_1(\mathfrak{h}),-\flat'(\mathfrak{h})\aleph'_2(\mathfrak{h}),-\flat'(\mathfrak{h})\aleph'_3(\mathfrak{h});-\flat'(\mathfrak{h})\tilde{\aleph}'_1(\mathfrak{h}),-\flat'(\mathfrak{h})\aleph'_2(\mathfrak{h}),-\flat'(\mathfrak{h})\tilde{\aleph}'_3(\mathfrak{h})\right),$$
$$\flat(\mathfrak{h})\odot\aleph''_{(\imath).gH}(\mathfrak{h})\ominus(-1)\flat'(\mathfrak{h})\odot\aleph'_{(\imath).gH}(\mathfrak{h})=\left(\flat(\mathfrak{h})\odot\aleph'(\mathfrak{h})\right)'_{(\imath).gH}.$$

The equation evaluated in this case is given as follows:

$$\left(\flat(\mathfrak{h})\odot\aleph'(\mathfrak{h})\right)'_{(\imath).gH}=\flat(\mathfrak{h})\odot\aleph''_{(\imath).gH}(\mathfrak{h})\ominus(-1)\flat'(\mathfrak{h})\odot\aleph'_{(\imath).gH}(\mathfrak{h}). \quad (8)$$

Now, integrating Equ. (8) from a to b, we get:

$$\int_a^b\left(\flat(\mathfrak{h})\odot\aleph'(\mathfrak{h})\right)'_{(\imath).gH}d\mathfrak{h}$$
$$=\int_a^b\flat(\mathfrak{h})\odot\aleph''_{(\imath).gH}(\mathfrak{h})d\mathfrak{h}\ominus(-1)\int_a^b\flat'(\mathfrak{h})\odot\aleph'_{(\imath).gH}(\mathfrak{h})d\mathfrak{h}.$$

Theorem 1 implies that:

$$\int_a^b\flat(\mathfrak{h})\odot\aleph''_{(\imath).gH}(\mathfrak{h})d\mathfrak{h}$$
$$=\flat(b)\odot\aleph'_{gH}(b)\ominus\flat(a)\odot\aleph'_{gH}(a)\oplus(-1)\int_a^b\flat'(\mathfrak{h})\odot\aleph'_{(\imath).gH}(\mathfrak{h})d\mathfrak{h}.$$

Now assume that $\aleph(\mathfrak{h})$ is $[(\imath)-gH]-$differentiable and $\aleph'(\mathfrak{h})$ is $[(u)-gH]-$differentiable. Given that $\flat(\mathfrak{h})>0$ and $\flat'(\mathfrak{h})<0$. Then

$$\flat(\mathfrak{h})\odot\aleph''_{(u).gH}(\mathfrak{h})\oplus\flat'(\mathfrak{h})\odot\aleph'_{(\imath).gH}(\mathfrak{h})$$
$$=\flat(\mathfrak{h})\odot\left(\aleph''_3(\mathfrak{h}),\aleph''_2(\mathfrak{h}),\aleph''_1(\mathfrak{h});\tilde{\aleph}''_3(\mathfrak{h}),\aleph''_2(\mathfrak{h}),\tilde{\aleph}''_1(\mathfrak{h})\right)$$
$$\oplus\flat'(\mathfrak{h})\odot\left(\aleph'_1(\mathfrak{h}),\aleph'_2(\mathfrak{h}),\aleph'_3(\mathfrak{h});\tilde{\aleph}'_1(\mathfrak{h}),\aleph'_2(\mathfrak{h}),\tilde{\aleph}'_3(\mathfrak{h})\right),$$
$$=\left(\flat(\mathfrak{h})\aleph''_3(\mathfrak{h}),\flat(\mathfrak{h})\aleph''_2(\mathfrak{h}),\flat(\mathfrak{h})\aleph''_1(\mathfrak{h});\flat(\mathfrak{h})\tilde{\aleph}''_3(\mathfrak{h}),\flat(\mathfrak{h})\aleph''_2(\mathfrak{h}),\flat(\mathfrak{h})\tilde{\aleph}''_1(\mathfrak{h})\right)$$
$$\oplus\left(\flat'(\mathfrak{h})\aleph'_3(\mathfrak{h}),\flat'(\mathfrak{h})\aleph'_2(\mathfrak{h}),\flat'(\mathfrak{h})\aleph'_1(\mathfrak{h});\flat'(\mathfrak{h})\tilde{\aleph}'_3(\mathfrak{h}),\flat'(\mathfrak{h})\aleph'_2(\mathfrak{h}),\flat'(\mathfrak{h})\tilde{\aleph}'_1(\mathfrak{h})\right),$$
$$\flat(\mathfrak{h})\odot\aleph''_{(u).gH}(\mathfrak{h})\oplus\flat'(\mathfrak{h})\odot\aleph'_{(\imath).gH}(\mathfrak{h})=\left(\flat(\mathfrak{h})\odot\aleph'(\mathfrak{h})\right)'_{(u).gH}.$$

The final result obtained in this case is given by:

$$\left(\flat(\mathfrak{h})\odot\aleph'(\mathfrak{h})\right)'_{(u).gH}=\flat(\mathfrak{h})\odot\aleph''_{(u).gH}(\mathfrak{h})\oplus\flat'(\mathfrak{h})\odot\aleph'_{(\imath).gH}(\mathfrak{h}). \quad (9)$$

Taking integral on both sides of Equ. (9) gives:

$$\int_a^b\left(\flat(\mathfrak{h})\odot\aleph'(\mathfrak{h})\right)'_{(u).gH}d\mathfrak{h}$$
$$=\int_a^b\flat(\mathfrak{h})\odot\aleph''_{(u).gH}(\mathfrak{h})d\mathfrak{h}\oplus\int_a^b\flat'(\mathfrak{h})\odot\aleph'_{(\imath).gH}(\mathfrak{h})d\mathfrak{h}.$$

From Theorem 1, it follows that:

$$\int_a^b\flat(\mathfrak{h})\odot\aleph''_{(u).gH}(\mathfrak{h})d\mathfrak{h}$$
$$=(-1)\flat(a)\odot\aleph'_{gH}(a)\ominus$$
$$(-1)\flat(b)\odot\aleph'_{gH}(b)\ominus_{gH}\int_a^b\flat'(\mathfrak{h})\odot\aleph'_{(\imath).gH}(\mathfrak{h})d\mathfrak{h}.$$

This completes the proof. Similarly, we can prove other cases. □

***Example* 1** Consider a TPFVF $\aleph(\mathfrak{h})=(5.6\mathfrak{h}^4,6\mathfrak{h}^4,8.5\mathfrak{h}^4;4.2\mathfrak{h}^4,6\mathfrak{h}^4,9.7\mathfrak{h}^4)$ and $\flat(\mathfrak{h})=4\mathfrak{h}^4$ for $\mathfrak{h}>1$. Since $\aleph(\mathfrak{h})$ is $[(\imath)-gH]$-differentiable and $\flat(\mathfrak{h}),\flat'(\mathfrak{h})>0$. Therefore, from Theorem 2 Case 1(i), we have the following expressions:

$$\flat(\mathfrak{h})\odot\aleph(\mathfrak{h})=(22.4\mathfrak{h}^8,24\mathfrak{h}^8,34\mathfrak{h}^8:16.8\mathfrak{h}^8,24\mathfrak{h}^8,38.8\mathfrak{h}^8)$$

and

$$(\flat(\mathfrak{h})\odot\aleph(\mathfrak{h}))'_{gH}=(179.2\mathfrak{h}^7,192\mathfrak{h}^7,272\mathfrak{h}^7;$$
$$134.4\mathfrak{h}^7,192\mathfrak{h}^7,310.4\mathfrak{h}^7). \quad (10)$$

Thus, Eq. (10) indicates that $\flat(\mathfrak{h})\odot\aleph(\mathfrak{h})$ is $[(\imath)-gH]$-differentiable TPFVF.

Now consider $\flat(\mathfrak{h})=\mathfrak{h}^{-2}$ and $\aleph(\mathfrak{h})=(5.6\mathfrak{h}^4,6\mathfrak{h}^4,8.5\mathfrak{h}^4;4.2\mathfrak{h}^4,6\mathfrak{h}^4,9.7\mathfrak{h}^4)$.

Since $\aleph(\mathfrak{h})$ and $\aleph'(\mathfrak{h})$ are $[(\imath)-gH]$-differentiable and $\flat(\mathfrak{h})>0$, $\flat'(\mathfrak{h})<0$. Therefore, Theorem 3 implies that:

$$(\flat(\mathfrak{h})\odot\aleph'(\mathfrak{h}))'_{gH}$$
$$=(112\mathfrak{h}^4,120\mathfrak{h}^4,170\mathfrak{h}^4;84\mathfrak{h}^4,120\mathfrak{h}^4,194\mathfrak{h}^4). \quad (11)$$

From Eq. (11), it is clear that $\flat(\mathfrak{h})\odot\aleph'(\mathfrak{h}$ is $[(\imath)-gH]$-differentiable.

4 Pythagorean fuzzy integral transforms

Integral transform is a powerful technique to solve different kinds of differential equations. With the help of this transformation technique, we can convert a differential equation into an algebraic equation or in the system of equations. In this section, we discuss two major integral transforms in the Pythagorean fuzzy context, namely, the PFLT and the PFFT.

4.1 The Pythagorean fuzzy Laplace transform

Definition 12 (Akram et al. (2022a)) Let $\aleph(\mathfrak{h})$ be a TPFVF of order $\beta > 0$ on $0 \le \mathfrak{h} < \infty$. Then, the PFLT of TPFVF $\aleph(\mathfrak{h})$ is expressed as follows:

$$V(q) = \mathscr{L}[\aleph(\mathfrak{h})] = \int_0^\infty e^{-q\mathfrak{h}} \odot \aleph(\mathfrak{h}) d\mathfrak{h}$$
$$= \lim_{T\to\infty} \int_0^T e^{-q\mathfrak{h}} \odot \aleph(\mathfrak{h}) d\mathfrak{h}, \quad q > 0.$$

If the limit exists.

The inverse PFLT of $V(q)$ is given by the following expression:

$$\mathscr{L}^{-1}[V(q)] = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} e^{q\mathfrak{h}} \odot V(q) dq,$$

with fixed $\delta \in \mathbb{R}$.

Definition 13 Let $\aleph(\mathfrak{h})$ be a triangular Pythagorean fuzzy piecewise continuous function and $\flat(\mathfrak{h})$ be a real-valued piecewise continuous function with $\mathfrak{h} > 0$. The convolution of $\aleph(\mathfrak{h})$ and $\flat(\mathfrak{h})$ is given by the subsequent expression:

$$(\aleph * \flat)(\mathfrak{h}) = \int_0^{\mathfrak{h}} \aleph(\tau) \flat(\mathfrak{h} - \tau) d\tau.$$

By substituting $\varepsilon = \mathfrak{h} - \tau$, we get:

$$(\aleph * \flat)(\mathfrak{h}) = \int_0^{\mathfrak{h}} \flat(\varepsilon) \aleph(\mathfrak{h} - \varepsilon) d\varepsilon = (\flat * \aleph)(\mathfrak{h}).$$

The aforesaid expression shows that convolution is commutative.

Theorem 4 (*Convolution Theorem*) *Suppose that $\aleph(\mathfrak{h})$ is a triangular Pythagorean fuzzy piecewise continuous function on $[0,\infty)$ of exponential order β and $\flat(\mathfrak{h})$ is a real-valued piecewise continuous function on $[0,\infty)$. Then, the PFLT of the convolution of $\aleph(\mathfrak{h})$ and $\flat(\mathfrak{h})$ is given by*:

$$\mathscr{L}[(\aleph * \flat)(\mathfrak{h})] = \mathscr{L}[\aleph(\mathfrak{h})] \odot \mathscr{L}[\flat(\mathfrak{h})], \qquad (Re(q) > \beta).$$

Theorem 5 *Assume that $\aleph(\mathfrak{h})$ is a triangular Pythagorean fuzzy piecewise continuous function on $[0,\infty)$ of exponential order β and $\aleph'_{gH}(\mathfrak{h})$ is piecewise continuous function in every finite closed interval $\mathbb{I} = [a,b]$. Furthermore, suppose that $\aleph(\mathfrak{h})$ and $\aleph'_{gH}(\mathfrak{h})$ are $gH-$differentiable, such that in the closed interval $\mathbb{I}$, the type of $gH-$differentiability does not alter. If $Re(q) > \beta$, then the PFLT of $\aleph'(\mathfrak{h})$ and $\aleph''_{gH}(\mathfrak{h})$ according to the $gH-$differentiability type is presented in Table 1.*

Theorem 6 *Suppose that the PFLT of TPFVFs $\aleph(\mathfrak{h})$, $\aleph'_{gH}(\mathfrak{h})$ exist. Then, the PFLT of the ${}^{CF}[gH]-$derivative of these functions of order $0 < \varrho \le 2$ according to the types of ${}^{CF}[gH]-$differentiability are as under*:

1. If $\aleph(\mathfrak{h})$ and ${}^{CF}_{gH}\mathscr{D}^{\varrho}\aleph(\mathfrak{h})$ are $[(\imath) - gH]-$differentiable TPFVFs, then
$$\mathscr{L}[{}^{CF}_{(\imath).gH}\mathscr{D}^{\varrho}_{0}\aleph(\mathfrak{h})] = q^{\varrho}V(q) \ominus q^{\varrho-1}\aleph(0) \ominus q^{\varrho-2}\aleph'_{gH}(0).$$
2. If $\aleph(\mathfrak{h})$ and ${}^{CF}_{gH}\mathscr{D}^{\varrho}\aleph(\mathfrak{h})$ are $[(\imath\imath) - gH]-$differentiable TPFVFs, then
$$\mathscr{L}[{}^{CF}_{(\imath\imath).gH}\mathscr{D}^{\varrho}_{0}\aleph(\mathfrak{h})] = q^{\varrho}V(q) \ominus_{gH} q^{\varrho-1}\aleph(0) \ominus q^{\varrho-2}\aleph'_{gH}(0).$$
3. If $\aleph(\mathfrak{h})$ is $[(\imath) - gH]-$differentiable TPFVF and ${}^{CF}_{gH}\mathscr{D}^{\varrho}\aleph(\mathfrak{h})$ is $[(\imath\imath) - gH]-$differentiable TPFVF, then
$$\mathscr{L}[{}^{CF}_{(\imath\imath).gH}\mathscr{D}^{\varrho}_{0}\aleph(\mathfrak{h})] = (-1)q^{\varrho-2}\aleph'_{gH}(0) \ominus_{gH} (-1)q^{\varrho}V(q) \oplus (-1)q^{\varrho-1}\aleph(0).$$
4. If $\aleph(\mathfrak{h})$ is $[(\imath\imath) - gH]-$differentiable TPFVF and ${}^{CF}_{gH}\mathscr{D}^{\varrho}\aleph(\mathfrak{h})$ is $[(\imath) - gH]-$differentiable TPFVF, then
$$\mathscr{L}[{}^{CF}_{(\imath\imath).gH}\mathscr{D}^{\varrho}_{0}\aleph(\mathfrak{h})] = (-1)q^{\varrho-2}\aleph'_{gH}(0) \ominus_{gH} (-1)q^{\varrho}V(q) \oplus (-1)q^{\varrho-1}\aleph(0).$$

Proof We only prove **3.**

Table 1 PFLT of $\aleph'_{gH}(\mathfrak{h})$ and $\aleph''_{gH}(\mathfrak{h})$

Case	$gH-$differentiability type		$\mathscr{L}[\aleph'_{gH}(\mathfrak{h})]$	$\mathscr{L}[\aleph''_{gH}(\mathfrak{h})]$
	$\aleph(\mathfrak{h})$	$\aleph'_{gH}(\mathfrak{h})$		
1	$(\imath)$	$(\imath)$	$qV(q) \ominus \aleph(0)$	$q^2V(q) \ominus q\aleph(0) \ominus \aleph'_{gH}(0)$
2	$(\imath)$	$(\imath\imath)$	$qV(q) \ominus \aleph(0)$	$(-1)\aleph'_{gH}(0) \ominus_{gH} (-1)q^2V(q) \oplus (-1)q\aleph(0)$
3	$(\imath\imath)$	$(\imath)$	$(-1)\aleph(0) \ominus_{gH} (-1)qV(q)$	$(-1)\aleph'_{gH}(0) \ominus_{gH} (-1)q^2V(q) \oplus (-1)q\aleph(0)$
4	$(\imath\imath)$	$(\imath\imath)$	$(-1)\aleph(0) \ominus_{gH} (-1)qV(q)$	$q^2V(q) \ominus q\aleph(0) \ominus \aleph'_{gH}(0)$

Assume that $\aleph(\mathfrak{h})$ is $[(\imath)-gH]-$differentiable TPFVF and $\aleph'(\mathfrak{h})$ is $[(\imath\imath)-gH]-$differentiable TPFVF. Then, using Definition 8 of ${}^{CF}[gH]-$derivative it gives:

$$ {}^{C}_{(\imath\imath).gH}\mathscr{D}^{\varrho}_{0}\aleph(\mathfrak{h}) = \frac{1}{\Gamma(2-\varrho)}\int_0^{\mathfrak{h}} (\mathfrak{h}-s)^{1-\varrho}\aleph''_{(\imath\imath.gH)}(s)ds. $$

Definition 13 implies that:

$$ {}^{C}_{(\imath\imath).gH}\mathscr{D}^{\varrho}_{0}\aleph(\mathfrak{h}) = \frac{1}{\Gamma(2-\varrho)}\odot\left(\mathfrak{h}^{1-\varrho} * \aleph''_{(\imath\imath).gH}(\mathfrak{h})\right). \tag{12} $$

Taking PFLT on both sides of Eq. (12) and using Theorem 4, we have:

$$ \mathscr{L}\left[{}^{C}_{(\imath\imath).gH}\mathscr{D}^{\varrho}_{0}\aleph(\mathfrak{h})\right] = \frac{1}{\Gamma(2-\varrho)}\odot\left(\mathscr{L}\left[\mathfrak{h}^{1-\varrho}\right]\odot\mathscr{L}\left[\aleph''_{(\imath\imath).gH}(\mathfrak{h})\right]\right). $$

Since $\mathscr{L}\left[\mathfrak{h}^{1-\varrho}\right] = \frac{\Gamma(2-\varrho)}{q^{2-\varrho}}$. Therefore, using Case 3 of Theorem 5, we have:

$$ \begin{aligned} &\mathscr{L}\left[{}^{C}_{(\imath\imath).gH}\mathscr{D}^{\varrho}_{0}\aleph(\mathfrak{h})\right] \\ &\quad = q^{\varrho-2}\left((-1)\aleph'_{gH}(0)\ominus_{gH}(-1)q^2V(q)\oplus(-1)q\aleph(0)\right), \\ &\mathscr{L}\left[{}^{CF}_{(\imath\imath).gH}\mathscr{D}^{\varrho}_{0}\aleph(\mathfrak{h})\right] \\ &\quad = (-1)q^{\varrho-2}\aleph'_{gH}(0)\ominus_{gH}(-1)q^{\varrho}V(q)\oplus(-1)q^{\varrho-1}\aleph(0). \end{aligned} $$

Hence, we obtained the desired result. Similarly, we can prove other cases. □

4.2 The Pythagorean fuzzy Fourier transform

Definition 14 Let $\aleph:\mathbb{I}\to\mathbb{R}_T$ be a TPFVF. The PFFT of $\aleph(\mathfrak{h})$ is defined by the following expression:

$$ V(\omega) = \mathfrak{F}[\aleph(\mathfrak{h})] = \int_{-\infty}^{\infty} e^{-i\omega\mathfrak{h}}\odot\aleph(\mathfrak{h})d\mathfrak{h}. $$

The inverse PFFT of $V(\omega)$ is expressed by the following expression:

$$ \mathfrak{F}^{-1}[V(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\mathfrak{h}}\odot V(\omega)d\omega = \aleph(\mathfrak{h}). $$

The existence of the fuzzy Fourier transform for the FVF is discussed in Gouyandeha et al. (2017). In this manuscript, we obtained a different result for integration by parts of the TPFVF $\aleph(\mathfrak{h})$ and real-valued continuous function $\flat(\mathfrak{h})$. Thus, we have to prove some important results for the PFFT.

Theorem 7 *Let $\aleph(\mathfrak{h})$ be a Pythagorean fuzzy continuous and absolutely integrable function, such that $\lim_{|\mathfrak{h}|\to\infty}\aleph^{(k)}(\mathfrak{h}) = 0$ for $k=0,1$. In addition, assume that $\aleph^{(j)}_{gH}(\mathfrak{h})$ is Pythagorean fuzzy absolutely integrable on $(-\infty,+\infty)$, for $0\le j\le 2$. Moreover, if $\aleph(\mathfrak{h})$ and $\aleph'_{gH}(\mathfrak{h})$ are $gH-$differentiable with the constraint that the type of $gH-$differentiability remains unalter in $(-\infty,+\infty)$. Then*

(i) If $\aleph(\mathfrak{h})$ is $gH-$differentiable then $\mathfrak{F}[\aleph'_{gH}(\mathfrak{h})] = i\omega\mathfrak{F}[\aleph(\mathfrak{h})]$.

(ii) If $\aleph(\mathfrak{h})$ and $\aleph'_{gH}(\mathfrak{h})$ are $gH-$differentiable then $\mathfrak{F}[\aleph''_{gH}(\mathfrak{h})] = \ominus_{gH}\omega^2\mathfrak{F}[\aleph(\mathfrak{h})]$.

Proof Suppose that $\aleph(\mathfrak{h})$ is $gH-$differentiable.

(i) If $\aleph(\mathfrak{h})$ is $[(\imath)-gH]-$differentiable, then Definition 14 implies that:

$$ \mathfrak{F}[\aleph'_{gH}(\mathfrak{h}) = \int_{-\infty}^{\infty} e^{-i\omega\mathfrak{h}}\odot\aleph'_{(\imath).gH}(\mathfrak{h})d\mathfrak{h}. \tag{13} $$

Using Case 1(ii) of Theorem 2, we have:

$$ \begin{aligned} \mathfrak{F}[\aleph'_{gH}(\mathfrak{h})] &= \lim_{T\to\infty}\int_{-T}^{T} e^{-i\omega\mathfrak{h}}\odot\aleph'_{(\imath).gH}(\mathfrak{h})d\mathfrak{h} \\ &= \lim_{T\to\infty}\left(e^{-i\omega T}\odot\aleph(T)\ominus e^{i\omega T}\odot\aleph(-T)\oplus(i\omega)\right. \\ &\qquad\left.\int_{-T}^{T} e^{-i\omega\mathfrak{h}}\odot\aleph(\mathfrak{h})d\mathfrak{h}\right). \end{aligned} $$

By the supposition $\lim_{|T|\to\infty}\aleph(T) = 0$, we have:

$$ \mathfrak{F}[\aleph'_{(\imath).gH}(\mathfrak{h}) = i\omega\mathfrak{F}[\aleph(\mathfrak{h})]. $$

Similarly, we can prove that if $\aleph(\mathfrak{h})$ is $[(\imath\imath)-gH]-$differentiable. Then, using Remark 1(3), it follows that:

$$ \mathfrak{F}[\aleph'_{(\imath\imath).gH}(\mathfrak{h}) = i\omega\mathfrak{F}[\aleph(\mathfrak{h})]. $$

(ii) Suppose that $\aleph(\mathfrak{h})$ and $\aleph'(\mathfrak{h})$ are $[(\imath)-gH]-$differentiable. Then, Definition 14 yields that:

$$ \mathfrak{F}[\aleph''_{(\imath).gH}(\mathfrak{h}) = \int_{-\infty}^{\infty} e^{-i\omega\mathfrak{h}}\odot\aleph''_{(\imath).gH}(\mathfrak{h})d\mathfrak{h}. \tag{14} $$

Equation (14) can be solved using Theorem 3

$$ \begin{aligned} \mathfrak{F}[\aleph''_{gH}(\mathfrak{h})] &= \lim_{T\to\infty}\int_{-T}^{T} e^{-i\omega\mathfrak{h}}\odot\aleph''_{(\imath).gH}(\mathfrak{h})d\mathfrak{h} \\ &= \lim_{T\to\infty}\left(e^{-i\omega T}\odot\aleph'_{gH}(T)\ominus e^{i\omega T}\odot\aleph'_{gH}(-T)\oplus(i\omega)\right. \\ &\qquad\left.\int_{-T}^{T} e^{-i\omega\mathfrak{h}}\odot\aleph'_{(\imath).gh}(\mathfrak{h})d\mathfrak{h}\right). \end{aligned} $$

But by the supposition $\lim_{|T|\to\infty}\aleph'(T) = 0$, we have:

$$ \mathfrak{F}[\aleph''_{(\imath).gH}(\mathfrak{h}) = (-1)\omega^2\mathfrak{F}[\aleph(\mathfrak{h})]. $$

From Remark 1, we have $(-1)\omega^2\mathfrak{F}[\aleph(\mathfrak{h})] = \ominus_{gH}\omega^2\mathfrak{F}[\aleph(\mathfrak{h})]$. Thus, we conclude that:

$$\mathfrak{F}[\aleph''_{(\imath).gH}(\mathfrak{h}) = \ominus_{gH}\omega^2\mathfrak{F}[\aleph(\mathfrak{h})].$$

This completes the proof. □

Lemma 1 *Suppose that k_0 is a TPFN and $\delta(\mathfrak{u})$ is a real Dirac delta function, then*

$$\mathfrak{F}[k_0 \odot \delta(\mathfrak{u})] = k_0.$$

Proof From Definition 14 of PFFT and Rehman (2011) it follows that:

$$\mathfrak{F}[k_0 \odot \delta(\mathfrak{u})] = k_0 \odot \int_{-\infty}^{\infty} e^{-i\omega\mathfrak{h}} \odot \delta(\mathfrak{u})d\mathfrak{u}$$
$$= k_0 \odot \mathfrak{F}[\delta(\mathfrak{u})] = k_0.$$

□

Lemma 2 *Let $k = (k_1, k_2, k_3; \tilde{k_1}, k_2, \tilde{k_3})$ be a TPFN. Then, the PFFT of k is given by $\mathfrak{F}[k] = k \odot (2\pi)\delta(\omega)$.*

Proof Using results in Rehman (2011), we have $\mathfrak{F}[1] = (2\pi)\delta(\omega)$.

$$\mathfrak{F}[(k_1, k_2, k_3; \tilde{k_1}, k_2, \tilde{k_3})]$$
$$= \Big(k_1(2\pi)\delta(\omega), k_2(2\pi)\delta(\omega), k_3(2\pi)\delta(\omega);$$
$$\tilde{k_1}(2\pi)\delta(\omega), k_2(2\pi)\delta(\omega), \tilde{k_3}(2\pi)\delta(\omega)\Big),$$
$$= (k_1, k_2, k_3; \tilde{k_1}, k_2, \tilde{k_3}) \odot (2\pi)\delta(\omega),$$
$$\mathfrak{F}[k] = k \odot (2\pi)\delta(\omega).$$

□

5 Pythagorean fuzzy solution of Pythagorean fuzzy partial fractional differential equation

In this section, we develop an analytical method to extract the Pythagorean fuzzy solution of PFPFDE using the PFLT and the PFFT. The understudy PFPFDE contains TPFVF in two variables. For this, we use the generalized Hukuhara partial derivatives with respect to both variables.

Consider the PFPFDE

$$\frac{\partial^{\varrho}\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{h}^{\varrho}} = p\frac{\partial^2\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^2}, \quad \mathfrak{h} > 0, \quad \mathfrak{u} \in (-\infty,\infty), \tag{15}$$

where $0 < \varrho \le 2$ and p is a real constant. Here, $\frac{\partial^{\varrho}\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{h}^{\varrho}}$ is supposed to be the Caputo generalized Hukuhara partial differentiable of order ϱ with respect to $\mathfrak{h}$ and $\frac{\partial\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}}$ is considered as Caputo generalized Hukuhara partial differentiable with respect to $\mathfrak{u}$.

The PFLT and the PFFT of TPFVF $\Lambda(\mathfrak{u},\mathfrak{h})$ with respect to $\mathfrak{h}$ and $\mathfrak{u}$, respectively, are used to extract the Pythagorean fuzzy solution of Equ. (15).

Applying PFLT with respect to $\mathfrak{h}$ to $\Lambda(\mathfrak{u},\mathfrak{h})$ provides:

$$\mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})] = \int_0^{\infty} e^{-q\mathfrak{h}} \odot \Lambda(\mathfrak{u},\mathfrak{h})d\mathfrak{h}. \tag{16}$$

The PFFT with respect to $\mathfrak{u}$ of $\Lambda(\mathfrak{u},\mathfrak{h})$, for any fixed $\mathfrak{u} \in \mathbb{R}$ is given by:

$$\mathfrak{F}_{\mathfrak{u}}[\Lambda(\mathfrak{u},\mathfrak{h})] = \int_{-\infty}^{\infty} e^{-i\omega\mathfrak{u}} \odot \Lambda(\mathfrak{u},\mathfrak{h})d\mathfrak{u}, \quad \text{for} \quad \mathfrak{h} > 0. \tag{17}$$

Similarly, we can define the inverse PFLT and the inverse PFFT using the definitions defined in Subsects. 4.1 and 4.2, respectively.

Let $\mathscr{L}\mathfrak{F}$ be the space of all TPFVFs $\Lambda(\mathfrak{u},\mathfrak{h})$, such that both the PFLT and the PFFT exist for them. Then, we develop the following notation:

$$\tilde{\Lambda}(\omega,q) := \mathfrak{F}_{\mathfrak{u}}\mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})]$$
$$= \int_{-\infty}^{\infty}\int_0^{\infty} e^{-q\mathfrak{h}}e^{-i\omega\mathfrak{u}} \odot \Lambda(\mathfrak{u},\mathfrak{h})d\mathfrak{h}d\mathfrak{u}, \quad \mathfrak{h} > 0. \tag{18}$$

Taking the inverse PFLT and the inverse PFFT, respectively, to Eq. (18) gives the solution of Equ. (15).

Theorem 8 *Consider the PFPFDE*

$$\frac{\partial^{\varrho}\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{h}^{\varrho}} = p\frac{\partial^2\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^2}, \quad \mathfrak{h} > 0, \quad \mathfrak{u} \in (-\infty,\infty), \tag{19}$$

with the Pythagorean fuzzy initial conditions:

$$\begin{cases} \Lambda(\mathfrak{u},0) = \mathfrak{g}(\mathfrak{u}), & 0 < \varrho \le 2, \\ \Lambda_{\mathfrak{h}}(\mathfrak{u},0) = 0, & 1 < \varrho \le 2, \end{cases} \tag{20}$$

where $\mathfrak{g}(\mathfrak{u})$ is a TPFVF. Assume that the PFFT for $\mathfrak{g}(\mathfrak{u})$ is $G(\omega)$, if exists. Then, the solution $\Lambda(\mathfrak{u},\mathfrak{h}) \in \mathscr{L}\mathfrak{F}$ is given by the following expression

$$\Lambda(\mathfrak{u},\mathfrak{h}) = \frac{1}{2\pi} \odot \int_{-\infty}^{\infty} G(\omega) \odot E_{\varrho}(-p\ \omega^2\mathfrak{h}^{\varrho})e^{-i\omega\mathfrak{u}}d\omega, \tag{21}$$

such that the integral on the right-hand side of Eq. (21) exists.

Proof Suppose that $\Lambda(\mathfrak{u},\mathfrak{h})$ and $\frac{\partial\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{h}}$ are $[(\imath) - gH]$-differentiable with respect to $\mathfrak{h}$. Applying the PFLT with respect to $\mathfrak{h}$ to Eq. (19) yields:

$$\mathscr{L}_{\mathfrak{h}}\left[\frac{\partial^{\varrho}\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{h}^{\varrho}}\right]=p\mathscr{L}_{\mathfrak{h}}\left[\frac{\partial^{2}\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^{2}}\right]. \tag{22}$$

Now, using Theorem 6 and Pythagorean fuzzy initial conditions (20), we get:

$$q^{\varrho}\mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})]\ominus q^{\varrho-1}\mathfrak{g}(\mathfrak{u})=p\frac{\partial^{2}}{\partial\mathfrak{u}^{2}}\mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})]. \tag{23}$$

We first apply the PFFT with respect to $\mathfrak{u}$ to Eq. (23) and then Theorem 7 implies that:

$$q^{\varrho}\tilde{\Lambda}(\omega,q)\ominus q^{\varrho-1}G(\omega)=\ominus_{gH}p\omega^{2}\tilde{\Lambda}(\omega,q)\Rightarrow$$
$$q^{\varrho}\tilde{\Lambda}(\omega,q)\oplus p\omega^{2}\tilde{\Lambda}(\omega,q)=q^{\varrho-1}G(\omega)$$

and

$$\tilde{\Lambda}(\omega,q)=G(\omega)\odot\frac{q^{\varrho-1}}{q^{\varrho}+p\omega^{2}}. \tag{24}$$

The inverse PFLT applied to Eq. (24) results:

$$\mathfrak{F}_{\mathfrak{u}}[\Lambda(\mathfrak{u},\mathfrak{h})]=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{e^{q\mathfrak{h}}G(\omega)\odot q^{\varrho-1}}{q^{\varrho}+p\omega^{2}}dq. \tag{25}$$

We know that the Laplace transform of the Mittag-Leffler function (Povstenko 2015), $E_{\varrho}(\mu\mathfrak{h}^{\varrho})$ is given by the following expression:

$$\mathscr{L}[E_{\varrho}(\mu\mathfrak{h}^{\varrho})]=\frac{q^{\varrho-1}}{q^{\varrho}-\mu},\quad(|\mu q^{\varrho-1}|<1).$$

Letting $\mu=-p\omega^{2}$ in Eq. (25), we conclude that:

$$\mathfrak{F}_{\mathfrak{u}}[\Lambda(\mathfrak{u},\mathfrak{h})]=G(\omega)\odot E_{\varrho}(-p\omega^{2}\mathfrak{h}^{\varrho}). \tag{26}$$

Taking the inverse PFFT to Eq. (26) gives the solution of given PFPFDE (19).

$$\Lambda(\mathfrak{u},\mathfrak{h})=\frac{1}{2\pi}\odot\int_{-\infty}^{\infty}G(\omega)\odot E_{\varrho}(-p\omega^{2}\mathfrak{h}^{\varrho})e^{-i\omega\mathfrak{u}}d\omega.$$

Similarly, the solution can be obtained for other cases of ${}^{CF}[gH]$-differentiability. □

Theorem 9 *Consider the Pythagorean fuzzy initial conditions for the PFPFDE (19):*

$$\begin{cases}\Lambda(\mathfrak{u},0)=k_{0}\delta(\mathfrak{u}), & 0<\varrho\leq 2,\\ \Lambda_{\mathfrak{h}}(\mathfrak{u},0)=0, & 1<\varrho\leq 2,\end{cases} \tag{27}$$

where k_0 is a TPFN and $\delta(\mathfrak{u})$ is a real Dirac delta function. Then, the solution $\Lambda(\mathfrak{u},\mathfrak{h})\in\mathfrak{F}\mathscr{L}$ is given by:

$$\Lambda(\mathfrak{u},\mathfrak{h})=\frac{k_{0}}{2\sqrt{p}\mathfrak{h}^{\frac{\varrho}{2}}}M\left(\frac{\varrho}{2};\frac{|\mathfrak{u}|}{\sqrt{p}\mathfrak{h}^{\frac{\varrho}{2}}}\right). \tag{28}$$

Proof Let $\Lambda(\mathfrak{u},\mathfrak{h})$ and $\frac{\partial\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{h}}$ be $[(\imath)-gH]$-differentiable with respect to $\mathfrak{h}$. Suppose $\mathfrak{g}(\mathfrak{u})=k_{0}\delta(\mathfrak{u})$ in Theorem 8 and using Lemma 1, we have:

$$\tilde{\Lambda}(\omega,q)=k_{0}\odot\frac{q^{\varrho-1}}{q^{\varrho}+p\omega^{2}}. \tag{29}$$

Taking the inverse PFLT to Eq. (29), it follows that:

$$\mathfrak{F}_{\mathfrak{u}}[\Lambda(\mathfrak{u},\mathfrak{h})]=\frac{k_{0}}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{e^{q\mathfrak{h}}\odot q^{\varrho-1}}{q^{\varrho}+p\omega^{2}}dq. \tag{30}$$

Taking under consideration the Laplace transform of the Mittag-Leffler function (Povstenko 2015), we get the following expression:

$$\mathfrak{F}_{\mathfrak{u}}[\Lambda(\mathfrak{u},\mathfrak{h})]=k_{0}\odot E_{\varrho}(-p\omega^{2}\mathfrak{h}^{\varrho}). \tag{31}$$

Taking inverse PFFT to Eq. (31) gives:

$$\Lambda(\mathfrak{u},\mathfrak{h})=\frac{k_{0}}{2\pi}\odot\int_{-\infty}^{\infty}E_{\varrho}(-p\omega^{2}\mathfrak{h}^{\varrho})\cos(\mathfrak{u}\omega)d\omega.$$

This is the solution of given PFPFDE.

Furthermore, if the sequence of the Pythagorean fuzzy integral transforms is altered and by applying the inverse PFFT to Eq. (29) gives:

$$\mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})]=\frac{k_{0}}{2\pi}\int_{-\infty}^{\infty}\cos(\mathfrak{u}\omega)\odot\frac{q^{\varrho-1}}{q^{\varrho}+p\omega^{2}}d\omega. \tag{32}$$

Using the results mentioned in Povstenko (2015), Eq. (32) is equivalent to:

$$\mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})]=\frac{k_{0}}{2\sqrt{p}}q^{\frac{\varrho}{2}-1}exp\left(-\frac{|\mathfrak{u}|}{\sqrt{p}}q^{\frac{\varrho}{2}}\right). \tag{33}$$

The inverse PFLT applied to Eq. (33) provides:

$$\Lambda(\mathfrak{u},\mathfrak{h})=\frac{k_{0}}{2\sqrt{p}}\mathscr{L}^{-1}\left[q^{\frac{\varrho}{2}-1}exp\left(-\frac{|\mathfrak{u}|}{\sqrt{p}}q^{\frac{\varrho}{2}}\right)\right]. \tag{34}$$

Since, $\mathscr{L}^{-1}\left[q^{\varrho-1}exp(-pq^{\varrho})\right]=\frac{1}{\mathfrak{h}^{\varrho}}M\left(\varrho;\frac{p}{\mathfrak{h}^{\varrho}}\right)$ (Povstenko 2015). Therefore, we conclude that:

$$\Lambda(\mathfrak{u},\mathfrak{h})=\frac{k_{0}}{2\sqrt{p}\mathfrak{h}^{\frac{\varrho}{2}}}M\left(\frac{\varrho}{2};\frac{|\mathfrak{u}|}{\sqrt{p}\mathfrak{h}^{\frac{\varrho}{2}}}\right). \tag{35}$$

Thus the desired result is obtained. □

5.1 Examples

In this section, we use the Theorem 8 and Theorem 9 as an application to solve two PFPFDEs by considering the types of their derivatives. We evaluate the Pythagorean fuzzy solution using the Pythagorean fuzzy integral transforms.

Example 2 Consider the PFPFDE:

$$\frac{\partial^{\frac{3}{2}} \Lambda(\mathfrak{u},\mathfrak{h})}{\partial \mathfrak{h}^{\frac{3}{2}}} = 7\frac{\partial^2 \Lambda(\mathfrak{u},\mathfrak{h})}{\partial \mathfrak{u}^2}, \quad \mathfrak{h} > 0, \quad \mathfrak{u} \in (-\infty, \infty), \tag{36}$$

with the Pythagorean fuzzy initial conditions:

$$\begin{cases} \Lambda(\mathfrak{u},0) = (3.5,4,7.5;2.8,4,8.6), \\ \quad \Lambda_{\mathfrak{h}}(\mathfrak{u},0) = 0. \end{cases} \tag{37}$$

Suppose that $\Lambda(\mathfrak{u},\mathfrak{h})$ and $\frac{\partial \Lambda(\mathfrak{u},\mathfrak{h})}{\partial \mathfrak{h}}$ are $[(\imath) - gH]$-differentiable with respect to $\mathfrak{h}$. Applying the PFLT with respect to $\mathfrak{h}$ to Eq. (36) gives:

$$\mathscr{L}_{\mathfrak{h}}\left[\frac{\partial^{\frac{3}{2}} \Lambda(\mathfrak{u},\mathfrak{h})}{\partial \mathfrak{h}^{\frac{3}{2}}}\right] = 7\mathscr{L}_{\mathfrak{h}}\left[\frac{\partial^2 \Lambda(\mathfrak{u},\mathfrak{h})}{\partial \mathfrak{u}^2}\right]. \tag{38}$$

Now, using Theorem 6 and Pythagorean fuzzy initial conditions (37), we get:

$$\begin{aligned} &q^{\frac{3}{2}}\mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})] \ominus q^{\frac{3}{2}-1}(3.5,4,7.5;2.8,4,8.6) \\ &\quad = 7\frac{\partial^2}{\partial \mathfrak{u}^2}\mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})]. \end{aligned} \tag{39}$$

Applying the PFFT with respect to $\mathfrak{u}$ to Eq. (39) and then Theorem 7 and Lemma 2 implies that:

$$\begin{aligned} &q^{\frac{3}{2}}\tilde{\Lambda}(\omega,q) \ominus q^{\frac{3}{2}-1}(3.5,4,7.5;2.8,4,8.6) \\ &\odot (2\pi)\delta(\omega) \\ &\quad = \ominus_{gH} 7\omega^2\tilde{\Lambda}(\omega,q), \\ q^{\frac{3}{2}}\tilde{\Lambda}(\omega,q) \oplus \quad &7\omega^2\tilde{\Lambda}(\omega,q) = q^{\frac{3}{2}-1}(3.5,4,7.5;2.8,4,8.6) \\ &\odot (2\pi)\delta(\omega) \end{aligned}$$

and

$$\tilde{\Lambda}(\omega,q) = (3.5,4,7.5;2.8,4,8.6) \odot (2\pi)\delta(\omega) \odot \frac{q^{\frac{3}{2}-1}}{q^{\frac{3}{2}} + 7\omega^2}. \tag{40}$$

The inverse PFLT applied to Eq. (40) implies that:

$$\begin{aligned} &\mathfrak{F}_{\mathfrak{u}}[\Lambda(\mathfrak{u},\mathfrak{h})] \\ &\quad = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \\ &\frac{e^{q\mathfrak{h}}(3.5,4,7.5;2.8,4,8.6) \odot (2\pi)\delta(\omega) \odot q^{\frac{3}{2}-1}}{q^{\frac{3}{2}} + 7\omega^2}dq. \end{aligned} \tag{41}$$

Using the Laplace transform of the Mittag-Leffler function (Povstenko 2015), we conclude that:

$$\begin{aligned} \mathfrak{F}_{\mathfrak{u}}[\Lambda(\mathfrak{u},\mathfrak{h})] &= (3.5,4,7.5;2.8,4,8.6) \\ &\odot (2\pi)\delta(\omega) \odot E_{\frac{3}{2}}(-7\omega^2\mathfrak{h}^{\frac{3}{2}}). \end{aligned} \tag{42}$$

Taking the inverse PFFT to Eq. (42) gives the solution of given PFPFDE.

$$\begin{aligned} &\Lambda(\mathfrak{u},\mathfrak{h}) \\ &\quad = \frac{1}{2\pi} \odot \int_{-\infty}^{\infty}(3.5,4,7.5;2.8,4,8.6) \\ &\odot (2\pi)\delta(\omega) \odot E_{\frac{3}{2}}(-7\omega^2\mathfrak{h}^{\frac{3}{2}})e^{-i\omega\mathfrak{u}}d\omega. \end{aligned}$$

Example 3 Consider the PFPFDE with initial conditions:

$$\frac{\partial^{\frac{1}{4}} \Lambda(\mathfrak{u},\mathfrak{h})}{\partial \mathfrak{h}^{\frac{1}{4}}} = 15\frac{\partial^2 \Lambda(\mathfrak{u},\mathfrak{h})}{\partial \mathfrak{u}^2}, \quad \mathfrak{h} > 0, \quad \mathfrak{u} \in (-\infty, \infty) \tag{43}$$

and

$$\begin{cases} \Lambda(\mathfrak{u},0) = (14.3,19.5,23.9;12.8,19.5,25.7)\delta(\mathfrak{u}), \\ \Lambda_{\mathfrak{h}}(\mathfrak{u},0) = 0, \end{cases} \tag{44}$$

where $\delta(\mathfrak{u})$ is a real Dirac delta function. Let $\Lambda(\mathfrak{u},\mathfrak{h})$ and $\frac{\partial \Lambda(\mathfrak{u},\mathfrak{h})}{\partial \mathfrak{h}}$ be $[(\imath) - gH]$-differentiable with respect to $\mathfrak{h}$. Assume that $k_0 = (14.3,19.5,23.9;12.8,19.5,25.7)$ in Theorem 9

$$\tilde{\Lambda}(\omega,q) = (14.3,19.5,23.9;12.8,19.5,25.7) \odot \frac{q^{\frac{1}{4}-1}}{q^{\frac{1}{4}} + 15\omega^2}. \tag{45}$$

Applying the inverse PFFT to Eq. (45) gives:

$$\begin{aligned} \mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})] &= \frac{(14.3,19.5,23.9;12.8,19.5,25.7)}{2\pi} \\ &\int_{-\infty}^{\infty}\cos(\mathfrak{u}\omega) \odot \frac{q^{\frac{1}{4}-1}}{q^{\frac{1}{4}} + 15\omega^2}d\omega. \end{aligned} \tag{46}$$

Using the results mentioned in Povstenko (2015), Eq. (46) is equivalent to:

$$\begin{aligned} &\mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})] \\ &\quad = \frac{(14.3,19.5,23.9;12.8,19.5,25.7)}{2\sqrt{15}}q^{\frac{1}{8}-1} \\ &exp\left(-\frac{|\mathfrak{u}|}{\sqrt{15}}q^{\frac{1}{8}}\right). \end{aligned} \tag{47}$$

The inverse PFLT applied to Eq. (47) provides:

$$\begin{aligned} \Lambda(\mathfrak{u},\mathfrak{h}) &= \frac{(14.3,19.5,23.9;12.8,19.5,25.7)}{2\sqrt{15}}\mathscr{L}^{-1} \\ &\left[q^{\frac{1}{8}-1}exp\left(-\frac{|\mathfrak{u}|}{\sqrt{15}}q^{\frac{1}{8}}\right)\right]. \end{aligned} \tag{48}$$

Since, $\mathscr{L}^{-1}\left[q^{\varrho-1}exp\left(-pq^{\varrho}\right)\right] = \frac{1}{\mathfrak{h}^{\varrho}}M\left(\varrho;\frac{p}{\mathfrak{h}^{\varrho}}\right)$ (Povstenko 2015). Therefore, we conclude that:

$$\Lambda(\mathfrak{u},\mathfrak{h}) = \frac{(14.3, 19.5, 23.9; 12.8, 19.5, 25.7)}{2\sqrt{15}\mathfrak{h}^{\frac{1}{8}}} M\left(\frac{1}{8}; \frac{|\mathfrak{u}|}{\sqrt{15}\mathfrak{h}^{\frac{1}{8}}}\right). \tag{49}$$

This is the required solution of given PFPFDE.

6 Application

In Sect. 5, we discussed the method to solve PFPFDE in detail. Now, we turn our attention to a recent real-life application of PFPFDE in COVID-19 vaccination. Namazi and Kulish (2015) considered the effectiveness of an anti-cancer medicine in the tumor. Keshavarz and Allahviranloo (2022), Keshavarz et al. (2022) discussed the diffusion of anti-cancer drug by fuzzy integral transform.

In our study, we develop a partial fractional diffusion equation to demonstrate the diffusion and effectiveness of different types of COVID-19 vaccination in the human body. The vaccine is delivered to the human body by an intramuscular injection. For this problem, we have the following PFPFDE, if the diffusion process is considered as fractals (such as random walk).

$$\frac{\partial^{2\mathcal{H}}\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{h}^{2\mathcal{H}}} = \mathcal{C}^{2(2\mathcal{H}-1)}\mathcal{D}^{2(1-\mathcal{H})}\frac{\partial^2\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^2}, \quad \mathfrak{h}>0, \quad 0<\mathcal{H}\leq 1. \tag{50}$$

The diffusion coefficient $\mathcal{D}$ indicates the diffusivity of the vaccine in the human body, $\mathcal{C}$ denotes the speed of the vaccine deliver to human body to resist the COVID-19 virus attack, $\mathcal{H}$ is the Hurst exponent with the condition $0<\mathcal{H}\leq 1$.

The effectiveness and dose of the vaccine vary from person to person depending upon the medical history and age. These factors indicate uncertainty and vagueness to the problem, and this uncertainty also affects the initial conditions. Under this consideration, a Pythagorean fuzzy model is more appropriate with all factors and can demonstrate the issues in a better and meaningful way.

Consider the initial conditions for our model expressed by Eq. (50):

$$\begin{cases} \Lambda(\mathfrak{u},0) = k_0\delta(\mathfrak{u}), & 0<\varrho\leq 2, \\ \Lambda_{\mathfrak{h}}(\mathfrak{u},0) = 0, & 1<\varrho\leq 2, \end{cases} \tag{51}$$

where k_0 is a TPFN and $\delta(\mathfrak{u})$ is a real Dirac delta function.

The solution of our proposed model (50) with the Pythagorean fuzzy initial conditions (51) is evaluated using Theorem 8 and Theorem 9 as under:

Let $\Lambda(\mathfrak{u},\mathfrak{h})$ and $\frac{\partial\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{h}}$ be $[(\iota)-gH]$-differentiable with respect to $\mathfrak{h}$. Taking PFLT with respect to $\mathfrak{h}$ to Eq. (50) gives:

$$\mathscr{L}_{\mathfrak{h}}\left[\frac{\partial^{2\mathcal{H}}\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{h}^{2\mathcal{H}}}\right] = \mathcal{C}^{2(2\mathcal{H}-1)}\mathcal{D}^{2(1-\mathcal{H})}\mathscr{L}_{\mathfrak{h}}\left[\frac{\partial^2\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^2}\right]. \tag{52}$$

Using Theorem 6 and initial conditions (51), we have:

$$q^{2\mathcal{H}}\mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})] - q^{2\mathcal{H}-1}k_0\delta(\mathfrak{u}) = \mathcal{C}^{2(2\mathcal{H}-1)}\mathcal{D}^{2(1-\mathcal{H})}\frac{\partial^2}{\partial\mathfrak{u}^2}\mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})]. \tag{53}$$

The PFFT applied to Eq. (53) provides:

$$q^{2\mathcal{H}}\tilde{\Lambda}(\omega,q) - q^{2\mathcal{H}-1}k_0 = \ominus_{gH}\mathcal{C}^{2(2\mathcal{H}-1)}\mathcal{D}^{2(1-\mathcal{H})}\omega^2\tilde{\Lambda}(\omega,q), \tag{54}$$

or

$$\tilde{\Lambda}(\omega,q) = k_0 \odot \frac{q^{2\mathcal{H}-1}}{q^{2\mathcal{H}} + \mathcal{C}^{2(2\mathcal{H}-1)}\mathcal{D}^{2(1-\mathcal{H})}\omega^2}. \tag{55}$$

Now, we change the order of inverse Pythagorean fuzzy integral transform; therefore, we first take the inverse PFFT to Eq. (55) that results:

$$\mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})] = \frac{k_0}{2\pi}\odot\int_{-\infty}^{\infty}\cos(\mathfrak{u}\omega)\odot\frac{q^{2\mathcal{H}-1}}{q^{2\mathcal{H}} + \mathcal{C}^{2(2\mathcal{H}-1)}\mathcal{D}^{2(1-\mathcal{H})}}d\omega. \tag{56}$$

Using the results mentioned in Povstenko (2015), Eq. (56) is equivalent to:

$$\mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})] = \frac{k_0}{2\mathcal{C}^{(2\mathcal{H}-1)}\mathcal{D}^{(1-\mathcal{H})}}\odot q^{\mathcal{H}-1}exp\left(-\frac{|\mathfrak{u}|}{\mathcal{C}^{(2\mathcal{H}-1)}\mathcal{D}^{(1-\mathcal{H})}}q^{\mathcal{H}}\right). \tag{57}$$

Now, we take the inverse PFLT to Eq. (57)

$$\Lambda(\mathfrak{u},\mathfrak{h}) = \frac{k_0}{2\mathcal{C}^{(2\mathcal{H}-1)}\mathcal{D}^{(1-\mathcal{H})}}\odot \mathscr{L}^{-1}\left[q^{\mathcal{H}-1}exp\left(-\frac{|\mathfrak{u}|}{\mathcal{C}^{(2\mathcal{H}-1)}\mathcal{D}^{(1-\mathcal{H})}}q^{\mathcal{H}}\right)\right] \tag{58}$$

Since $\mathscr{L}^{-1}\left[q^{\varrho-1}exp\left(-pq^{\varrho}\right)\right] = \frac{1}{\mathfrak{h}^{\varrho}}M\left(\varrho;\frac{p}{\mathfrak{h}^{\varrho}}\right)$ (Povstenko 2015). Therefore, we have

$$\Lambda(\mathfrak{u},\mathfrak{h}) = \frac{k_0}{2\mathcal{C}^{(2\mathcal{H}-1)}\mathcal{D}^{(1-\mathcal{H})}\mathfrak{h}^{\mathcal{H}}}\odot M\left(\mathcal{H};\frac{|\mathfrak{u}|}{\mathcal{C}^{(2\mathcal{H}-1)}\mathcal{D}^{(1-\mathcal{H})}\mathfrak{h}^{\mathcal{H}}}\right). \tag{59}$$

This is the solution of our proposed PFPFDE for COVID-19 vaccination.

We now discuss about two main types of COVID-19 vaccination, namely, AstraZeneca and Sinovac. AstraZeneca is a British manufactured vaccine, while Sinovac is Chinese vaccine. These both vaccines are very effective to resist the attack of COVID-19 virus.

Example 4 AstraZeneca is a British manufactured vaccine that is delivered to the human body through an intramuscular injection twice with a gap in first and second doses. The dosage of this vaccine is based on the medical history and age of a person. The final vaccine product in each $0.5ml$ dose is comprised of $3\mu g$ of Chimpanzee adenovirus. The recommended dose of a person of age 18 and above is $(2.95, 3, 4.52; 2.94, 3, 6)\mu g/ml$. World Health Organization (WHO) suggests an interval of $8-12$ weeks between the first and second doses.

Consider the PFPFDE

$$\frac{\partial \Lambda(\mathfrak{u},\mathfrak{h})}{\partial \mathfrak{h}} = \frac{\partial^2 \Lambda(\mathfrak{u},\mathfrak{h})}{\partial \mathfrak{u}^2}, \quad \mathfrak{h} > 0, \tag{60}$$

with the Pythagorean fuzzy initial condition

$$\begin{cases} \Lambda(\mathfrak{u},0) = (2.95,3,4.52;2.94,3,6) \odot \delta(\mathfrak{u}), \\ \qquad \Lambda_{\mathfrak{h}}(\mathfrak{u},0) = 0, \end{cases} \tag{61}$$

Let $\Lambda(\mathfrak{u},\mathfrak{h})$ and $\frac{\partial \Lambda(\mathfrak{u},\mathfrak{h})}{\partial \mathfrak{h}}$ be $[(\imath)-gH]$-differentiable with respect to $\mathfrak{h}$. Taking PFLT with respect to $\mathfrak{h}$ to Eq. (60) gives:

$$\mathscr{L}_{\mathfrak{h}}\left[\frac{\partial \Lambda(\mathfrak{u},\mathfrak{h})}{\partial \mathfrak{h}}\right] = \mathscr{L}_{\mathfrak{h}}\left[\frac{\partial^2 \Lambda(\mathfrak{u},\mathfrak{h})}{\partial \mathfrak{u}^2}\right]. \tag{62}$$

Using Theorem 6 and initial conditions (61), we have:

$$\begin{aligned} &q\mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})] - (2.95,3,4.52;2.94,3,6)\delta(\mathfrak{u}) \\ &\quad = \frac{\partial^2}{\partial \mathfrak{u}^2}\mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})]. \end{aligned} \tag{63}$$

The PFFT applied to Eq. (63) provides:

$$q\tilde{\Lambda}(\omega,q) - q(2.95,3,4.52;2.94,3,6) = \ominus_{gH}\omega^2\tilde{\Lambda}(\omega,q), \tag{64}$$

or

$$\tilde{\Lambda}(\omega,q) = (2.95,3,4.52;2.94,3,6) \odot \frac{1}{q} + \omega^2. \tag{65}$$

Now, we change the order of inverse Pythagorean fuzzy integral transform; therefore, we first take the inverse PFFT to Eq. (65) that results:

$$\begin{aligned} &\mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})] \\ &\quad = \frac{(2.95,3,4.52;2.94,3,6)}{2\pi} \odot \int_{-\infty}^{\infty} \cos(\mathfrak{u}\omega) \odot \frac{1}{q+1} d\omega. \end{aligned} \tag{66}$$

Using the results mentioned in Povstenko (2015), Eq. (66) is equivalent to:

$$\begin{aligned} \mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})] = &\frac{(2.95,3,4.52;2.94,3,6)}{2} \\ &\odot q^{-\frac{1}{2}} exp\left(-|\mathfrak{u}|q^{\frac{1}{2}}\right). \end{aligned} \tag{67}$$

Now, we take the inverse PFLT to Eq. (67)

$$\begin{aligned} \Lambda(\mathfrak{u},\mathfrak{h}) = &\frac{(2.95,3,4.52;2.94,3,6)}{2} \\ &\odot \mathscr{L}^{-1}\left[q^{-\frac{1}{2}} exp\left(-|\mathfrak{u}|q^{\frac{1}{2}}\right)\right]. \end{aligned} \tag{68}$$

Since $\mathscr{L}^{-1}\left[q^{\varrho-1}exp\left(-pq^{\varrho}\right)\right] = \frac{1}{\mathfrak{h}^{\varrho}}M\left(\varrho;\frac{p}{\mathfrak{h}^{\varrho}}\right)$ (Povstenko 2015). Therefore, we have

$$\Lambda(\mathfrak{u},\mathfrak{h}) = \frac{(2.95,3,4.52;2.94,3,6)}{2\mathfrak{h}^{\frac{1}{2}}} \odot M\left(\frac{1}{2};\frac{|\mathfrak{u}|}{\mathfrak{h}^{\frac{1}{2}}}\right). \tag{69}$$

Furthermore, $M\left(\frac{1}{2};z\right) = \frac{1}{\pi}\exp\left(\frac{-z^2}{4}\right)$ (Povstenko 2015), thus Eq. (69) takes the following form:

$$\Lambda(\mathfrak{u},\mathfrak{h}) = \frac{(2.95,3,4.52;2.94,3,6)}{2\pi\mathfrak{h}^{\frac{1}{2}}} \odot \exp\left(\frac{-|\mathfrak{u}|^2}{4\mathfrak{h}}\right). \tag{70}$$

$\Lambda(\mathfrak{u},\mathfrak{h})$ indicates the diffusion of AstraZeneca vaccine in a person.

The graphical representation for the diffusion of the Astrazeneca vaccine in the human body for $\mathfrak{u}=1$ and for different values of $\mathfrak{h}$ and r is illustrated in Figs. 1, 2, and 3. The r-cut representation of the lower and upper FVFs for the solution $\Lambda(\mathfrak{u},\mathfrak{h})$ is expressed by Fig. 1. Similarly, the lower and upper PFVFs for the Pythagorean fuzzy part of the solution $\Lambda(\mathfrak{u},\mathfrak{h})$ is viewed by Fig. 2. Finally, the complete solution of the Problem (69) is illustrated in Fig. 3.

Example 5 The Sinovac vaccine is prepared in Chinese laboratories. This vaccine is delivered to the human body by an intramuscular injection two times having a gap between first and second doses. WHO prescribed a gap of $2-4$ weeks between the both doses. The dose of Sinovac vaccine is recommended to adults of age 18 and above by WHO. The final vaccine product is composed of $3\mu g$ of

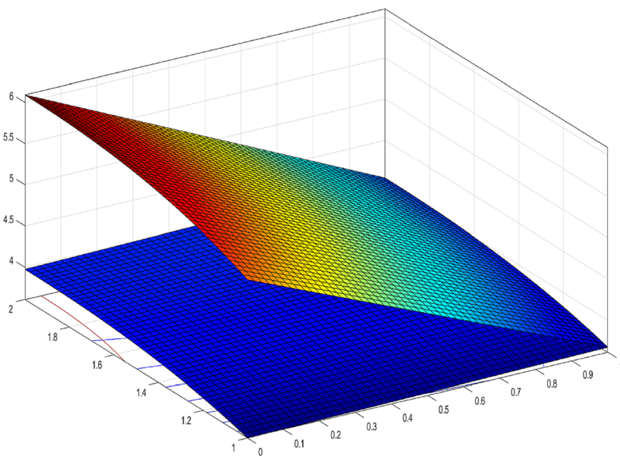

Fig. 1 r-cut representation of the fuzzy part of the solution (70)

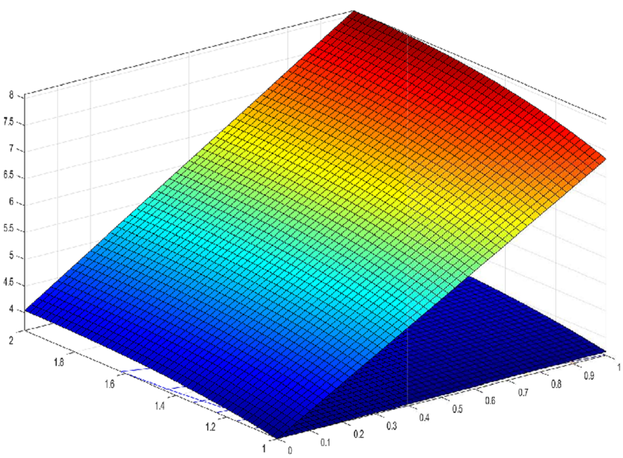

Fig. 2 r-cut representation of the Pythagorean fuzzy part of the solution (70)

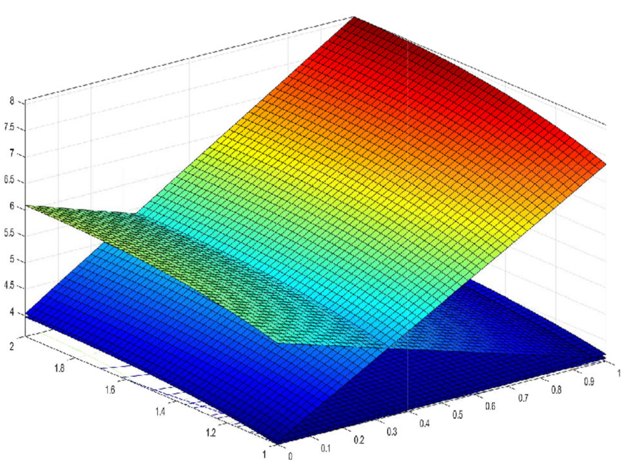

Fig. 3 r-cut representation for the solution of the diffusion of Astrazeneca vaccine

inactivated SARS-CoV-2 virus in each $0.5ml$ dose. Let $\Lambda(\mathfrak{u},\mathfrak{h})$ be the diffusion of Sinovac vaccine in the human body for the adults of age 18 and above with the dose $(2.85,3,5.2;2.79,3,6)\mu g/ml$ twice.

Consider PFPFDE

$$\frac{\partial^{\frac{2}{3}}\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{h}^{\frac{2}{3}}}=\frac{\partial^2\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^2},\quad \mathfrak{h}>0, \tag{71}$$

with the initial condition

$$\begin{cases}\Lambda(\mathfrak{u},0)=(2.85,3,5.2;2.79,3,6)\delta(\mathfrak{u}),\\ \Lambda_{\mathfrak{h}}(\mathfrak{u},0)=0,\end{cases}. \tag{72}$$

The solution of our proposed model (71) with the Pythagorean fuzzy initial conditions (72) is evaluated using Theorem 8 and Theorem 9 as under:

Let $\Lambda(\mathfrak{u},\mathfrak{h})$ and $\frac{\partial\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{h}}$ be $[(\iota)-gH]$-differentiable with respect to $\mathfrak{h}$. Taking PFLT with respect to $\mathfrak{h}$ to Eq. (71) gives:

$$\mathscr{L}_{\mathfrak{h}}\left[\frac{\partial^{\frac{2}{3}}\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{h}^{\frac{2}{3}}}\right]=\mathscr{L}_{\mathfrak{h}}\left[\frac{\partial^2\Lambda(\mathfrak{u},\mathfrak{h})}{\partial\mathfrak{u}^2}\right]. \tag{73}$$

Using Theorem 6 and initial conditions (72), we have:

$$\begin{aligned}&q^{\frac{2}{3}}\mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})]-q^{\frac{2}{3}-1}(2.85,3,5.2;2.79,3,6)\delta(\mathfrak{u})\\&\quad=\frac{\partial^2}{\partial\mathfrak{u}^2}\mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})].\end{aligned} \tag{74}$$

The PFFT applied to Eq. (74) provides:

$$q^{\frac{2}{3}}\tilde{\Lambda}(\omega,q)-q^{\frac{2}{3}-1}(2.85,3,5.2;2.79,3,6)=\ominus_{gH}\omega^2\tilde{\Lambda}(\omega,q), \tag{75}$$

or

$$\tilde{\Lambda}(\omega,q)=(2.85,3,5.2;2.79,3,6)\odot\frac{q^{\frac{2}{3}-1}}{q^{\frac{2}{3}}+\omega^2}. \tag{76}$$

Now, we change the order of inverse Pythagorean fuzzy integral transform; therefore, we first take the inverse PFFT to Eq. (76) that results:

$$\begin{aligned}\mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})]&=\frac{(2.85,3,5.2;2.79,3,6)}{2\pi}\odot\\&\int_{-\infty}^{\infty}\cos(\mathfrak{u}\omega)\odot\frac{q^{\frac{2}{3}-1}}{q^{\frac{2}{3}}+1}d\omega.\end{aligned} \tag{77}$$

Using the results mentioned in Povstenko (2015), Eq. (77) is equivalent to:

$$\begin{aligned}\mathscr{L}_{\mathfrak{h}}[\Lambda(\mathfrak{u},\mathfrak{h})]&=\frac{(2.85,3,5.2;2.79,3,6)}{2}\odot\\&q^{\frac{1}{3}-1}exp\left(-|\mathfrak{u}|q^{\frac{1}{3}}\right).\end{aligned} \tag{78}$$

Now, we take the inverse PFLT to Eq. (78)

$$\mathit{\Lambda}(\mathfrak{u},\mathfrak{h}) = \frac{(2.85,3,5.2;2.79,3,6)}{2} \odot \mathscr{L}^{-1}\left[q^{\frac{1}{3}-1} exp\left(-|\mathfrak{u}|q^{\frac{1}{3}}\right)\right] \tag{79}$$

Since $\mathscr{L}^{-1}\left[q^{\varrho-1} exp\left(-pq^{\varrho}\right)\right] = \frac{1}{\mathfrak{h}^{\varrho}} M\left(\varrho;\frac{p}{\mathfrak{h}^{\varrho}}\right)$ (Povstenko 2015). Therefore, we have

$$\mathit{\Lambda}(\mathfrak{u},\mathfrak{h}) = \frac{(2.85,3,5.2;2.79,3,6)}{2\mathfrak{h}^{\frac{1}{3}}} \odot M\left(\frac{1}{3};\frac{|\mathfrak{u}|}{\mathfrak{h}^{\frac{1}{3}}}\right). \tag{80}$$

But $M\left(\frac{1}{3};z\right) = 3^{\frac{2}{3}}\mathrm{Ai}\left(\frac{z}{3^{\frac{1}{3}}}\right)$, where $\mathrm{Ai}(z)$ is the Airy function (Povstenko 2015), thus Eq. (80) is equivalent to:

$$\mathit{\Lambda}(\mathfrak{u},\mathfrak{h}) = \frac{3^{\frac{2}{3}}(2.85,3,5.2;2.79,3,6)}{2\mathfrak{h}^{\frac{1}{3}}} \odot \mathrm{Ai}\left(\frac{|\mathfrak{u}|}{3^{\frac{1}{3}}\mathfrak{h}^{\frac{1}{3}}}\right). \tag{81}$$

$\mathit{\Lambda}(\mathfrak{u},\mathfrak{h})$ indicates the diffusion of Sinovac vaccine in the human body.

7 Conclusions

Fractional calculus is an important area of mathematical analysis. This is a generalization of the usual calculus that allows for non-integer order. It has become the focus of attention of mathematicians, physicists and engineers. A well-known and widely used method for solving fractional differential equations is Caputo fractional differentiation. This allows you to specify the number of integer derivatives at the initial point. This quantity is usually available and measurable. The concept of PFSs is a relatively new mathematical framework for fuzzy family and has a strong ability to deal with inaccuracies. In this paper, we have proposed a new analytical method using the PFLT and the PFFT. To this end, we have established the Riemann-Liouville fractional integral, the Riemann-Liouville fractional derivative, and the Caputo fractional derivative of the TPFVF. Furthermore, we have presented some basic results for the TPFVF integral. Furthermore, we have introduced the Fourier transform in the Pythagorean fuzzy environment. Finally, we have developed a Pythagorean fuzzy partial fractional diffusion model to study the diffusion of the COVID-19 vaccine in humans. Furthermore, we obtained the Pythagorean fuzzy solution of the proposed model under the generalized Hukuhara Caputo fractional partial differential using the PFLT and the PFFT. In this methodology, we have used the PFLT under generalized Hukuhara Caputo partial fractional differentiability and the PFFT under [gH - p]-differentiability. To this end, we established and proved some fundamental theorems of the PFLT and the PFFT under [gH - p]-differentiability. The validity of the introduced method has been examined by solving some relevant examples. Our proposed method is worthwhile and important for solving a PFFDE of order $0<\varrho\leq 1$ and PFPFDE of order $0<\varrho\leq 2$. Therefore, it is necessary to further examine this technique to extract the Pythagorean fuzzy solutions for such practical applications.

Data availability No data were used to support this study.

Declarations

Conflict of interest The authors declare no conflicts of interest.

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